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Solving Thurston's equation in a commutative ring

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Abstract

We show that solutions of Thurston's equation on triangulated 3-manifolds in a commutative ring carry topological information. We also introduce a homogeneous Thurston's equation and a commutative ring associated to triangulated 3-manifolds. It is shown that solutions to homogeneous equations over the real numbers are critical points of an entropy function.

1. *Introduction*

1.1. *Statement of results*

Given a triangulated oriented 3-manifold (or pseudo 3-manifold) $(M, \mathcal{T})$, Thurston's equation associated to $\mathcal{T}$ is a system of integer coefficient polynomials. Thurston [**16**] introduced his equation in the field $\mathbf{C}$ of complex numbers in order to find hyperbolic structures. Since then, there has been much research on solving Thurston's equation in $\mathbf{C}$, [**1**, **3**, **13**, **14**, **17**, **19**] and others. Since the equations are integer coefficient polynomials, one could attempt to solve Thurston's equation in a ring with identity. The purpose of this paper is to show that interesting topological results about the 3-manifolds can be obtained by solving Thurston's equation in a commutative ring with identity. For instance, by solving Thurston's equation in the field $\mathbf{Z}/3\mathbf{Z}$ of three elements, one obtains a result that was known to H. Rubinstein and S. Tillmann that a closed 1-vertex triangulated 3-manifold is not simply connected if each edge has even degree.

Theorem 1.1. *Suppose that $(M, \mathcal{T})$ is an oriented connected closed 3-manifold with a triangulation $\mathcal{T}$ and R is a commutative ring with identity. If Thurston's equation on $(M, \mathcal{T})$ is solvable in R, and $\mathcal{T}$ contains an edge which is a loop, then there exists a homomorphism from $\pi_1(M)$ to $\mathrm{PGL}(2, R)$ sending the loop to a non-identity element. In particular, M is not simply connected.*

We remark that the existence of an edge which is a loop cannot be dropped in the theorem. Indeed, it was observed in [**8**, **17**, **18**] that for simplicial triangulations $\mathcal{T}$ and any commutative ring R, there are always solutions to Thurston's equation. Theorem 1.1 for $R = \mathbf{C}$ was first proved by Segerman–Tillmann [**15**]. A careful examination of the proof of [**15**] shows that their method also works for any field R. However, for a commutative ring with zero divisors, the geometric argument breaks down. We prove Theorem 1.1 by introducing a homogeneous Thurston's equation (HTE) and studying its solutions. Theorem 1.1 prompts us to introduce the universal construction of a Thurston ring of a triangulated 3-manifold. Theorem 1.1 can be phrased in terms of the universal construction (see Theorem 6.2).

Thurston's equation can be defined for any ring (not necessary commutative) with identity (see § 2). We do not know if Theorem 1.1 holds in this case. The most interesting non-commutative rings for 3-manifolds are probably the algebras of 2×2 matrices with real or

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complex coefficients. Solving Thurston's equation in the algebra $M_{2\times 2}(\mathbf{R})$ has the advantage of linking hyperbolic geometry to $Ad(S^3)$ geometry. See [**4**] for related topics.

Motivated by Theorem 1.1, we propose the following two conjectures.

CONJECTURE 1. If M is a compact 3-manifold and $\gamma \in \pi_1(M) - \{1\}$, then there exists a finite commutative ring R with identity and a homomorphism from $\pi_1(M)$ to $\mathrm{PGL}(2, R)$ sending γ to a non-identity element.

CONJECTURE 2. If $M \neq S^3$ is a closed oriented 3-manifold, then there exists a 1-vertex triangulation $\mathcal{T}$ of M and a commutative ring R with identity so that Thurston's equation associated to $\mathcal{T}$ is solvable in R.

Conjecture 2 is supported by the main result in [**12**]. It states that if M is closed hyperbolic and $\mathcal{T}$ is a 1-vertex triangulation so that all edges are homotopically essential, then Thurston's equation on $\mathcal{T}$ is solvable in $\mathbf{C}$.

1.2. *Organization of the paper*

In § 2, we recall briefly triangulations of 3-manifolds and pseudo 3-manifolds and Thurston's equation. A HTE is introduced. In § 3, we recall the cross-ratio and the projective plane in a commutative ring. Theorem 1.1 is proved in § 4. In § 5, we show that solutions of Thurston's equation over the real numbers are related to critical points of an entropy function. In § 6, we introduce a universal construction of the Thurston ring of a triangulated 3-manifold. In the last section, some examples of solutions of Thurston's equation in finite rings are worked out.

2. *Preliminaries on triangulations and Thurston's equation*

All manifolds and tetrahedra are assumed to be oriented in this paper. We assume that all rings have the identity element.

2.1. *Triangulations*

A compact oriented triangulated pseudo 3-manifold $(M, \mathcal{T})$ consists of a disjoint union $X = \sqcup_i \sigma_i$ of oriented Euclidean tetrahedra σ_i and a collection of orientation-reversing affine homeomorphisms Φ between some pairs of codimension-1 faces in X. The pseudo 3-manifold M is the quotient space X/Φ and the simplices in $\mathcal{T}$ are the quotients of simplices in X. The *boundary* ∂M of M is the quotient of the union of unidentified codimension-1 faces in X. If $\partial M = \emptyset$, then we call M *closed*. The sets of all quadrilateral types (to be called *quads* for simplicity) and normal triangle types in $\mathcal{T}$ will be denoted by $\square = \square(\mathcal{T})$, and $\triangle = \triangle(\mathcal{T})$, respectively. See [**6**, **7**] or [**11**] for more details. The most important combinatorics ingredient in defining Thurston's equation is a $\mathbf{Z}/3\mathbf{Z}$ action on $\square$, which we recall now. If edges of an oriented tetrahedron σ are labelled by a, b, c so that opposite edges are labelled by the same letters (see Figure 1(a)), then the cyclic order $a \to b \to c \to a$ viewed at vertices is independent of the choice of the outward normal vectors at vertices and depends only on the orientation of σ. Since a quad in σ corresponds to a pair of opposite edges, this shows that there is a $\mathbf{Z}/3\mathbf{Z}$ action on the set of all quads in σ by cyclic permutations. If $q, q' \in \square$, then we use $q \to q'$ to indicate that q, q' are in the same tetrahedron so that q is ahead of q' in the cyclic order. The set of all i-simplices in $\mathcal{T}$ will be denoted by $\mathcal{T}^{(i)}$. If $s \in \mathcal{T}^{(i)}$ and $t \in \mathcal{T}^{(j)}$, then we use $t > s$ or $s < t$ to indicate that s is a face of t. Given an edge $e \in \mathcal{T}^{(1)}$, a tetrahedron $\sigma \in \mathcal{T}^{(3)}$ and a quad $q \in \square(\mathcal{T})$, we use $q \subset \sigma$ to indicate that q is inside σ and $q \sim e$ to indicate that q and e are in the same tetrahedron and q faces e (in the unidentified space X). See [**11**] for more

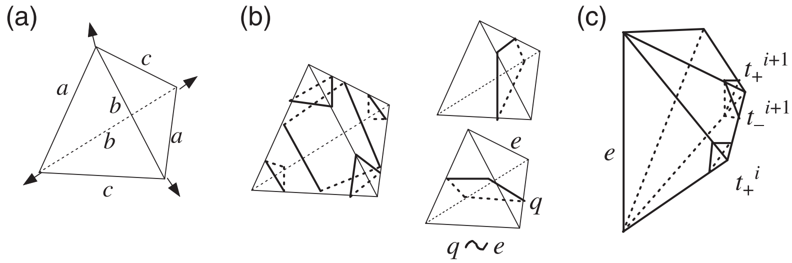

FIGURE 1. *Cyclic order on three quads in a tetrahedron.*

details. An edge $e \in \mathcal{T}^{(1)}$ is called *interior* if it is not in the boundary ∂M. In particular, if $(M, \mathcal{T})$ is a closed triangulated pseudo 3-manifold, then all edges are interior.

2.2. *Thurston's equation*

DEFINITION 2.1. Given a compact triangulated oriented pseudo 3-manifold $(M, \mathcal{T})$ and a ring R (not necessary commutative) with identity, a function $x : \square(\mathcal{T}) \to R$ is called a solution to Thurston's equation associated to $\mathcal{T}$ if

(a) whenever $q \to q'$ in $\square$,

$$x(q')(1 - x(q)) = (1 - x(q))x(q') = 1, \tag{1}$$

(b) for each interior edge $e \in \mathcal{T}^{(1)}$ so that $q_1, \ldots, q_n$ are quads facing e labelled cyclically around e,

$$x(q_1) \cdots x(q_n) = 1, \quad x(q_n) \cdots x(q_1) = 1. \tag{2}$$

Note that equation (1) implies that both $x(q)$ and $x(q) - 1$ are invertible elements in R with inverses $1 - x(q'')$ and $x(q')$ where $q'' \to q$. Since each $x(q)$ is invertible, equation (2) does not depend on the choice of the initial quad q_1. If the ring R is commutative, which will be assumed from now on, then we only need one equation in each of (1) and (2).

EXAMPLE 2.2. If $R = \mathbf{Z}/3\mathbf{Z} = \{0, 1, 2\}$ is the field of three elements, then a solution x to Thurston's equation must satisfy $x(q) = 2$ for all q. In this case, the first equation (1) holds automatically. Equation (2) at an edge e becomes $2^k = 1$, where k is the degree of e. Since $2^k = 1$ if and only if k is even, we conclude that Thurston's equation has a solution in $\mathbf{Z}/3\mathbf{Z}$ if and only if each interior edge has even degree.

A related homogeneous version of Thurston's equation incorporates the following definitions.

DEFINITION 2.3 (HTE). Suppose that $(M, \mathcal{T})$ is a compact oriented triangulated pseudo 3-manifold and R is a commutative ring with identity. A function $z : \square(\mathcal{T}) \to R$ is called a solution to the HTE if

(a) for each tetrahedron $\sigma \in \mathcal{T}$,

$$\sum_{q \subset \sigma} z(q) = 0, \tag{3}$$

(b) for each interior edge e so that the set of all quads facing it is $\{q_1, \dots, q_n\}$,

$$\prod_{i=1}^{n} z(q_i') = \prod_{i=1}^{n}(-z(q_i'')),$$

where $q_i \to q_i' \to q_i''$ or simply

$$\prod_{q\sim e} z(q') = \prod_{q\sim e}(-z(q'')), \tag{4}$$

where $q \to q' \to q''$.

Note that if z solves HTE and $k : \mathcal{T}^{(3)} \to R$ is any function, then $w(q) = k(\sigma)z(q) : \square \to R$, $q \subset \sigma$, is another solution to HTE.

Lemma 2.4. *Suppose that R is a commutative ring with identity and R^* is the group of all invertible elements in R.*

(a) *If $z : \square \to R^*$ solves HTE, then $x : \square \to R$ given by $x(q) = -z(q')/z(q'')$ solves Thurston's equation.*

(b) *If $x : \square \to R$ solves Thurston's equation, then there exists a solution $z : \square \to R^*$ to HTE so that, for all $q \in \square$, $x(q) = -z(q')/z(q'')$.*

(c) *If $z : \square \to R^*$ solves HTE, then $z(q'), -z(q'')$ and $z(q') + z(q'')(= -z(q))$ are in R^*. Furthermore, $w : \square \to R^2$ defined by $w(q) = \begin{pmatrix} z(q') \\ -z(q'') \end{pmatrix} \in R^2$ satisfies*

$$w(q') = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} w(q). \tag{5}$$

Proof. To see part (a), equation (2) follows immediately from (4) by division. To see that (1) holds, we use $z(q) = -z(q'') - z(q')$ for $q \to q' \to q''$ and calculate $x(q') = -z(q'')/z(q) = z(q'')/(z(q'') + z(q')) = 1/(1 + z(q')/z(q'')) = 1/(1 - x(q))$.

To see part (b), by definition, $x(q)$ and $1 - x(q)$ are invertible. Label the quads in each tetrahedron σ by q_1, q_2, q_3 so that $q_1 \to q_2 \to q_3$. We have $x(q_2) = 1/(1 - x(q_1)), x(q_3) = (x(q_1) - 1)/x(q_1)$. Define a map $z : \square \to R^*$ by $z(q_1) = 1 - x(q_1), z(q_2) = x(q_1), z(q_3) = -1$. Then, by definition $x(q) = -z(q')/z(q'')$ for all q and $\sum_{q\subset\sigma} z(q) = 0$ for each $\sigma \in \mathcal{T}^{(3)}$. Due to $x(q) = -z(q')/z(q'')$ and $\prod_{q\sim e} x(q) = 1$, we see that $\prod_{q\sim e} z(q') = \prod_{q\sim e}(-z(q''))$.

For part (c), the first statement follows from the definition that $z(q) \in R^*$. Now, $w(q') = \begin{pmatrix} z(q'') \\ -z(q) \end{pmatrix} = \begin{pmatrix} z(q'') \\ z(q')+z(q'') \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} z(q') \\ -z(q'') \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} w(q)$. □

3. *Cross-ratio and projective line in a commutative ring*

Let R be a commutative ring with identity and R^* be the group of invertible elements in R. Let $\mathrm{GL}(2, R)$ be the general linear group and $\mathrm{PGL}(2, R) = \mathrm{GL}(2, R)/\sim$, where $M \sim \lambda M, \lambda \in R^*$, be the associated projective group. The group $\mathrm{GL}(2, R)$ acts linearly from the left on $R^2 = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid a,$
$b, \in R\}$. Define the skew symmetric bilinear form $\langle, \rangle$ on R^2 by $\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \rangle = ad - bc$ which is the determinant of $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. We use the transpose to write $\begin{pmatrix} a \\ b \end{pmatrix}$ as $(a, b)^t$.

By the basic properties of the determinant, we have the following lemma.

Lemma 3.1. *Suppose $A, B, A_1, \dots, A_n \in R^2$ and $X \in \mathrm{GL}(2, R)$. Then*

(a) *$\langle A, B\rangle = -\langle B, A\rangle$ and $\langle XA, XB\rangle = \det(X)\langle A, B\rangle$;*
(b) *$\langle A_1, A_2\rangle A_3 + \langle A_2, A_3\rangle A_1 + \langle A_3, A_1\rangle A_2 = 0$;*

(c) *let* $R_{ijkl} = \langle A_i, A_j\rangle\langle A_k, A_l\rangle$; *then* $R_{ijkl} = R_{jilk} = R_{klij} = -R_{jikl} = -R_{ijlk}$ *and* $R_{ijkl} + R_{iklj} + R_{iljk} = 0$.

DEFINITION 3.2 (Cross-ratio). Suppose $A_1, \dots, A_4 \in R^2$. Then their cross ratio, denoted by $(A_1, A_2; A_3, A_4)$, is defined to be the vector $\begin{pmatrix} R_{1423} \\ R_{1324} \end{pmatrix} = \begin{pmatrix} R_{1423} \\ -R_{1342} \end{pmatrix} \in R^2$, where $R_{ijkl} = \langle A_i, A_j\rangle\langle A_k, A_l\rangle$.

For instance,

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \begin{pmatrix} -ay \\ -bx \end{pmatrix}. \tag{6}$$

By Lemma 3.1, we obtain the following corollary.

COROLLARY 3.3. *Suppose $A_1, \dots, A_n \in R^2$ and $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Then*

(a) $(A_1, A_2; A_3, A_4) = (A_3, A_4; A_1, A_2) = (A_2, A_1; A_4, A_3)$ *and* $(A_{i_1}, A_{i_2}; A_{i_3}, A_{i_4}) = (A_1, A_i; A_j, A_k)$;
(b) *if* $(A_1, A_2; A_3, A_4) = \begin{pmatrix} a \\ b \end{pmatrix}$, *then* $(A_2, A_1; A_3, A_4) = \begin{pmatrix} b \\ a \end{pmatrix}$;
(c) $(A_1, A_2; A_3, A_4) + (A_1, A_3; A_4, A_2) + (A_1, A_4; A_2, A_3) = 0$;
(d) $(A_1, A_3; A_4, A_2) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} (A_1, A_2; A_3, A_4)$ *and*

$$(A_1, A_4; A_2, A_3) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} (A_1, A_2; A_3, A_4); \tag{7}$$

(e) *if* $X \in \mathrm{GL}(2, R)$, *then* $(XA_1, XA_2; XA_3, XA_4) = \det(X)^2 (A_1, A_2; A_3, A_4)$,
(f) *if* $B, C \in R^2$ *and* $A_{n+1} = A_1$, *then*

$$\prod_{i=1}^{n} (\langle B, A_{i+1}\rangle\langle C, A_i\rangle) = \prod_{i=1}^{n} (\langle B, A_i\rangle\langle C, A_{i+1}\rangle).$$

The only part that is required to be proved is (d). In this case, we use $(A_1, A_4; A_2, A_3) = \begin{pmatrix} R_{1342} \\ -R_{1234} \end{pmatrix} = \begin{pmatrix} R_{1342} \\ R_{1423} + R_{1342} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} (A_1, A_2; A_3, A_4)$.

Corollary 3.3 shows that the cross-ratio $(A_1, A_2; A_3, A_4) = (A_i, A_j; A_k, A_l)$ whenever $\{i, j\} = \{1, 2\}$, $\{k, j\} = \{3, 4\}$ and (i, j, k, l) is a positive permutation of $(1, 2, 3, 4)$, that is, the cross-ratio depends only on the partition $\{i, j\} \sqcup \{k, l\}$ of $\{1, 2, 3, 4\}$ and the orientation of $(1, 2, 3, 4)$. This shows that if σ is an oriented tetrahedron so that its ith vertex is assigned a vector $A_i \in R^2$, then one can define the cross-ratio of a quad $q \subset \sigma$ to be $(A_i, A_j; A_k, A_l)$ where q corresponds to the partition $\{i, j\} \sqcup \{k, l\}$ of the vertex set $\{1, 2, 3, 4\}$ and (i, j, k, l) determines the orientation of σ. Furthermore, equations (5) and (7) show that if q corresponds to $\{1, 2\} \sqcup \{3, 4\}$, then q' corresponds to $\{1, 4\} \sqcup \{2, 3\}$.

EXAMPLE 3.4 (Solutions of HTE by cross-ratio). Given any compact triangulated pseudo 3-manifold $(M, \mathcal{T})$ and $f : \triangle(\mathcal{T}) \to R^2$ so that $f(t) = f(t')$ when two normal triangles t, t' share a common normal arc, we define a map $F : \square(\mathcal{T}) \to R^2$ by $F(q) = (f(t_1), f(t_2); f(t_3), f(t_4)) = \begin{pmatrix} z(q') \\ y(q') \end{pmatrix}$, where $t_1, \dots, t_4$ are the four normal triangles in the tetrahedron σ containing q so that q separates $\{t_1, t_2\}$ from $\{t_3, t_4\}$ and $t_1 \to t_2 \to t_3 \to t_4$ defines the orientation of σ. Then Corollary 3.3 shows that $z : \square \to R$ is a solution to HTE. Note that $y(q') = -z(q'')$ with $q \to q' \to q''$.

This example serves as a guide for us to solve Thurston's equation and HTE. Indeed, the goal is to solve Thurston's equation by writing each solution $x : \square \to R$ in terms of cross-ratio in the universal cover.

Let $R^2/\sim$ be the quotient space where $u \sim \lambda u$, $u \in R^2$ and $\lambda \in R^*$. If $x = (a,b)^t \in R^2$, then $[x] = [a,b]^t$ denotes the image of x in $R^2/\sim$.

DEFINITION 3.5. Let the projective line PR^1 be $\{A \in R^2 \,|\, \text{there exists } B \in R^2 \text{ so that } \langle A, B\rangle \in R^*\}/\sim$, where $A \sim \lambda A$ for $\lambda \in R^*$. A set of elements $\{A_1, \ldots, A_n\}$ (or $\{[A_1], \ldots, [A_n]\}$) in R^2 (or in PR^1) is called *admissible* if $\langle A_i, A_j\rangle \in R^*$ for all $i \neq j$. The *cross-ratio* of four points $\alpha_i, i = 1,2,3,4$ in PR^1, denoted by $[\alpha_1, \alpha_2; \alpha_3, \alpha_4]$, is the element $[(A_1, A_2; A_3, A_4)] \in R^2/\sim$ so that $A_i \in \alpha_i$. We also use $[A_1, A_2; A_3, A_4] \in PR^1$ to denote $[\alpha_1, \alpha_2; \alpha_3, \alpha_4]$.

PROPOSITION 3.6. *Suppose* $\{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}$.

(a) *Given an admissible set of three elements* $A_{i_1}, A_{i_2}, A_{i_3} \in R^2$ *and* $v = \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right) \in R^2$, *there exists a unique* $A_{i_4} \in R^2$ *so that* $(A_1, A_2; A_3, A_4) = v$. *Furthermore,* $A_1, \ldots, A_4$ *form an admissible set if and only if* $c_1, c_2, c_1 - c_2 \in R^*$.

(b) *Suppose* $A_1, \ldots, A_4, B_1, \ldots, B_4 \in R^2$ *so that both* $\{A_1, \ldots, A_4\}$ *and* $\{B_1, \ldots, B_4\}$ *are admissible and* $[A_1, A_2; A_3, A_4] = [B_1, B_2; B_3, B_2]$. *Then there exists a unique* $X \in \mathrm{PGL}(2,R)$ *so that* $[XA_i] = [B_i]$ *for all* i. *In particular, if* $Y \in \mathrm{GL}(2,R)$ *so that* $[YA_i] = [A_i]$, $i = 1,2,3$, *then* $Y = \lambda I$ *for* $\lambda \in R^*$.

Proof. To see the existence part of (a), by Corollary 3.3(a), we may assume for simplicity that $i_1 = 1$, $i_2 = 2$, $i_3 = 3$. Let $A_i = (a_i, b_i)^t$ and consider $X = (1/\langle A_1, A_2\rangle)\left[\begin{smallmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{smallmatrix}\right] \in \mathrm{GL}(2,R)$. Then $XA_1 = (1,0)^t$ and $XA_2 = (0,1)^t$. By Corollary 3.3(e), after replacing A_i by XA_i, we may assume that $A_1 = (1,0)^t$, $A_2 = (0,1)^t$. Then by identity (6), $(A_1, A_2; A_3, A_4) = (-a_3 b_4, -a_4 b_3)^t$. By the assumption that $\langle A_i, A_3\rangle \in R^*$ for $i = 1,2$, we see that $a_3, b_3 \in R^*$. It follows that $a_4 = -c_1/b_3$ and $b_4 = -c_2/a_3$. This shows that A_4 exists and is unique.

Now given that $\{A_1, A_2, A_3\}$ is admissible, the set $\{A_1, \ldots, A_4\}$ is admissible if and only if $\langle A_i, A_4\rangle \in R^*$. By the calculation above, $c_1 = -b_3 a_4$, $c_2 = -a_3 b_4$ and $c_1 - c_2 = \langle A_3, A_4\rangle$. If $\{A_1, \ldots, A_4\}$ is admissible, then $a_i, b_i \in R^*$. Therefore $c_1, c_2, c_1 - c_2 \in R^*$. If $\{c_1, c_2, c_1 - c_2\} \subset R^*$, then $a_i, b_i \in R^*$ for $i = 3,4$ and $\langle A_3, A_4\rangle \in R^*$. This implies $\langle A_i, A_4\rangle \in R^*$.

To see part (b), by the proof of part (a) and the assumption $\langle A_1, A_2\rangle, \langle B_1, B_2\rangle \in R^*$, after replacing A_i by YA_i and B_i by ZB_i for some $Y, Z \in \mathrm{GL}(2,R)$, we may assume that $A_1 = B_1 = (1,0)^t$ and $A_2 = B_2 = (0,1)^t$. Let $A_3 = (a,b)^t$, $A_4 = (a', b')^t$, $B_3 = (c,d)^t$ and $B_4 = (c', d')^t$. Then the admissible condition implies that $a, b, c, d, a', b', c', d' \in R^*$. Furthermore, $[A_1, A_2; A_3, A_4] = [B_1, B_2; B_3, B_4]$ implies that there exists $\lambda \in R^*$ so that $a'b = \lambda c'd$ and $ab' = \lambda cd'$. This shows that the matrix $X = \left[\begin{smallmatrix} c/a & 0 \\ 0 & d/b \end{smallmatrix}\right] \in \mathrm{GL}(2,R)$ satisfies $XA_1 = (c/a)B_1$, $XA_2 = (d/b)B_2$, $XA_3 = B_3$ and $XA_4 = \lambda(cd/ab)B_4$.

To see the uniqueness, say $YA_i = \lambda_i A_i$ for $\lambda_i \in R^*$. We claim that $\lambda_1 = \lambda_2 = \lambda_3$ and $Y = \lambda_1 I$. Indeed, by definition, $\det(Y)\langle A_i, A_j\rangle = \langle YA_i, YA_j\rangle = \lambda_i \lambda_j \langle A_i, A_j\rangle$. Since $\langle A_i, A_j\rangle \in R^*$, we obtain $\lambda_i \lambda_j = \det(Y)$. This implies that $\lambda_i = \lambda_1$ for $i = 2,3$. We conclude that $Y[A_1, A_2] = \lambda_1[A_1, A_2]$. Since the matrix $[A_1, A_2] \in \mathrm{GL}(2,R)$, it follows that $Y = \lambda_1 I$. □

4. *A proof of Theorem* 1.1

We will prove a slightly general theorem which holds for compact oriented pseudo 3-manifolds $(M, \mathcal{T})$. The main idea of the proof is based on the methods developed in [**13**, **15**, **17**, **19**] which construct a pseudo-developing map and the holonomy associated to a solution to Thurston's equation over $\mathbf{C}$.

Let M^* be M with a small regular neighbourhood of each vertex removed and let $\mathcal{T}^*$ be the ideal triangulation $\{s \cap M^* \,|\, s \in \mathcal{T}\}$ of the compact 3-manifold M^*.

THEOREM 4.1. *Suppose that* $(M, \mathcal{T})$ *is a connected triangulated pseudo 3-manifold with* $\partial M = \emptyset$ *and* R *is a commutative ring with identity so that Thurston's equation on* $\mathcal{T}$ *is solvable in* R*. Then each edge* $e \in \mathcal{T}^*$ *lifts to an arc in the universal cover* $\tilde{M}^*$ *of* M^* *joining different boundary components of* $\tilde{M}^*$*. Furthermore, if* M *is a closed connected 3-manifold so that there exists an edge* e *having the same end point, then there exists a representation of* $\pi_1(M)$ *into* $\mathrm{PGL}(2, R)$ *sending the loop* $[e]$ *to a non-identity element.*

4.1. *Definition of a pseudo-developing map*

Let $\pi : \tilde{M}^* \to M^*$ be the universal cover and $\tilde{\mathcal{T}}$ be the pull-back of the ideal triangulation $\mathcal{T}^*$ of M^* to $\tilde{M}^*$. We use $\tilde{\triangle}$ and $\tilde{\square}$ to denote the sets of all normal triangle types and quads in $\tilde{\mathcal{T}}$, respectively. The sets of all normal triangle types and quads in $\mathcal{T}^*$ are the same as those of $\mathcal{T}$ and will still be denoted by $\triangle$ and $\square$. The covering map π induces a surjection π_* from $\tilde{\triangle}$ and $\tilde{\square}$ to $\triangle$ and $\square$, respectively, so that $\pi_*(d_1) = \pi_*(d_2)$ if and only if d_1 and d_2 differ by a deck transformation element.

Suppose $x : \square \to R$ solves Thurston's equation on $\mathcal{T}$ and $z : \square \to R^*$ is an associated solution to HTE constructed by Lemma 2.4. Let $w : \square \to PR^1$ be the map $w(q) = [z(q'), -z(q'')]^t$, where $q \to q'$. Let $\tilde{x} = x\pi_*$, $\tilde{z} = z\pi_*$ and $\tilde{w} = w\pi_*$ be the associated maps defined on $\tilde{\square}$. By the construction, $\tilde{x}$ and $\tilde{z}$ are solutions to Thurston's equation and HTE on $\tilde{\mathcal{T}}$.

DEFINITION 4.2 (See [**13**, **15**, **17**, **19**]). Given a solution x to Thurston's equation on $(M, \mathcal{T})$, a map $\phi : \tilde{\triangle} \to PR^1$ is called a *pseudo-developing map* associated to x if

(a) whenever t_1, t_2 are two normal triangles in $\tilde{\triangle}$ sharing a normal arc, denoted by $t_1 \sim t_2$ in the sequel, then $\phi(t_1) = \phi(t_2)$,
(b) if t_1, t_2, t_3, t_4 are four normal triangles in a tetrahedron σ, then $\{\phi(t_1), \ldots, \phi(t_4)\}$ is admissible and the cross-ratio

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \tilde{w}(q), \tag{8}$$

where $t_1 \to t_2 \to t_3 \to t_4$ determines the orientation of the tetrahedron σ and $q \subset \sigma$ is the quad separating $\{t_1, t_2\}$ from $\{t_3, t_4\}$.

The following theorem is the main result in this section, which implies Theorem 4.1.

THEOREM 4.3. *Given any solution* x *to Thurston's equation on a connected pseudo 3-manifold* $(M, \mathcal{T})$ *with* $\partial M = \emptyset$*, there exists a pseudo-developing map associated to* x*. Furthermore, if* ϕ_1 *and* ϕ_2 *are two pseudo-developing maps associated to* x*, then there exists a unique* $A \in \mathrm{GL}(2, R)$ *so that* $\phi_1(t) = A\phi_2(t)$ *for all* $t \in \tilde{\Delta}$*.*

To prove it, we begin with a recall of Poincaré dual polyhedral decomposition of $\tilde{\mathcal{T}}$ and the edge path groupoid.

4.2. *Poincaré dual and edge paths*

The Poincaré dual polyhedral decomposition X of the ideal triangulation $\tilde{\mathcal{T}}$ is a CW complex defined as follows. Each simplex $s \in \tilde{\mathcal{T}}^{(k)}$ of dimension $k \geqslant 1$ corresponds to a $(3-k)$-dimensional (polyhedral) cell $D(s)$ in X so that for each tetrahedron $\sigma \in \tilde{\mathcal{T}}^{(3)}$, the intersection $D(s) \cap \sigma$ is the convex hull (in the natural linear structure on σ) of the barycentres of simplexes $\tau < \sigma$ with $\tau > s$. We use $|X|$ to denote the geometric realization of X and consider it as a subset of $\tilde{M}^*$. By the construction, for each tetrahedron σ, $\sigma \cap |X|$ is a strong deformation retract of σ; see Figure 2. Therefore, $|X|$ is a strong deformation retract of $\tilde{M}^*$. In particular, $|X|$ is simply connected.

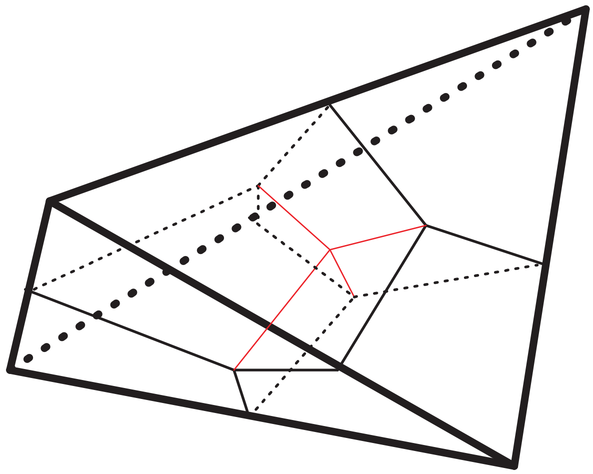

FIGURE 2 (colour online). *Poincaré dual decomposition in each tetrahedron.*

For an oriented edge e in X, we use orig(e) and end(e) to denote its starting and ending vertices, respectively, and use e^{-1} to denote the edge with the reversed orientation. An *edge path* γ in X is a finite sequence of oriented edges $e_1, \ldots, e_n$ so that $\text{orig}(e_{i+1}) = \text{end}(e_i)$. We write $\gamma = e_1 \cdots e_n$; call orig(e_1) and end(e_n) the starting and ending points of γ and denote them by orig(γ) and end(γ), respectively. The constant edge path (that is, $n = 0$) consists of a single vertex v with $\text{orig}(v) = \text{end}(v) = v$. If $\text{orig}(\gamma) = \text{end}(\gamma)$, then γ is called an edge loop based at orig(γ). The inverse of γ is the edge path $e_n^{-1}e_{n-1}^{-1} \cdots e_2^{-1}e_1^{-1}$, denoted by γ^{-1}. If $\delta = d_1 \cdots d_m$ is another edge path starting at $\text{orig}(d_1) = \text{end}(\gamma)$, then we define the product of γ and δ to be the edge path $\gamma\delta = e_1 \cdots e_n d_1 \cdots d_m$. Suppose that f is a 2-cell in X; we use $\partial f = e_1 \cdots e_k$ to denote a *boundary edge loop* of f where the initial edge e_1 and its orientation are arbitrarily chosen.

DEFINITION 4.4. Two edge paths α and β in X are related by an elementary move, denoted by $\alpha \sim \beta$, if one of the following holds:

(1) $\alpha = \beta\gamma\gamma^{-1}$, or $\beta = \alpha\gamma\gamma^{-1}$, or
(2) $\alpha = \beta\delta(\partial f)\delta^{-1}$ or $\beta = \alpha\delta(\partial f)\delta^{-1}$ for some 2-cell f in X.

A well-known fact from algebraic topology shows the following lemma.

LEMMA 4.5 (See [5]). *Suppose α and β are two edge paths in a simply connected CW complex X. Then the following two conditions are equivalent*:

(1) $\text{orig}(\alpha) = \text{orig}(\beta)$ *and* $\text{end}(\alpha) = \text{end}(\beta)$;
(2) *there exists a sequence of edge paths $\alpha_1 = \alpha$, $\alpha_2, \ldots, \alpha_n = \beta$ so that $\alpha_i \sim \alpha_{i+1}$ for all indices i.*

Let $X^{(0)}$ be the set of all 0-cells in X. Fix a 0-cell $\sigma_* \in X^{(0)}$ and let P be the set of all edge paths in X starting at σ_*.

COROLLARY 4.6. *For a simply connected CW complex X, if $F : P \to Y$ is a function so that whenever $\alpha \sim \beta$, $F(\alpha) = F(\beta)$, then $\hat{F}(\text{end}(\alpha)) = F(\alpha) : X^{(0)} \to Y$ is a well-defined function.*

Back to the Poincaré dual complex X, the 0-cells can be identified with tetrahedra $\sigma \in \tilde{\mathcal{T}}^{(3)}$, the oriented edges in X can be identified with the triple (σ_1, f, σ_2), where $f \in \tilde{\mathcal{T}}^{(2)}$ is a triangular face adjacent to σ_1, $\sigma_2 \in \tilde{\mathcal{T}}^{(3)}$ and σ_1 is the starting vertex. A 2-cell $D(e)$ in X dual to an edge $e \in \tilde{\mathcal{T}}^{(1)}$ has the boundary edge loop of the form $\epsilon_1 \cdots \epsilon_n$, where n is the degree of e, $\epsilon_i = (\sigma_i, f_i, \sigma_{i+1})$, where $f_i > e$ and $\sigma_{n+1} = \sigma_1$.

4.3. *Construction of map* $\hat{\Phi} : \tilde{\mathcal{T}}^{(3)} \to (PR^1)^4$

We will use Corollary 4.6 to define the map $\hat{\Phi}$ by constructing $\Phi : P \to (PR^1)^4$. Label the four normal triangle types in each tetrahedron $\sigma \in \tilde{\mathcal{T}}^{(3)}$ by $t_1(\sigma), t_2(\sigma), t_3(\sigma)$ and $t_4(\sigma)$ so that $t_1(\sigma), t_2(\sigma), t_3(\sigma), t_4(\sigma)$ determine the orientation of σ. Let the normal quad in σ separating $\{t_1(\sigma), t_2(\sigma)\}$ from $\{t_3(\sigma), t_4(\sigma)\}$ be $q_1(\sigma)$.

For the initial tetrahedron σ_*, let $\tilde{w}(q_1(\sigma_*)) = [\begin{smallmatrix} a \\ b \end{smallmatrix}] \in PR^1$. Define

$$\Phi(\sigma_*) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -b \\ -a \end{bmatrix} \right) \in (PR^1)^4. \tag{9}$$

By the construction, the cross-ratio of $[1,0]^t, [0,1]^t, [1,1]^t, [-b,-a]^t$ is $\tilde{w}(q_1(\sigma_*))$, that is, (8) holds for $q_1(\sigma_*)$.

By Lemma 2.4(b) and the fact that x solves Thurston's equation, we have $a, b, a-b \in R^*$. This shows that $\{[1,0]^t, [0,1]^t, [1,1]^t, [-b,-a]^t\}$ is an admissible set.

Suppose $\gamma = e_1 \cdots e_n$ is an edge path so that $\Phi(\gamma) = (a_1, a_2, a_3, a_4) \in (PR^1)^4$ is defined, $\{a_1, a_2, a_3, a_4\}$ is admissible and e is an edge with $\text{orig}(e) = \text{end}(\gamma) = \sigma_1$. We define $\Phi(\gamma e)$ as follows. Let the edge e (in X) be (σ_1, f, σ_2), where $f \in \tilde{\mathcal{T}}^{(2)}$ and σ_i are tetrahedra adjacent to f. Let the normal triangles in σ_1 (and σ_2) having a normal arc in f be t_{i_1}, t_{i_2}, t_{i_3} (and $s_{j_1}, s_{j_2}, s_{j_3}$ in σ_2) so that $t_{i_k} \sim s_{j_k}$, that is, t_{i_k} and s_{j_k} share a common normal arc in f. Define

$$\Phi(\gamma e) = (b_1, b_2, b_3, b_4) \in (PR^1)^4 \tag{10}$$

so that

$$b_{j_k} = a_{i_k} \tag{11}$$

for $k = 1, 2, 3$ and the last entry b_{j_4} ($\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$) is determined by the cross-ratio equation

$$[b_1, b_2; b_3, b_4] = \tilde{w}(q_1(\sigma_2)). \tag{12}$$

Since $\{b_{j_1}, b_{j_2}, b_{j_3}\}$ is admissible and $\tilde{w}(q_1(\sigma_2)) = [c_1, c_2]^t$ satisfies $c_1, c_2, c_1 - c_2 \in R^*$, by Proposition 3.6(a) there exists a unique b_{j_4} solving (12) so that $\{b_1, b_2, b_3, b_4\}$ is admissible. Note that by (7) and (5), equation (12) implies that if $q(\sigma_2)$ is any quad in σ_2 separating $\{s_i, s_j\}$ from $\{s_k, s_l\}$, we have the cross-ratio equation

$$[b_i, b_j; b_k, b_l] = \tilde{w}(q(\sigma_2)). \tag{13}$$

By the initial condition (9) and above, we can define Φ on any edge path starting at σ_*.

The basic property of the function Φ is the following lemma.

LEMMA 4.7. *Suppose that γ, δ are edge paths, e is an oriented edge and f is a 2-cell in X. Assume all edge path multiplications are defined below. Then*

(a) $\Phi(\gamma e e^{-1}) = \Phi(\gamma)$;
(b) $\Phi(\gamma \delta \delta^{-1}) = \Phi(\gamma)$;
(c) $\Phi(\gamma(\partial f)) = \Phi(\gamma)$;
(d) $\Phi(\gamma \delta (\partial f) \delta^{-1}) = \Phi(\gamma)$.

Proof. Part (a) follows from the definition (12) of Φ and the uniqueness part of Proposition 3.6(a). Indeed, suppose $\Phi(\gamma e e^{-1}) = (a'_1, a'_2, a'_3, a'_4)$. By definition, we have $a'_{i_k} = b_{j_k} = a_{i_k}$,

$k=1,2,3$. Now both cross-ratios $[a_1', a_2', a_3', a_4'] = \tilde{w}(q_1(\sigma_1)) = [a_1, a_2, a_3, a_4]$. Therefore by Proposition 3.6, all $a_i = a_i'$ for all i.

Part (b) follows from the induction. Write $\delta = e_1 \cdots e_k$. If $k=1$, then the result follows from part (a). Suppose it holds for $k-1$. Then, in the case of k, we have, by part (a),

$$\Phi(\gamma e_1 \cdots e_{k-1} e_k e_k^{-1} e_{k-1}^{-1} \cdots e_2^{-1} e_1^{-1}) = \Phi(\gamma e_1 \cdots e_{k-1} e_{k-1}^{-1} \cdots e_2^{-1} e_1^{-1}).$$

The right-hand side is $\Phi(\gamma)$ by the induction hypothesis.

Note by the same argument, part (d) follows from part (c) and part (b). It remains to prove part (c).

Suppose the 2-cell f is the dual $D(e)$ to an edge $e \in \tilde{\mathcal{T}}^{(1)}$ of degree n so that the boundary edge loop ∂f is $\epsilon_1 \epsilon_2 \cdots \epsilon_n$, where $\epsilon_i = (\sigma_{i-1}, f_i, \sigma_i)$, $f_i > e$, $\sigma_n = \sigma_0$ and $\mathrm{end}(\gamma) = \mathrm{orig}(\epsilon_1) = \sigma_0$. We will show that $\Phi(\gamma \epsilon_1 \epsilon_2 \cdots \epsilon_n) = \Phi(\gamma)$.

Let the end points of e be in the boundary components v_0 and v_∞ of $\tilde{M}^*$.

Case 1. In the special case where the labelling of the normal triangles $t_j(\sigma_i)$ are such that:

(1) $t_1(\sigma_i), t_2(\sigma_i), t_3(\sigma_i)$ have normal arcs in f_i;
(2) $t_1(\sigma_i)$ is in v_0 and $t_2(\sigma_i)$ is in v_∞;
(3) $t_1(\sigma_i) \sim t_1(\sigma_{i-1})$, $t_2(\sigma_i) \sim t_2(\sigma_{i-1})$, $t_3(\sigma_i) \sim t_4(\sigma_{i-1})$, $i = 0, 1, 2, \ldots, n$ and $\sigma_{-1} = \sigma_{n-1}$, $\sigma_0 = \sigma_n$.

Let $\Phi(\gamma \epsilon_1 \cdots \epsilon_i) = (a_{1,i}, a_{2,i}, a_{3,i}, a_{4,i}) \in (PR^1)^4$ for $i = 1, \ldots, n$, $\Phi(\gamma) = (a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0})$ and $\tilde{w}(q_1(\sigma_j)) = [c_j, d_j]^t \in PR^1$. Our goal is to show $a_{j,0} = a_{j,n}$ for $i = 1, 2, 3, 4$. By the definition that $q_1(\sigma_i)$ separates $t_1(\sigma_i), t_2(\sigma_i)$ from $t_3(\sigma_i), t_4(\sigma_i)$ and that $\tilde{z}$ solves HTE, we have the cross-ratio equation

$$[a_{1,i}, a_{2,i}; a_{3,i}, a_{4,i}] = [c_i, d_i]^t, \quad i = 0, 1, \ldots, n \tag{14}$$

$c_0 = c_n$, $d_0 = d_n$ and

$$\prod_{i=1}^n c_i = \prod_{i=1}^n d_i. \tag{15}$$

Now, by the construction $a_{1,i} = a_{1,i-1}$ and $a_{2,i} = a_{2,i-1}$ and $a_{3,i} = a_{4,i-1}$ for $i = 1, \ldots, n$, $a_{1,n} = a_{1,0}, a_{2,n} = a_{2,0}$ and $a_{3,0} = a_{4,n-1}$.

Choose $A \in \mathrm{GL}(2, R)$ so that A sends $(a_{1,0}, a_{2,0})$ to $([1,0]^t, [0,1]^t)$. Then by $a_{1,i} = a_{1,i-1}$ and $a_{2,i} = a_{2,i-1}$, we see that A sends $(a_{1,i}, a_{2,i})$ to $([1,0]^t, [0,1]^t)$ for all i. For simplicity, if $\Phi(\delta) = (x_1, x_2, x_3, x_4)$, then we use $A\Phi(\delta)$ to denote (Ax_1, Ax_2, Ax_3, Ax_4). Replacing Φ by $A\Phi$ will not effect the cross-ratio equation (14) and $\Phi(\gamma) = \Phi(\gamma \epsilon_1 \cdots \epsilon_n)$ holds if and only if $A\Phi(\gamma) = A\Phi(\gamma \epsilon_1 \cdots \epsilon_n)$. Therefore, we may assume, without loss of generality, that $a_{1,i} = [1,0]^t$ and $a_{2,i} = [0,1]^t$. In this case, we have

$$\Phi(\gamma \epsilon_1 \cdots \epsilon_i) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} \right) \quad \text{and} \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The cross-ratio equation (14) becomes

$$-y_{i+1} x_i = c_i, \quad \text{and} \quad -y_i x_{i+1} = d_i.$$

Then identity (15) implies $\prod_{i=1}^n x_i y_{i+1} = \prod_{i=1}^n y_i x_{i+1}$. Therefore, $y_{n+1} x_1 = y_1 x_{n+1}$, that is, $[x_{n+1}, y_{n+1}]^t = [x_1, y_1]^t \in PR^1$. Together with $[x_n, y_n]^t = [x_0, y_0]^t$, we have $\Phi(\gamma \epsilon_1 \cdots \epsilon_n) = \Phi(\gamma)$.

Case 2. The general case of different labelling of the normal triangles in σ_i follows from the case 1. Indeed, let Φ be the function associated to the general labelling and Φ_0 be the function associated to the special labelling in case 1. For each σ_i ($\sigma_0 = \sigma_n$), let τ_i be the permutation of $\{1, 2, 3, 4\}$ sending (x_1, x_2, x_3, x_4) to $(x_{\tau_i(1)}, x_{\tau_i(2)}, x_{\tau_i(3)}, x_{\tau_i(4)})$ which rearranges the labelling of quads in σ_i so that, for $i = 0, 1, \dots, n$,

$$\Phi(\gamma\epsilon_1\epsilon_2\cdots\epsilon_i) = \tau_i\Phi_0(\gamma\epsilon_1\epsilon_2\cdots\epsilon_i).$$

Now

$$\Phi(\gamma\epsilon_1\epsilon_2\cdots\epsilon_n) = \tau_n\Phi_0(\gamma\epsilon_1\epsilon_2\cdots\epsilon_n) = \tau_n\Phi_0(\gamma) = \tau_0\Phi_0(\gamma) = \Phi(\gamma),$$

where we have used case 1 in the second equality and $\tau_n = \tau_0$ in the third equality. □

4.4. *Proof of Theorem 4.3*

By Corollary 4.6, we have the induced map $\hat{\Phi} : \tilde{\mathcal{T}}^{(3)} \to (PR^1)^4$. Write

$$\hat{\Phi}(\sigma) = (\phi_1(\sigma), \phi_2(\sigma), \phi_3(\sigma), \phi_4(\sigma)).$$

Define the map

$$\phi : \tilde{\Delta} \longrightarrow PR^1$$

by $\phi(t_i(\sigma)) = \phi_i(\sigma)$. We claim that ϕ is a pseudo-developing map.

Indeed, if $t = t_i(\sigma)$ and $t' = t_j(\sigma')$ are two normal triangles sharing a common arc in their boundary, that is, $t \sim t'$, let f be the common codimension-1 face of σ and σ' which contains the $t \cap t'$. By definition, we have $\hat{\Phi}(\sigma) = (\phi_1(\sigma), \phi_2(\sigma), \phi_3(\sigma), \phi_4(\sigma))$ and $\hat{\Phi}(\sigma') = (\phi_1(\sigma'), \phi_2(\sigma'), \phi_3(\sigma'), \phi_4(\sigma'))$. Using identity (11), we conclude that $\phi(t) = \phi_i(\sigma) = \phi_j(\sigma') = \phi(t')$. This establishes condition (a) for pseudo-developing maps. The condition (b) for pseudo-developing maps follows from (13) and the functional equations for solutions to HTE for $\tilde{w}$ and Corollary 3.3(d).

The uniqueness of the pseudo-developing map follows from the definition. Indeed, if ϕ_1 and ϕ_2 are two pseudo-developing maps associated to a solution x to Thurston's equation, then, by Proposition 3.6(b), there exists $A \in \mathrm{GL}(2, R)$ so that, for all $i = 1, 2, 3, 4$,

$$\phi_1(t_i(\sigma_*)) = A\phi_2(t_i(\sigma_*)).$$

By condition (a) of the definition of pseudo-developing maps, we see

$$\phi_1(t_i(\sigma)) = A\phi_2(t_i(\sigma))$$

holds for the three normal triangles $t_{j_k}(\sigma)$ in each tetrahedron σ which has a codimension-1 face in common with σ_* and $t_{j_k}(\sigma) \sim t_{i_k}(\sigma_*)$. Now, by the cross-ratio equation (8) and Proposition 3.6(b), we conclude that

$$\phi_1(t_i(\sigma)) = A\phi_2(t_i(\sigma))$$

for $i = 1, 2, 3, 4$.

Inductively and using the fact that M is connected, we see $\phi_1 = A\phi_2$.

4.5. *The holonomy representation*

Suppose that x is a solution to Thurston's equation on $(M, \mathcal{T})$ in a ring R and $\phi : \tilde{\triangle} \to PR^1$ is an associated pseudo-developing map. Then there exists a homomorphism $\rho : \pi_1(M^*) \to \mathrm{PGL}(2, R)$ so that, for all $\gamma \in \pi_1(M^*)$, considered as a deck transformation group for the universal cover $\pi_* : \tilde{M}^* \to M^*$,

$$\phi(\gamma) = \rho(\gamma)\phi. \tag{16}$$

We call ρ a *holonomy representation* of x. It is unique up to conjugation in $\mathrm{PGL}(2, R)$. Here is the construction of ρ. Fix an element $\gamma \in \pi_1(M^*)$. By the construction, $\pi_1(M^*)$ acts on $\tilde{M}^*$, $\tilde{\mathcal{T}}^*$, $\tilde{\triangle}$ and $\tilde{\square}$ so that $\pi_*(\gamma) = \pi_*$ for $\gamma \in \pi_1(M^*)$. This implies

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = [\phi(\gamma t_1), \phi(\gamma t_2); \phi(\gamma t_3), \phi(\gamma t_4)]$$

for all normal triangles $t_1, \ldots, t_4$ in each tetrahedron σ in $\tilde{T}^{(3)}$ and

$$\psi(t(\sigma)) = \phi(\gamma(t(\sigma)))$$

is a pseudo-developing map associated to x. By the uniqueness of the pseudo-developing map, there exists an element $\rho(\gamma) \in \mathrm{PGL}(2, R)$ so that

$$\psi(t(\sigma)) = \rho(\gamma)\phi(t(\sigma)),$$

that is,

$$\phi(\gamma t(\sigma)) = \rho(\gamma)\phi(t(\sigma)).$$

Given $\gamma_1, \gamma_2 \in \pi_1(M^*)$, by definition, $\rho(\gamma_1\gamma_2)\phi = \phi(\gamma_1\gamma_2) = \rho(\gamma_1)\phi(\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)\phi$ and the uniqueness part of Proposition 3.6, we see that $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$, that is, ρ is a group homomorphism from $\pi_1(M^*)$ to $\mathrm{PGL}(2, R)$.

Note that the representation ρ is trivial if and only if $\phi(\gamma t) = \phi(t)$ for all $t \in \tilde{\triangle}$ and $\gamma \in \pi_1(M^*)$. In this case, the pseudo-developing map ϕ is defined on $\triangle \to PR^1$ so that $[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = [z(q'), -z(q'')]^t$. This was the construction in Example 3.1. In particular, the holonomy representations associated to solutions in Example 3.1 are trivial.

4.6. *A proof of Theorem 4.1*

Suppose otherwise that there exists an edge $e \in \mathcal{T}^*$ whose lift in $\tilde{M}^*$ is an edge e^* in $\tilde{\mathcal{T}}$ joining the same boundary component of $\tilde{M}^*$. Take a tetrahedron σ containing e^* as an edge and let t_1, t_2, t_3, t_4 be all normal triangles in σ so that t_1, t_2 are adjacent to e^*. By definition, the pseudo-developing map $\phi : \tilde{\triangle} \to PR^1$ satisfies the condition that $\{\phi(t_1), \ldots, \phi(t_4)\}$ is admissible. In particular, $\phi(t_1) \neq \phi(t_2)$. On the other hand, since e^* ends at the same connected component of $\partial\tilde{M}^*$ which is a union of normal triangles related by sharing common normal arcs, there exists a sequence of normal triangles $s_1 = t_1, s_2, \ldots, s_n = t_2$ in $\tilde{\triangle}$ so that $s_i \sim s_{i+1}$. In particular, $\phi(s_i) = \phi(s_{i+1})$. This implies that $\phi(t_1) = \phi(t_2)$, contradicting the assumption that $\phi(t_1) \neq \phi(t_2)$.

To prove the second part of Theorem 4.1 that M is a closed connected 3-manifold, we first note that $\pi_1(M^*)$ is isomorphic to $\pi_1(M)$ under the homomorphism induced by inclusion. We will identify these two groups and identify $\tilde{M}^*$ as a $\pi_1(M)$-invariant subset of the universal cover $\tilde{M}$ of M. If e is an edge in $\mathcal{T}$ ending at the same vertex v in $\mathcal{T}$, let $\gamma \in \pi_1(M, v)$ be the deck transformation element corresponding to the loop e. We claim that $\rho(\gamma) \neq id$ in $\mathrm{PGL}(2, R)$. Indeed, suppose that e^* is the lifting of e. Then by the statement just proved, e^* has two distinct vertices u_1 and u_2 in $\tilde{M}$ and $\phi(u_1) \neq \phi(u_2)$. By definition $\gamma(u_1) = u_2$. It follows that $\phi(u_2) = \phi(\gamma u_1) = \rho(\gamma)\phi(u_1)$. Since $\phi(u_1) \neq \phi(u_2)$, we obtain $\rho(\gamma) \neq id$. This ends the proof.

5. *A variational principle for solving homogeneous Thurston's equation over the real numbers*

We show that solutions of the HTE are related to the critical points of a Boltzmann entropy function of the form $\sum_{i=1}^n x_i \ln |x_i|$.

5.1. *$\mathbb{Z}_2$ taut angle structures*

Let us begin with the observation that if $z : \square \to \mathbb{R}$ is a solution to Thurston's equation, then the sign function $s : \square \to \{+1, -1\}$ of z given by $s(q) = \mathrm{sign}(z(q))$ satisfies the following conditions:

(a) for each $q \in \square$, exactly one of $s(q), s(q'), s(q'')$ is -1, and
(b) for each edge $e \in E$, $\prod_{q \sim e} s(q) = 1$.

One defines a $\mathbb{Z}_2$ taut angle structure on $(M,\mathcal{T})$ to be a function $s:\square(\mathcal{T})\to\{+1,-1\}$ satisfying (a) and (b). Therefore, the existence of a $\mathbb{Z}_2$ taut angle structure is a necessary condition for solving Thurston's equation over the real numbers. Fortunately, we have the following lemma.

LEMMA 5.1 (Baseilhac–Benedetti [1]). *For any triangulated closed pseudo 3-manifold $(M,\mathcal{T})$, there exists a $\mathbb{Z}_2$ taut angle structure.*

Indeed, Baseilhac–Benedetti proved a stronger result that each $(M,\mathcal{T})$ has a $\mathbb{Z}_2$ taut structure which assigns each codimension-1 face of $\mathcal{T}$ a normal vector so that, in each tetrahedron, there are exactly two normal vectors pointing inward. This later result can be seen by taking the 4-valent dual graph of the triangulation $\mathcal{T}$ and orienting the edges of the dual graph so that exactly two edges point towards each vertex. Now from $\mathbb{Z}_2$ taut structure, one constructs the associated $\mathbb{Z}_2$ taut angle structure by assigning -1 to those quads q facing the edges that are adjacent to two codimension-1 faces with both outward (or both inwards) normal vectors. In fact, a stronger result has been proved by Henry Segerman and us. Namely, given any embedded normal surface S in $(M,\mathcal{T})$, there exists a $\mathbb{Z}_2$ taut angle structure s for which the surface S is vertical, that is, all quads that appeared in S are assigned 1 in s.

5.2. *The Boltzmann entropy*

The Boltzmann entropy function $Bol:\mathbb{R}^{\square}\to\mathbb{R}$ is defined to be

$$\mathrm{Bol}(x)=\sum_{q\in\square}x(q)\ln(|x(q)|),$$

where $0\ln 0=0$. To set up a variational framework, let us fix a $\mathbb{Z}_2$ taut angle structure s. Define $\alpha_s:\square\to\mathbb{R}$ by $\alpha_s(q)=1$ if $s(q)=1$ and $\alpha_s(q)=-2$ if $s(q)=-1$. Define two non-empty closed convex sets C and W_s as follows:

$$C=\{x\in\mathbb{R}^{\square}\mid x(q)s(q)\geqslant 0, x(q)+x(q')+x(q'')=0,\ \text{for all } q\in\square\},\tag{17}$$

and

$$W_s=\left\{x\in C\,\middle|\,\sum_{q\sim e}x(q)=\sum_{q\sim e}\alpha_s(q), e\in E\right\}.$$

By the construction, α_s is in the relative interior $\mathrm{int}(W_s)$ of W_s. Let G be the restriction of the Boltzmann function Bol to C and F be the restriction of Bol to W_s. Evidently $F=G|_{W_s}$.

THEOREM 5.2. *For any closed triangulated pseudo 3-manifold $(M,\mathcal{T})$,*

(a) *G is convex in C;*
(b) *if $x_0\in int(W_s)$ is a critical point (that is, minimal point) of F on W_s, then x_0 is a solution to the HTE so that $x_0(q)\neq 0$ for all q;*
(c) *if $x_0\in\partial W_s$ is a minimal point of F on W_s, then there exists a function $l:E\to\mathbb{R}$ so that, for all tetrahedra σ which contain q,q',q'' with $s(q)=-1$,*

$$\exp\left(\sum_{e\sim q}l(e)\right)\geqslant\exp\left(\sum_{e\sim q'}l(e)\right)+\exp\left(\sum_{e\sim q''}l(e)\right),\tag{18}$$

that is, in the length l, each tetrahedron becomes a decorated ideal tetrahedron in the anti-de Sitter space.

Proof. To see the convexity of G, we write $G(x)$ as $\sum_{\sigma\in\mathcal{T}^{(3)}}\sum_{q\subset\sigma} x(q)\ln(|x(q)|)$, that is, $G(x)$ is the sum of function of the form $\phi(x_1,x_2,x_3)=\sum_{i=1}^{3} x_i\ln(|x_i|)$ defined on $\Delta=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_2+x_3=0, x_1,x_2\geqslant 0\}$. Thus it suffices to show that ϕ is convex in Δ. This is the same as showing $g(x_1,x_2)=x_1\ln x_1+x_2\ln x_2-(x_1+x_2)\ln(x_1+x_2)$ is convex in $\{(x_1,x_2)\mid x_1,x_2\geqslant 0\}$. Since the Hessian matrix of g is $(1/x_1x_2(x_1+x-2)^2)\begin{bmatrix} x_2^2 & x_1x_2 \\ x_1x_2 & x_1^2\end{bmatrix}$, which is semi-positive definite, the result follows.

To see (b), since $x_0\in int(W_s)$, by definition $x_0(q)\neq 0$ for all q. To check that x_0 solves the HTE, we use the critical point equation. Recall that the tangent space of W_s at x_0 can be identified with $TAS=\{v\in\mathbb{R}^{\square}\mid \forall\sigma\in\mathcal{T}^{(3)}, \sum_{q\subset\sigma} v(q)=0, \forall e\in E(\mathcal{T}), \sum_{q\sim e} v(q)=0\}$. Therefore, for all $v\in$ TAS, $(d/dt)|_{t=0}F(p+tv)=0$. Now, for each edge $e\in E$, define the vector $v_e:\square\to\mathbb{R}$ by $v_e=\sum_{q\sim e}(q')^*-(q'')^*$, where $q^*:\square\to\mathbb{R}$ is the map so that $q^*(q)=1$ and $q^*(p)=0$ for $p\neq q$. It is well known that $v_e\in TAS$; see, for instance, [**11**]. Thus, for each edge $e\in E(\mathcal{T})$,

$$0=\frac{d}{dt}\bigg|_{t=0} F(x_0+tv_e)=\sum_{q\sim e}(\ln(|x_0(q')|-\ln(|x_0(q'')|).$$

Therefore

$$\prod_{q\sim e}\left|\frac{x_0(q')}{x_0(q'')}\right|=1. \tag{19}$$

On the other hand, by the assumption that $x_0(q)s(q)>0$ for all q and that s is $\mathbb{Z}_2$ taut angle structure, we see that

$$\operatorname{sign}\left(\prod_{q\sim e}\frac{-x_0(q')}{x_0(q'')}\right)=\operatorname{sign}\left(\prod_{q\sim e}\frac{-s(q')}{s(q'')}\right)=\operatorname{sign}\left(\prod_{q\sim e}s(q)\right)=1.$$

Combining with (19), we see that $\prod_{q\sim e}x_0(q')=\prod_{q\sim e}(-x_0(q''))$. This shows that x_0 is a solution to the HTE.

To see the last part (c), we need to use the following important theorem from convex optimization. See, for instance, [**2**].

THEOREM 5.3. *Suppose $C\subset\mathbb{R}^n$ is convex, $f:C\to\mathbb{R}$ is a convex function and $h:\mathbb{R}^n\to\mathbb{R}^m$ is an affine map so that there exists a point $\alpha\in\operatorname{int}(C)$ with $h(\alpha)=0$. If $x_0\in D=\{x\in C\mid h(x)=0\}$ is a minimal point of $f|_D$, then there exists $\mu\in\mathbb{R}^m$ so that*

$$f(x)+(\mu,h(x))\geqslant f(x_0)$$

for all $x\in C$, where (u,v) is the standard inner product of u,v in $\mathbb{R}^m$.

Take C to be given by (17), $f=G$ and $h:\mathbb{R}^{\square}\to\mathbb{R}^E$ to be $h(x)(e)=\sum_{q\sim e}(x(q)-\alpha_s(q))$ in the above theorem. By the choice of $\alpha_s\in int(W_s)$, we have $h(\alpha_s)=0$. Therefore, there exists a function $l:E\to\mathbb{R}$ so that, for all $x\in C$,

$$G(x)-\sum_{e\in E} l(e)h(x)(e)\geqslant G(x_0).$$

This is the same as

$$\sum_{q\in\square}x(q)\ln(|x(q)|)-\sum_{e\in E}l(e)\sum_{q\sim e}(x(q)-\alpha_s(q))\geqslant\sum_{q}x_0(q)\ln(|x_0(q)|).$$

We can rewrite it as

$$\sum_{\sigma\in\mathcal{T}^{(3)}}\sum_{q\in\sigma}[x(q)\ln(|x(q)|)-x(q)\sum_{e\sim q}l(e)]\geqslant G(x_0)+(l,k_0) \tag{20}$$

for some constant $k_0 \in \mathbb{R}^m$. Since (20) holds for all $x \in C$ which is a product of spaces Δ, this shows, for all $\sigma \in \mathcal{T}^{(3)}$ and for all x with $\sum_{q \subset \sigma} x(q) = 0$ and $x(q)s(q) \geqslant 0$,

$$\sum_{q \in \sigma} x(q) \ln(|x(q)|) - \sum_{q \in \sigma} \sum_{e \sim q} l(e)x(q) \geqslant G(x_0) + (l, k_0). \tag{21}$$

Due to the homogeneity that, for any $\lambda > 0$,

$$\sum_{q \in \sigma} (\lambda x(q)) \ln(|\lambda x(q)|) - \sum_{q \in \sigma} \sum_{e \sim q} l(e)\lambda x(q) = \lambda \left[\sum_{q \in \sigma} x(q) \ln(|x(q)|) - \sum_{q \in \sigma} \sum_{e \sim q} l(e)x(q) \right],$$

we see (21) implies

$$\sum_{q \in \sigma} x(q) \ln(|x(q)|) - \sum_{q \in \sigma} \sum_{e \sim q} l(e)x(q) \geqslant 0, \tag{22}$$

for all x with $x(q)s(q) \geqslant 0$ and $\sum_{q \subset \sigma} x(q) = 0$.

Now we use the standard Legendre–Fenchel duality for the function $\phi(x_1, x_2, x_3)$ defined on Δ, that is, the following lemma (see [**10**, Lemma 2.7]).

LEMMA 5.4. *If* $s \ln(s) + t \ln(t) - (s + t) \ln(s + t) \geqslant as + bt + c(-s - t)$ *for all* $s, t \geqslant 0$, *then* $e^c \geqslant e^a + e^b$.

Applying this lemma to (22) with $s = x(q'), t = x(q'') \geqslant 0$, we obtain the result (18). □

Finally, we remark that the entropy function $f(t) = t \ln |t| : \mathbb{R} \to \mathbb{R}$ satisfies a pentagon relation (23) below. It suggests that one may produce a topological invariant of 3-manifolds using triangulations and the Boltzmann entropy *Bol*. The quantum version of the pentagon relation has been discovered recently by Kashaev, Vartanov and us in [**9**], which leads to a construction of a topological quantum field theory.

LEMMA 5.5. *Suppose* $a_1, a_2, a_3, b_1, b_2, b_3$ *are real numbers so that* $\sum_{i=1}^3 (a_i + b_i) = 0$, $\prod_{i=1}^3 |a_i| = \prod_{i=1}^3 |b_i|$ *and all but one of* $a_1, a_2, a_3, b_1, b_2, b_3$ *are positive. Then, for* $f(t) = t \ln |t|$,

$$\sum_{i,j=1}^{3} f(a_i + b_j) = \sum_{i=1}^{3} (f(a_i) + f(b_i)). \tag{23}$$

The proof is a direct calculation and will be omitted. It also follows from the limiting case of the pentagon relation for the Lobachevsky function $-\int_0^t \ln |2 \sin(u)| \, du$.

6. *A universal construction*

Recall that $(M, \mathcal{T})$ is a compact oriented triangulated pseudo 3-manifold. The boundary ∂M of M is triangulated by the subcomplex $\partial \mathcal{T} = \{s \cap \partial M \mid s \in \mathcal{T}\}$. An edge in $\mathcal{T}$ is called *interior* if it is not in $\partial \mathcal{T}$. The goal of this section is to introduce the *Thurston ring* $\mathcal{R}(\mathcal{T})$ and its homogeneous version $\mathcal{R}_h(\mathcal{T})$.

We will deal with quotients of the polynomial ring $\mathbf{Z}[\square]$ with $q \in \square$ as variables. As a convention, we will use $p \in \mathbf{Z}[\square]$ to denote its image in the quotient ring $\mathbf{Z}[\square]/\mathcal{I}$.

The 'ground' ring in the construction is the following. Let σ be an oriented tetrahedron so that $q \to q' \to q''$ are the three quads in it. Then the Thurston ring $\mathcal{R}(\sigma)$ is the quotient of the polynomial ring $\mathbf{Z}[q, q', q'']$ modulo the ideal generated by $q'(1 - q) - 1$, $q''(1 - q') - 1$, and $q(1 - q'') - 1$. Note that this implies in $\mathcal{R}(\sigma)$, $q' = 1/(1 - q)$ and $q'' = (q - 1)/q$ and, furthermore, $\mathcal{R}(\sigma) \cong \mathbf{Z}[x, 1/x, 1/(1 - x)]$, where x is an independent variable. Similarly, we defined $\mathcal{R}_h(\sigma)$ to be the quotient ring $\mathbf{Z}[q,q',q'']/(q+q'+q'')$, where $(q+q'+q'')$ is the ideal generated by $q+q'+q''$. Note that $\mathcal{R}_h(\sigma)(\sigma) \cong \mathbf{Z}[x,y]$ the polynomial ring in two independent variables.

Recall that the tensor product $R_1 \otimes R_2$ of two rings R_1 and R_2 is the tensor product of R_1 and R_2 considered as $\mathbf{Z}$ algebras.

Definition 6.1. Suppose that $(M,\mathcal{T})$ is a compact oriented pseudo 3-manifold. The *Thurston ring* $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$ is the quotient of the tensor product $\otimes_{\sigma\in\mathcal{T}^{(3)}}\mathcal{R}(\sigma)$ modulo the ideal generated by elements of the form W_e-1, where $W_e=\prod_{q\sim e} q$ for all interior edges e. The *homogeneous Thurston ring* $\mathcal{R}_h(\mathcal{T})$ is the quotient of $\otimes_{\sigma\in\mathcal{T}^{(3)}}\mathcal{R}_h(\sigma)$ modulo the ideal generated by elements of the form $U_e=\prod_{q\sim e} q' - \prod_{q\sim e}(-q'')$, $q\to q'\to q''$, for all interior edges e. The element $W_e=\prod_{q\sim e} q$ is called the *holonomy* of the edge e.

By the construction, given a commutative ring R with identity, Thurston's equation on $\mathcal{T}$ is solvable in R if and only if there exists a non-trivial ring homomorphism from $\mathcal{R}(\mathcal{T})$ to R. Therefore, Theorem 1.1 can be stated as follows.

Theorem 6.2. *Suppose that $(M,\mathcal{T})$ is a triangulated closed connected 3-manifold so that one edge in $\mathcal{T}$ is a loop. If $\mathcal{R}(\mathcal{T})\neq\{0\}$, then $\pi_1(M)\neq\{1\}$.*

Note that $\mathcal{R}(\mathcal{T})$ is also the quotient $\mathbf{Z}[\square]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $q'(1-q)-1$ for $q\to q'$ and W_e-1 for interior edges e, and $\mathcal{R}_h(\mathcal{T})$ is the quotient of $\mathbf{Z}[\square]/\mathcal{I}_h$, where $\mathcal{I}_h$ is the ideal generated by $q+q'+q''$ for $q\to q'\to q''$ and U_e for interior edges e. We remark that if there is $q\in\square$ so that $q=0$ in $\mathcal{R}(\mathcal{T})$, then $\mathcal{R}(\mathcal{T})=\{0\}$. Indeed, in this case the identity element $1=1-q(1-q'')$ is in the ideal $\mathcal{I}$, therefore $\mathcal{R}(\mathcal{T})=\{0\}$. In particular, if $\mathcal{T}$ contains an interior edge of degree 1 (that is, adjacent to only one tetrahedron), then $\mathcal{R}(\mathcal{T})=\{0\}$.

The relationship between $\mathcal{R}(\mathcal{T})$ and $\mathcal{R}_h(\mathcal{T})$ is summarized in the following proposition. To state it, recall that if S is a multiplicatively closed subset of a ring R, then the localization ring R_S of R at S is the quotient $R\times S/\sim$, where $(r_1,s_1)\sim(r_2,s_2)$ if there exists $s\in S$ so that $s(r_1s_2-r_2s_1)=0$. If $0\in S$, then $R_S=\{0\}$.

Proposition 6.3. *Let $\mathcal{S}=\{q_1\cdots q_m \mid q_i\in\square\}$ be the multiplicatively closed subset of all monomials in $\square$ in $\mathcal{R}_h(\mathcal{T})$. Then there exist a natural injective ring homomorphism $F:\mathcal{R}(\mathcal{T})\to\mathcal{R}_h(\mathcal{T})_S$ and a surjective ring homomorphism $G:\mathcal{R}_h(\mathcal{T})_S\to\mathcal{R}(\mathcal{T})$ so that $GF=id$. In particular, $\mathcal{R}(\mathcal{T})=\{0\}$ if and only if $\mathcal{R}_h(\mathcal{T})_S=\{0\}$.*

Proof. Define a ring homomorphism $F:\mathbf{Z}[\square]\to\mathcal{R}_h(\mathcal{T})_S$ by $F(q)=-q'/q''$ where $q\to q'\to q''$. We claim that $F(\mathcal{I})=\{0\}$ and thus F induces a homomorphism, still denoted by F, from $\mathcal{R}(\mathcal{T})$ to $\mathcal{R}_h(\mathcal{T})_S$. The generators of $\mathcal{I}$ are $q'(q-1)-1$ and W_e-1. If $q\to q'\to q''$, then $F(q'(1-q)-1)=-(q''/q)(1+q'/q'')-1=-(q+q'+q'')/q=0$. For an interior edge e, $F(\prod_{q\sim e} q-1)=(1/\prod_{q\sim e}(-q''))(\prod_{q\sim e} q'-\prod_{q\sim e}(-q''))=0$. To construct the inverse of F, we define $G:\mathbf{Z}[\square]\to\mathcal{R}(\mathcal{T})$ as follows. For each tetrahedron σ containing $q_1\to q_2\to q_3$ where q_1 is specified, define $G(q_1)=1-q_1, G(q_2)=q_1, G(q_3)=-1$ in $\mathcal{R}(\mathcal{T})$. By the construction, we have $G(q)=-q''G(q')$, $G(q')=-qG(q'')$ and $G(q'')=-q'G(q)$ for $q\to q'\to q''$ in $\square$ and $\sum_{q\subset\sigma} G(q)=0$ for each tetrahedron σ. We claim that $G(\mathcal{I}_h)=\{0\}$, that is, G induces a ring homomorphism, still denoted by $G:\mathcal{R}_h(\mathcal{T})\to\mathcal{R}(\mathcal{T})$. Indeed, we have just verified the first equation associated to each $\sigma\in\mathcal{T}$. For the second type of equation, given any interior edge e, due to $G(q')=-qG(q'')$, $G(\prod_{q\sim e} q'-\prod_{q\sim e}(-q''))=\prod_{q\sim e} G(q')-\prod_{q\sim e}(-G(q''))=[\prod_{q\sim e}(-G(q''))][\prod_{q\sim e} q-1]=0$. Note that by the construction of $\mathcal{R}(\mathcal{T})$, for $q\in\square$, then q and $1-q$ are invertible in $\mathcal{R}(\mathcal{T})$ with inverses $1-q''$ and q' where $q\to q'\to q''$. From the above calculation, we see that G induces a homomorphism from $\mathcal{R}_h(\mathcal{T})_S\to\mathcal{R}(\mathcal{T})$. To check $GF=id$, it suffices to see that $GF(q) = q$ for $q \in \square$. Let $q \to q' \to q''$, where $G(q'') = -1$, $G(q') = q$ and $G(q) = 1 - q = -qq''$. Then $GF(q) = G(-q'/q'') = -q/(-1) = q$, $GF(q') = G(-q''/q) = 1/(1-q) = q'$ and $GF(q'') = G(-q/q') = (q-1)/q = q''$.

To see the last statement, if $\mathcal{R}_h(\mathcal{T})_S = \{0\}$, then $\mathcal{R}(\mathcal{T}) = \{0\}$ since F is injective. On the other hand, if $\mathcal{R}(\mathcal{T}) = \{0\}$, then we claim that $q = 0$ in $\mathcal{R}_h(\mathcal{T})$ for some $q \in \square$. Indeed, if not, then due to $F(q) = -q'/q'' \neq 0$, we see that $q \neq 0$ in $\mathcal{R}(\mathcal{T})$. This contradicts the assumption. Therefore, the multiplicatively closed set S contains 0. Hence $\mathcal{R}_h(\mathcal{T})_S = \{0\}$. □

Note that if $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a $2 \to 3$ or $0 \to 2$ move, then there exists a natural ring homomorphism from $\mathcal{R}(\mathcal{T})$ to $\mathcal{R}(\mathcal{T}')$.

7. *Example of solving Thurston's equation in finite rings*

Suppose that $(M, \mathcal{T})$ is a closed oriented pseudo 3-manifold and R is a commutative ring with identity and $x : \square \to R$ solves Thurston's equation. If p is a prime number, let F_{p^n} be the finite field of p^n elements.

EXAMPLE 7.1. For $R = F_3$, then $x : \square \to F_3 - \{0, 1\}$ is the constant map $x(q) = 2$. Thus, as mentioned in § 1.1, Thurston's equation is solvable if and only if each edge has even degree.

EXAMPLE 7.2. For $R = F_5 = \{0, 1, 2, 3, 4\}$, we are looking for $x : \square \to \{2, 3, 4\}$. Due to $1/(1-2) = 4, 1/(1-4) = 3, 4 = 2^2, 3 = 2^3$ so that $2^4 = 1$, we can write $x(q) = 2^{z(q)}$ where $z \in \{1, 2, 3\}$. Thus Thurston's equation is solvable if and only if, for $q \to q' \to q''$, $(z(q), z(q'), z(q'')) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ so that, for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 4$.

EXAMPLE 7.3. For $R = F_{2^2} = \{0, 1, a, b\}$ where $b = a + 1 = a^2$ and $a^3 = 1$, we have $1/(1-a) = a$ and $1/(1-b) = b$. By writing solution x of Thurston's equation as $x(q) = a^{z(q)}$ where $z(q) \in \{1, 2\}$, we see that $z(q) = z(q')$ if $q \to q'$. Therefore, Thurston's equation is solvable if and only if there is $z : \mathcal{T}^{(3)} \to \{1, 2\}$ so that, for each edge e, $|\{\sigma \in \mathcal{T}^{(3)} | \sigma > e, z(\sigma) = 1\}| + 2|\{\sigma \in \mathcal{T}^{(3)} | \sigma > e, z(\sigma) = 2\}| = 0 \mod 3$.

EXAMPLE 7.4. For the field F_7, write $x(q) = 3^{z(q)}$. Then Thurston's equation is solvable if and only if $z : \square \to \{1, 2, 3, 4, 5\}$ satisfies that $(z(q), z(q'), z(q'')) \in \{(1, 1, 1), (5, 5, 5), (2, 3, 4), (3, 4, 2), (4, 2, 3)\}$ when $q \to q' \to q''$ and, for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 6$.

EXAMPLE 7.5. For the ring $\mathbf{Z}/9\mathbf{Z}$ (not F_{3^2}), since a solution $x(q)$ must satisfy $x(q)$ and $x(q) - 1$ are invertible, we conclude that $x(q) \in \{2, 5, 8\} = \{2, 2^5, 2^3\}$. Write $x(q) = 2^{z(q)}$. Therefore, Thurston's equation is solvable if and only if there is $z : \square \to \{1, 3, 5\}$ so that $(z(q), z(q'), z(q'')) \in \{(1, 3, 5), (3, 5, 1), (5, 1, 3)\}$ if $q \to q' \to q''$ and, for each edge e, $\sum_{q \sim e} z(q) = 0 \mod 6$. This implies that the degree of each edge must be even.

EXAMPLE 7.6. For the ring $\mathbf{Z}/15\mathbf{Z}$, the same argument as in Example 6.5 shows that Thurston's equation is solvable if and only if $x : \square \to \{2, 8, 14\}$ satisfies $(x(q), x(q'), x(q'')) \in \{(2, 14, 8), (14, 8, 2), (8, 2, 14)\}$ if $q \to q' \to q''$ and, for each edge e, $\prod_{q \sim e} x(q) = 0 \mod 15$.

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