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SIMPLE LOOPS ON SURFACES AND THEIR INTERSECTION NUMBERS

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Abstract

Given a compact orientable surface, we determine a complete set of relations for a function defined on the set of all homotopy classes of simple loops to be a geometric intersection number function. As a consequence, Thurston's space of measured laminations and Thurston's compactification of the Teichmüller space are described by a set of explicit equations. These equations are polynomials in the max-plus semi-ring structure on the real numbers. It shows that Thurston's theory of measured laminations is within the domain of tropical geometry.

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1. Introduction

Given a compact orientable surface $\Sigma = \Sigma_{g,r}$ of genus g with $r \geq 0$ boundary components, let $\mathbb{S} = \mathbb{S}(\Sigma)$ be the set of isotopy classes of essential simple loops on Σ. A function $f : \mathbb{S}(\Sigma) \to \mathbf{R}$ is called a *geometric intersection number function,* or simply *geometric function* if there is a measured lamination m on Σ so that $f(\alpha)$ is the measure of α in m. Geometric functions were introduced and studied by W. Thurston in his work on classification of surface homeomorphisms and

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his compactification of the Teichmüller spaces ([FLP], [Th]). The space of all geometric functions under the pointwise convergence topology is homeomorphic to Thurston's space of measured laminations $ML(\Sigma)$. Thurston showed that $ML(\Sigma)$ is homeomorphic to a Euclidean space and $ML(\Sigma)$ has a piecewise integral linear structure invariant under the action of the mapping class group. The projectivization of $ML(\Sigma)$ is Thurston's boundary of the Teichmüller space. The goal of the paper is to characterize explicitly all geometric functions on $\mathbb{S}(\Sigma)$. As a consequence, both $ML(\Sigma)$ and its projectivization are reconstructed explicitly in terms of an intrinsic $(\mathbf{Q}P^1, PSL(2,\mathbf{Z}))$ structure on $\mathbb{S}(\Sigma)$.

Very recently, tropical geometry on the max-plus semiring structure on the real numbers was introduced in [Mi], [RST] and others. The equations that we discovered to define the geometric intersection number functions are polynomials in the max-plus semiring. This shows that Thurston's space of measured laminations is a tropical variety and the method of tropical geometry could be used to study the geometry and topology of surfaces.

Theorem 1.1. *Suppose Σ is a compact orientable surface of negative Euler number. Then a real valued function f on $\mathbb{S}(\Sigma)$ is a geometric function if and only if for each essential subsurface $\Sigma' \cong \Sigma_{1,1}$ or $\Sigma_{0,4}$, the restriction $f|_{\mathbb{S}(\Sigma')}$ is geometric. Furthermore, geometric functions on $\mathbb{S}(\Sigma_{1,1})$ and $\mathbb{S}(\Sigma_{0,4})$ are characterized by two sets of homogeneous equations in a $(\mathbf{Q}P^1, PSL(2,\mathbf{Z}))$ structure on $\mathbb{S}(\Sigma)$.*

Recall that a subsurface $\Sigma' \subset \Sigma$ is *essential* if each non-null homotopic loop in Σ' is homotopically non-trivial in Σ. A loop in a surface is called *essential* if it is homotopically non-trivial. And a proper arc in a surface is *essential* if it is homotopically non-trivial relative to the boundary of the surface. It is well known that if each boundary component of Σ' is essential in Σ, then Σ' is essential.

Geometric functions and measures laminations haven been studied from many different points of views. Especially, they are identified with height functions and horizontal foliations associated to holomorphic quadratic forms on Σ ([Ga], [HM], [Ker1]). They are also related to the translation length functions of group action on $\mathbf{R}$-trees and the valuation theory ([Bu], [CM], [MS], [Par]). In [Bo1], measured laminations and hyperbolic metrics are considered as special cases of currents. As a consequence, Thurston's compactification is derived from a natural setting.

Our approach is combinatorial and is based on the notion of curve systems ([De], [FLP], [PH], [Th]). Recall that a *curve system* is a finite disjoint union of essential proper arcs and essential non-boundary parallel simple loops on the surface. Let $\mathbb{CS}(\Sigma)$ be the set of isotopy

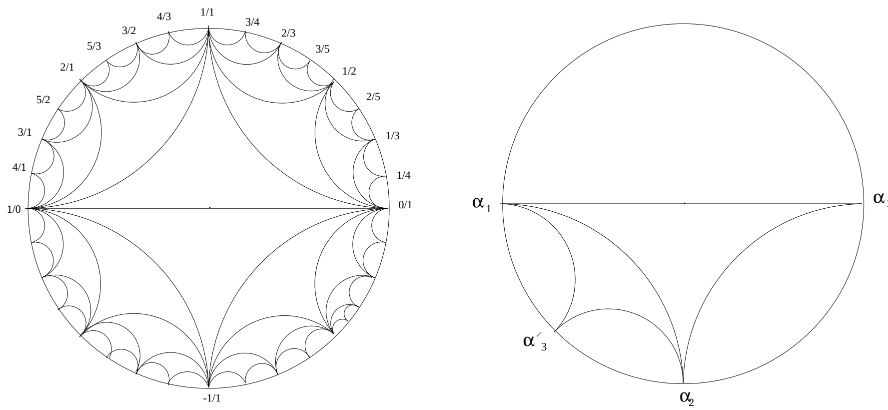

Figure 1

classes of curve systems on Σ. The space $\mathbb{CS}(\Sigma)$ was introduced by Dehn and rediscovered independently by Thurston. Dehn called the space the *arithmetic field* of the topological surface. Given two classes α, β in $\mathbb{CS}(\Sigma) \cup \mathbb{S}(\Sigma)$, their *geometric intersection number* $I(\alpha, \beta)$ is defined to be $\min\{|a \cap b| : a \in \alpha, b \in \beta\}$. The goal of the paper is to characterize those geometric functions f on $\mathbb{S}(\Sigma)$ so that $f(\alpha) = I(\alpha, \beta)(:= I_\beta(\alpha))$ for some fixed $\beta \in \mathbb{CS}(\Sigma)$ or $\beta \in ML(\Sigma)$.

In the rest of the paper, we will always assume that surfaces are connected, oriented and have negative Euler characteristic. Following Grothendieck, the *level* of the surface $\Sigma_{g,r}$ is defined to be $3g + r - 3$. Thus the level-0 surface is the 3-holed sphere $\Sigma_{0,3}$ and level-1 surfaces are the 1-holed torus $\Sigma_{1,1}$ and the 4-holed sphere $\Sigma_{0,4}$. Level-2 surfaces are the 2-holed torus or the 5-holed sphere.

Given a surface Σ, let $\mathbb{S}'(\Sigma) = \mathbb{CS}(\Sigma) \cap \mathbb{S}(\Sigma)$ be the set of isotopy classes of essential, non-boundary parallel simple loops in Σ. For level-1 surfaces $\Sigma = \Sigma_{1,1}$ and $\Sigma_{0,4}$, it is well known since the work of Dehn and Nielsen that there exists a bijection $\pi : \mathbb{S}'(\Sigma) \to \mathbb{QP}^1$ $(= \mathbb{Q} \cup \{\infty\})$ so that $p'q - pq' = \pm 1$ if and only if $I(\pi^{-1}(p/q), \pi^{-1}(p'/q')) = 1$ (for $\Sigma_{1,1}$) and 2 (for $\Sigma_{0,4}$). See figure 1. If Σ is a level-1 surface, Dehn-Nielsen's observation says that the set $\mathbb{S}'(\Sigma)$ has a natural modular structure invariant under the action of the mapping class group. This structure holds the key for us to understand the geometric intersection number function. We say that three distinct classes α, β, γ in $\mathbb{S}'(\Sigma)$ form a *triangle* if they correspond to the vertices of an ideal triangle in the modular relation under the map π, i.e., $I(\alpha, \beta) = I(\beta, \gamma) = I(\gamma, \alpha) = 1$ for $\Sigma_{1,1}$, and $I(\alpha, \beta) = I(\beta, \gamma) = I(\gamma, \alpha) = 2$ for $\Sigma_{0,4}$.

Theorem 1.2. *(i) For the 1-holed torus $\Sigma_{1,1}$, a function $f : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{Z}_{\geq 0}$ is a geometric intersection number function I_δ with $\delta \in \mathbb{CS}(\Sigma)$ if* *and only if the following hold.*

$$f(\alpha_1) + f(\alpha_2) + f(\alpha_3) = \max_{i=1,2,3}(2f(\alpha_i), f([\partial\Sigma_{1,1}])) \tag{1}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ *is a triangle, and*

$$f(\alpha_3) + f(\alpha_3') = \max(2f(\alpha_1), 2f(\alpha_2), f([\partial\Sigma_{1,1}])) \tag{2}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1, \alpha_2, \alpha_3')$ *are two distinct ideal triangles.*

$$f([\partial\Sigma_{1,1}]) \in 2\mathbf{Z}. \tag{3}$$

(ii) For the surface $\Sigma_{0,4}$ *with* $\partial\Sigma_{0,4} = b_1 \cup b_2 \cup b_3 \cup b_4$, *a function* $f : \mathbb{S}(\Sigma_{0,4}) \to \mathbf{Z}_{\geq 0}$ *is a geometric function* I_δ *for some* $\delta \in \mathbb{CS}(\Sigma)$ *if and only if for each ideal triangle* $(\alpha_1, \alpha_2, \alpha_3)$ *so that* (α_i, b_s, b_r) *bounds a* $\Sigma_{0,3}$ *in* $\Sigma_{0,4}$ *the following hold.*

$$\sum_{i=1}^{3} f(\alpha_i) \tag{4}$$

$$= \max_{1\leq i\leq 3; 1\leq s\leq 4}\left(2f(\alpha_i), 2f(b_s), \sum_{j=1}^{4} f(b_j), f(\alpha_i) + f(b_s) + f(b_r)\right)$$

$$f(\alpha_3) + f(\alpha_3') \tag{5}$$

$$= \max_{1\leq i\leq 2; 1\leq s\leq 4}\left(2f(\alpha_i), 2f(b_s), \sum_{j=1}^{4} f(b_j), f(\alpha_i) + f(b_s) + f(b_r)\right)$$

where $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1, \alpha_2, \alpha_3')$ *are two distinct ideal triangles,*

$$f(\alpha_i) + f(b_s) + f(b_r) \in 2\mathbf{Z}. \tag{6}$$

(iii) A characterization of real valued geometric functions $f : \mathbb{S}(\Sigma) \to \mathbf{R}_{\geq 0}$ *for* $\Sigma = \Sigma_{1,1}$ *and* $\Sigma_{0,4}$ *is given by equations (1),(2) (for* $\Sigma_{1,1}$*) and (10), (11) (for* $\Sigma_{0,4}$*).*

Theorem 1.2 is motivated by the tours. In fact for the torus $\Sigma_{1,0}$, a function on $\mathbb{S}(\Sigma_{1,0})$ is a geometric intersection number function if and only if it satisfies the triangular equality $f(\alpha_1) + f(\alpha_2) + f(\alpha_3) = \max_{i=1,2,3}(2f(\alpha_i))$ and $f(\alpha_3) + f(\alpha_3') = \max(2f(\alpha_1), 2f(\alpha_2))$.

Equations (1),(2),(4) and (5) in theorem 1.2 are polynomials in the max-plus semiring structure in $\mathbf{R}$ and are obtained as the degenerations of the trace identities for $SL(2, \mathbf{R})$ matrices. For instance, equations (1), (2) are the degenerations of $tr(A)tr(B)tr(AB) = tr^2(A) + tr^2(B) + tr^2(AB) - tr([A, B]) - 2$ and $tr(AB)tr(A^{-1}B) = tr^2(A) + tr^2(B) - tr([A, B]) - 2$.

Theorems 1.1 and 1.2 show that Thurston's theory of measured laminations is within the domain of tropical geometry. Thus tools from tropical geometry could be used to investigate geometry and topology of surfaces. On the other hands, equations (1), (2), (4) and (5) are semi-real algebraic. Indeed, the space defined by $\sum_{i=1}^{k} x_i = \max_{1\leq j\leq l}(y_j)$ is semi-real algebraic since it is equivalent to: $\prod_{j=1}^{l}(\sum_{i=1}^{k} x_i - y_j) = 0$, and $\sum_{i=1}^{k} x_i \geq y_j$, for all j. This seems to indicate that the space $ML(\Sigma)$ has a semi-real algebraic structure. Given a surface $\Sigma_{g,r}$, Thurston showed that there exists a finite set F consisting of $9g+4r-9$ elements in $\mathbb{S}(\Sigma)$ so that the projection map $\tau_F : ML(\Sigma) \to \mathbf{R}_{\geq 0}^{F}$ sending m to $I_m|_F$ is an embedding ([FLP]). As a consequence of theorems 1.1 and 1.2 we have,

Corollary 1.3. *For the surface $\Sigma_{g,r}$ of negative Euler number, there is a finite set F consisting of $9g+4r-9$ elements in $\mathbb{S}(\Sigma)$ so that the projection map τ_F is an embedding whose image is a tropical varierty.*

It is interesting to observe that the approach taken in the paper (also in [Lu1], [Lu3]) follows Grothendieck's philosophy of the "Teichmüller tower" where the "generators" are the level-1 surfaces $\Sigma_{1,1}$ and $\Sigma_{0,4}$ and the "relations" are the level-2 surfaces $\Sigma_{1,2}$ and $\Sigma_{0,5}$. See [Sch] and [Lu4] for more details. It also shows that the modular structure $(\mathbf{Q}P^1, PSL(2,\mathbf{Z}))$ is fundamental to the topology and geometry of surfaces.

The organization of the paper is as follows. In §2, we establish several basic properties of the curve systems. In particular, a multiplicative structure on $\mathbb{CS}(\Sigma)$ is introduced. In §3 and §4, we prove theorem 1.2. The proof in §4 is complicated due to the existence of eight different ideal triangulations of the surface $\Sigma_{0,4}$. In §5, we prove a reduction result. It reduces the general case to two surfaces: $\Sigma_{1,2}$ and $\Sigma_{0,5}$. In §6 and §7, we prove theorem 1.1 for surfaces $\Sigma_{1,2}$ and $\Sigma_{0,5}$. Proofs of the results in §2 are in §8.

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2. A Multiplicative Structure on Curve Systems

We work in the piecewise linear category. Surfaces are oriented and connected and have negative Euler numbers unless specified otherwise. A regular neighborhood of a submanifold X is denoted by $N(X)$. Regular neighborhoods are assumed to be small. The isotopy class of a curve system c will be denoted by $[c]$. Suppose $f : \mathbb{CS}(\Sigma) \to \mathbf{R}$ is a function and c is a curve system. We define $f(c)$ to be $f([c])$. In particular, $I(a,b) = I([a],[b])$. Homeomorphic manifolds X, Y are denoted by $X \cong Y$. Isotopic submanifolds c, d are denoted by $c \cong d$. If $m \in ML(\Sigma)$, I_m denotes the geometric intersection number function with respect to m. A class in $\mathbb{CS}(\Sigma_{g,r})$ is called a *3-holed sphere decomposition* (resp. an *ideal triangulation*) if it is the isotopy class of $3g+r-3$ (resp.

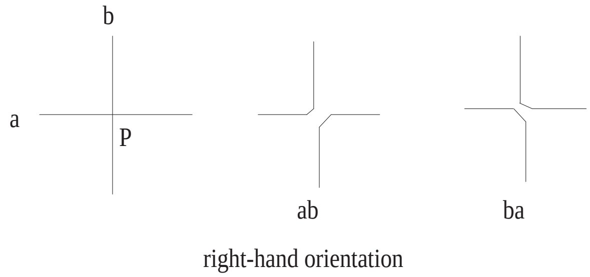


Figure 2

$6g+2r-6$) pairwise non-isotopic non-boundary parallel simple loops (resp. proper arcs). The numbers $3g+r-3$ and $6g+2r-6$ are maximal.

A convention : all surfaces drawn in this paper have the right-hand orientation in the front face.

2.1. A multiplicative structure on $\mathbb{CS}(\Sigma)$. Suppose a and b are two arcs in Σ intersecting transversely at one point P. Then the *resolution of $a \cup b$ at P from a to b* is defined as follows. Take any orientation on a and use the orientation on Σ to determine an orientation on b. This means that the orientation at P from the oriented a to the oriented b is the given orientation on the surface. Then resolve the intersection according to the orientations. The resolution is independent of the choice of the orientation on a. See figure 2. The resolution of $a \cup b$ from b to a is the opposition resolution.

Given two curve systems a, b on Σ with $|a \cap b| = I(a,b)$, the *multiplication ab* is defined to be the disjoint union of simple loops and arcs obtained by resolving all intersection points from a to b. It is shown in §8 (lemma 8.1) that ab is again a curve system whose isotopy class depends only on the isotopy classes of a, b. Given α,$\beta \in \mathbb{CS}(\Sigma)$, we define $\alpha\beta = [ab]$ where $a \in \alpha$, $b \in \beta$ so that $|a \cap b| = I(a,b)$. For $k \in \mathbf{Z}_{\geq 0}$, let α^k denote the self multiplication of α k times. The following theorem establishes the basic properties of the multiplication. See §8 for a proof.

Let $\mathbb{CS}_0(\Sigma)$ be the subset of $\mathbb{CS}(\Sigma)$ consisting of isotopy classes of curve systems which contain no arc components.

Theorem 2.1. *The multiplication $\mathbb{CS}(\Sigma) \times \mathbb{CS}(\Sigma) \to \mathbb{CS}(\Sigma)$ sends $\mathbb{CS}_0(\Sigma) \times \mathbb{CS}_0(\Sigma)$ to $\mathbb{CS}_0(\Sigma)$ and satisfies the following properties.*

(i) It is preserved by the action of the orientation preserving homeomorphisms.

(ii) If $I(\alpha,\beta)=0$, then $\alpha\beta = \beta\alpha$. Conversely, if $\alpha\beta = \beta\alpha$ and $\alpha \in \mathbb{CS}_0(\Sigma)$, then $I(\alpha,\beta)=0$.

(iii) If $\alpha \in \mathbb{CS}_0(\Sigma)$, $\beta \in \mathbb{CS}(\Sigma)$, then $I(\alpha,\alpha\beta) = I(\alpha,\beta\alpha) = I(\alpha,\beta)$ and $\alpha(\beta\alpha) = (\alpha\beta)\alpha$. If in addition that each component of α intersects β, then $\alpha(\beta\alpha) = \beta$.

(iv) If $[c_i] \in \mathbb{CS}(\Sigma)$ *so that* $|c_i \cap c_j| = I(c_i, c_j)$ *for* $i, j = 1, 2, 3$, $i \neq j$, $|c_1 \cap c_2 \cap c_3| = 0$, *and there is no contractible region in* $\Sigma - (c_1 \cup c_2 \cup c_3)$ *bounded by three arcs in* c_1, c_2, c_3, *then* $[c_1]([c_2][c_3]) = ([c_1][c_2])[c_3]$.

(v) For any positive integer k, $\alpha^k \beta^k = (\alpha\beta)^k$.

(vi) If α *is the isotopy class of a simple closed curve, then the positive Dehn twist along* α *sends* β *to* $\alpha^k \beta$ *where* $k = I(\alpha, \beta)$.

It follows from the definition that $I(\alpha, \gamma) + I(\beta, \gamma) \geq I(\alpha\beta, \gamma)$. Furthermore, theorem 2.1(iii) implies a stronger result that $I(\alpha\beta, \gamma) + I(\alpha, \gamma) \geq I(\beta, \gamma)$ when $\alpha, \beta \in CS_0(\Sigma)$. Indeed, $I(\alpha\beta, \gamma) + I(\alpha, \gamma) \geq I((\alpha\beta)\alpha, \gamma) \geq I(\beta\delta^2, \gamma) \geq I(\beta, \gamma)$ where δ consists of components of α which are disjoint from β.

2.2. The modular relation on $\mathbb{S}'(\Sigma_{1,1})$ and $\mathbb{S}'(\Sigma_{0,4})$. Given α, $\beta \in \mathbb{S}(\Sigma)$, we use $\alpha \perp \beta$ to denote $I(\alpha, \beta) = 1$ and use $\alpha \perp_0 \beta$ if $I(\alpha, \beta)$=2 so that their algebraic intersection number is zero. Suppose $\alpha \perp \beta$ or $\alpha \perp_0 \beta$. Take $a \in \alpha$, $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$. Then $N(a \cup b) \cong \Sigma_{1,1}$ if $\alpha \perp \beta$ and $N(a \cup b) \cong \Sigma_{0,4}$ if $\alpha \perp_0 \beta$. It follows from the definition that $\alpha\beta \perp \alpha, \beta$ if $\alpha \perp \beta$, and $\alpha\beta \perp_0 \alpha, \beta$ if $\alpha \perp_0 \beta$. Three distinct elements α, β, $\gamma \in \mathbb{S}'(\Sigma_{1,1})$ (resp. $\mathbb{S}'(\Sigma_{0,4})$) form a triangle if and only if $\gamma \in \{\alpha\beta, \beta\alpha\}$ so that $\alpha \perp \beta$ or $\alpha \perp_0 \beta$. In particular distinct triangles in equations (2), (4) in theorem 1.2 are $(\alpha_1, \alpha_2, \alpha_1\alpha_2)$ and $(\alpha_1, \alpha_2, \alpha_2\alpha_1)$ where $\alpha_1 \perp \alpha_2$ or $\alpha_1 \perp_0 \alpha_2$.

If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, we define $\alpha^{-n}\beta = \beta\alpha^n$ for $n \in \mathbf{Z}_{>0}$. It follows from proposition 2.1(c) that $\alpha^n(\alpha^m\beta) = \alpha^{n+m}\beta$ and $(\beta\alpha^n)\alpha^m = \beta\alpha^{n+m}$, for $n, m \in \mathbf{Z}$.

We use $\partial N(\alpha \cup \beta)$ to denote the set of isotopy classes of simple loops appeared in $\partial N(a \cup b)$ where $a \in \alpha$, $b \in \beta$ and $|a \cap b| = I(\alpha, \beta)$.

For $\Sigma = \Sigma_{1,1}$ or $\Sigma_{0,4}$, we can find an explicit bijection from $\mathbb{S}'(\Sigma)$ to $\hat{\mathbf{Q}}$ as follows. Take α, β in $\mathbb{S}'(\Sigma)$ so that $\alpha \perp \beta$ or $\alpha \perp_0 \beta$. Then each γ in $\mathbb{S}'(\Sigma)$ can be expressed uniquely as $\alpha^p\beta^q$ where $q \in \mathbf{Z}_{\geq 0}$, $p \in \mathbf{Z}$ and p, q are relatively prime. Define $\pi(\gamma) = p/q$ from $\mathbb{S}'(\Sigma)$ to $\hat{\mathbf{Q}}$. The map π is a bijection from the work of Dehn and Nielsen. Furthermore, suppose $\pi(\gamma_i) = p_i/q_i$, $i = 1, 2$. Then $p_1q_2 - p_2q_1 = \pm 1$ if and only if $\gamma_1 \perp \gamma_2$ or $\gamma_1 \perp_0 \gamma_2$.

Given two simple loops a, b, we use $a \perp b$ to denote $|a \cap b| = I(a, b) = 1$, and use $a \perp_0 b$ to denote $|a \cap b| = I(a, b) = 2$ and $[a] \perp_0 [b]$.

2.3. The restriction map and a gluing lemma. Suppose Σ' is an essential subsurface of Σ. We define the restriction map $R(= R^{\Sigma}_{\Sigma'})$: $\mathbb{CS}(\Sigma) \to \mathbb{CS}(\Sigma')$ as follows. Given α in $\mathbb{CS}(\Sigma)$, take $a \in \alpha$ so that $|a \cap \partial\Sigma'| = I(a, \partial\Sigma')$ and $a \cap \Sigma'$ contains no component parallel into $\partial\Sigma'$. We define $R(\alpha) = [a|_{\Sigma'}](:= \alpha|_{\Sigma'})$. The restriction map is well defined between the space of measured laminations $ML(\Sigma) \to ML(\Sigma')$ by the same construction. We use $m|_X$ to denote the restriction of a measure

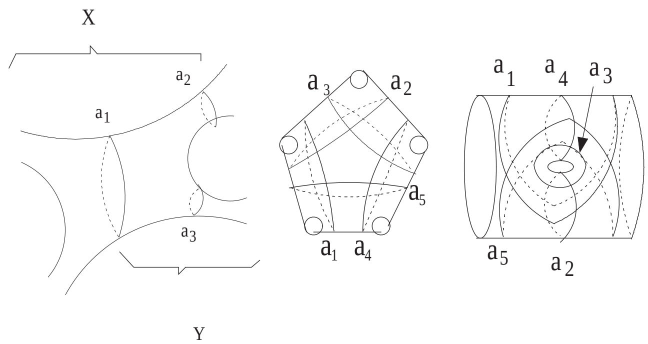


Figure 3

lamination m to the essential subsurface X. Note that if $X \subset Y$ and $Y \subset Z$ are essential subsurfaces, then $(m|_Y)|_X = m|_X$.

We say a subsurface X of Y *proper* if X is not homotopic to Y.

Lemma 2.2 *(Gluing along a 3-holed sphere) Suppose X and Y are essential proper subsurfaces in Σ so that $X \cap Y \cong \Sigma_{0,3}$ and $X \cup Y = \Sigma$. Then for any two elements $m_X \in ML(X)$, $m_Y \in ML(Y)$ with $m_X|_{X \cap Y} = m_Y|_{X \cap Y}$, there is a unique element $m \in ML(\Sigma)$ so that $m|_X = m_X$ and $m|_Y = m_Y$.*

Proof. Take a 3-holed sphere decomposition $\{a_1, ..., a_n\}$ of $X \cup Y$ so that $int(X \cap Y)$ is a component of $X \cup Y - \cup_{i=1}^n a_i$. Assume the isotopy classes of components of $\partial(X \cup Y)$ are $[a_1], ..., [a_k]$ where $k \leq 3$. By the construction, we also obtain a 3-holed sphere decompositions of X and Y using $a_1, ..., a_n$. Now we use the Dehn-Thurston coordinates ([FLP]) with respect to the 3-holed sphere decompositions for measured lamination spaces in surfaces X, Y and $X \cup Y$. Construct a measured lamination $m \in ML(X \cup Y)$ as follows. If $i > k$, then a_i lies in X or Y, but not both. We define the Dehn-Thurston coordinate of m at a_i for $i > k$ to be the Dehn-Thurston coordinate of m_X or m_Y at a_i. If $i \leq k$ and a_i is a boundary component of $X \cup Y$, then the Dehn-Thurston coordinate of m at a_i is that of m_X and m_Y at a_i. These two coordinates are the same due to the assumption $m_X|_{X \cap Y} = m_Y|_{X \cap Y}$. If $i \leq k$ and a_i is not a boundary component of $X \cup Y$, then a_i is a boundary component of X or Y. Say $a_i \subset \partial X$. Since both X and Y are proper subsurfaces, a_i is in Y and is not homotopic into ∂Y. We define the Dehn-Thurston coordinate of m at a_i to be that of m_Y at a_i. By the assumption that $m_X|_{X \cap Y} = m_Y|_{X \cap Y}$, we see that m is well defined with the required properties. The uniqueness follows from the basic property of Dehn-Thurston coordinates. q.e.d.

Remark 2.1. Note that if m_X and m_y are curve systems, then we obtain a curve system m in $X \cup Y$.

Remark 2.2. For surface with boundary, Mosher [Mo] has introduced a parametrization of $ML(\Sigma)$ using a ideal triangulation where the coordinates are the intersection numbers.

2.4. The pentagon relations. The following relationship among five simple loops will play an important role in the proof of theorem 1.1 for level-2 surfaces. Let Σ be either the 2-holed torus or the 5-holed sphere. Five distinct classes of simple loops $a_1, ..., a_5$ are said to form a *pentagon relation* if $I(a_i, a_j) = 0$ where indices $|i-j| \neq 1$ modulo 5.

Proposition 2.3. ([Lu3]) *If $a_1, ..., a_5$ form a pentagon relation, then up to a homeomorphism of Σ, the five classes are shown in the figure 3. Furthermore,*

(i) if $\Sigma = \Sigma_{0,5}$, then $(a_1a_2)(a_3a_4) = a_5$, $a_ia_{i+1} \cap a_ia_{i-1} = \emptyset$, and

(ii) if $\Sigma = \Sigma_{1,2}$ so that a_1, a_5 are separating, then $a_3a_2 = (a_1a_2)(a_3a_4^2)$.

The proof of the statements (i), (ii) will be given in §6.3 and §7.2. The proof of the first statement is in [Lu3].

3. The One-holed Torus

The goal of this section is to show theorem 1.2 for $\Sigma_{1,1}$. We restate the result in terms of the multiplicative structure as follows.

Theorem 3.1. *A function $f : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{Z}_{\geq 0}$ is the geometric intersection number function I_δ for some $\delta \in \mathbb{CS}(\Sigma_{1,1})$ if and only if for $\alpha \perp \beta$ and $\gamma = \alpha\beta$,*

$$f(\alpha) + f(\beta) + f(\gamma) = \max(2f(\alpha), 2f(\beta), 2f(\gamma), f(\partial\Sigma_{1,1})) \tag{7}$$

$$f(\alpha\beta) + f(\beta\alpha) = \max(2f(\alpha), 2f(\beta), f(\partial\Sigma_{1,1})) \tag{8}$$

$$f(\partial\Sigma_{1,1}) \in 2\mathbf{Z}. \tag{9}$$

Furthermore, a characterization of real-valued geometric functions $f : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{R}_{\geq 0}$ is given by equations (7), (8) above.

Proof. To see the necessity, we double the surface $\Sigma_{1,1}$ to obtain the genus two surface $\Sigma_{2,0} = \Sigma_{1,1} \cup_{id_\partial} \Sigma_{1,1}$. Then each $\gamma \in \mathbb{CS}(\Sigma_{1,1})$ corresponds to $\hat{\gamma} \in \mathbb{CS}(\Sigma_{2,0})$ whose restriction to both summands $\Sigma_{1,1}$ are γ. Let $\{d_i\}$ be a sequence of hyperbolic metrics on $\Sigma_{2,0}$ which pinch to $\hat{\gamma}$, i.e., there is a sequence of positive real numbers $\{\lambda_i\}$ so that

$$\lim_i \lambda_i l_{d_i}(\alpha) = I_{\hat{\gamma}}(\alpha)$$

for all $\alpha \in \mathbb{S}(\Sigma_{2,0})$ where $l_{d_i}(\alpha)$ is the length of the d_i-geodesic in the class α. Let $t_i(\alpha) = 2\cosh(l_{d_i}(\alpha)/2)$. It is shown in [FK], [Ke] (see also [Lu1]) that for $\alpha \perp \beta$ in $\mathbb{S}(\Sigma_{1,1})$ ($\subset \mathbb{S}(\Sigma_{2,0})$), one has the following identities:

$$t_i(\alpha)t_i(\beta)t_i(\alpha\beta) = t_i^2(\alpha) + t_i^2(\beta) + t_i^2(\alpha\beta) + t_i(\partial\Sigma_{1,1}) - 2$$

and

$$t_i(\alpha\beta)t_i(\beta\alpha) = t_i^2(\alpha) + t_i^2(\beta) + t_i(\partial\Sigma_{1,1}) - 2.$$

On the other hand, for $\alpha \in \mathbb{S}(\Sigma_{1,1})$, we have $I_{\hat{\gamma}}(\alpha) = I_{\gamma}(\alpha)$. Take the logarithm to the above two identities involving l_{d_i} and let i tend to infinity. By $\lim_i \lambda_i l_{d_i}(\alpha) = I_{\hat{\gamma}}(\alpha)$, we have

$$\lim_i \lambda_i \log(2\cosh(l_{d_i}/2)) = I_{\hat{\gamma}}(\alpha)/2.$$

Furthermore,

$$\lim_i \lambda_i \log\left(t_i^2(\alpha) + t_i^2(\beta) + t_i^2(\alpha\beta) + t_i(\partial\Sigma_{1,1}) - 2\right) =$$

$$\max(I_{\hat{\gamma}}(\alpha), I_{\hat{\gamma}}(\beta), I_{\hat{\gamma}}(\alpha\beta), I_{\hat{\gamma}}(\partial\Sigma_{1,1})/2).$$

Thus we obtain (7) in theorem 3.1. The same argument establishes (8). Equation (9) is evident. q.e.d.

Remark 3.1. To derive equation (7) directly from the trace identity

$$tr(A)tr(B)tr(AB) = tr^2(A) + tr^2(B) + tr^2(AB) - tr([A,B]) - 2$$

where $A, B \in SL(2, \mathbf{R})$, we assume that A, B, AB correspond to three simple closed geodesics forming a triangle in $\mathbb{S}$. Then $tr(A)tr(B)tr(AB) > 0$ and $tr([A,B]) < 0$ (see [GiM] for instance). In particular, we obtain

$$|tr(A)||tr(B)||tr(AB)| = tr^2(A) + tr^2(B) + tr^2(AB) + |tr([A,B])| - 2.$$

Now the length $l(A)$ of the geodesic in the class A satisfies

$$|tr(A)| = 2\cosh(l(A)/2).$$

The degeneration of it becomes

$$f(A) + f(B) + f(AB) = \max(2f(A), 2f(B), 2f(AB), f([A,B]))$$

which is equation (7).

To show that the conditions (7)−(9) are also sufficient, we begin with a function $f : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{Z}_{\geq 0}$ satisfying equations (7),(8),(9). Our first observation is that if f, g are two functions on $\mathbb{S}(\Sigma_{1,1})$ satisfying equation (8) so that $f = g$ on $\{a, b, ab, \partial\Sigma_{1,1}\}$ where $a \perp b$, then $f = g$ due to the modular configuration. Below, we will construct $\delta \in \mathbb{CS}(\Sigma_{1,1})$ so that f and I_δ have the same values at $\{a, b, ab, \partial\Sigma_{1,1}\}$ for some $a \perp b$.

We consider two cases: $\min\{f(\alpha) : \alpha \in \mathbb{S}'(\Sigma_{1,1})\} = 0$, or > 0.

Case 1. There is $\alpha \in \mathbb{S}'(\Sigma)$ so that $f(\alpha) = 0$. In this case, if $\beta \perp \alpha$ and $\gamma = \alpha\beta$, then $f(\beta) = f(\gamma)$. Indeed, by equation (7),

$$f(\beta) + f(\gamma) = \max(2f(\beta), 2f(\gamma), f(\partial\Sigma_{1,1}))$$

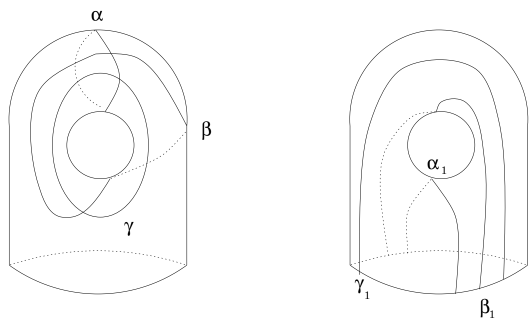


Figure 4

$$\geq \max(2f(\beta), 2f(\gamma))$$

Thus $f(\beta) = f(\gamma)$. In particular, $f(\beta) \geq \frac{1}{2}f(\partial\Sigma_{1,1})$. We construct the curve system δ as follows. Let $\Sigma' = \Sigma_{1,1} - int(N(a))$ where $a \in \alpha$. Then $\Sigma' \cong \Sigma_{0,3}$. Curve systems on $\Sigma_{0,3}$ with boundary components $\partial\Sigma_{0,3} = b_1 \cup b_2 \cup b_3$ are well understood. Namely, $\mathbb{S}(\Sigma_{0,3}) = \{b_1, b_2, b_3\}$ and each $\delta \in \mathbb{CS}(\Sigma_{0,3})$ is uniquely determined by $\pi(\delta) = (I_{b_1}(\delta), I_{b_2}(\delta), I_{b_3}(\delta))$. Furthermore, each triple of non-negative integers whose sum is even is of the form $\pi(\delta)$ and $\pi(\delta\delta') = \pi(\delta)+\pi(\delta')$. Let $\delta' \in \mathbb{CS}(\Sigma')$ ($\subset \mathbb{CS}(\Sigma_{1,1})$) so that $I(\delta', \partial\Sigma_{1,1}) = f(\partial\Sigma_{1,1})$ and $I(\delta', \alpha) = 0$. Let $\delta = \delta'\alpha^k$ in $\mathbb{CS}(\Sigma_{1,1})$ where $k = f(\beta) - \frac{1}{2}f(\partial\Sigma_{1,1})$. Then I_δ and f have the same values at $\{\alpha, \beta, \gamma, \partial\Sigma_{1,1}\}$ by the construction. Thus $f = I_\delta$.

Case 2. Suppose $\min\{f(\alpha) : \alpha \in \mathbb{S}'(\Sigma_{1,1})\} > 0$. Let α, β be the classes in $\mathbb{S}'(\Sigma_{1,1})$ with $\alpha \perp \beta$ so that $f(\alpha) + f(\beta) + f(\alpha\beta) = \min\{f(\alpha') + f(\beta') + f(\alpha'\beta') : \alpha' \perp \beta'\}$. We first claim that

$$f(\alpha) + f(\beta) + f(\alpha\beta) = f(\partial\Sigma_{1,1}).$$

To see this, let $\gamma = \alpha\beta$ and assume without loss of generality that $f(\alpha) \geq f(\beta) \geq f(\gamma) > 0$ (since $\{\alpha, \beta, \gamma\}$ is symmetric). Suppose the claim is false. Then equation (7) shows that $f(\alpha) + f(\beta) + f(\gamma) > f(\partial\Sigma_{1,1})$ and $f(\alpha) + f(\beta) + f(\gamma) = 2f(\alpha)$. This implies $f(\alpha) = f(\beta) + f(\gamma) > \max(f(\beta), f(\gamma))$ due to $f(\beta), f(\gamma) > 0$. It follows $f(\partial\Sigma_{1,1}) < f(\alpha) + f(\beta) + f(\gamma) = 2f(\alpha)$. Now consider equation (8) for α ($=\beta\gamma$) and α'($= \gamma\beta$). We obtain $f(\alpha) + f(\alpha') = \max(2f(\beta), 2f(\gamma), f(\partial\Sigma_{1,1}))$ $< 2f(\alpha)$. Thus $f(\alpha') < f(\alpha)$. This shows that $f(\alpha') + f(\beta) + f(\gamma) < f(\alpha) + f(\beta) + f(\gamma)$ which contradicts the choice of $\{\alpha, \beta, \gamma\}$. Thus the claim holds.

Now equation (7) shows that $f(\alpha), f(\beta), f(\gamma)$ satisfy the triangular inequalities (the sum of two is no less than the third) and their sum is an even number. Thus there exist integers $x, y, z \in \mathbf{Z}_{\geq 0}$ so that $f(\alpha) = y + z$, $f(\beta) = z + x$, and $f(\gamma) = x + y$. Let $\alpha_1\beta_1\gamma_1$ in $\mathbb{CS}(\Sigma)$ be the ideal triangulation so that $I(\alpha,\alpha_1) = I(\beta,\beta_1) = I(\gamma,\gamma_1) = 0$ (see figure 4). Define $\delta = \alpha_1^x\beta_1^y\gamma_1^z$. Then $f = I_\delta$ on the four element set $\{\alpha,\beta,\gamma,\partial\Sigma_{1,1}\}$ by the construction. Indeed, by the construction of δ, $f = I_\delta$ on the set $\{\alpha,\beta,\gamma\}$. By the claim $f(\partial\Sigma_{1,1}) = 2x+2y+2z$ and by the construction $I_\delta(\partial\Sigma_{1,1}) = 2x+2y+2z$. Thus $f = I_\delta$.

To establish theorem 3.1 for real valued function $f : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{R}$, we need the following lemmas.

Lemma 3.2. *Suppose $a,b,c \in \mathbf{R}$. The equation $x+a = \max(2x, x+b, c)$ has solutions in x over $\mathbf{R}$ if and only if $a \geq \max(b, c/2)$. If it has solutions, then the set of all solutions is given by*

(i) $\{c-a, a\}$ in the case $a > b$,

(ii) the closed interval $[c-a, a]$ in the case $a = b$.

Furthermore,

(iii) if x_1 is a solution, then $c - x_1$ is also a solution,

(iv) if x_1 and x_2 are solutions so that $x_1 + x_2 = c$, then $\max(x_1, x_2, b) = a$.

Proof. If $a \geq \max(b, c/2)$, then $x = a$ is a solution. If x' is a solution, then since $x'+a \geq x'+b$, we have $a \geq b$. Also $x'+a \geq 2x'$ and $x'+a \geq c$. Thus $a \geq x' \geq c-a$. This shows $a \geq c/2$, i.e., $a \geq \max(b, c/2)$. This shows that the equation has a solution if and only if $a \geq \max(b, c/2)$.

Now to prove (i) and (ii), if $a > b$, then the equation becomes $x+a = \max(2x, c)$ with $a \geq c/2$. Thus the solutions are $\{c-a, a\}$. If $a = b$, then one checks easily that all solutions are points in $[c-a, a]$. Parts (iii) and (iv) follow from (i) and (ii). q.e.d.

Lemma 3.3. *Suppose $x_1, x_2, x_3, x_4 \in \mathbf{Z}_{\geq 0}$ so that*

$$x_1 + x_2 + x_3 = \max(2x_1, 2x_2, 2x_3, x_4).$$

Then there is a function $g : \mathbb{S}(\Sigma_{1,1}) \to \mathbf{Z}$ satisfying equations (7),(8) and a triangle $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{S}'(\Sigma_{1,1})$ so that $g(\alpha_i) = x_i$, $i = 1,2,3$, and $g(\partial\Sigma_{1,1}) = x_4$.

Proof. Take any three elements $\alpha_1, \alpha_2, \alpha_3 = \alpha_1\alpha_2$ in $\mathbb{S}(\Sigma_{1,1})$ forming a triangle, i.e., $\alpha_1 \perp \alpha_2$. We define g on α_i and $\partial\Sigma_{1,1}$ as required. Now extend g through the neighboring triangles, say $\alpha_1, \alpha_2, \alpha_3' = \alpha_2\alpha_1$ by using equation (8). Thus, we need to verify that the equation (7) for g on the neighboring triangles $\alpha_1, \alpha_2, \alpha_3'$ still holds. By definition $g(\alpha_3') = x_3'$ where $x_3' = \max(2x_1, 2x_2, x_4) - x_3$. We first note that $x_3' \geq 0$ since $x_i + x_j \geq x_k$ for $\{i,j,k\} = \{1,2,3\}$ by the given condition on $x_i's$. Next, consider $x_1 + x_2 + x_3 = \max(2x_1, 2x_2, 2x_3, x_4)$ as an equation in x_3. The equation takes the form $x + x_1 + x_2 = \max(2x, x + x_3', c)$ where $c = \max(2x_1, 2x_2, x_4)$. By lemma 3.2(iii), since x_3 is a solution, $x_3' = c - x_3$ is another solution, i.e., equation (7) holds for g on the neighboring triangles. q.e.d.

We now show that equations (7), (8) characterize real-valued geometric functions. Evidently, any geometric functions satisfies the equations (7), (8). Conversely, suppose that f is a solution to equations (7), (8). Fix a ideal triangle $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{S}'(\Sigma_{1,1})$. Note that the rational solutions of the equation $x_1 + x_2 + x_3 = \max(2x_1, 2x_2, 2x_3, x_4)$ are dense in the set of solutions over $\mathbf{R}_{\geq 0}$. By lemma 3.3, there is a sequence of functions g_n from $\mathbb{S}(\Sigma_{1,1})$ to $2\mathbf{Z}_{\geq 0}$ solving equations (7), (8) and a sequence of numbers $k_n \in \mathbf{Q}_{\geq 0}$ so that $\lim_n k_n g_n(x) = f(x)$ for $x \in \{\alpha_1, \alpha_2, \alpha_3, \partial\Sigma_{1,1}\}$. By equation (8), we have $\lim_n k_n g_n(x) = f(x)$ for all $x \in \mathbb{S}(\Sigma_{1,1})$. On the other hand, we have $g_n = I_{\delta_n}$ for some $\delta_n \in \mathbb{S}(\Sigma_{1,1})$ by theorem 3.1 for curve systems. Thus $f = I_m$ where $m = \lim_n k_n \delta_n \in ML(\Sigma_{1,1})$ by definition. q.e.d.

4. The Four-holed Sphere

The goal of this section is to show theorem 1.2 for the 4-holed sphere $\Sigma_{0,4}$. The basic ideal of the proof is the same as that in §3. But the proof is considerably longer and more complicated due to the existence of eight non-homeomorphic ideal triangulations of the four-holed sphere. We restate the theorem in terms of the multiplicative structure in $\mathbb{S}(\Sigma)$ below.

Theorem 4.1. *For the 4-holed sphere $\Sigma_{0,4}$ with boundary components b_1, b_2, b_3 and b_4, a function $f : \mathbb{S}(\Sigma_{0,4}) \to \mathbf{Z}_{\geq 0}$ is the geometric intersection number function I_δ for some $\delta \in \mathbb{CS}(\Sigma)$ if and only if for $\alpha_1 \perp_0 \alpha_2$ with $\alpha_3 = \alpha_1\alpha_2$ so that (α_i, b_s, b_r) bounds a $\Sigma_{0,3}$ in $\Sigma_{0,4}$,*

$$\sum_{i=1}^{3} f(\alpha_i) = \max_{1\leq i\leq 3; 1\leq s\leq 4}\left(2f(\alpha_i), 2f(b_s), \sum_{j=1}^{4} f(b_j), f(\alpha_i) + f(b_s) + f(b_r)\right), \tag{10}$$

$$f(\alpha_1\alpha_2) + f(\alpha_2\alpha_1) = \max_{1\leq i\leq 2; 1\leq s\leq 4}\left(2f(\alpha_i), 2f(b_s), \sum_{j=1}^{4} f(b_j), f(\alpha_i) + f(b_s) + f(b_r)\right), \tag{11}$$

$$f(\alpha_i) + f(b_s) + f(b_r) \in 2\mathbf{Z}. \tag{12}$$

Furthermore, real valued geometric functions on $\mathbb{S}(\Sigma_{0,4})$ are characterized by the equations (10),(11).

Proof. The necessity of the equations (10),(11) follows from the same argument as that in §3 using the degenerations of the trace relations for geodesic length functions. To be more precise, it is shown in [Lu1] that for any hyperbolic metric d on $\Sigma_{0,4}$ with geodesic boundary or cusp ends, then $t(\alpha) = 2\cosh(l_d(\alpha)/2)$ satisfies:

$$t(\alpha_1)t(\alpha_2)t(\alpha_3) + 4$$
$$= \sum_{i=1}^{3} t^2(\alpha_i) + \sum_{s=1}^{4} t^2(b_s) + \prod_{s=1}^{4} t(b_s) + \frac{1}{2}\sum_{i=1}^{3}\sum_{s=1}^{4} t(\alpha_i)t(b_s)t(b_r)$$

and

$$t(\alpha_1\alpha_2)t(\alpha_2\alpha_1) = \sum_{i=1}^{2} t^2(\alpha_i) + \sum_{s=1}^{4} t^2(b_s) +$$
$$\prod_{s=1}^{4} t(b_s) + \frac{1}{2}\sum_{i=1}^{2}\sum_{s=1}^{4} t(\alpha_i)t(b_s)t(b_r) - 4$$

where α_i, b_s, b_r bound a 3-holed sphere. Now the degenerations of the above two equations are equations (10), (11). Since the number of end points of a curve system is even, equation (12) holds for curve systems.

To show that the conditions are also sufficient, suppose a function $f : \mathbb{S} \to \mathbf{Z}_{\geq 0}$ satisfies equations (10),(11),(12). By the structure of the modular relation, we conclude that f is determined by $f|_{\{\alpha,\beta,\alpha\beta,b_1,b_2,b_3,b_4\}}$ where $\alpha \perp_0 \beta$. Thus it suffices to construct $\delta \in \mathbb{CS}(\Sigma)$ so that f and I_δ have the same values on $\{\alpha, \beta, \alpha\beta, b_1, ..., b_4\}$.

Note that equation (12) implies both $\sum_{i=1}^{4} f(b_i)$ and $\sum_{i=1}^{3} f(\alpha_i)$ are even numbers. Indeed, suppose α, b_1, b_2 bound a 3-holed sphere, then $\sum_{i=1}^{4} f(b_i) = (f(b_1) + f(b_2) + f(\alpha)) + (f(b_3) + f(b_4) + f(\alpha)) - 2f(\alpha)$ shows, by (12), that $\sum_{i=1}^{4} f(b_i)$ is even. The same argument shows that $\sum_{i=1}^{3} f(\alpha_i)$ is even.

To construct the curve system δ, we shall consider two cases: $\min\{f(\alpha) : \alpha \in \mathbb{S}'(\Sigma_{0,4})\} = 0$ or > 0.

Case 1. Suppose $f(\alpha) = 0$ for some $\alpha \in \mathbb{S}'(\Sigma_{0,4})$. Choose β so that $\beta \perp_0 \alpha$ and $\gamma = \alpha\beta$. Then $f(\beta) = f(\gamma)$ due to equation (10) that $f(\beta) + f(\gamma) \geq \max(2f(\beta), 2f(\gamma))$. Assume without loss of generality that (α, b_1, b_2), (β, b_1, b_3) bound $\Sigma_{0,3}$'s in $\Sigma_{0,4}$. Construct a curve system $\delta' \in \mathbb{CS}(\Sigma_{0,4})$ so that $I(\delta', \alpha) = 0$, $I(\delta', b_i) = f(b_i)$ as shown in Fig. 5. The existence of δ' is due to the classification of curve systems on $\Sigma_{0,3}$ and the equation (12) that $f(b_1)+f(b_2), f(b_3)+f(b_4)$ are even numbers. Let $k = \frac{1}{2}(f(\beta)-\max(f(b_1), f(b_2))-\max(f(b_3), f(b_4)))$. Then equations (10), (12) for $\alpha \perp_0 \beta$, $f(\alpha) = 0$ and $f(\beta) = f(\gamma)$ show that $k \geq 0$ and that $k \in \mathbf{Z}$. Let $\delta = \delta'\alpha^k \in \mathbb{CS}(\Sigma_{0,4})$. Then by the construction, I_δ and f have the same values on the set $\{\alpha, \beta, \alpha\beta, b_1, ..., b_4\}$.

Case 2. Assume $f(\alpha) \geq 1$ for all $\alpha \in \mathbb{S}'(\Sigma_{0,4})$. Let $\{\alpha, \beta, \gamma\}$ be a ideal triangle in $\mathbb{S}(\Sigma)$ so that $f(\alpha) + f(\beta) + f(\gamma)$ achieves the minimal values among all such triples. Assume without loss of generality that

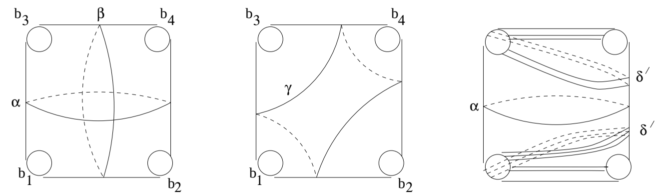


Figure 5

(α, b_1, b_2), (β, b_1, b_3) bound $\Sigma_{0,3}$ in $\Sigma_{0,4}$ and that $f(\alpha) \geq f(\beta) \geq f(\gamma)$. We claim that

$$f(\alpha) + f(\beta) + f(\gamma) = A \tag{13}$$

where $A = \max_{1 \leq s \leq 4}(2f(b_s),\ \Sigma_{s=1}^4 f(b_s),\ f(\alpha) + f(b_1) + f(b_2), f(\alpha) + f(b_3) + f(b_4),\ f(\beta)$
$+f(b_1) + f(b_3), f(\beta) + f(b_2) + f(b_4),\ f(\gamma) + f(b_1) + f(b_4),\ f(\gamma) + f(b_2) + f(b_3))$. Indeed, if otherwise, by equation (10), $f(\beta), f(\gamma) > 0$ and (13), we obtain $f(\alpha) + f(\beta) + f(\gamma) > A$ and $f(\alpha) = f(\beta) + f(\gamma)$. In particular, $f(\alpha) > f(\beta), f(\gamma)$, and $2f(\alpha) > A$. Applying equation (11) to α, α' where $\{\alpha, \alpha'\} = \{\beta\gamma, \gamma\beta\}$, we obtain $f(\alpha) + f(\alpha') = \max(2f(\beta), 2f(\gamma), A')$ where $A' \leq A < 2f(\alpha)$. Thus $f(\alpha) + f(\alpha') < 2f(\alpha)$, i.e., $f(\alpha') < f(\alpha)$. This contradicts the choice of (α, β, γ).

Under the assumption (13), we now construct $\delta \in \mathbb{CS}(\Sigma_{0,4})$ so that f and I_δ have the same values on $\{\alpha, \beta, \gamma, b_1, ..., b_4\}$. For simplicity, we still assume that (α, b_1, b_2) and (β, b_1, b_3) bound $\Sigma_{0,3}$ but do not assume that $f(\alpha) \geq f(\beta) \geq f(\gamma)$ below.

By symmetry, since $f(\alpha) + f(\beta) + f(\gamma) = A$, it suffices to consider the following three subcases: (2.1) $f(\alpha) + f(\beta) + f(\gamma) = \Sigma_{s=1}^4 f(b_s)$; (2.2) $f(\alpha) + f(\beta) + f(\gamma) = 2f(b_1)$; and (2.3) $f(\alpha) + f(\beta) + f(\gamma) = f(\alpha) + f(b_1) + f(b_2)$. The corresponding curve system δ in $\mathbb{CS}(\Sigma_{0,4})$ will be constructed as follows. First, we construct a ideal triangulation $\tau = \tau_1 ... \tau_6$ of $\Sigma_{0,4}$. Then the curve system δ is taken to be of the form $\tau_1^{x_1} ... \tau_6^{x_6}$, $x_i \in \mathbf{Z}_{\geq 0}$.

Case 2.1. $f(\alpha) + f(\beta) + f(\gamma) = \sum_{s=1}^4 f(b_s)$. The ideal triangulation τ is shown in figure 6 where the locations of α, β, γ are indicated. The conditions that f and I_δ have the same values on $\{\alpha, \beta, \gamma, b_1, ..., b_4\}$ are given by the following systems of linear equations in x_i.

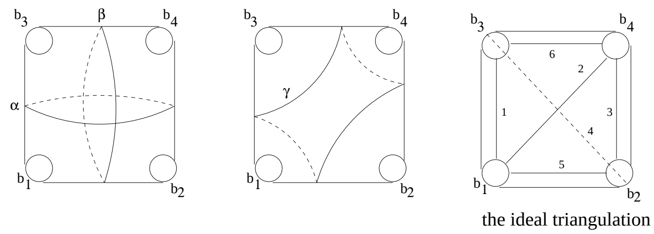


Figure 6

$$\begin{aligned} x_1 + x_2 + x_5 &= f(b_1) \\ x_3 + x_4 + x_5 &= f(b_2) \\ x_1 + x_4 + x_6 &= f(b_3) \\ x_2 + x_3 + x_6 &= f(b_4) \\ x_1 + x_2 + x_3 + x_4 &= f(\alpha) \\ x_2 + x_4 + x_5 + x_6 &= f(\beta) \\ x_1 + x_3 + x_5 + x_6 &= f(\gamma) \end{aligned}$$

Note that $f(\alpha) + f(\beta) + f(\gamma) = \Sigma_{s=1}^4 f(b_s)$ is a consequence of (13). Thus, it is essentially a systems of six equations in six variables. The solution is

$$\begin{aligned} x_1 &= (f(b_1) + f(b_3) - f(\beta))/2 \\ x_2 &= (f(b_1) + f(b_4) - f(\gamma))/2 \\ x_3 &= (f(b_2) + f(b_4) - f(\beta))/2 \\ x_4 &= (f(b_2) + f(b_3) - f(\gamma))/2 \\ x_5 &= (f(b_1) + f(b_2) - f(\alpha))/2 \\ x_6 &= (f(b_3) + f(b_4) - f(\alpha))/2 \end{aligned}$$

It remains to show that $x_i \in \mathbf{Z}_{\geq 0}$. First of all $x_i \in \mathbf{Z}$ due to equation (12). To see $x_i \geq 0$, say $x_1 \geq 0$, for definiteness, we use equation (10) that $f(\alpha)+f(\beta)+f(\gamma) \geq f(\beta)+f(b_2)+f(b_4)$. But $f(\alpha)+f(\beta)+f(\gamma) = \Sigma_{s=1}^4 f(b_s)$. Thus, $f(b_1) + f(b_3) \geq f(\beta)$, i.e., $x_1 \geq 0$. The proof of the rest of the cases $x_i \geq 0$ is similar. (The solutions x_i are found as follows: x_1 is the number of arcs joining b_1, b_3 in the 3-holed sphere $\Sigma_{0,3}$ bounded by b_1, b_3, β, etc.).

Case 2.2. $f(\alpha)+f(\beta)+f(\gamma) = 2f(b_1)$. The curve system δ is based on the ideal triangulation τ as shown in figure 7. We obtain the following system of linear equations in x_i

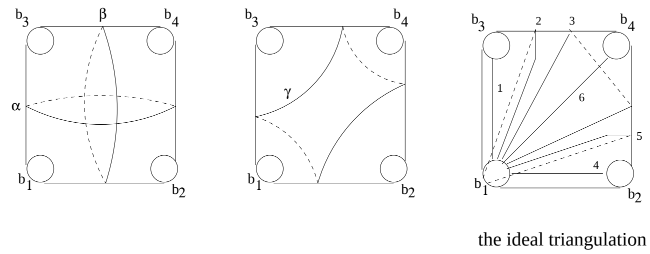


Figure 7

$$
\begin{aligned}
x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 + x_6 &= f(b_1) \\
x_4 &= f(b_2) \\
x_1 &= f(b_3) \\
x_6 &= f(b_4) \\
x_1 + 2x_2 + 2x_3 + x_6 &= f(\alpha) \\
2x_3 + x_4 + 2x_5 + x_6 &= f(\beta) \\
x_1 + 2x_2 + x_4 + 2x_5 &= f(\gamma)
\end{aligned}
$$

The solution is,

$$
\begin{aligned}
x_1 &= f(b_3) \\
x_2 &= (f(b_1) - f(b_3) - f(\beta))/2 \\
x_3 &= (f(b_1) - f(b_4) - f(\gamma))/2 \\
x_4 &= f(b_2) \\
x_5 &= (f(b_1) - f(b_2) - f(\alpha))/2 \\
x_6 &= f(b_4)
\end{aligned}
$$

To see that $x_i \in \mathbf{Z}_{\geq 0}$, we note that $x_i \in \mathbf{Z}$ by equation (12). To show $x_i \geq 0$, say $x_2 \geq 0$, we use equation (10) and the assumption that $f(\alpha) + f(\beta) + f(\gamma) = 2f(b_1)$. Thus $2f(b_1) \geq f(\beta) + f(b_3) + f(b_1)$, i.e., $x_2 \geq 0$. By symmetry, $x_3, x_5 \geq 0$.

Case 2.3. $f(\alpha)+f(\beta)+f(\gamma) = f(\alpha)+f(b_1)+f(b_2)$, i.e., $f(\beta)+f(\gamma) = f(b_1) + f(b_2)$. We first observe that many inequalities follow from the assumption. To simplify the notions, we use $\Delta = \{(a_1, a_2, a_3) \in \mathbf{R}_{\geq 0} : a_i + a_j \geq a_k, i \neq j \neq k \neq i\}$.

q.e.d.

Lemma 4.2. *Under the assumption* $f(\alpha) + f(\beta) + f(\gamma) = f(\alpha) + f(b_1) + f(b_2)$*, we have*

(i) $(f(\alpha), f(b_1), f(b_2)) \in \Delta$*;*

(ii) $f(\alpha) \geq f(b_3) + f(b_4)$*;*

(iii) $f(\beta) + f(b_1) \geq f(b_3)$*, and* $f(\beta) + f(b_2) \geq f(b_4)$*;*

(iv) $f(\gamma)+f(b_1)\geq f(b_4)$ *and* $f(\gamma)+f(b_2)\geq f(b_3)$.

Proof. To see (i), since $(f(\alpha),f(\beta),f(\gamma))\in\Delta$, thus $f(\alpha)\leq f(\beta)+f(\gamma)=f(b_1)+f(b_2)$. On the other hand, equation (10) shows that $f(\alpha)+f(\beta)+f(\gamma)\geq 2f(b_i)$, for $i=1,2$. Thus $f(\alpha)+f(b_i)\geq f(b_j)$ for $\{i,j\}=\{1,2\}$. To see (ii), we use $f(\alpha)+f(\beta)+f(\gamma)\geq\Sigma_{s=1}^4 f(b_s)$ and the assumption. To see $f(\beta)+f(b_1)\geq f(b_3)$ in part (iii), we use (13) that $f(\alpha)+f(\beta)+f(\gamma)\geq f(\gamma)+f(b_2)+f(b_3)$. Now $f(\alpha)+f(\beta)+f(\gamma)\leq f(\beta)+f(\gamma)+f(\beta)+f(\gamma)=f(\beta)+f(\gamma)+f(b_1)+f(b_2)$. Thus the result follows. The rest of the inequalities in (iii),(iv) are proved by the same argument. q.e.d.

To construct the curve system δ, we shall consider nine subcases due to the different situations: $(f(\beta),f(b_i),f(b_j))\in\Delta$, $f(\beta)+f(b_i)\geq f(b_j)$ for $(i,j)\in\{(1,3),(3,1),(2,4),(4,2)\}$. The nine subcases are listed in figure 8. The (i,j)-th subcase corresponds to the i-th row and j-th column in figure 8. Due to symmetry, the (i,j)-th subcase and the (j,i)-th subcase are essentially the same. We shall consider six subcases: (1,1), (1,2), (1,3), (2,2), (2,3), (3,3). The corresponding ideal triangulations and the system of linear equations are listed below.
Subcase (1,1). $f(b_1)\geq f(\beta)+f(b_3)$ and $f(b_2)\geq f(\beta)+f(b_4)$.

$$\begin{aligned}
x_1+x_2+x_3+2x_5&=f(b_1)\\
x_1+x_2+x_4+2x_6&=f(b_2)\\
x_3&=f(b_3)\\
x_4&=f(b_4)\\
2x_2+x_3+x_4+2x_5+2x_6&=f(\alpha)\\
x_1+x_2&=f(\beta)\\
x_1+x_2+x_3+x_4+2x_5+2x_6&=f(\gamma)
\end{aligned}$$

The solution is,

$$\begin{aligned}
x_1&=(f(b_1)+f(b_2)-f(\alpha))/2\\
x_2&=(f(\alpha)+f(\beta)-f(\gamma))/2\\
x_3&=f(b_3)\\
x_4&=f(b_4)\\
x_5&=(f(b_1)-f(\beta)-f(b_3))/2\\
x_6&=(f(b_2)-f(\beta)-f(b_4))/2
\end{aligned}$$

The solutions x_i are in $\mathbf{Z}_{\geq 0}$ by lemma 4.2, equation (12) and the assumption $(x_5,x_6\geq 0)$.

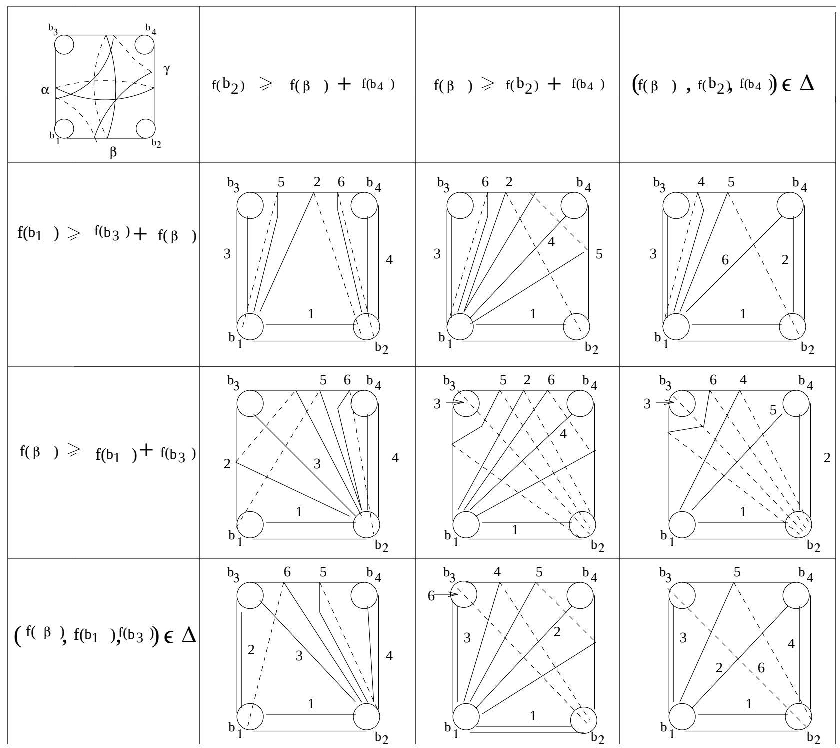


Figure 8

Subcase (1.2). $f(b_1) \geq f(\beta) + f(b_3)$ and $f(\beta) \geq f(b_2) + f(b_4)$.

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &= f(b_1)\\
x_1 + x_2 &= f(b_2)\\
x_3 &= f(b_3)\\
x_4 &= f(b_4)\\
2x_2 + x_3 + x_4 + 2x_5 + 2x_6 &= f(\alpha)\\
x_1 + x_2 + x_4 + 2x_5 &= f(\beta)\\
x_1 + x_2 + x_3 + 2x_6 &= f(\gamma)
\end{aligned}
$$

The solution is,

$$\begin{aligned} x_1 &= (f(b_1)+f(b_2)-f(\alpha))/2 \\ x_2 &= (f(b_2)+f(\alpha)-f(b_1))/2 \\ x_3 &= f(b_3) \\ x_4 &= f(b_4) \\ x_5 &= (f(\beta)-f(b_2)-f(b_4))/2 \\ x_6 &= (f(b_1)-f(b_3)-f(\beta))/2 \end{aligned}$$

The solutions are in $\mathbf{Z}_{\geq 0}$ by lemma 4, equation (12) and the assumption.
Subcase (1.3). $f(b_1) \geq f(\beta)+f(b_3)$ and $(f(\beta), f(b_2), f(b_4)) \in \Delta$.

$$\begin{aligned} x_1+x_3+2x_4+x_5+x_6 &= f(b_1) \\ x_1+x_2+x_5 &= f(b_2) \\ x_3 &= f(b_3) \\ x_2+x_6 &= f(b_4) \\ x_2+x_3+2x_4+2x_5+x_6 &= f(\alpha) \\ x_1+x_5+x_6 &= f(\beta) \\ x_1+x_2+x_3+2x_4+x_5 &= f(\gamma) \end{aligned}$$

The solution is,

$$\begin{aligned} x_1 &= (f(b_1)+f(b_2)-f(\alpha))/2 \\ x_2 &= (f(b_2)+f(b_4)-f(\beta))/2 \\ x_3 &= f(b_3) \\ x_4 &= (f(b_1)-f(b_3)-f(\beta))/2 \\ x_5 &= (f(\alpha)+f(\beta)-f(b_1)-f(b_4))/2 \\ x_6 &= (f(b_4)+f(\beta)-f(b_2))/2 \end{aligned}$$

By the same argument as in the previous cases, all x_i except possibly x_5 are in $\mathbf{Z}_{\geq 0}$. It remains to show that $x_5 \in \mathbf{Z}_{\geq 0}$. Indeed, $f(\alpha)+f(\beta)-f(b_1)-f(b_4) = (f(\alpha)+f(\beta)+f(\gamma))-(f(\gamma)+f(b_1)+f(b_4))$. Thus, by equations (10), (12), $x_5 \in \mathbf{Z}_{\geq 0}$.
Subcase (2.2). $f(\beta) \geq f(b_1)+f(b_3)$ and $f(\beta) \geq f(b_2)+f(b_4))$.

$$\begin{aligned} x_1+x_2+x_4+2x_6 &= f(b_1) \\ x_1+x_2+x_3+2x_5 &= f(b_2) \\ x_3 &= f(b_3) \\ x_4 &= f(b_4) \\ 2x_2+x_3+x_4+2x_5+2x_6 &= f(\alpha) \\ x_1+x_2+x_3+x_4+2x_5+2x_6 &= f(\beta) \\ x_1+x_2 &= f(\gamma) \end{aligned}$$

The solution is

$$
\begin{aligned}
x_1 &= (f(b_1) + f(b_2) - f(\alpha))/2 \\
x_2 &= (f(\alpha) + f(\gamma) - f(\beta))/2 \\
x_3 &= f(b_3) \\
x_4 &= f(b_4) \\
x_5 &= (f(\beta) - f(b_1) - f(b_3))/2 \\
x_6 &= (f(\beta) - f(b_2) - f(b_4))/2
\end{aligned}
$$

The solutions x_i's are in $\mathbf{Z}_{\geq 0}$ by lemma 4.2, equations (10), (12) and the assumption.

Subcase (2.3). $f(\beta) \geq f(b_1) + f(b_3)$ and $(f(\beta), f(b_2), f(b_4)) \in \Delta$.

$$
\begin{aligned}
x_1 + x_4 + x_5 &= f(b_1) \\
x_1 + x_2 + x_3 + x_4 + 2x_6 &= f(b_2) \\
x_3 &= f(b_3) \\
x_2 + x_5 &= f(b_4) \\
x_2 + x_3 + 2x_4 + x_5 + 2x_6 &= f(\alpha) \\
x_1 + x_3 + x_4 + x_5 + 2x_6 &= f(\beta) \\
x_1 + x_2 + x_4 &= f(\gamma)
\end{aligned}
$$

The solution is

$$
\begin{aligned}
x_1 &= (f(b_1) + f(b_2) - f(\alpha))/2 \\
x_2 &= (f(b_2) + f(b_4) - f(\beta))/2 \\
x_3 &= f(b_3) \\
x_4 &= (f(\alpha) + f(b_1) - f(\beta) - f(b_4))/2 \\
x_5 &= (f(b_1) + f(b_4) - f(\gamma))/2 \\
x_6 &= (f(\beta) - f(b_1) - f(b_3))/2
\end{aligned}
$$

To show that the solutions are in $\mathbf{Z}_{\geq 0}$, it suffices to show that $x_4 \in \mathbf{Z}_{\geq 0}$ (the rest of the $x_i \in \mathbf{Z}_{\geq 0}$ follows from equations (10),(12), and the assumption). For x_4, we express x_4 as $\frac{1}{2}((f(\alpha) + f(\beta) + f(\gamma)) - (f(\beta) + f(b_2) + f(b_4))$. Thus x_4 is in $\mathbf{Z}_{\geq 0}$ by equations (10) and (12).

Subcase (3.3). Both $(f(\beta), f(b_1), f(b_3))$ and $(f(\beta), f(b_2), f(b_4))$ are in Δ.

The equation is,

$$\begin{aligned}
x_1 + x_2 + x_3 + x_5 &= f(b_1)\\
x_1 + x_4 + x_5 + x_6 &= f(b_2)\\
x_3 + x_6 &= f(b_3)\\
x_2 + x_4 &= f(b_4)\\
x_2 + x_3 + x_4 + 2x_5 + x_6 &= f(\alpha)\\
x_1 + x_2 + x_5 + x_6 &= f(\beta)\\
x_1 + x_3 + x_4 + x_5 &= f(\gamma)
\end{aligned}$$

The solution is,

$$\begin{aligned}
x_1 &= (f(b_1) + f(b_2) - f(\alpha))/2\\
x_2 &= (f(b_4) + f(\beta) - f(b_2))/2\\
x_3 &= (f(b_1) + f(b_3) - f(\beta))/2\\
x_4 &= (f(b_2) + f(b_4) - f(\beta))/2\\
x_5 &= (f(\alpha) - f(b_3) - f(b_4))/2\\
x_6 &= (f(b_3) + f(\beta) - f(b_1))/2
\end{aligned}$$

By equations (10), (12), the solutions are in $\mathbf{Z}_{\geq 0}$.

This ends the proof of theorem 4.1 for curve systems. The proof of the characterization of real valued geometric functions on $\mathbb{S}(\Sigma_{0,4})$ is the same as that in §3. Indeed, first of all, the rational solutions of $\Sigma_{i=1}^{3} x_i = \max_{1\leq i\leq 3; 1\leq j\leq 4}(2x_i, 2y_j, \Sigma_{k=1}^{4} y_k, x_1+y_1+y_2, x_1+y_3+y_4, x_2+y_1+y_3, x_2+y_2+y_4, x_3+y_1+y_4, x_3+y_2+y_3)$ are dense in the set of solutions over $\mathbf{R}_{\geq 0}$. Also if we consider $f(\alpha_1\alpha_2)$ as an unknown in equation (10), it becomes $x + a = \max(2x, x + b, c)$ where $c = f(\alpha_1\alpha_2) + f(\alpha_2\alpha_1)$ (by equation (11)). Thus, by lemma 3.2, we see that the corresponding lemma 3.3 holds for $\Sigma_{0,4}$. This shows that equations (10),(11) characterize the geometric functions.

Remark 4.1. The proof actually shows that except for at most four adjacent ideal triangles, equations (7), (10) become triangular equalities

$$\sum_{i=1}^{3} f(\alpha_i) = \max_{i=1}^{3}(f(\alpha_i)) \text{ when } f = I_\delta \text{ for } \delta \in \mathbb{CS}.$$

As a consequence of the discussion in the last paragraph and lemma 3.2(ii), we obtain,

Corollary 4.3. *(i) Suppose* $\alpha_1 \perp_0 \alpha_2$ *in* $\mathbb{S}(\Sigma_{0,4})$ *so that* $(\alpha_1\alpha_2, b_1, b_2)$ *bounds a* $\Sigma_{0,3}$*, then* $f(\alpha_1) + f(\alpha_2) = \max(f(\alpha_1\alpha_2), f(\alpha_2\alpha_1), f(b_1) + f(b_2), f(b_3) + f(b_4))$.

(ii) Suppose $\alpha_1 \perp \alpha_2$ *in* $\mathbb{S}(\Sigma)$. *Then* $f(\alpha_1) + f(\alpha_2) = \max(f(\alpha_1\alpha_2), f(\alpha_2\alpha_1))$.

In particular, if $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, then

$$f(\alpha\beta) \leq f(\alpha) + f(\beta).$$

Combining theorems 3.1, 4.1, we obtain the following useful consequence.

Corollary 4.4. (Eventual linear property) *Suppose $f : \mathbb{S}(\Sigma) \to \mathbf{R}_{\geq 0}$ satisfies equations (7),(8),(10),(11) and $\alpha \perp \beta$, or $\alpha \perp_0 \beta$ in $\mathbb{S}(\Sigma)$. Then $f(\alpha^n\beta)$ is convex in $n \in \mathbf{Z}$. Furthermore, there is an integer N so that for $n \geq N$, $f(\alpha^n\beta) = f(\alpha^{n-1}\beta) + f(\alpha)$ and $f(\beta\alpha^n) = f(\beta\alpha^{n-1}) + f(\alpha)$.*

Remark 4.2. It will be shown in §8 that $f(\alpha^n\beta)$ is convex in $n \in \mathbf{Z}$ for all $\alpha, \beta \in \mathbb{CS}_0(\Sigma)$. This seems to be an analogy with the fact that the geodesic length functions are convex along the Thurston's earthquake paths ([Ker2], [Wo]). We would like to thank P. Schmutz for drawing our attention to the convexity property. The operation $\alpha^n\beta$ is similar to the extension of the earthquake from the Teichmüller space to the measured lamination space. See [Bo3], [Pa1], [Pa2] also §8 for more discussion.

Proof. Since α, β lie in an essential subsurface homeomorphic to either $\Sigma_{1,1}$ or $\Sigma_{0,4}$, we may assume that $\Sigma \cong \Sigma_{1,1}$ or $\Sigma_{0,4}$. We shall consider the case $\alpha \perp_0 \beta$ only (the other case is similar and simpler). Let $x_n = f(\alpha^n\beta)$, $n \in \mathbf{Z}$. Since $\alpha^n\beta \perp_0 \alpha$ with $\alpha(\alpha^n\beta) = \alpha^{n+1}\beta$, we obtain following two equations for the sequence $\{x_n\}$ by equations (10),(11):

$$x_{n+1} + x_n + f(\alpha) = \max(2x_{n+1}, 2x_n, x_{n+1} + b_{n+1}, x_n + b_n, c) \tag{14}$$

where $b_{2n} = b_0$ and $b_{2n+1} = b_1$, and

$$x_{n+1} + x_{n-1} = \max(2x_n, x_n + b_n, c). \tag{15}$$

Now by (15), $x_{n+1} + x_{n-1} \geq 2x_n$. Thus $f(\alpha^n\beta)$ is convex in n. To show that x_n is linear in n for $|n|$ large, we shall consider $n > 0$ only (the other case is similar). By convexity, x_n is monotonic for n large. If $\lim_n x_n = \infty$, then $x_{n+1} \geq x_n > \max(b_n, c, c/2)$ for n large. Thus for n large, (14) becomes, $x_{n+1} = x_n + f(\alpha)$. If $\lim_n x_n = L$ is a finite number, take the limit to the equations (14) and (15). We obtain:

$$2L + f(\alpha) = \max(2L, L + b_\infty, c)$$

and

$$2L = \max(2L, L + b_\infty, c)$$

where $b_\infty = \max(b_0, b_1)$. Thus $f(\alpha) = 0$. By (14), this shows $x_n = x_{n+1}$ for all n, i.e., $f(\alpha^n\beta) = f(\alpha^{n-1}\beta) + f(\alpha)$. q.e.d.

5. A Reduction Theorem

The goal of this section is to show that theorem 1.1 for all surfaces follows from theorem 1.1 for level-2 surfaces. The basic tools used in the proof is a theorem of [HT] that the complex of 3-holed sphere decomposition is connected.

Theorem 5.1. *Assume theorem 1.1 holds for level-2 surfaces. Then for any surface Σ of level at least 3, each real valued function $f : \mathbb{S}(\Sigma) \to$ $\mathbf{R}$ satisfying conditions (7), (8), (10) and (11) is the geometric intersection number function I_m for some $m \in ML(\Sigma)$.*

Proof. We will construct the measured lamination $m \in ML(\Sigma)$ as follows. Take any 3-holed sphere decomposition $P = \{a_1, ..., a_n\}$ of Σ. For each non-boundary parallel component a_i of P, let $X(P, i)$ be the level-1 subsurface in the components of $\Sigma - \cup_{j \neq i} int(N(a_j))$. By the construction $[a_i] \in \mathbb{S}'(X(P, i))$ and isotopy classes of boundary components of $X(P, i)$ are $[a_j]$'s. By theorems 3.1 and 4.1, there exists a measured lamination m_i on $X(P, i)$ so that the restriction of f to $\mathbb{S}(X(P, i))$ is I_{m_i}. Now if a_i, a_j, a_k bound a 3-holed sphere in the decomposition P, then $X(P, i)$ and $X(P, j)$ intersect in a level-0 surface. Thus, by the gluing lemma 2.2, we can find a measured lamination m_{ij} on the level-2 surface $X(P, i) \cup X(P, j)$ whose restrictions to $X(P, i)$ and $X(P, j)$ are m_i and m_j respectively. Note that the restriction condition is automatically satisfied in our case. By repeat using the gluing lemma 2.2, we obtain a measured lamination m_P on Σ so that

$$f(\alpha) = I(m_P, \alpha)$$

for all $\alpha \in \mathbb{S}(\Sigma)$ so that α intersects at most one a_i's.

We claim that $m_P = m_{P'}$ for any other 3-holed sphere decomposition P' of Σ. Assuming the claim, we obtain the measured lamination $m = m_P \in \mathbb{S}(\Sigma)$ so that $f = I_m$.

To prove the claim, we will use the following theorem of Hatcher-Thurston [HT]. Recall that two 3-holed sphere decompositions $P = \{a_1, ..., a_n\}$ and $P' = \{b_1, ..., b_n\}$ are related by a *move* if there exists an index i so that $b_j = a_j$ for all $j \neq i$ and either $b_i \cap a_i = \emptyset$, or $b_i \perp a_i$ or $b_i \perp_0 a_i$.

Theorem 5.2 ([HT]). *Any two 3-holed sphere decompositions are related by a finite number of moves.*

By this theorem, it suffices to show that if P' is obtain from P by one move so that $b_i \perp a_i$ or $b_i \perp_0 a_i$, then $m_P = m_{P'}$. For simplicity let us assume without loss of generality that $i = 1$. Since $X(P, 1) = X(P', 1)$, by definition $m_P|_{X(P,1)} = m_{P'}|_{X(P',1)}$. Let $a_2, ..., a_k$, $k \leq 5$, be all simple loops which appear as the boundary components of $X(P,1)$. By the construction of m_P, $m_{P'}$ and definition of P', for $i \geq k+1$, the Dehn-Thurston coordinates of m_P and $m_{P'}$ at a_i (with respect to P) are the same. Thus to show $m_P = m_{P'}$, it suffices to show that the Dehn-Thurston coordinates of m_P and $m_{P'}$ at a_i, for $i \leq k$, are the same. To this end, fix $i \leq k$, let Y be the level-2 subsurface of Σ which is the component of $\Sigma - \cup_{j \neq 1,i} int(N(a_j))$. By the construction, $X(P,1) \subset Y$ and $[a_i] \in \mathbb{S}(Y)$. Indeed, $\{a_1, a_i\}$ forms a 3-holed sphere decomposition P'' of Y as shown in figure 9. We will show that $m_P|_Y = m_{P'}|_Y$ which implies that the Dehn-Thurston coordinate of m_P and $m_{P'}$ are the same. To this end, consider the restriction $f|_{\mathbb{S}(Y)}$. By the hypothesis that theorem 1.1 holds for level-2 surfaces, there exists a measured lamination $m_Y \in ML(Y)$ so that $f|_{\mathbb{S}(Y)} = I_{m_Y}$. Now let us compare m_Y with $m_P|_Y$ and $m_{P'}|_Y$. By the construction, $m_Y|_{X(P,1)} = m_P|_{X(P,1)}$ and $m_Y|_{X(P'',i)} = m_P|_{X(P'',i)}$. Thus by the uniqueness part of lemma 2.2, we see that $m_Y = m_P|_Y$. By the same argument, $m_Y = m_{P'}|_Y$. This shows in turn that $m_P = m_{P'}$. q.e.d.

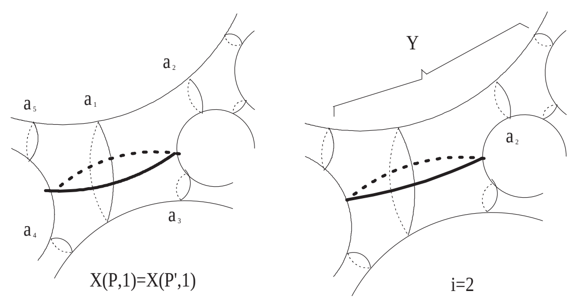


Figure 9

5.1. A Proof of Corollary 1.3. To prove the corollary 1.3 for the surface $\Sigma = \Sigma_{g,r}$ with $\partial\Sigma = b_1 \cup ... \cup b_r$, we choose a 3-holed sphere decomposition $\alpha = \alpha_1 .... \alpha_n$ for Σ where $n = 3g + r - 3$, $\alpha_i \in \mathbb{S}(\Sigma)$. For each index i, choose $\beta_i \in \mathbb{S}'(\Sigma)$ so that $I(\beta_i, \alpha_j) = 0$ for $j \neq i$ and $\beta_i \perp \alpha_i$ or $\beta_i \perp_0 \alpha_i$. We call the set $F = \{\alpha_i, \beta_i, \alpha_i\beta_i, b_j | i = 1, ..., n, j = 1, ...r\}$ a *Thurston basis* of the measured lamination space. It is shown in [FLP] that the map $\tau_F : ML(\Sigma) \to R^F_{\geq 0}$ sending m to $I_m|_F$ is an embedding (In [FPL], the set F is taken to be $\{\alpha_i, \beta_i, \alpha_i\beta_i^2, b_j\}$. But the proof works for our case as well). We shall show that the image of τ_F is a tropical variety. By theorem 1.2, the result holds for level-1 surfaces. Now if Σ is a surface of level at least 2, we decompose $\Sigma = X \cup Y$ so that (i) the levels of X and Y are smaller than that of Σ, (ii) $X \cap Y \cong \Sigma_{0,3}$ and (iii) the components a_1, a_2, a_3 of $\partial(X \cap Y)$ represent elements, say, α_1, α_2, α_3 in F (α_2 may be the same as α_3). Let $F_X = F \cap \mathbb{S}(X)$ and $F_Y = F \cap \mathbb{S}(Y)$. There are two possibilities: either α_1, α_2, α_3 are pairwise distinct or $\alpha_2 = \alpha_3$ ($\neq \alpha_1$). In the first case, then F_X and F_Y

are Thurston bases for X and Y by condition (iii) and the definition. Let $\tau_{F_X}(m) = (x_1, ..., x_k)$ and $\tau_{F_Y}(m) = (y_1, ..., y_l)$ so that $x_i = I_m(\alpha_i)$, $y_i = I_m(\alpha_i)$ for $i = 1, 2, 3$. By the induction hypothesis, both images $Imag(\tau_{F_X})$ and $Imag(\tau_{F_Y})$ are tropical varieties. Now by lemma 2.2, each $m \in ML(\Sigma)$ is determined by its restriction on X and Y. Thus $Imag(\tau_F) = \{(x_1, ..., x_k; y_1, ..., y_l) \in Imag(\tau_{F_X}) \times Imag(\tau_{F_Y})$: $x_i = y_i$, $i = 1, 2, 3\}$. Thus the result follows by the induction hypothesis. In the second case that $\alpha_2 = \alpha_3$, one of the surfaces X, Y, say, X is $\Sigma_{1,1}$. Then F_X is a Thurston basis for X and $F_Y \cup \{[a_2], [a_3]\}$ is a Thurston basis for Y. Let $\tau_{F_X}(m) = (x_1, ..., x_k)$ and $\tau_{F_Y}(m) = (y_1, ..., y_l)$ so that $x_i = I_m(\alpha_i)$, $i = 1, 2$, and $y_1 = I_m(\alpha_1)$, $y_2 = I_m([a_2])$, $y_3 = I_m([a_3])$. By the same argument as above (using $x_1 = y_1$, $x_2 = y_2 = y_3$), the result follows.

The rest of the paper will focus on proving theorem 1.1 for level-2 surfaces.

6. The 5-holed Sphere

We will prove theorem 1.1 for $\Sigma_{0,5}$ in this section. Given a non-zero function $f : \mathbb{S}(\Sigma_{0,5}) \to \mathbf{R}$ satisfying (10) and (11), we will construct a measured lamination $m \in ML(\Sigma_{0,5})$ so that $f(x) = I_m(x)$ for all $x \in \mathbb{S}(\Sigma_{0,5})$.

The proof goes as follows. Let $\{a, b\}$ be a 3-holed sphere decomposition of the surface $\Sigma_{0,5}$ and let X, Y be the 4-holed spheres in $\Sigma_{0,5}$ bounded by a, b respectively. By the construction $X \cup Y = \Sigma_{0,5}$ and $X \cap Y = \Sigma_{0,3}$. By theorem 1.2 for the 4-holed spheres, there are $m_X \in ML(X)$ and $m_Y \in ML(Y)$ so that $f|_{\mathbb{S}(X)} = I_{m_X}$ and $f|_{\mathbb{S}(Y)} = I_{m_Y}$. By the construction I_{m_X} and I_{m_Y} are the same when restricted to $\mathbb{S}(X \cap Y)$. Thus, by the gluing lemma 2.2, there exists a measured lamination $m \in ML(\Sigma_{0,5})$ whose restrictions to X and Y are m_X and m_y. In particular we have $f(x) = I_m(x)$ for all $x \in \mathbb{S}(X) \cup \mathbb{S}(Y)$. Our goal is to show that $f = I_m$. To this end, let

$$\mathbb{L}(a, b) = \{x \in \mathbb{S}(\Sigma_{0,5}) | f(x) = I_m(x)\}.$$

We have $\mathbb{S}(X) \cup \mathbb{S}(Y) \subset \mathbb{L}(a, b)$. We will prove $f = I_m$ by showing that $\mathbb{L}(a, b) = \mathbb{S}(\Sigma_{0,5})$ for some choices of $\{a, b\}$.

First note that by theorem 4.1, the function I_m also satisfies the same set of equations (10) and (11). In particular, equation (11) says that the value of f (and I_m) at xy, where $x \perp_0 y$, is determined by the values of f (and I_m) on $\{x, y, yx\} \cup \partial N(x \cup y)$. This can be stated as,

Lemma 6.1. *If $x \perp_0 y$ in $\mathbb{S}(\Sigma_{0,5})$ so that $\{x, y, yx\} \cup \partial N(x \cup y) \subset \mathbb{L}(a, b)$. Then $xy \in \mathbb{L}(a, b)$.*

We will prove that $\mathbb{L}(a,b) = \mathbb{S}(\Sigma_{0,5})$ by establishing the following propositions.

Proposition 6.2. *Suppose* $\{x \in \mathbb{S}(\Sigma_{0,5}) | I(x,a) \leq 2, I(x,b) \leq 2\} \subset \mathbb{L}(a,b)$. *Then* $\mathbb{L}(a,b) = \mathbb{S}(\Sigma_{0,5})$.

Proposition 6.3. *Suppose there exists* $c \in \mathbb{S}(\Sigma_{0,5})$ *so that* $c \perp_0 a$, $c \perp_0 b$ *and* $\{ac, cb, acb\} \subset \mathbb{L}(a,b)$. *Then* $\{x \in \mathbb{S}(\Sigma_{0,5}) | I(x,a) \leq 2, I(x,b) \leq 2\} \subset \mathbb{L}(a,b)$.

Proposition 6.4. *Given a non-zero* f *satisfying (10) and (11), then there exist* a, b, c *in* $\mathbb{S}(\Sigma_{0,5})$ *so that the condition in Proposition 6.3 holds.*

6.1. A proof of proposition 6.2. The proof is based on the following lemma. A preliminary version of the lemma can be found in [Li].

Lemma 6.5. *Suppose* c *is a curve system in a surface* Σ *and* $x \in \mathbb{S}(\Sigma)$ *so that* $I(x, c_1) \geq 3$ *for a component* c_1 *of* c. *Furthermore, assume there are three points* p_1, p_2, p_3 *in* $x \cap c_1$ *which are adjacent along* c_1 *so that their intersection signs are* $(+,-,+)$ *or* $(-,+,-)$ *as shown in figure 10. Then there exist* $y, z \in \mathbb{S}(\Sigma)$ *with* $y \perp_0 z$ *so that the following properties hold:*
(i) $x = yz$
(ii) $\max(I(y,c), I(z,c), I(zy,c)) < I(x,c)$
(iii) for each component b *of* $\partial N(y \cup z)$, $I(b,c) < I(x,c)$.

Proof. We may assume that $|x \cap c| = I(x,c)$. There are two possibilities for the joining of p_i's along x as shown in figures 10(b) and 10(c). However, these two cases are symmetric and we will consider only the case (b) in figure 10.

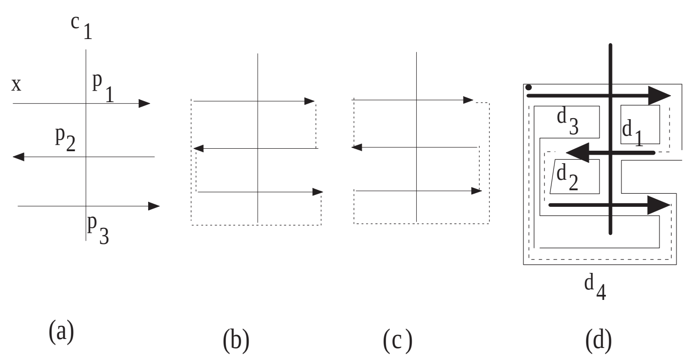


Figure 10

Let $[p_1, p_2]$ and $[p_2, p_3]$ be the closed intervals in c_1 bounded by p_1, p_2 and p_2, p_3 so that the interiors of the intervals are disjoint from x. Then the regular neighborhood $N(x \cup [p_1, p_2] \cup [p_2, p_3])$ is a subsurface homeomorphic to the 4-holed sphere. Since $|x \cap c| = I(x,c)$, the subsurface $N(x \cup [p_1, p_2] \cup [p_2, p_3])$ is essential in Σ. In particular, the boundary components d_1, d_2, d_3, d_4 of the subsurface are essential, i.e., in $\mathbb{S}(\Sigma_{0,5})$. One checks easily from the construction that $I(d_i, c) < I(x, c)$. See figure 10.

Now construct two simple loops y, z in $N(x \cup [p_1, p_2] \cup [p_2, p_3])$ as shown in figure 11 so that $y \perp_0 z$ and $x = yz$. By the construction, we have $y, z \in \mathbb{S}(\Sigma_{0,5})$. Furthermore, one sees that conclusion (ii) follows from figure 11 and conclusion (iii) follows from $N(y \cup z) = N(x \cup [p_1, p_2] \cup [p_2, p_3])$. q.e.d.

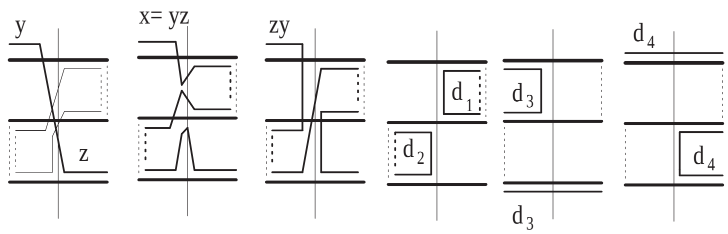


Figure 11

Now to prove proposition 6.2, take $c = a \cup b$ in the lemma 6.5. Suppose $x \in \mathbb{S}(\Sigma_{0,5})$ so that $I(x, a) \geq 3$. Since x is separating, any three adjacent intersection points of $x \cap a$ along a must have intersection signs $(+, -, +)$ or $(-, +, -)$. Thus by lemma 6.5, induction on $I(x, a) + I(x, b)$ we conclude that proposition 6.2 holds.

6.2. A proof of proposition 6.3. Suppose $x \in \mathbb{S}(\Sigma_{0,5})$ so that $I(x, a) = I(x, b) = 2$. We will show that $x \in \mathbb{L}(a, b)$. This will establish proposition 6.3 since $l(x, d)$ is always even. Consider the 3-holed sphere decomposition $\{a, b\}$ and the associated Dehn-Thurston coordinate. We see that both c and x have the same intersection coordinates. This implies that

$$x = x_{n,m}$$

for some $n, m \in \mathbf{Z}$ where

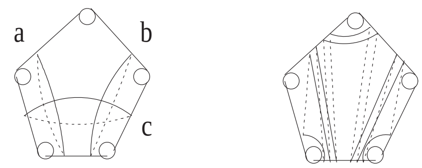
$$x_{n,m} = a^n c b^m.$$

Recall that $a^n c = c a^{-n}$ and $c a^n = a^{-n} c$ if $n < 0$. Also, the complement of the union of three systems of curves $a^{|n|}, c, b^{|m|}$ contains no contractible region. Thus by theorem 2.1(iv), the multiplication of $a^{|n|}, c, b^{|m|}$ is associative, i.e., $x_{n,m}$ is well defined. By theorem 2.1(iii), we have for any $k \in \mathbf{Z}$,

$$a^k x_{n,m} = x_{n+k,m}, \quad \text{and} \quad x_{n,m} a^k = x_{n-k,m}$$

and

$$x_{n,m} b^k = x_{n,m+k} \quad \text{and} \quad b^k x_{n,m} = x_{n,m-k}.$$



$x = a^n c b^m$

Figure 12

By the construction $x_{n,m} \perp_0 a$ and $x_{n,m} \perp_0 b$. This implies that $\partial N(x_{n,m} \cup a)$ and $\partial N(x_{n,m} \cup b)$ are subsets of $\mathbb{L}(a, b)$. Now by lemma 6.5, and $x_{n+1,m} = ax_{n,m}$ and $x_{n,m}a = x_{n-1,m}$, we conclude the following,

(i) if $\{x_{n,m}, x_{n-1,m}\} \subset \mathbb{L}(a, b)$, then $x_{n+1,m} \in \mathbb{L}(a, b)$,

(ii) if $\{x_{n,m}, x_{n+1,m}\} \subset \mathbb{L}(a, b)$, then $x_{n-1,m} \in \mathbb{L}(a, b)$.

Similarly,

(iii) if $\{x_{n,m}, x_{n,m-1}\} \subset \mathbb{L}(a, b)$, then $x_{n,m+1} \in \mathbb{L}(a, b)$,

(iv) if $\{x_{n,m}, x_{n,m+1}\} \subset \mathbb{L}(a, b)$, then $x_{n,m-1} \in \mathbb{L}(a, b)$.

These four properties together with the assumption that $x_{0,0} = c, x_{1,0} = ac$, $x_{0,1} = cb$ and $x_{1,1} = acb$ are in $\mathbb{L}(a, b)$ imply that $x_{n,m} \in \mathbb{L}(a, b)$ for all n, m. Thus proposition 6.3 follows.

6.3. A proof of proposition 6.4. Let the boundary components of $\Sigma_{0,5}$ be $b_1, b_2, ..., b_5$ and $M = \max\{f(b_i)|1 \le i \le 5\}$. Choose a 3-holed sphere decomposition $\{a, b\}$ of $\Sigma_{0,5}$ so that

$$\min(f(a), f(b)) > M \tag{16}$$

To see that (16) can be achieved, we first find $d \in \mathbb{S}(\Sigma_{0,5})$ so that $f(d) > 0$. Let $\{a', b'\}$ be a 3-holed sphere decomposition of $\Sigma_{0,5}$ so that $a' \perp_0 d$ and $b' \perp_0 d$.

Now by the eventual linearity (corollary 4.4), we see that

$$\lim_{n\to\infty} f(D_d^n(a')) = \lim_{n\to\infty} f(D_d^n(b')) = \infty$$

where $D_d^n(a') = d^{2n}a'$ is the Dehn twist n times along d applied to a'. Thus, (16) can be achieved by taking $a = D_d^n(a')$ and $b = D_d^n(b')$ for large n.

Next, we need the lemma below.

Lemma 6.6. *Suppose $\{a_1, a_2, a_3, a_4, a_5\}$ forms a pentagon in $\Sigma_{0,5}$ so that*

$$f(a_2) + f(a_3) > f(a_1a_2) + f(a_3a_4) + 2M. \tag{17}$$

Then

$$f(a_2a_3) = f(a_2) + f(a_3). \tag{18}$$

Proof. Recall that the pentagon relation means $a_i \cap a_j = \emptyset$ if $|i-j| \neq 1$ and $a_i \perp_0 a_{i+1}$ where indices are counted modulo 5. Furthermore we have

$$(a_1a_2)(a_3a_4) = a_5 \tag{19}$$

and

$$a_3a_2 \cap a_1a_2 = \emptyset \quad \text{and} \quad a_3a_2 \cap a_3a_4 = \emptyset. \tag{20}$$

See figures 13, 14.

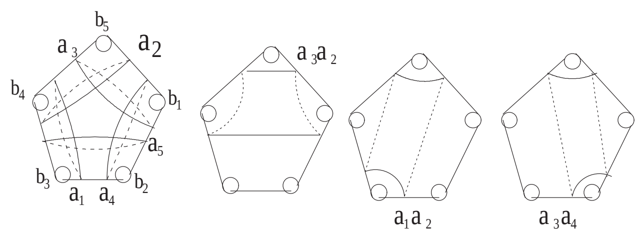


Figure 13

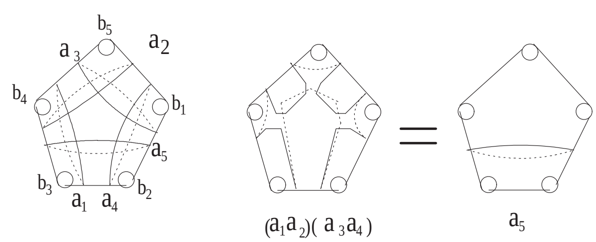


Figure 14

By corollary 4.3 applied to $f|_{\mathbb{S}(Z)}$ where Z is the 4-holed sphere bounded by a_5, we obtain

$$f(a_2)+f(a_3) = \max(f(a_2a_3), f(a_3a_2), f(a_5)+f(b_5), f(b_1)+f(b_4)) \tag{21}$$

We claim that (21) implies (18). Indeed,

$$f(b_1) + f(b_4) \leq 2M < f(a_2) + f(a_3)$$

where the last inequality is due to (17).

By the pentagon relation (19) and corollary 4.3(c) that $f(\alpha\beta) \leq f(\alpha) + f(\beta)$ when $\alpha \perp_0 \beta$, we have

$$f(a_5) \leq f(a_1a_2) + f(a_3a_4).$$

By the assumption (17), we have

$$f(a_5) + f(b_5) \leq f(a_5) + M \leq f(a_1a_2) + f(a_3a_4) + M < f(a_2) + f(a_3).$$

Finally to see that $f(a_2)+f(a_3)>f(a_3a_2)$, we consider the 4-holed sphere bounded by a_3a_2. See figure 13. By (20), the 4-holed sphere contains a_1a_2 and a_3a_4 where $a_1a_2 \perp_0 a_3a_4$. By corollary 4.3, we have

$$f(a_1a_2)+f(a_3a_4)$$

$$=\max(f(a_1a_2a_3a_4), f(a_3a_4a_1a_2), f(a_3a_2)+f(b_5), f(b_2)+f(b_3)).$$

In particular, it says that

$$f(a_3a_2)+f(b_5)\le f(a_1a_2)+f(a_3a_4).$$

Using (17), we conclude that $f(a_3a_2)<f(a_2)+f(a_3)$. Thus, (21) implies (18). q.e.d.

Now to prove proposition 6.4, take a pentagon $\{a_1,a_2,a_3,a_4,a_5\}$ so that $a_1=a$ and $a_4=b$. For any positive integer $n\in\mathbf{Z}$, define

$$a_2(n)=a_2a_1^n, \quad \text{and} \quad a_3(n)=a_4^na_3.$$

Then $\{a_1,a_2(n),a_3(n),a_4\}$ still form a part of a pentagon relation, i.e.,

$$a_1\cap a_3(n)=a_2(n)\cap a_4=\emptyset$$

and

$$a_1\perp_0 a_2(n)\perp_0 a_3(n)\perp_0 a_4.$$

In particular, we still have $a_2(n),a_3(n)\in\mathbb{L}(a,b)$.

By the eventual linearity property (corollary 4.4), we see there is a positive integer N so that when $n\ge N-1$,

$$f(a_2(n))=f(a_2(n-1))+f(a) \tag{22}$$

and

$$f(a_3(n))=f(a_3(n-1))+f(b). \tag{23}$$

Define $c=a_2(N)a_3(N)$. We claim that $\{c,ac,cb,acb\}\subset\mathbb{L}(a,b)$. To this end, we first show,

$$f(c)=f(a_2(N))+f(a_3(N)) \tag{24}$$

$$f(ac)=f(a_2(N+1))+f(a_3(N)) \tag{25}$$

$$f(cb)=f(a_2(N))+f(a_3(N+1)) \tag{26}$$

and

$$f(acb)=f(a_2(N+1))+f(a_3(N+1)). \tag{27}$$

Note that $a_2(n-1)=a_1a_2(n)$ and $a_2(n-1)=a_3(n)a_4$.

To see (24), use lemma 6.6 for $a_1, a_2(N), a_3(N), a_4$. We see that (17) follows from the linearity condition (22), (23) and (16). Thus (24) follows from (18).

The rest of the identities (25)-(27) are proved in the same way.

Since $\{a_1, a_2(n), a_3(n), a_4\} \subset \mathbb{L}(a,b)$, it follows that equations (22)-(27) and lemma 6.6 hold if we replace f by l_m. Therefore, $\{c, ca, bc, bca\} \subset \mathbb{L}(a,b)$. Indeed, to check $c \in \mathbb{L}(a,b)$, we have $f(c) = f(a_2(N)) + f(a_3(N)) = l_m(a_2(N)) + l_m(a_3(N)) = l_m(c)$. The same argument shows that f and l_m are equal on $\{ca, bc, cab\}$. This proves proposition 6.4.

7. The 2-holed Torus

We will prove theorem 1.1 for the 2-holed torus in this section. The basic idea of the proof is the same as that in §6. Given a non-zero function $f : \mathbb{S}(\Sigma_{1,2}) \to \mathbf{R}_{\geq 0}$ satisfying (7), (8), (10) and (11), we will construct $m \in ML(\Sigma_{1,2})$ so that $f = I_m$.

Let $\{a, b\}$ be a 3-holed sphere decomposition of $\Sigma_{1,2}$ so that a is separating. Let X, Y be the 1-holed torus and the 4-holed sphere bounded by a and b respectively. Then $\Sigma_{1,2} = X \cup Y$ and $X \cap Y = \Sigma_{0,3}$. By the same argument as used in §6, we produce $m \in ML(\Sigma_{1,2})$ so that $I_m(x) = f(x)$ for all $x \in S(X) \cup S(Y)$. Define

$$\mathbb{L}(a,b) = \{x \in \mathbb{S}(\Sigma_{1,2}) | f(x) = I(m,x)\}.$$

We will prove that for some choices of $\{a,b\}$, $\mathbb{L}(a,b) = \mathbb{S}(\Sigma_{1,2})$. This will establish $f = I_m$.

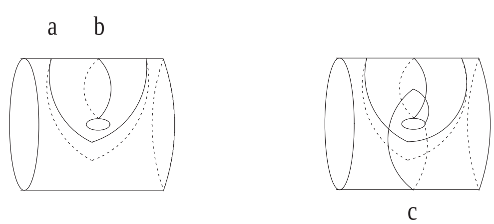


Figure 15

Since both f and I_m satisfy the set of equations (8) and (11) we conclude that,

Lemma 7.1. *If $x \perp y$ or $x \perp_0 y$ so that $\{x, y, yx\} \cup \partial N(x \cup y) \subset \mathbb{L}(a,b)$, then $xy \in \mathbb{L}(a,b)$.*

Furthermore, we have $\mathbb{S}(X) \cup \mathbb{S}(Y) \subset \mathbb{L}(a,b)$ by the construction. Now the following three propositions imply $\mathbb{L}(a,b) = \mathbb{S}(\Sigma_{1,2})$.

Proposition 7.2. *Suppose there exists $c \in \mathbb{S}(\Sigma_{1,2})$ so that $c \perp_0 a$, $c \perp b$ and $\{c, ca, bc, bca\} \subset \mathbb{L}(a,b)$, then $\{x \in \mathbb{S}(\Sigma_{1,2}) | I(x,a) \leq 2, I(x,b) \leq 1\} \subset \mathbb{L}(a,b)$.*

Proposition 7.3 *Given $f : \mathbb{S}(\Sigma_{1,2}) \to \mathbf{R}_{\geq 0}$ satisfying (7), (8), (10) and (11), there exists a 3-holed sphere decomposition $\{a, b\}$ and $c \in \mathbb{S}(\Sigma_{1,2})$ so that the condition in proposition 7.2 holds.*

Proposition 7.4. *If $\{x \in \mathbb{S}(\Sigma_{1,2}) | I(x, a) \leq 2, I(x, b) \leq 1\} \subset \mathbb{L}(a, b)$, then $\mathbb{L}(a, b) = \mathbb{S}(\Sigma_{1,2})$.*

The rest of the section will prove these propositions.

7.1. A proof of proposition 7.2. The proof is exactly the same as that of proposition 6.3. We provide a sketch of the argument. Each $x \in \mathbb{S}(\Sigma_{1,2})$ so that $I(x, a) = 2$ and $I(x, b) = 1$ is of the form

$$x = x_{n,m} = a^n c b^m$$

for some $n, m \in \mathbf{Z}$. This can be seen by considering the twisting part of the Dehn-Thurston coordinates with respect to the 3-holed sphere decomposition $\{a, b\}$. Now by the construction $a \perp_0 x_{n,m}$, $b \perp x_{n,m}$, $\partial N(a \cup x_{n,m}) \subset \mathbb{L}(a, b)$, and $\partial N(b \cup x_{n,m}) \subset \mathbb{L}(a, b)$. Furthermore, we have $a^k x_{n,m} = x_{n+k,m}$, $x_{n,m} a^k = x_{n-k,m}$, $x_{n,m} b^k = x_{n,m+k}$, and $b^k x_{n,m} = x_{n,m-k}$. These identities together with lemma 7.1 and the assumption that $x_{0,0}, x_{0,-1}, x_{-1,0}, x_{-1,-1} \in \mathbb{L}(a, b)$ imply that $x_{n,m} \in \mathbb{L}(a, b)$ for all n, m.

7.2. A proof of proposition 7.3. We begin with the following lemma

Lemma 7.5. *Suppose $\{a_1, a_2, a_3, a_4, a_5\}$ forms a pentagon in $\Sigma_{1,2}$ so that a_1, a_5 are separating. Suppose further that*

$$f(a_2) + f(a_3) > f(a_1 a_2) + f(a_3 a_4^2). \tag{28}$$

Then

$$f(a_2 a_3) = f(a_2) + f(a_3). \tag{29}$$

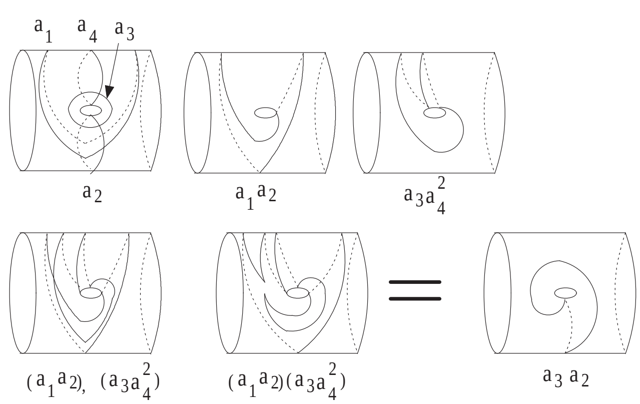


Figure 16

Proof. Note that the pentagon relation (proposition 2.3) says that

$$a_3a_2 = (a_1a_2)(a_3a_4^2) \tag{30}$$

See figure 16 for a proof. Also that $a_1a_2 \perp a_3a_4^2$. By corollary 4.3, (30) and (28), we obtain

$$f(a_3a_2) = f((a_1a_2)(a_3a_4^2)) \leq f(a_1a_2) + f(a_3a_4^2) < f(a_2) + f(a_3). \tag{31}$$

By corollary 4.3, we have

$$f(a_2) + f(a_3) = \max(f(a_2a_3), f(a_3a_2)) \tag{32}$$

Thus by (31) and (32), we conclude that (29) holds. q.e.d.

Now to prove proposition 7.3, we find a 3-holed sphere decomposition $\{a, b\}$ (with a separating) of $\Sigma_{1,2}$ so that $f(a)f(b) > 0$. This can be achieved by the same method as in §6.3 using Dehn twists and the eventual linearity of the intersection numbers. Take a pentagon relation $\{a_1 = a, a_2, a_3, a_4 = b, a_5\}$. For any $n \in \mathbf{Z}_{\geq 0}$, define

$$a_2(n) = a_2a_1^n$$

and

$$a_3(n) = a_4^n a_3.$$

Then $\{a_1, a_2(n), a_3(n), a_4\}$ still form a part of a pentagon relation, i.e.,

$$a_1 \cap (a_3(n) \cup a_4) = a_2(n) \cap a_4 = \emptyset$$

and

$$a_1 \perp_0 a_2(n), a_2(n) \perp a_3(n), a_3(n) \perp a_4.$$

Furthermore, $a_2(n), a_3(n) \in \mathbb{L}(a, b)$. By the eventual linearity property (corollary 4.4), we find an integer N so that for $n \geq N - 1$,

$$f(a_2(n)) = f(a_2(n-1)) + f(a) \tag{33}$$

and

$$f(a_3(n)) = f(a_3(n-1)) + f(b). \tag{34}$$

Now take $c = a_2(N)a_3(N)$. We claim that $\{c, ca, bc, bca\} \subset \mathbb{L}(a, b)$. Indeed, we will prove

$$f(c) = f(a_2(N)) + f(a_3(N)) \tag{35}$$

$$f(ac) = f(a_2(N-1)) + f(a_3(N)) \tag{36}$$

$$f(bc) = f(a_2(N)) + f(a_3(N+1)) \tag{37}$$

and

$$f(bca) = f(a_2(N+1)) + f(a_3(N+1)). \tag{38}$$

To see (35), we use lemma 7.5. for $\{a_1, a_2(N), a_3(N), a_4\}$ as the first four elements in the pentagon relation. Now $a_1 a_2(N) = a_2(N-1)$ and $a_3(N)a_4^2 = a_3(N-2)$. Thus (28) is a consequence of (33), (34) and $f(a), f(b) > 0$. Therefore (35) is a consequence of (29). The other identities (36)-(38) are proved in the same way.

Since $\{a_1, a_2(n), a_3(n), a_4\} \subset \mathbb{L}(a,b)$, it follows that equation (33)-(38) and lemma 7.5 hold if we replace f by l_m. Therefore $\{c, ca, bc, bca\} \subset \mathbb{L}(a,b)$. Indeed, to check $c \in \mathbb{L}(a,b)$, we have $f(c) = f(a_2(N)) + f(a_3(N)) = l_m(a_2(N)) + l_m(a_3(N)) = l_m(c)$. The same argument shows that f and l_m are equal on $\{ca, bc, cab\}$. This prove proposition 7.3.

7.3. A proof of proposition 7.4. The key step of the proof is in the following,

Lemma 7.6. *Suppose* $\{x \in \mathbb{S}(\Sigma_{1,2}) | I(x,a) + I(x,b) \leq 4\} \subset \mathbb{L}(a,b)$. *Then* $\mathbb{L}(a,b) = \mathbb{S}(\Sigma_{1,2})$.

Proof. We use induction on $|x| := I(x, ab)(= I(x,a) + I(x,b))$ to prove the lemma. Assume that $|x \cap (ab)| = |x|$. Suppose inductively that for $|x| < n$, $x \in \mathbb{L}(a,b)$. Now in the case of $|x| = n \geq 5$, one of $I(x,a)$ or $I(x,b)$, say $I(x,a)$, is at least 3. The proof below works for $I(x,b) \geq 3$.

There are two cases we must consider: either there are three points p_1, p_2, p_3 in $x \cap a$ adjacent along a so that their intersection signs are $+, -, +$ or $-, +, -$, or there exists a pair of points p_1, p_2 in $x \cap a$ adjacent along a so that their intersection signs are the same.

In the first case, by lemma 6.5, the induction hypothesis and lemma 7.1, we conclude that $x \in \mathbb{L}(a,b)$.

In the second case, let $[p_1, p_2]$ be the closed interval in a bounded by p_1, p_2 so that $x \cap [p_1, p_2] = \{p_1, p_2\}$. Due to $|x \cap a| = I(x,a)$, we see that the subsurface $N(x \cup [p_1, p_2])$ is an essential 1-holed torus in $\Sigma_{1,2}$. Let y, z be the simple loops in the subsurface $N(x \cup [p_1, p_2])$ as shown in figure 17. We have $y \perp z$, $x = yz$, and $\max(I(y,ab), I(z,ab), I(zy,ab)) < I(x,ab)$. Thus to establish $x \in \mathbb{L}(a,b)$, by the induction hypothesis that $y, z, zy \in \mathbb{L}(a,b)$ and lemma 7.1, we only need to show that

$$\partial N(y \cup z) \in \mathbb{L}(a,b). \tag{39}$$

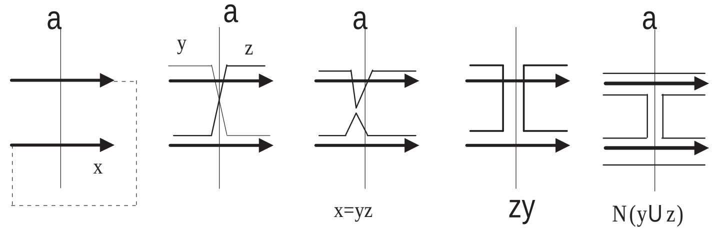


Figure 17

By the construction of y, z, we have $I(y, ab) + I(z, ab) \leq I(x, ab)$. Thus, one of $I(y, ab)$ or $I(z, ab)$, say $I(y, ab) \leq I(x, ab)/2 = n/2$.

Since $y \perp z$, the simple loop y is non-separating. Consider the 4-holed sphere Σ' obtained by cutting $\Sigma_{1,2}$ open along y. Let b_1, b_2 be the boundary components of Σ' corresponding to y. By the construction $\partial N(y \cup z)$ is in Σ'. Let $d = \Sigma' \cap ab$. The arc system d consists of $I(y, ab) \leq n/2$ many components. Each arc in d either joins b_1 to b_2 or has two end points on b_i for $i = 1, 2$. Furthermore there is a component of d (corresponding to $[p_1, p_2]$) joining b_1 to b_2 due to the intersection sign assumption.

Let $m' \in ML(\Sigma')$ so that $f|_{\mathbb{S}(\Sigma')} = I_{m'}$ and $m'' = m|_{\Sigma'}$. We will show that $m' = m''$. In particular, this implies that $f(\partial N(y \cup z)) = I_m(\partial N(y \cup z))$, i.e., (39) holds.

For the two measured laminations m', m'' in Σ', by the induction hypothesis, $I(m', t) = I(m'', t)$ for all $t \in \mathbb{S}(\Sigma')$ so that $I(t, d) < I(x, ab) = n$.

We will verify $m' = m''$ by showing (i) for each boundary component t of Σ', $I(t, a) < n$ and, (ii) there exist three curves s_1, s_2, s_3 in $\mathbb{S}(\Sigma')$ forming a triangle in the modular configuration so that $I(s_i, d) < n$. Since a measured laminations on $\Sigma_{0,4}$ is determined by its values on $\{s_1, s_2, s_3\} \cup \partial\Sigma'$, this implies $m' = m''$.

To verify the first condition (i), let t be a component of $\partial\Sigma'$. If $t = b_1$ or b_2, then $I(t, d) = I(y, ab) < I(x, a)$. If $t \neq b_1, b_2$, then $I(t, d) = 0$.

To verify the second condition (ii), by the classification of ideal triangulations of the 4-holed sphere, we can write $d = t_1^{k_1} t_2^{k_2} t_3^{k_3} t_4^{k_4}$ where t_1, t_2, t_3, t_4 are disjoint as shown in figure 18 and $k_i \geq 0$ so that $k_1 + k_2 + k_3 + k_4 = I(y, ab) \leq n/2$. Furthermore, $k_1 + k_2 > 0$ due to the existence of an arc in d joining b_1, b_2.

Assume without loss of generality that $k_2 \leq k_1$. Let s_1, s_2, s_3 form a triangle in $\mathbb{S}'(\Sigma')$ as shown in figure 18 where $I(s_1, t_3 t_4) = 0$ and $I(s_3, t_1) = 0$.

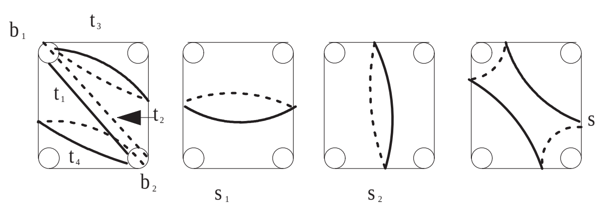


Figure 18

We have,

$$I(s_1, d) = k_1 + k_2 \leq \sum_{i=1}^{4} k_i \leq n/2 < n,$$

$$I(s_2, d) = k_1 + k_2 + 2k_3 + 2k_4 < 2\sum_{i=1}^{4} k_i \le n,$$

$$I(s_3, d) = 2k_2 + 2k_3 + 2k_4 < 2\sum_{i=1}^{4} k_i \le n.$$

This concludes the proof of lemma 7.6. q.e.d.

We now prove proposition 7.4. By lemma 7.6, it suffices to show if $x \in \mathbb{S}(\Sigma_{1,2})$ so that $I(x, ab) \le 4$, then $x \in \mathbb{L}(a, b)$.

The proof of lemma 7.6 shows that we may assume (i) $I(x, a), I(x, b) \le 2$ and (ii) there is no pair of points p_1, p_2 of $x \cap b$ adjacent along b so that their intersection signs are the same.

Let us analysis the various situations that could occur to x. First of all, since a is a separating simple loop, $I(x, a)$ has to be an even integer. It follows that $I(x, a)$ are 0, 2 or 4.

If $I(x, a) = 0$, then $x \in \mathbb{S}(X)$. Thus $x \in \mathbb{L}(a, b)$ by definition.

If $I(x, a) = 4$, then due to $I(x, a)+I(x, b) \le 4$, it follows that $I(x, b) = 0$. Thus $b \in \mathbb{S}(Y)$. Therefore, $b \in \mathbb{S}(Y)$ by definition again.

Finally, if $I(x, a) = 2$, then $I(x, b) = 0$, or 1, or 2. The case $I(x, b) = 0$ implies that $x \in \mathbb{L}(a, b)$. If $I(x, b) = 1$, then the hypothesis of the proposition 7.4 says $x \in \mathbb{L}(a, b)$. If $I(x, b) = 2$, due to the assumption that $I(x, a) = 2$, the intersection $x \cap X$ consists of an arc. Now by the classification of arcs in a 1-holed torus X, we see that all intersection signs of $x \cap b$ are the same. This implies that the two points in $x \cap b$ have the same intersection sign. But this was ruled out by the assumption (ii) above.

8. A proof of theorem 2.1

We begin with the following

Lemma 8.1. *(i) If a and b are curve systems with $|a \cap b| = I(a, b)$, then the disjoint union ab is a curve system.*

(ii) Suppose a, b and c are curve systems in Σ so that $|a \cap b| = I(a, b)$, $|b \cap c| = I(b, c)$, $|c \cap a| = I(c, a)$ and $|a \cap b \cap c| = 0$. If there is no contractible region in $\Sigma - (a \cup b \cup c)$ which is either bounded by three arcs in a, b and c respectively, or by four arcs in a, b, c and $\partial\Sigma$ respectively (see figure 20(a)), then $|c \cap ab| = I(c, ab)$.

Proof. (i) If ab is not a curve system, then there exists either (1) a simple closed curve s in ab and an annulus D with $\partial D = s \cup d$ where d is a boundary component of Σ or (2) a simple closed curve or a proper arc s in ab and a disc D in Σ so that either (2.1) $\partial D = s$ or (2.2) $\partial D = s \cup d$ where $s \cap d = \partial s = \partial d$ and d is an arc in $\partial\Sigma$. Using the inner most disk argument, by replacing s and finding another component of ab in $int(D)$ if necessary, we may assume that $ab \cap int(D) = \emptyset$. Take a small regular neighborhood $N(a \cup b)$ of $a \cup b$ to be $N(a) \cup N(b)$. We assume the resolutions of $a \cap b$ are taken place inside $N(a) \cap N(b)$. Thus $int(D)$ contains a finite number of connected components $R_0, R_1, ..., R_n$ of $\Sigma - \text{int}(N(a) \cup N(b))$, where $R_i \neq \emptyset$, and $R_i \cap \partial\Sigma = \emptyset$, for $i \geq 1$, and $R_0 = \emptyset$ in case D is a disc in $int(\Sigma)$, and R_0 is the region which intersects $\partial\Sigma$ in the other cases. Furthermore, R_0 is a disc if D is a disc intersecting $\partial\Sigma$ and is an annulus if D is an annulus. Each region R_i ($i \geq 1$) is a disc since otherwise there would be at least two boundary components of R_i in $int(D)$. This would contradict the assumption that $int(D) \cap ab = \emptyset$. Call a point in $\partial N(a) \cap \partial N(b)$ a *corner* of $N(a \cup b)$. Each point p in $a \cap b$ corresponds to four corners in $\partial N(p)$ where $N(p)$ is the connected component of $N(a) \cap N(b)$ containing p. Join opposite corners in $\partial N(p)$ by an arc in $int(N(p))$ so that it avoids one of the resolutions of $a \cap b$ at p. We call the arc a *bridge* between the corners. A corner of $\partial N(a \cup b)$ in a region R_i is called a *vertex* of R_i. Vertices of R_i decompose ∂R_i into *edges.* Each edge is either in $\partial N(a)$, or in $\partial N(b)$, or in $\partial\Sigma$. There is at most one edge which is in R_0. If two edges have a vertex in common, they cannot be both in $N(a)$ (resp. in $N(b)$). Thus for $i \geq 1$, there are even number of edges in R_i. Each region R_i with $i \geq 1$, must have at least four edges since $|a \cap b| = I(a, b)$ (if there were regions with only two edges, then the region provides a Whitney disc for $a \cup b$). More importantly, the definition of the resolution implies the following alternating principle: if v and v' are two vertices joint by an edge in R_i so that the edge is either in $N(a)$ or in $N(b)$, then exactly one of the bridges from v or v' still lies in D (see figure 19(b)).

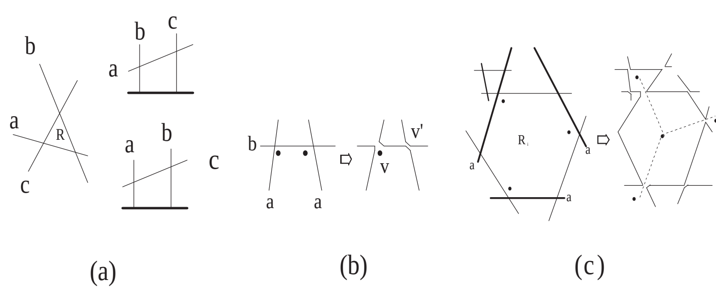


Figure 19

Form a graph G in D by putting a 0-cell in each $int(R_i)$. Joint two 0-cells of $int(R_i)$ and $int(R_j)$ by a 1-cell in D if there are opposite vertices in R_i and R_j so that their bridge is in D (the 1-cell is an extension of the bridge). These 1-cells are chosen to be pairwise disjoint except at the end points. By the construction, if D is a disc, the graph G is homotopic to D since each region R_i is a disc; if D is an annulus, the region R_0 is an annulus, thus the graph G is again homotopic to a disc. In both cases, G is a tree. Therefore either G is a point or G contains two 0-cells of valency one. However by the construction, each region R_i ($i \geq 1$) has at least four edges and thus corresponds to a 0-cell of valency at least two by the alternating principle. Thus the graph G must be a point. Therefore, there is only one region R_0 which has at most one vertex by the alternating principle. This contradicts the condition that $|a \cap b| = I(a, b)$.

(ii) Suppose the result is false. Then there is a disc $D \subset \Sigma$ so that either (1) ∂D is a union of two arcs s and t with $s \cap t = \partial s = \partial t$, $s \subset c$ and $t \subset ab$, or (2) ∂D is a union of three arcs s, t, u so that each pair of arcs intersect at one end point and $s \subset c$, $t \subset ab$, and $u \subset \partial\Sigma$. By taking the inner most disc if necessary, we may assume that $int(D) \cap (c \cup ab) = \emptyset$. Let $N(ab) = N(a) \cup N(b)$, $N(a \cap b) = N(a) \cap N(b)$, and $R_0, R_1, ..., R_n$ be the set of components of $\Sigma - (c \cup N(a) \cup N(b))$ which are contained in D. We set R_0 to be the region so that $R_0 \cap c \neq \emptyset$. Then $R_0 \cap u \neq \emptyset$ if $u \neq \emptyset$. Furthermore, $R_i \cap (c \cup \partial\Sigma) = \emptyset$ for $i \geq 1$. By the assumption that $\mathrm{int}(D) \cap (c \cup ab) = \emptyset$, each region R_i is a disc. Use the same argument as in the part (i), each region R_i ($i \geq 1$) has at least four sides and adjacent vertices in ∂R_i ($i \geq 0$) satisfy the alternating principle. Form the same type of graph G in D based on the combinatorics of the regions R_i as in the part (i). Since each region R_i is contractible, the graph G is a tree. Thus G is either a point or contains two vertices of valency one. The later case is impossible by the alternating property. Thus G is a point. Thus, there is only one region R_0 in D which has exactly one vertex. This is equivalent to the condition that there is a contractible region in $\Sigma - (a \cup b \cup c)$ which is bounded by three arcs in a, b, and c, or by four arcs in a, b, c, and $\partial\Sigma$. Thus we obtain a contradiction. q.e.d.

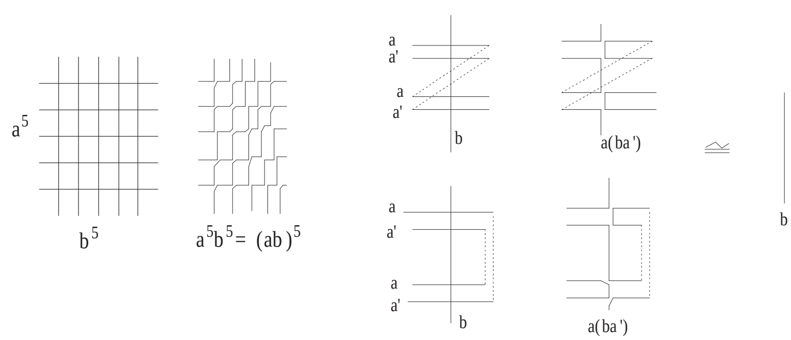


Figure 20

Theorem 2.1. *The multiplication* $\mathbb{CS}(\Sigma) \times \mathbb{CS}(\Sigma) \to \mathbb{CS}(\Sigma)$ *sends* $\mathbb{CS}_0(\Sigma) \times \mathbb{CS}_0(\Sigma)$ *to* $\mathbb{CS}_0(\Sigma)$ *and satisfies the following properties.*

(i) It is invariant under the action of the orientation preserving homeomorphisms.

(ii) If $I(\alpha,\beta)$ =0, then $\alpha\beta=\beta\alpha$. Conversely, if $\alpha\beta=\beta\alpha$ and $\alpha\in\mathbb{CS}_0(\Sigma)$, then $I(\alpha,\beta)$=0.

(iii) If $\alpha\in\mathbb{CS}_0(\Sigma)$, $\beta\in\mathbb{CS}(\Sigma)$, then $I(\alpha,\alpha\beta)=I(\alpha,\beta\alpha)=I(\alpha,\beta)$ and $\alpha(\beta\alpha)=(\alpha\beta)\alpha$. If in addition that each component of α intersects β, then $\alpha(\beta\alpha)=\beta$.

(iv) If $[c_i]\in\mathbb{CS}(\Sigma)$ so that $|c_i\cap c_j|=I(c_i,c_j)$ for $i,j=1,2,3$, $i\neq j$, $|c_1\cap c_2\cap c_3|=0$, and there is no contractible region in $\Sigma-(c_1\cup c_2\cup c_3)$ bounded by three arcs in c_1, c_2, c_3, then $[c_1]([c_2][c_3])=([c_1][c_2])[c_3]$.

(v) For any positive integer k, $(\alpha^k\beta^k)=(\alpha\beta)^k$.

(vi) If α is the isotopy class of a simple closed curve, then the positive Dehn twist along α sends β to $\alpha^k\beta$ where $k=I(\alpha,\beta)$.

Proof. Properties (i), (v) and (vi) follow from the definition (see figure 20(a)). Property (iv) follows from lemma 8.1(ii). Indeed, by the lemma, both $([c_1][c_2])[c_3]$ and $[c_1]([c_2][c_3])$ are obtained by simultaneously resolving all intersection points in $c_1\cup c_2\cup c_3$ from c_1 to c_2, c_2 to c_3, and c_1 to c_3. To see (iii), take a and a' to be in α with $a\cap a'=\emptyset$ (two nearby parallel copies), and $b\in\beta$ with $|a\cap b|=|a'\cap b|=I(a,b)$. Then, since α is closed, a,a', and b satisfy the condition in lemma 8.1(ii). Thus $\alpha(\beta\alpha)=(\alpha\beta)\alpha$ follows. Also by lemma 8.1(ii), $I(\alpha,\alpha\beta)=|a\cap a'b|=|a\cap b|=I(\alpha,\beta)$ where $|a\cap a'b|=|a\cap b|$ follows from the definition. The equality $I(\alpha,\beta\alpha)=I(\alpha,\beta)$ follows similarly. If each component of α intersects β, then figure 20(b) shows that $\alpha(\beta\alpha)=\beta$. Indeed, it suffices to consider two adjacent intersection points P_1, P_2 along a component of a. Figure 20(b) shows that the multiplication $a(ba')$ is the same as finger moves on b. Thus $\alpha(\beta\alpha)=\beta$. It remains to show (ii). Clearly if $I(\alpha,\beta)=0$, then $\alpha\beta=\beta\alpha$. Conversely, suppose $\alpha\in CS_0(\Sigma)$ and $\beta\in CS(\Sigma)$ with $\alpha\beta=\beta\alpha$. We decompose α as a disjoint union $\alpha_1\alpha_2$ where $I(\alpha_1,\beta)=0$ and each component of α_2 intersects β. Now since α_1 is disjoint from both α_2 and β, we have $\beta(\alpha_1\alpha_2)=\alpha_1(\beta\alpha_2)$. Thus, by $\alpha\beta=\beta\alpha$, we obtain $\alpha_2\beta=\beta\alpha_2$. Since each component of α_2 intersects β, by property (iii), $\beta=\alpha_2(\beta\alpha_2)=\alpha_2(\alpha_2\beta)=(\alpha_2)^2\beta$ where the last equality follows from property (iv). Now by property (iii), $I(\beta,\beta)=I(\beta,(\alpha_2^2)\beta)=I(\beta,\alpha_2^2)$ $=2I(\beta,\alpha_2)\neq 0$. This is a contradiction. q.e.d.

Remark 8.1. Properties (ii), (iii) and (iv) are similar to the commutative, the inverse, and the associative laws in group theory. Indeed, if each component of a curve system $\alpha\in\mathbb{CS}_0$ intersects both β and γ and $\beta\alpha=\gamma\alpha$, then (iii) implies that $\beta=\alpha(\beta\alpha)=\alpha(\gamma\alpha)=\gamma$.

8.1. Convexity of intersection numbers.

Corollary 8.2. *Suppose $\alpha,\beta\in\mathbb{CS}_0(\Sigma)$, then $I(\alpha\beta,\beta\alpha)=2I(\alpha,\beta)$. In particular, we have*

(i) $(\alpha\beta)(\beta\alpha)=\beta^2\gamma$ where γ is disjoint from both α and β, and

(ii) $I(\delta,\alpha\beta)+I(\delta,\beta\alpha)\geq 2I(\delta,\beta)$.

(iii) The function $I(\alpha^n\beta, \delta)$ is convex in n.

Proof. Choose $x, x' \in \alpha$, $y, y' \in \beta$ so that $|x \cap y| = I(x, y)$ and x', y' are parallel copies of x, y. Then by the definition of the multiplication, there are no bi-gons in $xy \cup y'x'$ and $|xy \cap y'x'| = 2I(x, y)|$. Thus $(xy)(y'x')$ is a representative of $(\alpha\beta)(\beta\alpha)$. An easy calculation shows that $(xy)(y'x') \cong y^2 z$ where z consists of components of x disjoint from y.

To see the statement (iii), let $\beta' = \alpha^n\beta$. Then the convexity follows from (ii) where we replace β by β'. q.e.d.

Call the set $\{\alpha^n\beta \colon n \in \mathbf{Z}\}$ a *horocycle* in the space $\mathbb{CS}_0(\Sigma)$. It follows from the corollary 8.2 (iii)) that I_δ is convex along horocycles. This seems to be an analogy of the earthquake paths. See Bonahon [Bo3] and Papadopoulos [Pa1] for more details.

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