# ON TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

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**Abstract**

*We characterize hyperbolic metrics on compact triangulated surfaces with boundary using a variational principle. As a consequence, a new parameterization of the Teichmüller space of a compact surface with boundary is produced. In the new parameterization, the Teichmüller space becomes an explicit open convex polytope. Our results can be considered as a generalization of the simplicial coordinate of Penner [P1], [P2] for hyperbolic metrics with cusp ends to the case of surfaces with geodesic boundary. It is conjectured that the Weil-Petersson symplectic form can be expressed explicitly in terms of the new coordinate.*

## 1. Introduction

*1.1*

The purpose of this article is to produce a new parameterization of the Teichmüller space of a compact surface with nonempty boundary so that the lengths of the boundary components are fixed. In this new parameterization, the Teichmüller space becomes an explicit open convex polytope. Our result is motivated by the articles [L], [R], and [Lu1] for hyperbolic, Euclidean, and spherical cone metrics on closed triangulated surfaces. In these approaches, constant curvature metrics are identified with the critical points of some natural energy functions. The energy functions used in [L] and [Lu1] can be constructed by the cosine laws for hyperbolic and spherical triangles. The cosine law for right-angled hyperbolic hexagons produces the energy for the current work. All these energies are related to the dilogarithm function. Our work can be considered as the counterpart to, and a generalization of, Penner's work on the decorated Teichmüller space of cusped surfaces (see [P2], [P1]). Indeed, the recent work of Mondello [Mo] shows that by taking a sequence of hyperbolic metrics with geodesic boundary converging to a cusped metric, under appropriate normalization, the limit of the coordinate introduced in this article is the simplicial coordinate introduced by Penner [P2]. Furthermore, the statements of results in our article are very similar to

DUKE MATHEMATICAL JOURNAL
Vol. 139, No. 3, 
Received 19 January 2006. Revision received 2 November 2006.
2000 *Mathematics Subject Classification*. Primary 32G15; Secondary 57M50.

the theorems of Penner on simplicial coordinates. Our results establish a link between the works of Leibon [L], Rivin [R], and others to the work of Penner [P1], [P2].

As a convention in this article, all surfaces are assumed to be compact and connected with nonempty boundary and to have negative Euler characteristic unless stated otherwise. A hyperbolic metric on the compact surface is assumed to have totally geodesic boundary.

*1.2*

We begin with a brief review of the Teichmüller spaces. Suppose that $S$ is a compact surface of nonempty boundary and has negative Euler characteristic. It is well known that there are hyperbolic metrics with totally geodesic boundary on the surface $S$. Two such hyperbolic metrics are *isotopic* if there is an isometry isotopic to the identity between them. The space of all isotopy classes of hyperbolic metrics on $S$, denoted by $T(S)$, is called the Teichmüller space of the surface $S$. We are interested in the subspace of $T(S)$ with prescribed boundary lengths. To be precise, let the boundary components of $S$ be $b_1, \dots, b_r$. Assign the $i$th boundary component $b_i$ a positive number $l_i$, and let $l = (l_1, \dots, l_r)$. Then the *bordered Teichmüller space T(S, l)* is the subset of $T(S)$ consisting of those isotopy classes of metrics such that the length of $b_i$ in the metrics is $l_i$. The space $T(S, l)$ has been used recently in calculation of the Weil-Petersson volume of the moduli spaces of curves in [M]. Using a 3-holed sphere decomposition of the surface $S$ and the associated Fenchel-Nielsen coordinate, it is known (see [IT] or [Bu]) that $T(S, l)$ is diffeomorphic to $(\mathbf{R} \times \mathbf{R}_{>0})^N$ for some integer $N$.

One can decompose the surface $S$ into a union of hexagons instead of 3-holed spheres. These decompositions are called *ideal triangulations* of the surface. They are also called trivalent ribbon graphs in the dual setting. The main result of the article (Theorem 1.2) gives a natural parameterization of the bordered Teichmüller space $T(S, l)$ using an ideal triangulation.

*1.3*

We now set up the framework by recalling ideal triangulations and right-angled hyperbolic hexagons. A *colored hexagon* is a hexagon such that three of its nonpairwise adjacent edges are designated as $x$-edges, while the other three edges are the $y$-edges. Let $X$ be a finite disjoint union of colored hexagons. Identify all $y$-edges in $X$ in pairs by homeomorphisms. The quotient space $S$ is a compact surface (possibly disconnected) with an *ideal triangulation* $T$. The *edges* and 2-*cells* of the triangulation $T$ are the images of $y$-edges and hexagons in $X$ under the quotient map. The quotient of each $x$-edge is called an *x-arc* in $T$. We use $C(S, T)$, $E = E(S, T)$, and $F = F(S, T)$ to denote the sets of all $x$-arcs, all edges, and all 2-cells in $T$, respectively. It is easy to see that every compact surface with negative Euler characteristic and nonempty boundary admits an ideal triangulation.

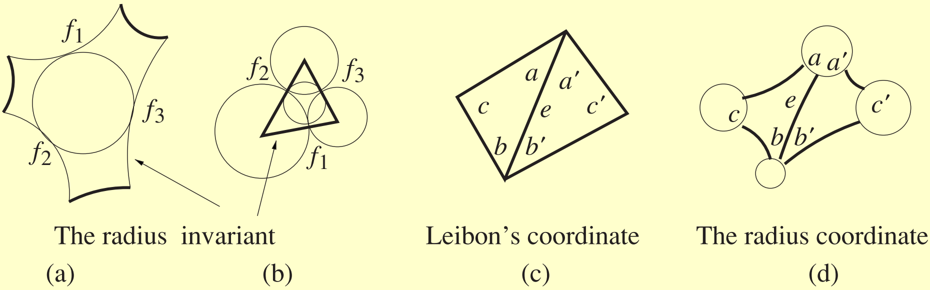


Figure 1.1

Suppose that $H$ is a colored right-angled hyperbolic hexagon with three $y$-edges $e_1, e_2, e_3$ and three $x$-edges $f_1$, $f_2$, $f_3$ such that $f_i$ is the opposite edge of $e_i$. We call $f_i$ the edge *facing* $e_i$ and $f_j$ an edge *adjacent* to $e_i$ for $j \neq i$. It is well known that the hexagon $H$ is determined up to isometry-preserving coloring by the three lengths $l(e_1), l(e_2), l(e_3)$ of the $y$-edges. Furthermore, these three lengths $l(e_1), l(e_2), l(e_3)$ can take any assigned positive numbers (see [Bu]). We define the *radius invariant* of the edge $e_i$ to be the number $(1/2)(l(f_j) + l(f_k) - l(f_i))$, where $\{i, j, k\} = \{1, 2, 3\}$ and $l(f_i)$ is the length of $f_i$. This definition is motivated by the circle-packing construction. The counterpart of a hexagon is the triangle of edge lengths $l(f_1), l(f_2), l(f_3)$, along with inner angles $e_1, e_2, e_3$, such that $e_i$ is facing $f_i$. In this case, the invariants $(1/2)(l(f_j) + l(f_k) - l(f_i))$ are the radii of three pairwise tangent circles whose centers are the vertices of the triangle (see Figure 1.1(b)). The radius invariants play the pivotal role in this article and serve as a coordinate for the bordered Teichmüller space $T(S, l)$.

There is a natural one-to-one correspondence between an ideally triangulated compact surface with boundary and a triangulated closed surface. Namely, for a triangulated closed surface $(S', T')$, let $S$ be the compact surface obtained from $S'$ by removing a small open regular neighborhood of the union of all vertices. Then the triangulation $T'$ induces an ideal triangulation $T$ of the surface $S$. Under this correspondence, vertices of $T'$ correspond to boundary components of $S$ and edges of $T'$ correspond to edges of $T$. The 2-cells (hexagons) of $T$ correspond to triangles in $T'$. The $x$-arcs in $T$ correspond to angles (or corners) in $T'$. The radius invariant of an edge in $T$ introduced below is also the counterpart of the edge invariant introduced by Leibon [L] for hyperbolic metrics on triangulated closed surfaces. Here, Leibon's invariant $\phi$ assigns an edge the sum of the four angles adjacent to the edge subtracting the sum of the two angles facing the edge; that is,

$$\phi(e) = a + a' + b + b' - c - c',$$

where $a, b, a', b'$ are angles adjacent to $e$ and $c, c'$ are the angles facing $e$ (see Figure 1.1(c)). On the other hand, the radius coordinate is also the analogy to Penner's simplicial coordinate (see [P2]).

*1.4*

Fix an ideal triangulation $T$ of a compact surface $S$. Each hyperbolic metric $d$ on $S$ produces a *length function* $l_d : E \to \mathbf{R}_{>0}$ which assigns each edge $e$ in $T$ the length of the shortest geodesic arc homotopic to $e$ relative to the boundary $\partial S$. It is known that two hyperbolic metrics $d, d'$ on $S$ are isotopic if and only if $l_d = l_{d'}$. Furthermore, any function $l : E \to \mathbf{R}_{>0}$ can be realized as $l_d$ for some hyperbolic metric $d$ with totally geodesic boundary by gluing (see [U]). We call $l_d$ the *length coordinate* of the metric $d$. Thus, the length coordinate parameterizes the Teichmüller space $T(S)$ by $E^{\mathbf{R}_{>0}}$. However, the image of the bordered Teichmüller space $T(S, l)$ inside $E^{\mathbf{R}_{>0}}$ is complicated.

The radius coordinate of a hyperbolic metric $d$ on an ideally triangulated surface $(S, T)$ is defined as follows. The triangulation $T$ is isotopic to a geometric ideal triangulation $T^*$ in $d$-metric such that each edge in $T^*$ is a geodesic segment orthogonal to the boundary $\partial S$. In particular, these edge $e^*$'s decompose the surface $S$ into a union of right-angled hyperbolic hexagons. Each edge $e^*$ in $T^*$ is adjacent to one or two hyperbolic hexagons (the 2-cells in $T^*$). We define the *radius invariant* of the edge $e$, denoted by $z(e)$, to be the sum of the radius invariants of the corresponding edge $e^*$ in hyperbolic hexagons adjacent to it; that is,

$$z(e) = \frac{a + a' + b + b' - c - c'}{2},$$

where $a, a', b, b'$ are the lengths of the $x$-arcs adjacent to $e^*$ and $c, c'$ are the lengths of $x$-arcs facing $e^*$ (see Figure 1.1(d)). The radius coordinate of the metric is the function $z : E \to \mathbf{R}$. Our main results are the following theorems.

THEOREM 1.1

*Suppose that $(S, T)$ is a compact ideal triangulated surface. Then each hyperbolic metric with totally geodesic boundary on the surface $S$ is determined up to isotopy by its radius coordinate.*

To state the result for bordered Teichmüller space, we have to introduce the notion of *edge cycle* in the ideal triangulation $T$. By definition, an edge cycle is an edge loop in the dual cellular decomposition of the ideal triangulation. To be more precise, an edge cycle is a collection of ordered edges $e_1, \ldots, e_k$ and 2-cells $f_1, \ldots, f_k$ in $T$ such that for each index $i$, counted modulo $k$, $e_i$ and $e_{i+1}$ are adjacent to the 2-cell $f_i$ in $T$. For simplicity, we use $\{e_1, \ldots, e_k\}$ to denote an edge cycle by suppressing the 2-cells. A *fundamental edge cycle* is an edge cycle such that each edge in $T$ appears at most twice in the cycle. Each boundary component of the surface $S$ corresponds to a fundamental edge cycle by counting edges adjacent to the boundary component cyclically. We call these *boundary edge cycles*.

THEOREM 1.2
*Suppose that $(S, T)$ is a compact ideally triangulated surface with $r$ boundary components and $l = (l_1, \ldots, l_r) \in \mathbf{R}^r_{>0}$. Let $E$ be the set of all edges in the triangulation $T$. Then the radius coordinate is a real analytic diffeomorphism from the bordered Teichmüller space $T(S, (l_1, \ldots, l_r))$ to the convex polytope $\{z : E \to \mathbf{R} \mid$ so that (1.1) and (1.2) hold$\}$ Here, for each fundamental edge cycle $e_1, \ldots, e_k$,*

$$\sum_{i=1}^{k} z(e_i) > 0, \tag{1.1}$$

*and for the boundary edge cycle $e_1, \ldots, e_k$ corresponding to the $j$th boundary component,*

$$\sum_{i=1}^{k} z(e_i) = l_j. \tag{1.2}$$

This theorem is the analogy of [P2, Theorem 5.4]. It seems highly likely that the Weil-Petersson symplectic form on the bordered Teichmüller space $T(S, l)$ can be expressed explicitly in terms of the radius coordinate (see [Mo] and [B]).

One interesting consequence of Theorem 1.2 concerns the cell decompositions of the Teichmüller space, first observed in [Mo].

Recall that the *arc complex* of a compact surface $S$ is the following simplicial complex, denoted by $A(S)$. The vertices of $A(S)$ are isotopy classes $[a]$ of proper arcs $a$ in $S$ which are homotopically nontrivial relative to the boundary of $S$. A simplex in $A(S)$ is a collection of distinct vertices $[a_1], \ldots, [a_k]$ such that $a_i \cap a_j = \emptyset$ for all $i \neq j$. For instance, the isotopy class of an ideal triangulation corresponds to a simplex of maximal dimension in $A(S)$. The nonfillable subcomplex $A_\infty(S)$ of $A(S)$ consists of those simplexes $([a_1], \ldots, [a_k])$ with $a_i \cap a_j = \emptyset$ such that one component of $S - \bigcup_{i=1}^{k} a_i$ is not simply connected. The simplexes in $A(S) - A_\infty(S)$ are called *fillable*. Let $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$ be the geometric realization space whose points are of the form $x = \sum_{i=1}^{k} c_i[a_i]$, where $c_i > 0$, so that $([a_1], \ldots, [a_k])$ is a fillable simplex. Now, take a point $x = \sum_{i=1}^{k} c_i[a_i]$ in $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$. Let $([a_1], \ldots, [a_n])$ be an ideal triangulation containing the fillable simplex $([a_1], \ldots, [a_k])$. Assign each edge $[a_i]$ the positive number $z_i = c_i$ if $i \leq k$ and zero otherwise. Then this assignment $z$ satisfies condition (1.1) in the ideal triangulation $([a_1], \ldots, [a_n])$. By Theorem 1.2, there exists a hyperbolic metric on $S$ whose radius coordinate is $z$. As a consequence, each point in $(|A(S)| - A_\infty(S)|) \times \mathbf{R}_{>0}$ is the radius coordinate of some hyperbolic metric in an ideal triangulation.

On the other hand, by [EP], [Ko], and [U], each hyperbolic metric on $S$ produces a unique point in $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$.

To be more precise, we have the following.

THEOREM 1.3 (see [EP], [U])
*Suppose that $S$ is a compact surface with boundary together with a hyperbolic metric. Then there is an ideal triangulation such that the radius coordinate of the metric in the ideal triangulation is nonnegative. Furthermore, the set of all edges in the ideal triangulation with positive radius coordinate forms a fillable simplex in $A(S)$, and the simplex is unique.*

Define a continuous map

$$\Phi : T(S) \to |A(S) - A_\infty(S)| \times \mathbf{R}_{>0}$$

by sending a metric to the point $\sum_i z_i[a_i]$, where $[a_1], \ldots, [a_n]$ is a preferred ideal triangulation associated to the metric produced by Ushijima's theorem (see [U]) and $z_i$ is the radius coordinate. Combining Theorems 1.2 and 1.3, one obtains the following result.

COROLLARY 1.4 ([Mo, Theorem 2.14])
*For any compact surface with boundary and of negative Euler characteristic, the map*

$$\Phi : T(S) \to |A(S) - A_\infty(S)| \times \mathbf{R}_{>0}$$

*is a homeomorphism equivariant under the action of the mapping class group. In particular, the map $\Phi$ produces a natural cellular decomposition of the moduli space of surfaces with boundary.*

*1.5*
The strategy of proving Theorems 1.1 and 1.2 is the following. By a *length structure* on the ideal triangulated surface $(S, T)$ we mean a map $x : C(S, T) \to \mathbf{R}_{>0}$ assigning each $x$-arc a positive number. Length structure is the counterpart of angle structure on closed triangulated surfaces first introduced by Colin de Verdière [C1] and later by Rivin [R]. The radius invariant of a length structure $x$ is the function $D_x : E \to \mathbf{R}$, assigning each edge $e$ the value $(1/2)\big(\sum_{w\in I} x(w) - \sum_{w'\in II} x(w')\big)$, where $I$ is the set of all $x$-arcs adjacent to $e$ and $II$ is the set of all $x$-arcs facing $e$. Note that each hyperbolic metric $d$ on $(S, T)$ induces a length structure by measuring the lengths of $x$-arcs in the hyperbolic ideal triangulation $T^*$ isotopic to $T$. The radius coordinate of the metric $d$ is the radius invariant of its length structure.

For each length structure $x$, we define an energy $V(x)$ by using the cosine law for hyperbolic hexagons. The energy is a strictly concave function of $x$. Given a function $z : E \to \mathbf{R}$, the set of all length structures with $z$ as radius invariant is a bounded convex set (which may be empty) $L(S, T, z)$. We prove that the maximum point of the strictly concave function $V| : L(S, T, z) \to \mathbf{R}$ is exactly the length structure derived from a hyperbolic metric on the surface $S$. Since a strictly concave function on a convex set has at most one critical point, this establishes Theorem 1.1. To prove Theorem 1.2, we show that if the edge invariant $z : E \to \mathbf{R}$ satisfies conditions (1.1) and (1.2), then $L(S, T, z) \neq \phi$ and the maximum point of $V| : L(S, T, z) \to \mathbf{R}$ always exists. The necessity of conditions (1.1) and (1.2) can be verified easily.

We note that there are now different proofs of Theorems 1.1 and 1.2 in [Lu3]. The new proof of Theorem 1.1 uses the Legendre transform of the energy function used in this article. Theorem 1.2 can be deduced from Theorem 1.1 by analyzing the map sending the length coordinate to the radius coordinate. It is motivated by Thurston's original proof of the circle-packing theorem in [T] and the proof in [MR].

*1.6*

The techniques used in this article are related to and motivated by the seminal work of Colin de Verdière [C1] on variational principle on triangulated surfaces. Important works on the subject have been done by Rivin [R], Brägger [Br], Leibon [L], and others for circle packing, as well as for singular Euclidean and singular hyperbolic structures on surfaces. In [R], Rivin used the Lagrangian multipliers' method in the variational approach to find flat metrics. Our work follows the strategy developed in [L] and also the framework in [R] on Lagrangian multipliers. In these works, the energy functions are all related to the 3-dimensional volume. (The energy functional used by Colin de Verdière was discovered by using the Schläfli formula for tetrahedra; see [C2].) In [Lu3], we observe that all these energy functions can be constructed using the cosine law and Legendre transform. Furthermore, the cosine law produces continuous families of energy functions for variational framework on triangulated surfaces. As a consequence, Theorems 1.1 and 1.2 are special cases in a continuous family of rigidity theorems. Whether this is related to the quantum phenomena (e.g., quantum Teichmüller theory) is not clear to us. Another rich source of energy functions for variational principles on triangulated surfaces has been discovered recently by Bobenko and Springborn [BS] using discrete integrable systems. Also of note is the recent work of Ren Guo [G].

Parameterization of the Teichmüller space using metric ribbon graph has recently been used extensively (see, e.g., the solution of the Witten conjecture in [Kon] and the stability of the homology of the mapping class group in [H]). In the metric ribbon graph approach, the key lies in the singular flat metrics arising from Jenkins-Strebel differentials. The approach in this article can be considered as a counterpart to metric ribbon graph theory, but one using hyperbolic metrics instead of flat metrics. In the case of cusped ends, this has been achieved in the work of Penner [P1], [P2].

There are many works on constructing coordinates for Teichmüller spaces. Besides the works [P1], [P2], and [U] mentioned above, Bonahon [B] constructed a very nice parameterization of the Teichmüller space of a compact surface with boundary using the Bonahon-Thurston shearing cocycles. Recently, Kashaev [K] introduced a coordinate for the space $T(S, l) \times H^1(S, \mathbf{R})$. Other related works are the papers of Schlenker [S] and Springborn [Sp]. The exact relationship between the result in this article and in the works of Bonahon, Kashaev, and Schlenker is not clear to us, but it deserves further study. A fascinating question, suggested by a referee, is whether there is a geometric interpretation of the energy function used in this article in terms of a hyperbolic volume of some hyperideal simplices (see [S] for more details).

*1.7*

The rest of this article is organized as follows. In Section 2, we recall the cosine law and establish some of the basic properties. The energy of a right-angled hyperbolic hexagon is introduced and is shown to be a strictly concave function. In Section 3, we prove Theorems 1.1 and 1.2.

## 2. The cosine law of a hyperbolic right-angled hexagon

We establish some of the basic properties of the cosine law for hyperbolic right-angled hexagons in this section. In particular, the *capacity* of a right-angled hyperbolic hexagon is defined. Some of the basic properties of the capacity function are established. We do not know the geometric meaning of the capacity.

For simplicity, we assume that the indices $i, j, k$ are pairwise distinct in this section.

*2.1*

Given a colored hyperbolic right-angle hexagon with $y$-edge lengths $y_1, y_2, y_3$, let $x_1, x_2, x_3$ be the lengths of $x$-edges, so that the $x_i$th edge is opposite to the $y_i$th edge. The cosine law relating the lengths' $x_i$'s with $y_j$'s states that

$$\cosh(y_i) = \frac{\cosh x_i + \cosh x_j \cosh x_k}{\sinh x_j \sinh x_k}, \tag{2.1}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

The partial derivatives of $y_i$ as a function of $x = (x_1, x_2, x_3)$ are given by the following lemma.

LEMMA 2.1

*Let $\{i, j, k\} = \{1, 2, 3\}$. We have the following:*

(a) (sine law) $\sinh(x_i)/\sinh(y_i)$ *is independent of the index* $i$*; in particular,* $A_{ijk} = A_{123}$*, where* $A_{ijk} = \sinh(y_i)\sinh x_j \sinh x_k$*;*

(b) $\frac{\partial y_i}{\partial x_i} = \sinh(x_i)/A_{ijk} = A\sinh(y_i)$*, where* $A > 0$ *is independent of indices;*

(c) $\frac{\partial y_i}{\partial x_j} = -\frac{\partial y_i}{\partial x_i}\cosh y_k$.

*Proof*

The proof is a simple exercise in calculus (see, e.g., [Lu2]).

Introduce a new variable $t_i = (x_j + x_k - x_i)/2$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $x_i = t_j + t_k$. The space of all colored hyperbolic right-angled hexagons parameterized by the new coordinate $t = (t_1, t_2, t_3)$ becomes $H_3 = \{(t_1, t_2, t_3) \in \mathbf{R}^3 \mid t_i + t_j > 0\}$. We consider $y_i = y_i(t)$ as a smooth function defined on $H_3$. □

COROLLARY 2.2

*The length function* $y_i = y_i(t)$ *on* $H_3$ *satisfies the following:*

(a) *the differential* 1*-form* $w = \sum_{i=1}^{3} \ln\cosh(y_i/2)\,dt_i$ *is closed in the open set* $H_3$*;*

(b) *the function* $\theta(t) = \int_{(0,0,0)}^{t} w$ *is strictly concave on* $H_3$.

*Remark*. The differential 1-form $w$ in Corollary 2.2(a) has logarithmic singularity at the point $(0, 0, 0)$. Thus, the integral in Corollary 2.2(b) is well defined. This can also be seen in Proposition 2.3.

*Proof of Corollary 2.2*

To show part (a), it suffices to prove $\frac{\partial(\ln\cosh(y_i/2))}{\partial t_j}$ is symmetric in $i \neq j$. By Lemma 2.1 and $x_i = t_j + t_k$, the partial derivative is found to be

$$\begin{aligned}
\frac{\partial}{\partial t_j}\Big(\ln\cosh\Big(\frac{y_i}{2}\Big)\Big) &= \frac{1}{2}\tanh\Big(\frac{y_i}{2}\Big)\frac{\partial y_i}{\partial t_j} \\
&= \frac{1}{2}\tanh\Big(\frac{y_i}{2}\Big)\Big(\frac{\partial y_i}{\partial x_i} + \frac{\partial y_i}{\partial x_k}\Big) \\
&= \frac{1}{2}\tanh\Big(\frac{y_i}{2}\Big)\frac{\partial y_i}{\partial x_i}\big(1 - \cosh(y_j)\big) \\
&= \frac{1}{2}\tanh\Big(\frac{y_i}{2}\Big)A\sinh(y_i)\big(1 - \cosh(y_j)\big) \\
&= \frac{A}{2}\frac{\sinh(y_i/2)}{\cosh(y_i/2)}\Big(2\sinh\Big(\frac{y_i}{2}\Big)\cosh\Big(\frac{y_i}{2}\Big)\Big)\Big(-2\sinh^2\Big(\frac{y_j}{2}\Big)\Big) \\
&= -2A\sinh^2\Big(\frac{y_i}{2}\Big)\sinh^2\Big(\frac{y_j}{2}\Big), \qquad (2.2)
\end{aligned}$$

where $\frac{\partial y_i}{\partial x_i} = A\sinh(y_i)$ is given by Lemma 2.1(b). The last expression in (2.2) is symmetric in $i, j$. This establishes (a).

To see part (b), due to part (a) and to the simple connectivity of $H_3$, the function $\theta(t)$ is well defined. To check the convexity, we calculate the Hessian of $\theta(t)$. The Hessian matrix is $\left[\frac{\partial^2\theta}{\partial t_r \partial t_s}\right]_{3\times 3} = \left[\frac{\partial}{\partial t_r}(\ln\cosh(y_s/2))\right]_{3\times 3}$. The diagonal entries of the Hessian can be calculated using Lemma 2.1 as follows:

$$\begin{aligned}\frac{\partial}{\partial t_i}\left(\ln\cosh\left(\frac{y_i}{2}\right)\right) &= \frac{1}{2}\tanh\left(\frac{y_i}{2}\right)\frac{\partial y_i}{\partial t_i}\\ &= \frac{1}{2}\tanh\left(\frac{y_i}{2}\right)\left(\frac{\partial y_i}{\partial x_j}+\frac{\partial y_i}{\partial x_k}\right)\\ &= \frac{1}{2}\tanh\left(\frac{y_i}{2}\right)\frac{\partial y_i}{\partial x_i}\left(-\cosh(y_k)-\cosh(y_j)\right)\\ &= \frac{1}{2}\frac{\sinh(y_i/2)}{\cosh(y_i/2)}\sinh(y_i)A\left(-2\sinh^2\left(\frac{y_k}{2}\right)-2\sinh^2\left(\frac{y_j}{2}\right)-2\right)\\ &= -2A\sinh^2\left(\frac{y_i}{2}\right)\left(\sinh^2\left(\frac{y_j}{2}\right)+\sinh^2\left(\frac{y_k}{2}\right)+1\right). \qquad (2.3)\end{aligned}$$

By (2.2), (2.3), and $A > 0$, the matrix $-\left[\frac{\partial^2\theta}{\partial t_r \partial t_s}\right]$ is a diagonally dominated matrix; that is,

$$-\frac{\partial^2\theta}{\partial t_i \partial t_i} > \left|\frac{\partial^2\theta}{\partial t_i \partial t_j}\right| + \left|\frac{\partial^2\theta}{\partial t_i \partial t_k}\right|.$$

Since a diagonally dominated matrix is positive definite, it follows that the Hessian matrix is negative definite and the function $\theta(t)$ is strictly concave. $\square$

The next proposition relates $\theta(t)$ with the dilogarithm function. Let $\Lambda_1(u) = \int_0^u \ln\cosh(s)\,ds$, and let $\Lambda_2(u) = \int_0^u \ln\sinh(s)\,ds$. Both are continuous in $\mathbf{R}$ and are related to the dilogarithm function and the Lobachevsky function.

PROPOSITION 2.3
*The function $\theta(t)$ is*

$$2\theta(t) = \Lambda_1(t_1+t_2+t_3) + \sum_{i=1}^{3}\Lambda_1(t_i) - \Lambda_2(t_1+t_2) - \Lambda_2(t_2+t_3) - \Lambda_2(t_3+t_1). \quad (2.4)$$

*Proof*
We verify that the derivatives of the functions on both sides of (2.4) are the same. Note that $2\frac{\partial\theta(t)}{\partial t_i} = \ln\cosh^2(y_i/2)$. By the cosine law (2.1) and the identity $\cosh^2(u/2) =$

$(\cosh u + 1)/2$, we have

$$\cosh^2\left(\frac{y_i}{2}\right) = \frac{\cosh(x_i) + \cosh(x_j)\cosh(x_k) + \sinh(x_j)\sinh(x_k)}{2\sinh(x_j)\sinh(x_k)}$$
$$= \frac{\cosh(x_i) + \cosh(x_j + x_k)}{2\sinh(x_j)\sinh(x_k)}$$
$$= \frac{\cosh((x_1 + x_2 + x_3)/2)\cosh((x_j + x_k - x_i)/2)}{\sinh(x_j)\sinh(x_k)}$$
$$= \frac{\cosh(t_1 + t_2 + t_3)\cosh(t_i)}{\sinh(t_i + t_j)\sinh(t_i + t_k)}.$$

This shows that

$$2\frac{\partial\theta(t)}{\partial t_i} = \ln\cosh(t_1 + t_2 + t_3) + \ln\cosh(t_i) - \ln\sinh(t_i + t_j) - \ln\sinh(t_i + t_k). \tag{2.5}$$

Evidently, the right-hand side of (2.5) is the partial derivative of the right-hand side of (2.4) with respect to the variable $t_i$. Since both functions vanish at $(0, 0, 0)$, this proves the proposition. □

Both functions $\Lambda_1(u)$ and $\Lambda_2(u)$ are continuous in $\mathbf{R}$. Thus, the function $\theta(t)$ has a continuous extension, still denoted by $\theta(t)$, to the closure of $H_3$ in $\mathbf{R}^3$; that is, $\theta(t)$ is well defined on $\overline{H_3} = \{(t_1, t_2, t_3) \in \mathbf{R}^3 \mid t_i + t_j \geq 0 \text{ for all } i \neq j\}$. The next result studies the behavior of the function $\theta(t)$ near the boundary of $H_3$ and near infinity.

PROPOSITION 2.4
*The function $\theta(t)$ defined on $\overline{H_3} = \{t \in \mathbf{R}^3 \mid t_i + t_j \geq 0\}$ is nonnegative and bounded. Furthermore, for any point $a \in \partial H_3$ and any point $p \in H_3$,*

$$\lim_{s\to 0}\frac{d}{ds}\big(\theta((1-s)a + sp)\big) = \infty. \tag{2.6}$$

*Proof*
Let $f(s) = 2\theta((1-s)a + sp)$, and let $t_i = (1-s)a_i + sp_i$. In the following calculation, the indices are counted modulo 3. Then by (2.5),

$$\begin{aligned}\frac{df(s)}{ds} &= \sum_{i=1}^{3} 2\frac{\partial\theta}{\partial t_i}(p_i - a_i)\\ &= \ln\cosh\Big(\sum_{i=1}^{3} t_i\Big)\sum_{i=1}^{3}(p_i - a_i) + \sum_{i=1}^{3}\ln\cosh(t_i)(p_i - a_i)\\ &\quad - \sum_{i=1}^{3}\ln\Big(\sinh(t_i + t_{i+1})\sinh(t_i + t_{i-1})\Big)(p_i - a_i)\\ &= -\sum_{i=1}^{3}\ln\sinh(t_i + t_{i+1})(p_i + p_{i+1} - a_i - a_{i+1}) + A(s), \qquad (2.7)\end{aligned}$$

where $A(s) = \ln\cosh\left(\sum_{i=1}^{3} t_i\right)\sum_{i=1}^{3}(p_i - a_i) + \sum_{i=1}^{3}\ln\cosh(t_i)(p_i - a_i)$, so that $\lim_{s\to 0} A(s)$ exists in $\mathbf{R}$.

To understand $\lim_{s\to 0} f(s)$, we discuss three cases according to the location of the boundary point $a$:

(1) only one of $a_i + a_{i+1}$, for $i = 1, 2, 3$, is zero;

(2) exactly two of three numbers $a_i + a_{i+1}$ are zero;

(3) all $a_i$'s are zero.

Note that $\lim_{s\to 0}(t_i + t_j) = a_i + a_j$.

*Case 1*. Say that $a_1 + a_2 = 0$ and that $a_2 + a_3, a_3 + a_1 > 0$. Then by (2.7) and $\lim_{s\to 0}(t_i + t_j) > 0$ for $(i, j) \neq (1, 2)$,

$$\frac{df(s)}{ds} = -\ln\sinh(t_1 + t_2)(p_1 + p_2 - a_1 - a_2) + A_1(s),$$

where $\lim_{s\to 0} A_1(s)$ exists in $\mathbf{R}$. Due to $p_1 + p_2 > 0$, $a_1 + a_2 = 0$, and $\lim_{s\to 0}\ln\sinh(t_1 + t_2) \to -\infty$, it follows that (2.6) holds.

*Case 2*. Say that $a_1 + a_2 = a_2 + a_3 = 0$, and say that $a_3 + a_1 > 0$. Then by (2.7),

$$\begin{aligned}\frac{df(s)}{ds} &= -\ln\sinh(t_1 + t_2)(p_1 + p_2 - a_1 - a_2)\\ &\quad - \ln\sinh(t_2 + t_3)(p_2 + p_3 - a_2 - a_3) + A_2(s),\end{aligned}$$

where $\lim_{s\to 0} A_2(s)$ exists in $\mathbf{R}$. Due to $p_i + p_j > a_i + a_j = 0$ and $\lim_{s\to 0}(t_i + t_j) = 0$ for $(i, j) = (1, 2), (2, 3)$, it follows again that (2.6) holds.

*Case 3*. Let $a_1 = a_2 = a_3 = 0$. Then we have

$$\frac{df(s)}{ds} = -\sum_{i=1}^{3}\ln\sinh(t_i + t_{i+1})(p_i + p_{i+1} - a_i - a_{i+1}) + A_3(s),$$

where $\lim_{s\to 0} A_3(s)$ exists in $\mathbf{R}$. Since $p_i + p_j > a_i + a_j = 0$ and $\lim_{s\to 0}(t_i + t_j) = 0$ for all $i$, $j$, then (2.6) holds again.

To see that the function $2\theta(t)$ is bounded in $\overline{H_3}$, let us consider, for each $u > 0$, the minimum and maximum values $m(u)$ and $M(u)$ of $2\theta(t)$ on the triangle $X_u = \{(t_1, t_2, t_3) \mid t_1 + t_2 + t_3 = u, t_i + t_j \geq 0\}$. Since the function $2\theta(t)$ is strictly concave in $X_u$ and $2\theta(t)$ is symmetric in $t_1, t_2, t_3$, its minimum point is achieved at the vertices of $X_u$ and its unique maximum point is invariant under the permutations of $t_1, t_2, t_3$. Thus, $M(u) = 2\theta(u/3, u/3, u/3)$ and $m(u) = 2\theta(u, 0, 0)$; that is, $M(u) = \Lambda_1(u) + 3\Lambda_1(u/3) - 3\Lambda_2(2u/3)$ and $m(u) = 2\Lambda_1(u) - 2\Lambda_2(u)$. Since $m(u) > 0$, then $\theta(t) \geq 0$. On the other hand, $M(u)$ is known to be bounded in $[0, \infty)$. Thus, $\theta(t)$ is bounded on $\{t \mid t_i + t_j \geq 0, \text{ where } i \neq j\}$. $\square$

## 3. Proofs of Theorems 1.1 and 1.2

Suppose that $(S, T)$ is an ideally triangulated surface obtained by identifying $y$-edges of colored hexagons $\tilde{P}_1, \ldots, \tilde{P}_n$ in pairs by homeomorphisms $\phi_{ij}$'s. Let $E = \{e_1, \ldots, e_m\}$, let $F = \{P_1, \ldots, P_n\}$, and let $C(S, T) = \{w_1, \ldots, w_{3n}\}$ be the sets of all edges, 2-cells, and $x$-arcs in $T$, respectively. Here, the quotient of $\tilde{P}_i$ is $P_i$. Each 2-cell $P_i$ contains exactly three $x$-arcs $w_{i_1}, w_{i_2}, w_{i_3}$ in $\partial S$. We say that $w_{i_1}, w_{i_2}, w_{i_3}$ bound the 2-cell and that $w_{i_j}$ is an $x$-arc of $P_i$. An $x$-arc $w$ is said to *facing* (resp., *adjacent to*) an edge $e$ if there is a hexagon $\tilde{P}$ and an $x$-edge $\tilde{w}$ facing (or adjacent to) a $y$-edge $\tilde{e}$ in $\tilde{P}$, so that $w$ and $e$ are the quotients of $\tilde{w}$ and $\tilde{e}$. If $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are three $y$-edges in a hexagon $\tilde{P}_i$, we call their quotient edges $e_1, e_2, e_3$ the *edges* of the 2-cell $P_i$. Note that it may occur that $e_1 = e_2$.

Recall that a length structure on $(S, T)$ is a function $x : C(S, T) \to \mathbf{R}_{>0}$. Geometrically, a length structure is the same as a realization of each hexagon $\tilde{P}_i$ by a hyperbolic right-angled hexagon (by measuring the lengths of the $x$-edges). There is no guarantee that the gluing homeomorphism $\phi_{ij}$ identifies two $y$-edges of $\tilde{P}_k$'s of the same length. Thus, a length structure does not correspond to a metric on the surface. Hyperbolic metrics on the surface $S$ are the same as those length structures, so that all $\phi_{ij}$'s identify pairs of edges of the same lengths. These length structures are said to be induced from hyperbolic metrics. We consider the Teichmüller space $T(S)$ as the subset of the space of all length structures under this identification. The goal of this article is to characterize $T(S)$ as the critical point of a natural energy function.

Given a length structure $x : C(S, T) \to \mathbf{R}_{>0}$, we define its *t-coordinate* $t = t_x : C(S, T) \to \mathbf{R}$ by

$$t(w) = \frac{1}{2}\big(x(w') + x(w'') - x(w)\big), \tag{3.1}$$

where $w, w', w''$ are the $x$-arcs in the 2-cell containing $w$. The length structure $x$ can be recovered from its $t$-coordinate $t$ by $x(w) = t(w') + t(w'')$. The radius invariant of a length structure $x$ is $z = z_x : E \to \mathbf{R}$, given by

$$z(e) = t(w) + t(w'), \tag{3.2}$$

where $w$ and $w'$ are the $x$-arcs facing the edge $e$. Note that this definition coincides with the definition of radius invariants introduced in Section 1.4 when $x$ is induced from a hyperbolic metric.

The space of all length structures parameterized by their $t$-coordinate is $L_t(S, T) = \{t = (t_1, \ldots, t_{3n}) \in \mathbf{R}^3 \mid t_i + t_j > 0$ whenever $x$-arcs $w_i$ and $w_j$ are inside a 2-cell in $T\}$. We define the *energy* $V(t)$ of a length structure $t \in L_t(S, T)$ to be

$$V(t) = \sum_{\{w_i, w_j, w_k\} \text{ bounds a 2-cell}} \theta(t_i, t_j, t_k). \tag{3.3}$$

Geometrically, for a length structure corresponding to a collection of hyperbolic right-angled hexagons, its energy is the sum of the values of $\theta$-function at its hexagons.

Given a function $z : E \to \mathbf{R}$, let $L_t(S, T, z)$ be the set of all length structures (in $t$-coordinates), so that its radius invariant is $z$; that is, $L_t(S, T, z) = \{t \in \mathbf{R}^{3n} \mid t_i + t_j > 0$ when $w_i$, $w_j$ are in a 2-cell, and $t_i + t_j = z(e)$ when $x$-arcs $w_i$ and $w_j$ are facing $e\}$.

LEMMA 3.1

*If $L_t(S, T, z) \neq \emptyset$, the energy function $V| : L_t(S, T, z) \to \mathbf{R}$ is strictly concave, so that the critical points of $V|$ are exactly the length structures induced from hyperbolic metrics.*

*Proof*

The concavity follows from the concavity of $\theta(t)$. To identify the critical points, we use the Lagrangian multiplier to $V : L(S, T) \to \mathbf{R}$ subject to a set of linear constraints $t_i + t_j = z(e)$ when $w_i$, $w_j$ are facing $e$. At a critical point $q$ of $V|$, there exists a function $h : E \to \mathbf{R}$ (the Lagrangian multipliers), so that for all indices $i$,

$$\frac{\partial V}{\partial t_i}(q) = h(e), \tag{3.4}$$

where the $x$-arc $w_i$ is facing the edge $e$. Suppose that the $x$-arc $w_i$ lies in the 2-cell $P_r$, so that $\tilde{e}$ is the $y$-edge of $\tilde{P}_r$ corresponding to $e$. We realize all hexagons $\tilde{P}_i$ by hyperbolic right-angled hexagons with $x$-edge lengths given by the length structure $q$. Then by Corollary 2.2, $\frac{\partial V}{\partial t_i} = \ln\cosh(l(\tilde{e})/2)$. Together with (3.4), this shows that the length $l(\tilde{e})$ of $\tilde{e}$ in the hyperbolic hexagon $\tilde{P}_r$ depends only on the quotient edge $e$ in $T$; that is, the gluing homeomorphism $\phi_{ij}$ identifies pairs of $y$-edges of the same hyperbolic lengths. Thus, the length structure $q$ is induced from a hyperbolic metric on the surface. Conversely, suppose that we have a length structure $q$ induced from a hyperbolic metric. Then by defining the Lagrangian multipliers $h(e)$ to be $\ln\cosh(l(e)/2)$, we see that (3.4) holds. Since the constraints are linear functions, it follows that the point $q$ is a critical point of $V|$. □

*3.1. Proof of Theorem 1.1*

The proof of Theorem 1.1 is now simple. Since a strictly concave function on a convex set has at most one critical point, by Lemma 3.1, we see that Theorem 1.1 holds. □

*Remark.* Another way to prove Theorem 1.1 uses the Legendre transform of the function $\theta$ (see [Lu3]). By definition, the Legendre transform $\eta(u_1, u_2, u_3)$ of $\theta(t_1, t_2, t_3)$ is a strictly concave function in variable $u = (u_1, u_2, u_3)$, where $u_i = \ln\cosh(y_i/2)$, so that $\frac{\partial \eta}{\partial u_i} = t_i$. Now, for a hyperbolic metric on the triangulated surface $(S, T)$ with lengths of edges $x = (x_1, \ldots, x_m)$, let $u = (\ln\cosh(x_1/2), \ldots, \ln\cosh(x_m/2))$. Define $W(u)$ to be the sum of the values of $\eta$ at $(u_i, u_j, u_k)$, where the $i$th, $j$th, and $k$th edges bound a hexagon. Then, by definition, $W$ is a smooth strictly concave function such that the gradient of $W$ is the radius coordinate $z(x)$ of the metric. It is well known that for a smooth strictly concave function $W$ defined in an open convex set in $\mathbf{R}^N$, the gradient $\bigtriangledown W$ is injective. This gives a different proof of Theorem 1.1.

*3.2. Proof of Theorem 1.2*

The proof of Theorem 1.2 breaks into two parts. In the first part, we show that if $L_t(S, T, z) \neq \emptyset$, the maximum point of $V|$ exists in $L_t(S, T, z)$. In the second part, we prove that $L_t(S, T, z) \neq \emptyset$ if and only if condition (1.1) in Theorem 1.2 holds.

*3.3*

To prove the first part, by Proposition 2.3 the function $V : L_t(S, T) \to \mathbf{R}$ can be extended continuously to the closure $\overline{L_t(S, T)}$ of $L_t(S, T) \subset \mathbf{R}^{3n}$. On the other hand, the set $L_t(S, T, z)$ is bounded. Indeed, we have the following lemma.

LEMMA 3.2

*Suppose that $\{e_1, f_1, e_2, f_2, e_3, \ldots, e_k, f_k\}$ forms an edge cycle in $T$, where edges $e_i, e_{i+1}$ are adjacent to the hexagon $f_i$ for all $i$ and $e_{k+1} = e_1$. Then*

$$\sum_{i=1}^{k} z(e_i) = \sum_{i=1}^{k} x(w_{n_i}), \tag{3.5}$$

*where $w_{n_i}$ is the x-arc in the hexagon $f_i$ adjacent to both $e_i$ and $e_{i+1}$ with indices counted modulo $k$. In particular, $\sum_{i+1}^{k} z(e_i) > 0$ for all edge cycles. If the length* *structure $x$ is induced from a hyperbolic metric, then*

$$\sum_{i=1}^{k} z(e_i) = l_j \tag{3.6}$$

*for the boundary cycle $e_1, \ldots, e_k$ associated to the jth boundary component of length $l_j$.*

*Proof*
The proof is a simple calculation using the following identity. Namely, the sum of two $t$-coordinates $t_i = (1/2)(x_j + x_k - x_i)$ and $t_j = (1/2)(x_i + x_k - x_j)$ is $x_k$, where $x_k$ is the edge adjacent to both $y_i$th and $y_j$th edges. Thus, (3.5) follows from the above identity and the definition of the edge cycles. The identity (3.6) follows from (3.5) and the definition of the boundary length. □

COROLLARY 3.3
(a) *The space $L_t(S, T, z)$ is bounded.*
(b) *If $z : E \to \mathbf{R}$ is a radius coordinate associated to a hyperbolic metric, then (1.1) and (1.2) hold.*

*Proof*
Part (b) follows from (3.5) and (3.6). To see part (a), we consider the $x$-coordinate of length structures. Take a length structure $x : C(S, T) \to \mathbf{R}_{>0}$ with radius invariant $z$. Each $x$-arc $w$ is in some boundary component $b_i$ of the surface. Thus, by (3.5),

$$0 \le x(w) \le \sum_{w' \subset b_i} x(w') = \sum_{j=1}^{k} z(e_{n_j}),$$

where $e_{n_1}, \ldots, e_{n_k}$ is the boundary edge cycle associated to $b_i$. This shows that $x(w)$ is bounded. □

By Corollary 3.3, the energy function $V|$ can be extended continuously to the compact closure $\overline{L_t(S, T, z)}$. In particular, it has a maximum point $p = (p_1, \ldots, p_{3n})$ (considered as a $t$-coordinate) in $\overline{L_t(S, T, z)}$. We claim that the maximum point $p$ is in $L_t(S, T, z)$.

We prove the claim by contradiction. Suppose otherwise, by definition, that there are pairs of $x$-arcs, say, $w_1$ and $w_2$, in a 2-cell, so that $p_1 + p_2 = 0$. A 2-cell $P$ in $T$ is said to be *degenerated* with respect to $p$ if there are two $x$-arcs $w_i$, $w_j$ in $P$ such that $p_i + p_j = 0$. Let $I$ be the set of all degenerated 2-cells, and let $II$ be the set of all nondegenerated 2-cells. Take a point $q \in L_t(S, T, z)$. For each 2-cell $P$ in the triangulation with $x$-arcs $w_i$, $w_j$, $w_k$, consider the limit

$$h(P) = \lim_{s \to 0} \frac{d}{ds}\big(\theta((1-s)p_i + sq_i, (1-s)p_j + sq_j, (1-s)p_k + sq_k)\big).$$

By Proposition 2.3, this limit is finite if $P \in II$; it is the positive infinite if $P \in I$. By the assumption that $I \neq \emptyset$, it follows that

$$\lim_{s \to 0} \frac{d}{ds}\big(V((1-s)p + sq)\big) = \sum_{P \in I} h(P) + \sum_{P \in II} h(P) = \infty. \tag{3.7}$$

On the other hand, since $p$ is the maximum point, the function $V((1-s)p+sq)$ has a maximum point at $s = 0$. Thus, $\limsup_{s \to 0} \frac{d}{ds}\big(V((1-s)p + sq)\big) \leq 0$. This is a contradiction of (3.7).

By the claim, $p \in L_t(S, T, z)$. By Lemma 3.1, it follows that $p$ is induced by a hyperbolic metric. To summarize, we have shown that if $L_t(S, T, z) \neq \emptyset$, then there exists a hyperbolic metric with radius invariant $z$.

*3.4*

The necessity of conditions (1.1) and (1.2) follows from Corollary 3.3(b). To finish the proof of Theorem 1.2, it remains to show the following.

LEMMA 3.4

*Given a function $z : E \to \mathbf{R}$ such that (1.1) holds, then $L(S, T, z) \neq \emptyset$.*

*Proof*

Let us consider length structures parameterized by the $x$-coordinate. Here, $x : C(S, T) \to \mathbf{R}$, so that $x_i = x(w_i)$ and $x = (x_1, \ldots, x_{3n})$. Let the set of all edges be $E = \{e_1, \ldots, e_m\}$ and $z_i = z(e_i)$. By definition, $L(S, T, z) = \{x \in \mathbf{R}^{3n} \mid$ therefore, (3.8) and (3.9) hold$\}$, and thus,

$$\sum_{i \in I} x_i - \sum_{i \in J} x_j = 2z(e) \quad \text{for each edge } e, \tag{3.8}$$

where $\{w_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$ are the sets of $x$-arcs adjacent to and facing the edge $e \in E$, respectively, and

$$x_i > 0 \quad \text{for all } i. \tag{3.9}$$

Consider the set $D = \{(y_1, \ldots, y_m) \in \mathbf{R}^m \mid y_i + y_j \geq y_k$ whenever $e_i, e_j, e_k$ form the edges of a 2-cell$\}$ and the linear programming problem $\min \left\{ \sum_{i=1}^{n} y_i z_i \;\middle|\; y \in D \right\}$. The dual linear programming problem is $\max\{0 \mid x \in L(S, T, z)\}$ by the construction. By the duality theorem of linear programming (in fact, Farkas's lemma suffices in this case; see, e.g., [BL]), $L(S, T, z) \neq \emptyset$ if and only if for each nonzero vector $y \in D$, $\sum_{i=1}^{n} y_i z_i > 0$.

The set $D$ is a cone in $\mathbf{R}^m$. Furthermore, the inequalities $y_i + y_j \geq y_k$ and $y_k + y_i \geq y_j$ imply that $y_i \geq 0$. Thus, $D = \{y = (y_1, \ldots, y_m) \in \mathbf{R}^m \mid y_i \geq 0$, and $y_i + y_j \geq y_k$ whenever $e_i, e_j, e_k$ form the edges of a 2-cell$\}$. This shows that $D$ can be identified with the space of all measured laminations on the surface $S$ where $y_i$'s are the geometric intersection coordinates. By the work of Thurston (see, e.g., [Mos]), it is known that every measured lamination on $S$ considered as a vector in $D$ is a nonnegative linear combination of those vectors in $D$ associated to essential simple loops. Furthermore, these essential simple loops can be assumed to intersect each edge $e \in E$ in at most two points. It follows that each one of these simple loops corresponds to a fundamental edge cycle by counting the edges intersecting it. In particular, each fundamental cycle $c = (e_{n_i}, \ldots, e_{n_k})$ in the triangulation $T$ corresponds to base vector $v_c = (y_1, \ldots, y_n)$, where $y_i = 0$ if $i \neq n_j$, and where $y_i = 1$ if $i = n_j$. The above discussion shows that each vector in the cone $D$ is a nonnegative linear combination of the base vectors $v_c$'s. Condition (1.1) says that $\sum_{i=1}^{n} y_i z_i$ is positive at every base vector $v_c$. It follows that for all $y \in D - \{0\}$, $\sum_{i=1}^{n} y_i z_i > 0$. □

Thus, Theorem 1.2 is proved. □

*Acknowledgments.* We thank the referees for their careful reading of the manuscript and for their nice suggestions, especially for pointing out the relationship between our work and that of Penner. We also thank Mondello for his preprint [Mo], which establishes the exact relationship between the radius coordinate in this article and the simplicial coordinate of Penner.

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