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A DILOGARITHM IDENTITY ON MODULI SPACES OF CURVES

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To Michael Freedman on the occasion of his sixtieth birthday

Abstract

We establish an identity for closed hyperbolic surfaces whose terms depend on the dilogarithms of the lengths of simple closed geodesics in all 3-holed spheres and 1-holed tori in the surface.

1. introduction

1.1. Statement of results. In [**7**], McShane established a remarkable identity for the lengths of simple closed geodesics in hyperbolic surfaces with cusp ends. Generalizations of McShane's identity to hyperbolic surfaces with non-empty geodesic boundaries and cone singularities were carried out independently by Mirzakhani [**9**] and Tan et al [**12**]. Furthermore, Mirzakhani found striking applications of her identity to the computation of the volumes of moduli spaces of bordered Riemann surfaces. The goal of this paper is to establish a McShane type identity on *any* closed hyperbolic surface. The identity that we produce involves the dilogarithm of the lengths of simple closed geodesics in all 1-holed tori and 3-holed spheres in the surface. Our work is motivated by [**1**, **7**, **9**, **12**]. We expect that the identity found here will have applications towards the study of the moduli space of curves.

For a hyperbolic surface F, a compact embedded subsurface $\Sigma \subset F$ is said to be *geometric* if the boundaries of Σ are geodesic and *proper* if the inclusion map $i : \Sigma \to F$ is injective. Furthermore call a surface *simple* if it is a 3-holed sphere or 1-holed torus. Our main result is the following:

Theorem 1.1. *There exist functions f and g defined on the moduli space of hyperbolic 3-holed spheres and 1-holed tori involving the dilogarithm such that for any closed orientable hyperbolic surface of genus $g \geq 2$*

$$\sum_P f(P) + \sum_T g(T) = 8\pi^2(g-1) \tag{1}$$

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where the first sum is over all properly embedded geometric 3-holed spheres $P \subset F$, the second sum is over all properly embedded geometric 1-holed tori $T \subset F$.

We remark that theorem 1.1 can be extended to hyperbolic surfaces with geodesic boundary and cusp ends and to non-orientable surfaces without too much difficulty. These will be carried out in a separate paper [**10**]; see also [**5**].

We have been informed by G. McShane that he and D. Calegari have obtained results similar to theorem 1.1.

The two functions f, g in theorem 1.1 are defined using dilogarithms as follows. Let $\mathcal{L}(x)$ be the Roger's dilogarithm function defined for $x < 1$ by

$$\mathcal{L}(x) = -\int_0^x \frac{\log(1-z)}{z} dz + \frac{1}{2}\log(|x|)\log(1-x). \tag{2}$$

It is the only function satisfying the pentagon relation that for $x, y \in (0,1)$,

$$\mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(1-xy) + \mathcal{L}(\frac{1-x}{1-xy}) + \mathcal{L}(\frac{1-y}{1-xy}) = \frac{\pi^2}{2}. \tag{3}$$

Suppose P is a hyperbolic 3-holed sphere with geodesic boundaries of lengths l_1, l_2, l_3. Let m_i be the length of the shortest path from the l_{i+1}-th boundary to the l_{i+2}-th boundary ($l_4 = l_1$, $l_5 = l_2$). Then

(4)

$$f(P) := 4\sum_{i \neq j}\Big[2\mathcal{L}(\frac{1-x_i}{1-x_iy_j}) - 2\mathcal{L}(\frac{1-y_j}{1-x_iy_j}) - \mathcal{L}(y_j) - \mathcal{L}(\frac{(1-y_j)^2x_i}{(1-x_i)^2y_j})\Big]$$

where $x_i = e^{-l_i}$ and $y_i = \tanh^2(m_i/2)$.

Suppose T is a hyperbolic 1-holed torus with geodesic boundary. For any non-boundary parallel simple closed geodesic A of length a in T, let m_A be the distance between ∂T and A. Then

$$\begin{aligned} g(T) := 4\pi^2 + 8\sum_A \Big[2\mathcal{L}(\frac{1-x_A}{1-x_Ay_A}) - 2\mathcal{L}(\frac{1-y_A}{1-x_Ay_A}) - 2\mathcal{L}(y_A) \\ - \mathcal{L}(\frac{(1-y_A)^2x_A}{(1-x_A)^2y_A})\Big] \end{aligned} \tag{5}$$

where $x_A = e^{-a}$ and $y_A = \tanh^2(m_A/2)$ and the sum is over all simple closed geodesics A in $T - \partial T$.

1.2. Basic idea of the proof. The key idea is to use ergodicity of the geodesic flow to decompose the unit tangent bundle $S(F)$ of a closed hyperbolic surface F according to properly embedded geometric 1-holed tori and 3-holed spheres. The decomposition is measure theoretic in the sense that we will ignore a measure zero set in $S(F)$. Here is the

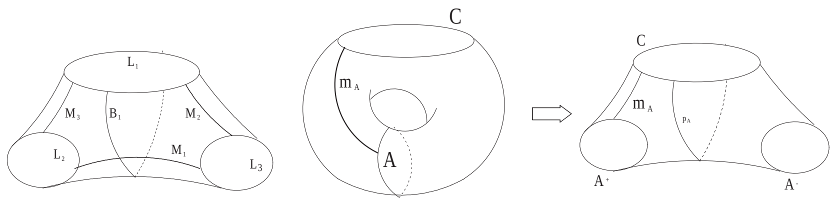

Figure 1. 3-holed spheres and 1-holed tori

decomposition. For a unit tangent vector $v \in S(F)$, consider the unit speed geodesic rays $g_v^+(t)$ and $g_v^-(t)$ $(t \geq 0)$ determined by $\pm v$. If the vector v is generic, then both rays will self intersect transversely by the ergodicity of the geodesic flow. This vector v will determine a canonical graph $G(v)$ as follows. Consider the path $A_t = g_v^-([0,t]) \cup g_v^+([0,t])$ for $t > 0$ obtained by letting the geodesic rays g_v^- and g_v^+ grow at equal speed from time 0 to t. Let $t_+ > 0$ be the smallest positive number so that A_{t_+} is not a simple arc; without loss of generality, we may assume that $g_v^+(t_+) \neq g_v^-(t_+)$ by ignoring a set of measure zero. Say $g_v^+(t_+) \in g_v^-([0,t_+]) \cup g_v^+([0,t_+))$. Next, let $t_- > t_+$ be the next smallest time so that $g_v^-(t_-) \in g_v^-([0,t_-)) \cup g_v^+[0,t_+])$. The union $g_v^-([0,t_-]) \cup g_v^+([0,t_+])$ is the graph, denoted by $G(v)$ associated to v. Its Euler characteristic is -1. The graph $G(v)$ is contained in a unique properly embedded geometric subsurface $\Sigma(v)$ which is either a 1-holed torus or a 3-holed sphere in F. Furthermore either the graph $G(v)$ is a deformation retract of $\Sigma(v)$, or $\Sigma(v)$ is a 1-holed torus so that $\Sigma(v) - G(v)$ is a union of two annuli (figure 2(c)). By abuse of notation, we will say in this case that $G(v)$ is also a spine for $\Sigma(v)$. In this way, each vector $v \in S(F)$ has its base point in a unique geometric 1-holed torus T or a 3-holed sphere P so that $G(v)$ is a spine for the subsurface. This gives a decomposition of $S(F)$. It remains to calculate for a simple hyperbolic surface Σ the volume of the set of all unit tangent vectors v in $S(\Sigma)$ so that $G(v)$ is a spine for Σ. It turns out this can be explicitly calculated using the dilogarithm function.

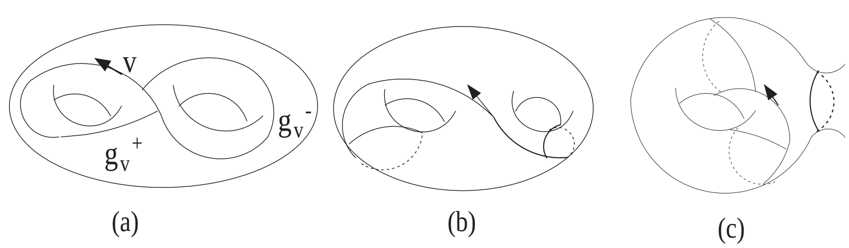

Figure 2. Creation of spines

1.3. Plan of the paper. In section 2, we describe how to decompose the unit tangent bundle $S(F)$ of the surface F. In section 3, for simple subsurfaces $\Sigma \subset F$, we identify the subset of the unit tangent vectors in $S(\Sigma)$ which do not generate spines for Σ with subsets of $S(\mathbb{H}^2)$. In section 4, we derive the formula for the measure of the set studied in section 3, thereby giving the formulas for f and g.

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2. Decomposing the unit tangent bundle of the surface

The unit tangent bundle of a hyperbolic surface X will be denoted by $S(X)$. Recall that F is a closed orientable hyperbolic surface. Due to the ergodicity of the geodesic flow, we will always assume that vectors $v \in S(F)$ are generic, that is, both forward and backward geodesic rays $g_v^{\pm}$ intersect themselves and every closed geodesic, and furthermore that v does not lie on a critical set where for some $T > 0$, $g_v^+[0,T] \cup g_v^-[0,T]$ is simple and $g_v^+(T) = g_v^-(T)$. For a generic unit tangent vector $v \in S(F)$, $G(v)$ is the graph of Euler characteristic -1 constructed in §1.2.

Proposition 2.1. *The graph $G(v)$ is contained in a unique simple geometric embedded subsurface $\Sigma(v) \subset F$.*

Proof. Cutting the surface F open along $G(v)$, we obtain a surface whose metric completion $\hat{F}$ is a compact hyperbolic surface with convex boundary. The boundary of $\hat{F}$ consists of piecewise simple geodesic loops (corresponding to $G(v)$). If $\hat{\gamma}$ is a piecewise simple geodesic loop in $\partial\hat{F}$, it is freely homotopic to a simple closed geodesic γ in $\hat{F}$ which is a component of the boundary of $core(\hat{F})$, the convex core of $\hat{F}$. Furthermore $\hat{\gamma}$ and γ are disjoint by convexity. Therefore, $\hat{\gamma}$ and γ bound a convex annulus exterior to $core(\hat{F})$ and $G(v)$ is disjoint from $core(\hat{F})$. The simple geometric subsurface $\Sigma(v)$ containing $G(v)$ is the union of these convex annuli bounded by $\hat{\gamma}$ and γ. The Euler characteristic of $\Sigma(v)$ is -1 by the construction. The surface $\Sigma(v)$ is unique. Indeed, if $\Sigma' \neq \Sigma \subset F$ is a simple geometric subsurface so that $G(v) \subset \Sigma'$, then Σ' has a boundary component say B which intersects one of the boundaries γ of Σ transversely. Therefore, B must intersect the other boundary $\hat{\gamma}$ of the convex annulus described earlier. Hence it intersects $G(v)$, which contradicts $G(v) \subset \Sigma'$. q.e.d.

As a consequence, we have produced the following decomposition of the unit tangent bundle $S(F)$. Given a simple geometric subsurface Σ in F, let $W(\Sigma) = \{v \in S(F) | G(v) \subset \Sigma\}$. Then by proposition 2.1, we have the following decomposition

$$S(F) = Z \bigsqcup \bigsqcup_P W(P) \bigsqcup_T W(T)$$

where Z is a set of measure zero and the union is over all geometric 3-holed spheres P and 1-holed tori T in F.

Let μ be the measure on $S(\mathbb{H}^2)$ invariant under the geodesic flow so that $\mu(S(F)) = -4\pi^2\chi(F)$. This measure comes from the Haar measure on $PSL(2, \mathbf{R})$. Take the μ measure of the above decomposition, we obtain the main identity in Theorem 1.1,

$$\mu(S(F)) = \sum_P \mu(W(P)) + \sum_T \mu(W(T)). \tag{6}$$

The focus of the rest of the paper is to calculate the volume of $W(\Sigma)$ for simple surfaces Σ.

We end this section with a related simpler decomposition of $S(F)$ indexed by the set of all simple closed geodesics. Given a generic unit tangent vector v, the geodesic ray g_v^+ intersects itself. Let $t_1 > 0$ be the first time so that $g_v^+(t_1) \in g_v^+([0, t_1))$, say $g_v^+(t_1) = g_v^+(t_2)$ for some $0 \le t_2 < t_1$. Then $g_v^+|_{[t_2,t_1]}$ is a simple loop freely homotopic to a simple closed geodesic s in F. See [**3**]. Denote $g_v^+([t_2, t_1])$ by $Lop(v)$. For any given simple closed geodesic s in F, let $U(s) = \{v \in S(F) | Lop(v) \cong s\}$. Then we obtain a decomposition $S(F) = \mathbf{Z}' \bigsqcup \bigsqcup_s U(s)$ where the disjoint union is indexed by the simple closed geodesics s and $\mu(\mathbf{Z}') = 0$. The associated identity is $\mu(S(F)) = \sum_s \mu(U(s))$. However, we are not able to calculate $\mu(U(s))$.

3. Identifying the sets in the decomposition

We will investigate the sets $W(P)$ and $W(T)$ (defined in §2) by studying their complements $V(P)$ and $V(T)$ in $S(P)$ and $S(T)$ in this section. This is due to the fact that for a simple geometric surface Σ, $\mu(W(\Sigma)) = 4\pi^2 - \mu(V(\Sigma))$.

The following notations and conventions will be used. All surfaces are assumed to be oriented so that their boundaries have the induced orientation. Geodesics are always parameterized by the arc length. A geodesic path $s : [a, b] \to X$ is called a *geodesic loop* based at $s(a)$ if $s(a) = s(b)$. It may not be a *closed geodesic* unless $s'(a) = s'(b)$. A *simple* geodesic loop satisfies the additional condition that $s|_{[a,b)}$ is injective. Two paths $\alpha_i : ([a_i, b_i], \{a_i, b_i\}) \to (X, \partial X)$, $i = 0, 1$, are *homotopic,* denoted by $\alpha_0 \cong \alpha_1$, if there is a homotopy $H : ([0, 1] \times [0, 1], \{0, 1\} \times [0, 1]) \to (X, \partial X)$ so that $H(t, i) = \alpha_i(a_i + t(b_i - a_i))$ for $i = 0, 1$. Two loops α_i, $i = 0, 1$, with the same base point $p = \alpha_i(a_i) = \alpha_i(b_i)$, which are *relatively homotopic with respect to p,* will be denoted by $\alpha_0 \cong \alpha_1$ rel{p}.

3.1. Preliminaries on convex surfaces. Suppose X is a compact connected convex hyperbolic surface, i.e., ∂X is convex in X. We will always identify the universal cover $\tilde{X}$ of X with a convex subset of the hyperbolic plane $\mathbb{H}^2$.

Proposition 3.1. *Suppose X is a compact convex hyperbolic surface.*

(1) *If X is an annulus, then any geodesic path s in X joining different boundary components of X is simple. Furthermore, if t is another such geodesic path with $t \neq s$, then $s \cup t$ is diffeomorphic to the union of a closed interval J and the graph of the sine function over J;*

(2) *If $s \cong t$ are two geodesic paths in X joining different boundary components of X and t is simple, then s is simple;*

(3) *If s is a geodesic loop based at $p \in \partial X$ so that $s \cong t$ rel$\{p\}$ where t is a simple topological loop, then s is a simple loop.*

Proof. We will need the following simple lemma whose proof is omitted, see figure 3(a).

Lemma 3.2. *Suppose γ is a hyperbolic isometry of $\mathbb{H}^2$ with axis A and g is a geodesic intersecting A transversely. Then $\gamma^n(g) \cap g = \emptyset$ for all $n \in \mathbf{Z} - \{0\}$.*

To see part (1), let c be the unique simple closed geodesic in X. Then s must intersect c. Lifting s and c to the universal cover and using the above lemma, we see that any two distinct lifts of s in $\tilde{X}$ are disjoint. Thus s is simple. The second statement follows from the fact that geodesics in a hyperbolic annulus which intersect the closed geodesic spiral towards each other. See figure 3(b).

To see part (2), suppose not. Then there exist two distinct lifts s_1 and s_2 of $s : [0, d] \to X$ in $\tilde{X}$ so that the interiors of s_1 and s_2 intersect. Let s join distinct boundary components a and b of X, and $\tilde{a_i}$ and $\tilde{b_i}$ be the lifts of a and b so that $s_i(0) \in \tilde{a_i}$ and $s_i(d) \in \tilde{b_i}$. Since $s \cong t$, by the homotopy lifting theorem, there exist two distinct lifts t_1 and t_2 of t in $\tilde{X}$ so that t_i joins $\tilde{a_i}$ to $\tilde{b_i}$.

We claim that interiors of t_1 and t_2 intersect. This will contradict the assumption that t is simple. To see the claim, first we note that $\tilde{a_1}$ is disjoint from $\tilde{a_2}$. For otherwise, $s_2 = \gamma^n(s_1)$ for a deck transformation element γ corresponding to the boundary a of X. Furthermore, due to convexity both s_1 and s_2 intersect the axis of γ. Thus by lemma 3.2, s_1 is disjoint from s_2, which contradicts the assumption. By the same argument we see that $\tilde{b_1}$ is disjoint from $\tilde{b_2}$. By assumption, $a \cap b = \emptyset$, hence $\tilde{a_1}, \tilde{a_2}, \tilde{b_1}, \tilde{b_2}$ are four distinct convex curves in $\tilde{X}$. Let

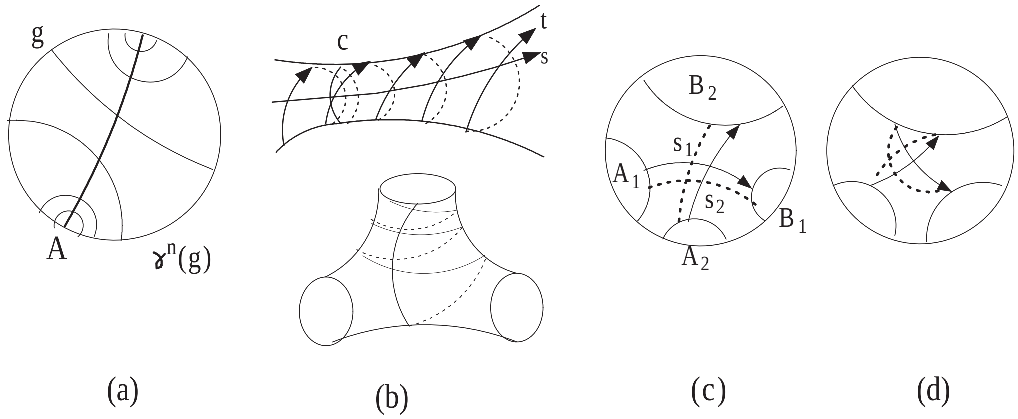

Figure 3. Lifting and disjointness

A_1, A_2, B_1, B_2 be the four disjoint half-spaces in $\mathbb{H}^2$ bounded by these four convex curves. Let S^1_∞ be the circle at infinity of the hyperbolic plane. Then $s_1 \cap s_2 \neq \emptyset$ is equivalent that $\overline{A_1} \cap S^1_\infty$ and $\overline{B_1} \cap S^1_\infty$ are in different components of $S^1_\infty - \overline{A_2} \cup \overline{B_2}$. This in turn implies that the interiors t_1 and t_2 intersect (figure 3(c)). Thus part (2) holds.

The proof of part (3) is very similar to that of (2). We omit the details. It is based on the simple fact that if s_1 and s_2 are two intersecting geodesic arcs in $\tilde{X}$ each joining different boundary components, and $t_i \cong s_i$ rel$\{\partial s_i\}$ for $i = 1, 2$, then t_1 intersects t_2. See figure 3(d). q.e.d.

For a geometric subsurface $\Sigma \subset F$ and generic $v \in S(\Sigma)$, define $G_\Sigma(v)$ as follows. Let $t_1 > 0$ be the smallest number so that the geodesic segment $g_v^-[0, t_1] \cup g_v^+[0, t_1]$ either intersects $\partial\Sigma$ or intersects itself, and without loss of generality, assume that $g_v^-(t_1) \neq g_v^+(t_1)$. Say this occurs in the ray g_v^+, i.e., $g_v^+(t_1) \in \partial\Sigma \cup g_v^-([0, t_1]) \cup g_v^+([0, t_1))$. Let $t_2 \geq t_1$ be the next smallest number so that $g_v^-(t_2) \in \partial\Sigma \cup g_v^-([0, t_2)) \cup g_v^+([0, t_1])$. Define $G_\Sigma(v) = g_v^-([0, t_2]) \cup g_v^+([0, t_1])$. By the construction, if $G(v) \subset \Sigma$, then $G_\Sigma(v) = G(v)$. In general, $G_\Sigma(v)$ is not a subset of $G(v) \cap \Sigma$. For generic v in $V(P)$ or $V(T)$, examples of $G_P(v)$ and $G_T(v)$ that are not equal to $G(v)$ are shown in figure 4(b), (c). In this notation, the complement of $W(\Sigma)$ in $S(\Sigma)$ is given by

$$V(\Sigma) = \{v \in S(\Sigma) | G_\Sigma(v) \text{ is either a simple arc or is homotopic to } S^1\}.$$

3.2. Vectors v in $V(P)$ so that $G_P(v)$ is a simple arc. We begin by recalling the beautiful work of M. Bridgeman [**1**] relevant to our setting. Given a compact hyperbolic surface X with geodesic boundary and a geodesic path $\alpha : ([0, a], \{0, a\}) \to (X, \partial X)$ so that $\alpha'(0)$ and $\alpha'(a)$ are perpendicular to ∂X, let

$$H(\alpha) = \{s'(t) | s : ([0, b], \{0, b\}) \to (X, \partial X) \text{ is a geodesic such that } s \cong \alpha\}.$$

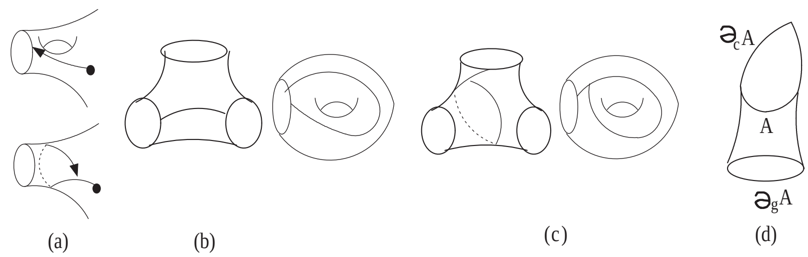

Figure 4. $G_\Sigma(v)$ in simple surfaces

Theorem 3.3. *(Bridgeman) The measure $\mu(H(\alpha))$ of $H(\alpha)$ is $4\mathcal{L}(\frac{1}{\cosh^2(l(\alpha)/2)})$ where $l(\alpha)$ is the length of α.*

A very nice short proof of this can be found in [**2**]. If we use α^{-1} to denote the reversed path $\alpha^{-1}(t) = \alpha(a-t)$, then the measures of $H(\alpha^{-1})$ and $H(\alpha)$ are the same. For simplicity, we use $H(\alpha^{\pm 1})$ to denote $H(\alpha) \cup H(\alpha^{-1})$.

The main result in this subsection is to prove:

Proposition 3.4. *Suppose P is a hyperbolic 3-holed sphere with geodesic boundary components L_1, L_2, L_3. Let B_i (respectively M_i) be the non-trivial shortest geodesic path from L_i to L_i (repectively L_{i+1} to L_{i+2}) (see figure 1). Then*

(1) $\{v \in S(P) | G_P(v) \text{ is a simple arc}\} \subset \cup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$.
(2) $\cup_{i=1}^3 (H(M_i^{\pm 1}) \cup H(B_i^{\pm 1})) \subset V(P)$.

Proof. To see part(1), by assumption, the geodesic path $G_P(v)$ is a simple arc with end points in ∂P. It is well known that any simple path $s : ([0,1], \{0,1\}) \to (P, \partial P)$ is homotopic to M_i, B_i, or a point. The path $G_P(v)$ cannot be homotopic to a point since it is a geodesic path. Thus the conclusion follows.

To see part (2), let $s : [0,a] \to P$ be a geodesic path homotopic to M_i or B_i. If $s \cong M_i$, by proposition 3.1(2), s is simple. Thus $s'(t) \in V(P)$. If $s \cong B_i$, we claim that there exists $b \in (0,a)$ so that $s|_{[0,b]}$ and $s|_{[b,a]}$ are simple arcs. First of all, the paths s and M_i intersect in exactly one point. Indeed, if $|s \cap M_i| \geq 2$, then there is a lift $\tilde{s}$ of s in the universal cover so that $\tilde{s}$ intersects two distinct lifts a_1 and a_2 of M_i. By the homotopy lifting theorem, we find a lift $\tilde{B}$ of B_i so that both $\tilde{B}$ and $\tilde{s}$ start and end at the same boundary components of $\tilde{P}$. Then $\tilde{B}$ intersects a_1 and a_2, i.e., B_i intersects M_i at two points. This is impossible. Furthermore, since B_i intersects M_i, s must also intersect M_i. It follows that s intersects M_i in exactly one point, say $s(b) \in M_i$ for some $b \in (0,a)$. We claim that $s|_{[0,b]}$ and $s|_{[b,a]}$ are both simple arcs. Indeed, let X be the convex annulus obtained by cutting P open along

M_i. Both paths $s|_{[0,b]}$ and $s|_{[b,a]}$ are geodesics in X joining different boundary components of X. Thus, by proposition 3.1, both of them are simple arcs in X.

We now finish the proof of part (2) by showing that for any $t \in (0, a)$, $s'(t) \in V(P)$. By proposition 3.1(1) applied to the two simple arcs $s|_{[0,b]}$ and $s|_{[b,a]}$, the curve s is homeomorphic to the union of a closed interval J and the graph of the sine function over J. This implies that s does not contain any spine graph $G(v)$ of Euler characteristic -1 constructed in §1.2, see figure 3(b). Thus $s'(t) \in V(P)$.

q.e.d.

3.3. Lassos and one-corner convex annuli. For a hyperbolic 3-holed sphere P, if $v \in V(P) - \cup_{i=1}^{3}(H(M^{\pm 1}) \cup H(B_i^{\pm 1}))$, then $G_P(v)$ is a lasso (see figure 4(c)).

Definition 3.5. (Lassos) Let X be a hyperbolic surface with boundary. A positive *lasso* on X is a geodesic path $\alpha : [T_1, T_2] \to X$ such that

(1) $\alpha(T_1) \in \partial X$,
(2) α is injective on $[T_1, T_2)$, and
(3) $\alpha(T_3) = \alpha(T_2)$ for some $T_1 < T_3 < T_2$.

Call $\alpha(T_1)$ the *base point*, $\alpha(T_2) = \alpha(T_3)$ the *knot*, $\alpha|_{[T_3,T_2]}$ the *loop*, and $\alpha(\frac{T_2+T_3}{2})$ the *midpoint* of the lasso. If $0 \le t < \frac{T_2+T_3}{2}$, we say $\alpha'(t)$ *generates* the lasso α. A negative lasso β is a geodesic path so that $\beta(-t)$ is a positive lasso.

Proposition 3.6. *Suppose $\alpha : [T_1, T_2] \to \Sigma$ is a positive lasso in a geometric subsurface $\Sigma \subset F$ with knot $\alpha(T_3) = \alpha(T_2)$. Then $G_\Sigma(\alpha'(t)) = \alpha([T_1, T_2])$ if and only if $T_1 \le t < \frac{T_2+T_3}{2}$.*

Proof. By the definition of $G_\Sigma(v)$ in §3.1 and §1.2, we see that if $T_1 \le t < \frac{T_2+T_3}{2}$, $G_\Sigma(\alpha'(t)) = \alpha([T_1, T_2])$ and if $t \in (\frac{T_2+T_3}{2}, T_2]$, then $G_\Sigma(a'(t)) \cap \alpha([T_1, T_3)) \ne \alpha([T_1, T_3))$. q.e.d.

A *one-corner annulus* A is a convex annulus so that $\partial A = \partial_g A \cup \partial_c A$ where $\partial_g A$ is a closed geodesic and $\partial_c A$ is a geodesic loop with one corner. See figure 4(d). A geodesic is called *maximum* if it is not properly contained in another geodesic.

Proposition 3.7. *Let A be a one-corner convex annulus with corner point b.*

(1) Each lasso α in a compact convex hyperbolic surface X is contained in a unique one-corner annulus A so that the base point of α is the corner point. The midpoint of α is the point in α that is closest to $\partial_g A$ in A.

(2) Each maximum geodesic path in A from the corner point b is (i) a simple arc joining b to $\partial_g A$, or (ii) spirals toward $\partial_g A$, or (iii) contains a lasso. It contains a lasso if and only if it joins b to $\partial_c A$.

Proof. To see part (1), let γ be a lasso with base point b in a convex surface X. Let α be the simple closed geodesic homotopic to the loop of γ. By cutting X open along γ and using the convex core argument as in §2.1, we see that α is disjoint from γ. We claim there is a simple geodesic loop β in X based at b so that (1) β is freely homotopic α with $\beta \cap \alpha = \emptyset$, and (2) γ is contained in the one-corner annulus A bounded by β and α. Indeed, cutting the surface X open along the lasso γ we obtain a convex surface Y (possibly disconnected) whose boundary contains copies of γ. Let q_1 and q_2 be the preimages of b in Y and let s be the piecewise geodesic arc in the boundary of Y joining q_1 to q_2 so that s contains preimages of the loop of γ. Since Y is convex, there exists a unique shortest geodesic path ρ in Y joining q_1 to q_2 so that $\rho \cong s$ rel$(\{q_1, q_2\})$. The simple loop β is the image of ρ in X. By the construction, $\beta \cap \gamma = \{b\}$ and $\beta \cong \alpha$. Furthermore, $\beta \cap \alpha = \emptyset$ by the same argument that $\gamma \cap \alpha = \emptyset$. Let A be the one-corner convex annulus bounded by β and α in X. By the construction, points in γ near b are in the convex side of β. Thus $\gamma \cap int(A) \neq \emptyset$. But γ is disjoint from α, therefore $\gamma \subset A$. To characterize the midpoint of the lasso γ, we lift γ to the universal cover of A. The second part of (1) follows.

To see part (2), suppose $\gamma(t)$ is a maximum geodesic path in A with $\gamma(0) = b$. If γ is simple on $[0, \infty)$, then γ spirals toward $\partial_g A$. Otherwise, there exists $T < \infty$ so that $\gamma|_{[0,T)}$ is injective and $\gamma(T) \in \gamma([0,T)) \cup \partial A$. If $\gamma(T) \in \partial_g A$, then γ is a simple geodesic joining b to $\partial_g A$. By the Gauss-Bonnet theorem, there is no hyperbolic bi-gon disk. It follows that $\gamma(T)$ cannot be in $\partial_c A$ (for otherwise, γ and an arc in $\partial_c A$ bound a bi-gon). The remaining case is that $\gamma(T) \in \gamma([0,T))$, i.e., $\gamma|_{[0,T]}$ is a lasso in A. The last statement in part (2) follows from proposition 3.1(1). q.e.d.

Note that if α and β are two lassos so that α is positive and β is negative, then by definition $\alpha'(t) \neq \beta'(t')$ for all parameters t, t'. Furthermore, the involution map $\mathbf{A}(v) = -v$ in $S(X)$ sends tangent vectors to positive lassos to tangent vectors of negative lassos. Thus it suffices to calculate the measure of tangents to positive lassos.

Given $v \in V(P) - \bigcup_{i=1}^{3}(H(M_i^{\pm 1}) \bigcup H(B_i^{\pm 1}))$, the graph $G_P(v)$ is a lasso generated by v. Since its loop is simple, it is freely homotopic to $L_i^{\pm 1}$ for some i. For i, j, k distinct, let $W(L_i^{\pm}, M_j)$ be $\{v \in S(P) - \cup_{s=1}^{3}(H(M_s^{\pm 1}) \cup H(B_s^{\pm 1}))|\ G_P(v)$ is a positive lasso whose loop is homotopic to $L_i^{\pm}$ and whose base point is in $L_k\}$.

We have:

Lemma 3.8. *The set $V(P) - \cup_{i=1}^{3}(H(M_i^{\pm 1}) \cup H(B_i^{\pm 1}))$ can be decomposed as*

$$\bigsqcup_{i \neq j}\big(W(L_i, M_j) \bigcup W(L_i^{-1}, M_j)\big) \bigsqcup \mathbf{A}\big(\bigsqcup_{i \neq j}(W(L_i, M_j) \bigcup W(L_i^{-1}, M_j))\big).$$

In particular, by theorem 3.3 and proposition 3.4, we have

$$\mu(V(P)) = 8\sum_{i=1}^{3}\Big(\mathcal{L}\left(\frac{1}{\cosh^2(m_i/2)}\right) + \mathcal{L}\left(\frac{1}{\cosh^2(p_i/2)}\right)\Big) + 4\sum_{i\neq j}\mu(W(L_i, M_j)),$$

where m_i, p_i are the lengths of M_i and B_i respectively.

Proof. The decomposition follows from the above discussion. We claim that $W(L_i, M_j)$ and $W(L_i^{-1}, M_j)$ are related by an isometry of P. Indeed, the hyperbolic 3-holed sphere P admits an orientation reversing isometry R so that $R|_{M_i} = id$ and R interchanges the two hexagons obtained by cutting P open along M_i's. In particular, R reverses the orientation of each boundary component. Therefore, the derivative of R sends $W(L_i, M_j)$ to $W(L_i^{-1}, M_j)$. This implies that $\mu(W(L_i, M_j)) = \mu(W(L_i^{-1}, M_j))$. q.e.d.

3.4. Understanding the set $W(L_i, M_j)$. The circle at infinity of the hyperbolic plane $\mathbb{H}^2$ is denoted by $S^1_\infty = \mathbf{R} \cup \{\infty\}$. Given $x \neq y \in \mathbb{H}^2 \cup S^1_\infty$, let $G[y, x]$ be the oriented geodesic from y to x and $S(G[y, x])$ be the set of unit tangent vectors to $G[y, x]$.

We will focus on understanding the set $W(L_2, M_3)$ in this section. To this end, let $\Pi : \tilde{P} \to P$ be the universal cover of a hyperbolic 3-holed sphere P so that $\tilde{L}_i$ are boundary components of $\tilde{P}$ with $\Pi(\tilde{L}_i) = L_i$ and $d(\tilde{L}_i, \tilde{L}_j) = d(L_i, L_j)$ where $\partial P = L_1 \cup L_2 \cup L_3$. Let γ_i be the deck transformation element of Π corresponding to L_i (with the induced orientation from P) so that $\gamma_i(\tilde{L}_i) = \tilde{L}_i$. We will identify $\tilde{P}$ with a subset of $\mathbb{H}^2$ and normalize so that $\tilde{L}_2 = G[\infty, 0]$, $\tilde{L}_1 = G[c, d]$, $\gamma_2(\tilde{L}_1) = G[1, d/c]$ and $\gamma_2(z) = c^{-1}z$ where $1 < d/c < c < d$. Furthermore, we may assume that $\tilde{L}_3 = G[c', d']$ with $d/c < c' < d' < c$. See figure 6(a) and, for instance, [**4**] for the construction. Define the subset Ω of $S(\mathbb{H}^2)$ as follows. Given $x, y \in \mathbf{R}$ with $0 < x < 1$ and $c < y < d$, let q be the intersection point $G[y, x] \cap \tilde{L}_1$ and let p be the point on $G[y, x]$ that is closest to $\tilde{L}_2 = G[\infty, 0]$, i.e., the Euclidean ray $0p$ is tangent to $G[y, x]$. See figure 6(a). Then,

$$\Omega = \{v \in S(\mathbb{H}^2) | v \in S(G[q, p]), 0 < x < 1, c < y < d\}. \tag{7}$$

The main result in this subsection is the following:

Proposition 3.9. *Let Π_* be the derivative of the covering map $\Pi : \tilde{P} \to P$. Then $\Pi_*|_\Omega$ is a bijection from Ω to $W(L_2, M_3)$. In particular, $\mu(W(L_2, M_3)) = \mu(\Omega)$.*

Proof. We will construct a continuous map $\Phi : W(L_2, M_3) \to \Omega$ so that $\Pi_* \Phi = id$ and show that Φ is onto.

To construct Φ, take a vector $v \in W(L_2, M_3)$. By assumption, $\gamma = G_P(v)$ is a lasso generated by v so that its base point b is in L_1 and its loop is homotopic to L_2. By proposition 3.7, γ is contained a unique one-corner annulus $A \subset P$ so that $\partial_g A = L_2$.

Since $\Pi| : \tilde{L}_2 \to L_2$ is the universal cover and L_2 is a strong deformation retract of A, we can identify the universal cover $\tilde{A}$ of A as a subset of $\mathbb{H}^2$ so that $\tilde{L}_2 \subset \tilde{A}$ is a boundary component, see figure 5(a). Let the other boundary component of $\tilde{A}$ be $\tilde{\alpha}$, which is a piecewise geodesic from ∞ to 0 parameterized by the arc length. We claim that $|\tilde{\alpha} \cap \tilde{L}_1| = 1$. Indeed, since there exists a simple path δ in A from $b \in L_1$ to L_2 so that $\delta \cong M_3$ in P, by lifting this path to $\tilde{A}$, we see that $\tilde{\alpha} \cap \cup_{n \in \mathbf{Z}} \gamma_2^n(\tilde{L}_1) \neq \emptyset$. But $\gamma_2(\tilde{\alpha}) = \tilde{\alpha}$, thus $\tilde{\alpha} \cap \tilde{L}_1 \neq \emptyset$. Let $\beta = \partial_c A$. Since $|\beta \cap L_1| = 1$ and β is not freely homotopic to L_1^n for any $n \in \mathbf{Z}$, we have $\tilde{\alpha} \cap \tilde{L}_1$ consists of one point, denoted by q. See figure 5(a). Let $\tilde{\gamma}$ be the lifting of γ in $\tilde{A}$ starting at q and u be the unit tangent vector to $\tilde{\gamma}$ so that $\Pi_*(u) = v$. We claim that $u \in \Omega$. Consider the complete geodesic $G[y, x]$ that contains $\tilde{\gamma}$ where $x, y \in \mathbf{R}$. Since $G[y, x] \cap \tilde{L}_1 = \{q\}$, we have $c < y < d$. Since $\tilde{\gamma} \subset \tilde{A}$ and $\Pi(G[q, \gamma_2(q)]) = \beta$, it follows that $x < d/c$ or $x > d$. We cannot have $x \in [1, d/c]$ since otherwise the maximum geodesic path γ^* containing γ in P will be homotopic to $B_1^{\pm}$, contradicting $v \notin H(B_1^{\pm})$. The same argument shows that $x \leq 0$ is impossible due to $v \notin \cup_i H(M_i^{\pm})$. Furthermore, since the loop of γ is not homotopic to L_2^{-1}, $x > d$ is impossible. Therefore, $0 < x < 1$. Next, we claim that $u \in S(G[q, p])$ (defined in (7)). Indeed, since v generates γ, $v = \gamma'(t)$ where $\gamma(t)$ is before the mid-point of the lasso γ. By proposition 3.7, the mid-point of γ is the image under Π_* of the unit tangent at p of $G[y, x]$. It follows that $u \in S(G[q, p])$. This shows $u \in \Omega$. Define $\Phi : W(L_2, M_3) \to \Omega$ by $\Phi(v) = u$. By the construction $\Pi_* \Phi = id$. In particular, Φ is injective.

To see Φ is onto, take a vector $v \in \Omega$. We will show $\Pi_*(v) \in W(L_2, M_3)$.

Lemma 3.10. *Let $\tilde{\beta} = G[q, \gamma_2(q)]$ be the geodesic in $\mathbb{H}^2$ from $q \in \tilde{L}_1$ to $\gamma_2(q) \in \gamma_2(\tilde{L}_1)$. Then $\Pi(\tilde{\beta}) = \beta$ is a simple geodesic loop in P based at $b = \Pi(q)$.*

Proof. By proposition 3.1, it suffices to show that that $\beta \simeq \delta$ rel(b) for some simple loop δ at b. Consider the shortest path $a_1 = G[q, q_1]$ from q to $\tilde{L}_2$ where $q_1 \in \tilde{L}_2$. Since $dist(\tilde{L}_1, \tilde{L}_2) = dist(L_1, L_2)$, the projection $\Pi(a_1)$ is homotopic to the shortest path M_3 from L_1 to L_2. Thus, by proposition 3.1, $\Pi(a_1)$ is a simple arc from L_1 to L_2. Now by construction, $\tilde{\beta}$ and the path $a_1 * G[q_1, \gamma_2(q_1)] * \gamma_2(a_1^{-1})$ have the same end points in $\mathbb{H}^2$. Thus $\beta \simeq \Pi(a_1) * \Pi(G[q_1, \gamma_2(q_1)]) * \Pi(a_1^{-1})$ rel(b). Since $\Pi(a_1)$ is an embedded arc whose interior is disjoint from $\Pi(G[q_1, \gamma_2(q_1)])$

($=L_2$), by a small perturbation, the loop $\Pi(a_1) * L_2 * \Pi(a_1^{-1})$ is relatively homotopic to a simple loop δ based at b. It follows that $\beta \simeq \delta$ rel(b) where δ is simple. See figure 5(b). q.e.d.

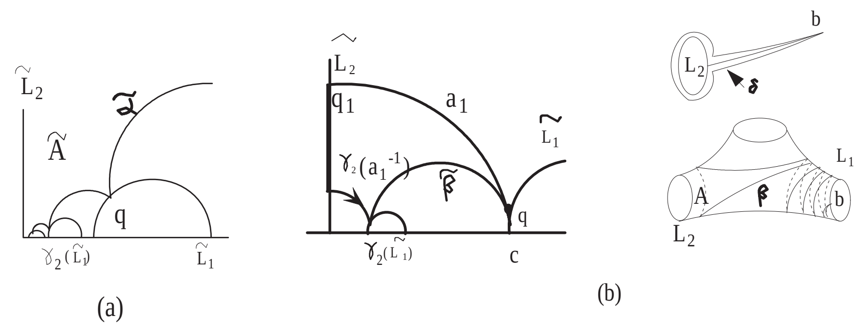

Figure 5. Homotopic loops are simple

Since the simple loop β is disjoint from L_2 and is homotopic to L_2, there is a one-corner convex annulus A in P bounded by β and L_2. We identify the universal cover $\tilde{A}$ of A (via Π^{-1}) with the convex region in $\mathbb{H}^2$ bounded by $\tilde{L}_2$ and the simple path $\tilde{\alpha} = \cup_{n\in\mathbf{Z}}\gamma_2^n(\tilde{\beta})$ as before, see figure 5(a). By construction, $G[q,p] \subset \tilde{A}$. Let $G[q,r]$ be the maximum geodesic containing $G[q,p]$ in $\tilde{A}$. Then $\Pi(G[q,r])$ is a maximum geodesic in A starting at $\Pi(q)$. By proposition 3.7 and the assumption that $x > 0$, $\Pi(G[q,r])$ contains a positive lasso γ whose midpoint is $\Pi(p)$. Since $v \in S(G[q,p])$, by proposition 3.6, $\Pi_*(v)$ generates the lasso γ. Now by the construction, the loop of γ is homotopic to $\partial_g A = L_2$ and the base of γ is in L_1. Due to $0 < x < 1$, the maximum geodesic $\Pi(G[q,r])$ containing γ is not homotopic to $B_1^{\pm}$, $M_2^{\pm}$ or $M_3^{\pm}$. Thus $\Pi_*(v) \in W(L_2, M_3)$.
q.e.d.

3.5. The lasso function $La(l_i, m_j) = \mu(W(L_i, M_j))$.

Corollary 3.11. *$\mu(W(L_i, M_j))$ depends only on the lengths l_i of L_i and m_j of M_j.*

We may assume without loss of generality that $i = 2, j = 3$. By proposition 3.8, $\mu(W(L_2, M_3)) = \mu(\Omega)$, which depends only on c, d. Now by construction, $\gamma_2(z) = z/c$, hence $c = e^{l_2}$. By the hyperbolic distance formula

$$d(G[x_1, x_2], G[x_3, x_4]) = \sqrt{\coth^{-1}\left(\frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}\right)}$$

(with $x_1 < x_2 < x_3 < x_4$) and $m_3 = d(\tilde{L}_2, \tilde{L}_1) = d(G[0,\infty], G[c,d])$, we obtain $d = e^{l_2}\coth^2(m_3/2)$. Thus the corollary follows.

We define the *lasso function* $La(l_i, m_j)$ to be

$$La(l_i, m_j) := \mu(W(L_i, M_j)).$$

3.6. Vectors in $V(T)$ for a 1-holed torus T. Let T be a hyperbolic 1-holed torus with geodesic boundary. Then $v \in V(T)$ if and only if $G_T(v) \cap \partial T \neq \emptyset$. Cutting T open along $G_T(v)$ gives a convex hyperbolic annulus and there is a unique simple closed geodesic $A \subset T$ which is disjoint from $G_T(v)$. Hence $V(T)$ decomposes into the infinite disjoint union $V(T) = \bigsqcup_{\{A\}} V_A(T)$ where the union is over the set of all simple closed non-boundary parallel geodesics A and

$$V_A(T) = \{v \in V(T) \mid G_T(v) \cap A = \emptyset\}.$$

Let P_A be the 3-holed sphere obtained by cutting T along A so that $\partial P = L_1 \cup L_2 \cup L_3$ where $L_1 = \partial T$, $L_2 = A^+$ and $L_3 = A^-$ are two copies of A. See figure 1. Let B_i and M_i be the shortest paths in P_A, as in Proposition 3.4. Then, using similar arguments as in the previous subsections, we have that $V_A(T)$ is the disjoint union

$$H(B_1^{\pm 1}) \bigsqcup \bigsqcup_{i \neq j \neq 1 \neq i} (W(L_i, M_j) \cup W(L_i^{-1}, M_j))$$
$$\bigsqcup \mathbf{A}(\bigsqcup_{i \neq j \neq 1 \neq i} (W(L_i, M_j) \cup W(L_i^{-1}, M_j))).$$

The hyperelliptic involution on T induces an isometric involution of P_A sending L_2 to L_3 and fixing L_1. Therefore, $\mu(W(L_3, M_2)) = \mu(W(L_2, M_3))$. It follows that

$$\mu(V_A(T)) = \mu(H(B_1^{\pm 1})) + 8\mu(W(L_2, M_3)).$$

Let the simple closed geodesic A have length a. Let m_A be the distance from A to ∂T and p_A be the length of the shortest non-trivial path in $T - A$ from ∂T to itself. Then using Bridgeman's theorem, the lasso function and $g(T) := \mu(W(T)) = \mu(S(T)) - \mu(V(T)) = 4\pi^2 - \sum_A \mu(V_A(T))$, we obtain,

$$g(T) = 4\pi^2 - 8 \sum_A (\mathcal{L}(\frac{1}{\cosh^2(p_A/2)}) + La(a, m_A)). \tag{8}$$

4. Calculating the lasso function $La(l, m)$

By §3.4 and theorem 3.3, it remains to calculate $\mu(W(L_i, M_j))$, that is, $\mu(\Omega)$.

Recall that the invariant measure on the unit tangent bundle $S(\mathbb{H}^2)$ in local coordinates can be written as $\frac{2dxdydu}{(x-y)^2}$, where $x \neq y \in \mathbf{R}$ and $u \in \mathbf{R}$, so that the oriented geodesic determined by $v \in S(\mathbb{H}^2)$ is $G[x, y]$ and u is the signed distance from the base point of v to the highest

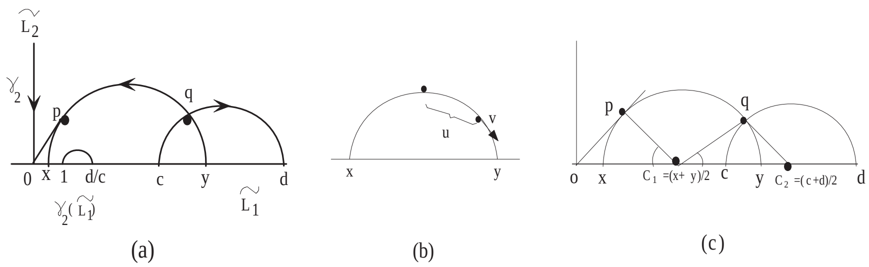

Figure 6. Coordinates for $S(\mathbb{H}^2)$

point in the semicircle $G[x, y]$ (in the Euclidean plane). See figure 6(b) below.

We will show that the volume of Ω defined by (7) is given by the identity (13).

4.1. Deriving the volume formula for Ω. We will establish

Proposition 4.1. *The volume of Ω is given by*

$$\int_0^1 \Big(\int_c^d \frac{\ln |\frac{y(x-c)(x-d)}{x(y-c)(y-d)}|}{(y-x)^2} dy \Big) dx. \tag{9}$$

Proof. The hyperbolic distance $d(e^{i\phi}, e^{-i\psi})$ between $e^{i\phi}, e^{-i\psi} \in \mathbb{H}^2$ is $\ln \cot(\phi/2) + \ln \cot(\psi/2)$ where $\phi, \psi \in (0, \pi/2)$. Let $C_1 = \frac{x+y}{2}$ be the Euclidean center of the semicircle $G[x, y]$ and ψ and ϕ be the angles $\angle 0C_1p$ and $\angle qC_1y$ as shown in figure 6(c). Then by the definition of μ, we see that $\mu(\Omega)$ is given by

$$\begin{aligned} &\int_0^1 \Big(\int_c^d \frac{2 \ln \cot(\psi/2) + 2 \ln \cot(\phi/2)}{(y-x)^2} dy \Big) dx \\ &= \int_0^1 \Big(\int_c^d \frac{\ln[\cot^2(\psi/2) \cot^2(\phi/2)]}{(y-x)^2} dy \Big) dx. \end{aligned} \tag{10}$$

We calculate $\cot^2(\psi/2)$ and $\cot^2(\phi/2)$ using the cosine law for Euclidean triangles $\Delta 0pC_1$ and ΔC_1C_2q where $C_2 = \frac{c+d}{2}$ is the center of the semicircle $G[c, d]$.

For the angle ψ, since the triangle $\Delta 0pC_1$ is right-angled, we obtain $\cos(\psi) = \frac{y-x}{y+x}$ and

$$\cot^2(\frac{\psi}{2}) = \frac{y}{x}. \tag{11}$$

To find $\cot(\phi/2)$, we use

Lemma 4.2. *Suppose the lengths of a Euclidean triangle are l, m, n so that the angle facing the edge of length l is θ. Then*

$$\cot^2(\theta/2) = \frac{(m+n+l)(m+n-l)}{(m+l-n)(n+l-m)}.$$

This follows from the cosine law $\cos(\theta) = \frac{m^2+n^2-l^2}{2mn}$ and $\cot^2(\theta/2) = \frac{1+\cos(\theta)}{1-\cos(\theta)}$.

For the angle ϕ, the edge lengths of the Euclidean triangle $\Delta q C_1 C_2$ are $n = \frac{y-x}{2}$, $m = \frac{c+d-x-y}{2}$, and $l = \frac{d-c}{2}$ so that ϕ is facing the edge of length l. Now using $l+m+n = d-x, m+n-l = c-x, l+n-m = y-c, l+m-n = d-y$ and lemma 4.2, we obtain that

$$\cot^2(\phi/2) = \left|\frac{(x-c)(x-d)}{(y-c)(y-d)}\right|. \tag{12}$$

Putting (11), (12) into (10), we obtain (9). q.e.d.

4.2. Evaluation of the integral (9). The evaluation of the integral is similar to the work in [**1**]. Recall the Roger's dilogarithm $\mathcal{L}(x)$ satisfies $\mathcal{L}(0) = 0$ and $2\mathcal{L}'(x) = \frac{\ln|x|}{x-1} - \frac{\ln|x-1|}{x}$, for $x < 1$.

Proposition 4.3. *If $d > c > 1$, then*

$$\begin{aligned}\int_0^1 \Big(\int_c^d \frac{\ln\left|\frac{y(x-c)(x-d)}{x(y-c)(y-d)}\right|}{(y-x)^2} dy\Big) dx \\ = 2\Big[\mathcal{L}(\frac{d-1}{d}) - \mathcal{L}(\frac{c-1}{c}) + 2\mathcal{L}(\frac{c-1}{c-d}) - 2\mathcal{L}(\frac{c}{c-d})\Big].\end{aligned} \tag{13}$$

Proof. Let $R = \left|\frac{y(x-c)(x-d)}{x(y-c)(y-d)}\right|$ and write (13) as $\int_0^1 \int_c^d \frac{\ln R}{(x-y)^2} dy dx$. For simplicity, we drop the constant term in the indefinite integrals in the lemma below.

Lemma 4.4.

$$\begin{aligned}\int \frac{\ln R}{(x-y)^2} dy = {} & \frac{\ln\left|\frac{(x-c)(x-d)}{x}\right|}{x-y} + (\frac{1}{x-y} - \frac{1}{x}) \ln|y| \\ & + (\frac{1}{x} - \frac{1}{x-c} - \frac{1}{x-d}) \ln|y-x| \\ & + (-\frac{1}{x-y} + \frac{1}{x-c}) \ln|y-c| \\ & + (-\frac{1}{x-y} + \frac{1}{x-d}) \ln|y-d|.\end{aligned}$$

The proof is standard integration by parts calculus and will be omitted.

Lemma 4.5. *Let* $W(x) = \int_c^d \frac{\ln R}{(x-y)^2} dy$. *Then*

$$W(x) = (\frac{\ln|\frac{x-d}{d}|}{x} - \frac{\ln|\frac{x}{d}|}{x-d}) - (\frac{\ln|\frac{x-c}{c}|}{x} - \frac{\ln|\frac{x}{c}|}{x-c}) + 2(\frac{\ln|\frac{x-c}{x-d}|}{x-d} - \frac{\ln|\frac{x-d}{x-c}|}{x-c}).$$

Proof. By lemma 4.4, we can write $W(x)$ as

(14)
$$\begin{aligned}
&(\frac{1}{x-d} - \frac{1}{x-c})\ln(|\frac{(x-c)(x-d)}{x}|) \\
&\quad + (\frac{1}{x-d} - \frac{1}{x})\ln|d| - (\frac{1}{x-c} - \frac{1}{x})\ln|c| \\
&\quad + (\frac{1}{x} - \frac{1}{x-c} - \frac{1}{x-d})(\ln|x-d| - \ln|x-c|) \\
&\quad + (-\frac{1}{x-d} + \frac{1}{x-c})\ln|d-c| \\
&\quad - \lim_{y\to c}(-\frac{1}{x-y} + \frac{1}{x-c})\ln|y-c| \\
&\quad + \lim_{y\to d}(-\frac{1}{x-y} + \frac{1}{x-d})\ln|y-d| - (-\frac{1}{x-c} + \frac{1}{x-d})\ln|c-d|.
\end{aligned}$$

Now both limits appearing in (14) are zero since $\lim_{t\to 0} t\ln|t| = 0$. Thus, by rewriting (14)and after regrouping according to $\frac{1}{x}$, $\frac{1}{x-c}$ and $\frac{1}{x-d}$, we obtain,

$$\begin{aligned}
W(x) &= \frac{1}{x}(-\ln|d| + \ln|c| + \ln|x-d| - \ln|x-c|) \\
&\quad + \frac{1}{x-c}(-\ln|x-c| - \ln|x-d| + \ln|x| - \ln|c| - \ln|x-d| \\
&\qquad\qquad + \ln|x-c| + \ln|d-c| + \ln|d-c|) \\
&\quad + \frac{1}{x-d}(\ln|x-c| + \ln|x-d| - \ln|x| + \ln|d| - \ln|x-d| \\
&\qquad\qquad + \ln|x-c| - \ln|d-c| - \ln|d-c|) \\
&= \frac{1}{x}(\ln|\frac{x-d}{d}| - \ln|\frac{x-c}{c}|) + \frac{1}{x-c}(\ln|\frac{x}{c}| - 2\ln|\frac{x-d}{c-d}|) \\
&\quad + \frac{1}{x-d}(-\ln|\frac{x}{d}| + 2\ln|\frac{x-c}{d-c}|) \\
&= (\frac{\ln|\frac{x-d}{d}|}{x} - \frac{\ln|\frac{x}{d}|}{x-d}) - (\frac{\ln|\frac{x-c}{c}|}{x} - \frac{\ln|\frac{x}{c}|}{x-c}) \\
&\quad + 2(\frac{\ln|\frac{x-c}{x-d}|}{x-d} - \frac{\ln|\frac{x-d}{x-c}|}{x-c}).
\end{aligned}$$

q.e.d.

Now to finish the proof of proposition 4.3, following [**1**], for $a \neq b$ let $J(x,a,b) = 2\mathcal{L}(\frac{x-b}{a-b})$ so that $J'(x,a,b) = \frac{dJ(x,a,b)}{dx} = \frac{\ln|\frac{x-b}{a-b}|}{x-a} - \frac{\ln|\frac{x-a}{b-a}|}{x-b}$. By lemma 4.5, it follows that $W(x) = J'(x,0,d) - J'(x,0,c) + 2J'(x,d,c)$.

Therefore

$$\begin{aligned}
&\int_0^1 \int_c^d \frac{\ln R}{(x-y)^2} dy dx \\
&= \int_0^1 W(x) dx \\
&= J(1,0,d) - J(0,0,d) - J(1,0,c) + J(0,0,c) + 2J(1,d,c) - 2J(0,d,c)
\end{aligned}$$

But $J(0,0,k) = 2\mathcal{L}(1)$, thus it follows that (13) holds. q.e.d.

Proposition 4.6. *The lasso function* $La(l,m) = 2(\mathcal{L}(y) - \mathcal{L}(\frac{1-x}{1-xy}) + \mathcal{L}(\frac{1-y}{1-xy}))$ *where* $x = e^{-l}$ *and* $y = \tanh^2(m/2)$.

Proof. By §3.5, $La(l,m)$ is given by (13) with $c = 1/x$ and $d = \frac{1}{xy}$. Now, $\frac{d-1}{d} = 1-xy$, $\frac{c-1}{c} = 1-x$, $\frac{c-1}{c-d} = \frac{-r}{1-r}$ where $r = \frac{y(1-x)}{y-1}$ and $\frac{c}{c-d} = -\frac{y}{1-y}$. Furthermore, the Roger's dilogarithm satisfies $\mathcal{L}(1-u) = \pi^2/6 - \mathcal{L}(u)$ and $\mathcal{L}(-\frac{u}{1-u}) = -\mathcal{L}(u)$ for $0 < u < 1$ (see [**11**]). It follows that $\mathcal{L}(\frac{c-1}{c-d}) = \mathcal{L}(\frac{-r}{1-r}) = -\mathcal{L}(r) = -\mathcal{L}(\frac{y(1-x)}{1-xy})$ and $\mathcal{L}(\frac{c}{c-d}) = \mathcal{L}(-\frac{y}{1-y}) = -\mathcal{L}(y)$.

Thus the right-hand-side of (13) divided by 2 is

$$\begin{aligned}
&\mathcal{L}(1-xy) - \mathcal{L}(1-x) - 2\mathcal{L}(\frac{y(1-x)}{1-xy}) + 2\mathcal{L}(y) \\
&\quad = \pi^2/6 - \mathcal{L}(xy) - \pi^2/6 + \mathcal{L}(x) - 2\mathcal{L}(\frac{y(1-x)}{1-xy}) + 2\mathcal{L}(y) \\
&\quad = \mathcal{L}(x) - \mathcal{L}(xy) + 2\mathcal{L}(y) - 2\mathcal{L}(\frac{y(1-x)}{1-xy})
\end{aligned}$$

Using a variation of the pentagon relation (3) (see [**11**]) that $\mathcal{L}(xy) - \mathcal{L}(x) - \mathcal{L}(y) + \mathcal{L}(\frac{x(1-y)}{1-xy}) + \mathcal{L}(\frac{y(1-x)}{1-xy}) = 0$, we can write the above as

$$= \mathcal{L}(y) + \mathcal{L}(\frac{x(1-y)}{1-xy}) - \mathcal{L}(\frac{y(1-x)}{1-xy}).$$

Since $\frac{x(1-y)}{1-xy} = 1 - \frac{1-x}{1-xy}$ and $\mathcal{L}(1-u) = \pi^2/6 - \mathcal{L}(u)$, the result follows. q.e.d.

4.3. The functions $f(P)$ and $g(T)$.

Corollary 4.7. *(Equation for $f(P)$) Suppose P is a hyperbolic 3-holed sphere of boundary lengths l_i's so that the lengths of M_i is m_i. Let*

$x_i = e^{-l_i}$ *and* $y_i = \tanh^2(m_i/2)$. *Then*

$$f(P) = 4\sum_{i\neq j}\Big[2\mathcal{L}(\frac{1-x_i}{1-x_iy_j}) - 2\mathcal{L}(\frac{1-y_j}{1-x_iy_j}) - \mathcal{L}(y_j) - \mathcal{L}(\frac{(1-x_i)^2y_j}{(1-y_j)^2x_i})\Big]$$

Proof. Recall that by definition and lemma 3.8, $f(P) = \mu(W(P)) = \mu(S(P)) - \mu(V(P)) = 4\pi^2 - [\sum_{i=1}^3(\mu(H(M^{\pm1}) + \mu(H(B^{\pm1})) + 4\sum_{i\neq j}\mu(W(L_i, M_j))]$. Using Bridgeman's theorem and the definition of the lasso function, we obtain,

$$f(P) = 4\pi^2 - 8\Big[\sum_{i=1}^{3}(\mathcal{L}(\frac{1}{\cosh^2(m_i/2)}) + \mathcal{L}(\frac{1}{\cosh^2(p_i/2)})) + \frac{1}{2}\sum_{i\neq j} La(l_i, m_j)\Big].$$

Now since $\frac{1}{\cosh^2(m_j/2)} = 1 - \frac{1}{\tanh^2(m_j/2)} = 1 - y_j$, we have $\mathcal{L}(\frac{1}{\cosh^2(m_i/2)}) = \mathcal{L}(1-y_i) = \pi^2/6 - \mathcal{L}(y_i)$. To find p_k, we cut P open along L_1, L_2, L_3 and B_k to obtain a right-angled hyperbolic pentagon whose edge lengths, listed according to the cyclic order along the boundary, are $l_i/2, m_j, s_1, p_k/2, s_2$. The cosine law for pentagons says $\cosh(p_k/2) = \sinh(l_i/2)\cdot\sinh(m_j)$. Write $\sinh^2(m_j) = 4\frac{\tanh^2(m_j/2)}{(1-\tanh^2(m_j/2))^2}$ and $\sinh^2(l_i/2) = \frac{(1-e^{l_i})^2}{4e^{l_i}}$; then we obtain $\frac{1}{\cosh^2(p_k/2)} = \frac{(1-y_j)^2x_i}{(1-x_i)^2y_j}$. Putting all these into the above identity for $f(P)$ and using proposition 4.6, we obtain the formula for $f(P)$. q.e.d.

The formula (5) for $g(T)$ is computed in exactly the same way using identity (8) and proposition 4.6. We leave the details to the reader.

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