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A CHARACTERIZATION OF SPHERICAL POLYHEDRAL SURFACES

Feng Luo

Abstract

A spherical polyhedral surface is a triangulated surface obtained by isometric gluing of spherical triangles. For instance, the boundary of a generic convex polytope in the 3-sphere is a spherical polyhedral surface. This paper investigates these surfaces from the point of view of inner angles. A rigidity result is obtained. A characterization of spherical polyhedral surfaces in terms of the triangulation and the angle assignment is established.

1. Introduction

1.1. In an attempt to understand the geometric triangulations of closed 3-manifolds with constant sectional curvature metrics, we are led to the study of spherical polyhedral surfaces. These are metrics obtained by taking a finite collection of spherical triangles and identifying their edges in pairs by isometries. In particular, they are spherical cone metrics on a surface together with a geometric triangulation. For instance, the link of a vertex in a 3-dimensional geometric triangulation is a spherical polyhedral surface. In [**Lu1**], we have initiated an approach to find constant curvature metrics on triangulated closed 3-manifolds using dihedral angles as parameters. This leads us to investigate spherical polyhedral surfaces from the inner angle point of view. For a spherical polyhedral surface, its *edge invariant* associates each edge of the triangulation the sum of the two inner angles facing the edge. The main result of the paper gives a characterization of the spherical polyhedral metrics in terms of the edge invariant. To be more precise, we prove that if two spherical polyhedral surfaces with isomorphic triangulations have the same edge invariant, then they are isometric. We also establish an existence result on spherical polyhedral surfaces when the edge invariants take values in $[0, \pi]$. Similar results for Delaunay triangulations of surfaces in the Euclidean or hyperbolic cone metrics have been worked out beautifully by Rivin [**Ri1**] and Leibon [**Le**]. Our approach follows the strategies in [**Ri1**], [**Le**] by using a different energy function.

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1.2. We now set up the frame work. Suppose S is a closed surface and T is a triangulation of the surface. Here by a triangulation we mean the following: take a finite collection of triangles and identify their edges in pairs by homeomorphisms. Let V, E, F be the sets of all vertices, edges and triangles in the triangulation T respectively. If a, b are two simplices in the triangulation T, we use $a < b$ to denote that a is a face of b. The set of *corners* of T is $\{(e, f)|e \in E, f \in F$ so that $e < f\}$ and is denoted by $C(S, T)$. By a *spherical angle structure* on the triangulated surface (S, T) we mean a map $x : C(S, T) \to (0, \pi)$ so that for each $f \in T$ and the three edges e_1, e_2, e_3 of f, the numbers $x_i = x(e_i, f)$, $i = 1, 2, 3$, form the inner angles of a spherical triangle. A *spherical polyhedral metric* on the triangulated surface (S, T) is a map $l : E \to (0, \pi)$ so that for each triangle f and its three edges e_1, e_2, e_3, the three numbers $l_i = l(e_i)$, $i = 1, 2, 3$, form the edge lengths of a spherical triangle. Evidently, given any spherical polyhedral metric, there is a natural spherical angle structure associated to it by measuring its inner angles. One of the goals in the paper is to characterize the set of all spherical polyhedral metrics inside the space of all spherical angle structures. To this end, we introduce the notion of the *edge invariant* D_x of the spherical angle structure x. The edge invariant D_x is the map defined on the set of all edges E so that the its value at an edge is the sum of the two inner angles facing the edge, i.e., $D_x(e) = x(e, f) + x(e, f')$ where $f, f' \in F$ and $e < f, e < f'$ (it may occur that $f = f'$).

Theorem 1.1. *Given any triangulated closed surface and a real valued function D defined on the set of all edges of the triangulation, there is at most one spherical polyhedral metric having D as the edge invariant.*

An interesting consequence of Theorem 1.1 says that if two convex spherical polytopes in S^3 have the same combinatorial triangulation so that their edge invariants are the same, then these two polytopes are isometric in S^3.

Theorem 1.2. *Given any triangulated closed surface and a function $D : E \to (0, \pi)$ so that there is a spherical angle structure having D as the edge invariant, then there exists a spherical polyhedral metric having D as the edge invariant function.*

The existence of spherical angle structures with given edge invariant is a linear programming problem and can be checked algorithmically. The following theorem has been proved by R. Guo [**Gu**].

Theorem 1.3 (Guo [**Gu**]). *Given any triangulated closed surface and any function $D : E \to (0, \pi)$, there is a spherical angle structure having D as the edge invariant if and only if for any subset X of triangles in* *the triangulation,*

$$\pi|X| < \sum_{e \in E(X)} D(e),$$

where $E(X)$ is the set of all edges of triangles in X and $|X|$ is the number of triangles in X.

We remark that a slightly stronger version of Theorem 1.2 can also be established for edge invariants $D(E) \subset (0, \pi]$. See Theorem 2.1.

The space of all spherical polyhedral metrics on (S, T), denoted by $CM(S, T)$ is an open convex polytope of dimension $|E|$, the number of edges. The space of all positive functions on the set of all edges E is denoted by $\mathbf{R}^E_{>0}$. The map $\Pi : CM(S, T) \to \mathbf{R}^E_{>0}$ sending a cone metric to its edge invariant is evidently a smooth map between two open cells of the same dimension. Theorem 1.1 shows that the map is injective (in fact it is a local diffeomorphism). Theorems 1.2 and 1.3 show that the image of the subset of $CM(S, T)$ with edge invariant $D : E \to (0, \pi)$ under Π is a convex polytope. An interesting question is whether the image of Π is an open convex polyhedron in $\mathbf{R}^E_{>0}$. The situation is a bit similar to Thurston's proof of his circle packing theorem for triangulated surfaces of negative Euler characteristic ([**Th**]).

The strategy of proving theorems 1.1 and 1.2 goes as follows. For each spherical triangle, we introduce the concept of *capacity* of the triangle. The capacity is a strictly convex function defined on the space of all spherical triangles parametrized by the inner angles. We define the capacity of a spherical angle structure to be the sum of the capacities of its triangles. Then the capacity defines a strictly convex function on the space $AS(S, T)$ of all spherical angle structures on (S, T). Given an edge invariant $D : E \to (0, \infty)$, we consider the subset $AS(S, T; D)$ of $AS(S, T)$ consisting of all spherical angle structures with D as the edge invariant. We prove that the critical points of the capacity function restricted to the subspace $AS(S, T; D)$ are exactly the spherical polyhedral metrics on (S, T). Since a strictly convex function cannot have more than one critical point, Theorem 1.1 follows. For Theorem 1.2, we show that the capacity function which has a natural continuous extension to the compact closure of $AS(S, T; D)$ cannot achieve its minimal points in the boundary. Thus the minimal point of the capacity exists in $AS(S, T; D)$ when $D : E \to (0, \pi)$.

1.3. The study of geometric structures on triangulated surfaces from the variational point of view has appeared in many works, [**BS**], [**CV**], [**Le**], [**Ri1**] and others. In [**Ri1**] Rivin studied the Euclidean cone metrics and Leibon [**Le**] worked out the Delaunay triangulations for hyperbolic surfaces. Results similar to theorems 1.1, 1.2 and 1.3 were proved for Euclidean and hyperbolic geometric triangulations in [**Ri1**] and [**Le**]. The approach in this paper follows the work in [**Ri1**] and [**Le**]. In [**Ri1**] and [**Le**], the "capacity" of a Euclidean and a hyperbolic triangle was introduced. They are all related to the volume in hyperbolic spaces. It turns out the capacities introduced in [**Ri1**], [**Le**] and in our current work can be summarized in one sentence. Namely, given a spherical, or a Euclidean or hyperbolic triangle in the Riemann sphere considered as the infinity of the hyperbolic 3-space, there are three circles bounding the triangle. The capacity of the triangle is essentially (up to multiplication and addition of constants) the hyperbolic volume of the convex hull of the intersection points of these three circles. The explicit expressions of the capacities are (3.9) and (3.10). For Euclidean triangles, the ideal hyperbolic convex polytopes are ideal tetrehedra; for hyperbolic triangles, they are ideal hyperbolic prisms; and for spherical triangles, they are ideal hyperbolic octahedra. In our case, we first discovered the capacity of a spherical triangle through the derivative of the cosine law and later realized that it is again a hyperbolic volume. It turns out for spherical triangle, Peter Doyle [**Le**] defined a different capacity (see (3.10)). Doyle's capacity of a spherical triangle is the volume of the hyperbolic tetrahedron which is the convex hull of four points consisting of the three vertices of the spherical triangle and the Euclidean center in the Poincare model (where the spherical triangles are bounded by great circles).

From this point of view, given a triangulated surface (S, T), there are five linear programming problems and variational problems associated to the surface. The linear programming problems are related to the angle structures and the variational problems are the critical points of the "capacities". To begin, let us introduce some concepts. An *angle structure* on a triangulated surface (S, T) assigns each corner of (S, T) a number in $(0, \pi)$, called the inner angle. A *hyperbolic* (*or spherical, or Euclidean*) *angle structure* is an angle structure so that each triangle with the angle assignments is hyperbolic (or spherical, or Euclidean). Euclidean angle structures were first defined by Rivin in [**Ri1**] who called them *locally Euclidean structures.* The basic examples of hyperbolic (or spherical, or Euclidean) angle structures are hyperbolic (or spherical, Euclidean) cone metrics with a geometric triangulation by measuring the inner angles. Given an angle structure $x : C(S,T) \to \mathbf{R}_{>0}$, we define its *edge invariant*, denoted by $D_x : E \to \mathbf{R}_{>0}$, to be the sum of two opposite facing angles and its *Delaunay invariant* $\mathcal{D}_x : E \to \mathbf{R}_{>0}$ to be $\mathcal{D}_x(e) = c + d + f + g - a - b$ where a, b are the two angles facing the edge e and c, d, f, g are the four angles having e as an edge. An angle structure is called *Delaunay* if its Delaunay invariant $\mathcal{D}_x$ is non-negative. For Euclidean angle structures, the Delaunay invariant and the edge invariant are related by $2D_x + \mathcal{D}_x = 2\pi$. For a spherical, Euclidean or hyperbolic cone metric with a geometric triangulation, its underlying angle structure is Delaunay if and only if the triangulation satisfies the empty circumcircle property, i.e., the interior of the circumcircle of each triangle does not contain any vertices.

The five linear programming problems associated to the triangulated surface (S, T) are as follows. Namely, the spaces of all hyperbolic angle structures with prescribed edge invariant D or Delaunay invariant $\mathcal{D}$, the spaces of all spherical angle structures with prescribed edge invariant D or Delaunay invariant $\mathcal{D}$, and the space of all Euclidean angle structures with prescribed Delaunay invariant $\mathcal{D}$. We denote these five convex polytopes by $AH(S,T;D)$, $AH(S,T;\mathcal{D})$, $AS(S,T;D)$, $AS(S,T;\mathcal{D})$ and $AE(S,T;\mathcal{D})$. In the recent work of R. Guo [**Gu**], he has found the necessary and sufficient conditions for these spaces to be non-empty. The works of Rivin and Leibon dealt with the spaces $AE(S,T;D)$ and $AH(S,T;\mathcal{D})$ and used the capacity given by formula (3.10). Our paper addresses the space $AS(S,T;D)$ using capacity (3.9). There remain the problems on the existence and uniqueness of constant curvature cone metrics in the spaces $AS(S,T;\mathcal{D})$ and $AH(S,T;D)$. We remark that the associated energies for these problems have been found. Namely, for the space $AS(S,T;\mathcal{D})$, Peter Doyle [**Le**] associated the capacity function given by (3.10) and observed that the critical points of the capacity are exactly the spherical cone metrics with geodesic triangulations. The capacity function for the space $AH(S,T;D)$ is given by (3.9) and it is easy to prove that the critical points of the energy are the hyperbolic cone metrics. However, in both cases the capacity functions are no longer convex or concave. It is a very interesting problem to establish the existence of the critical points of the capacity function in these cases. Furthermore, it is also interesting to know if the critical points are unique in the case of hyperbolic cone metrics in $AH(S,T;D)$.

In our recent work [**Lu1**], we proposed a generalization of the above setup for closed triangulated 3-manifolds by introducing the 3-dimensional angle structure and its volume. The link of a vertex in a 3-dimensional angle structure is a spherical angle structure on the 2-sphere. We tend to think that the Delaunay condition for angle structures in dimension-3 is related to the edge invariant D being in the interval $[0,\pi]$ for surfaces. This is the motivation of the study in this paper. Another motivation of the study is that a spherical angle structure on surface is a 2-dimensional simple model of the 3-dimensional project in [**Lu1**]. Theorems 1.1 and 1.2 give some positive evidences for the 3-dimensional project in [**Lu1**]. In this comparison, the resolution of the Milnor conjecture on volume of simplexes in [**Lu2**] ([**Ri2**] has a new proof) can be considered as the counterpart of Proposition 3.1 in dimension-3.

1.4. The paper is organized as follows. In section 2, we recall some known facts about the derivatives of the cosine laws. We also introduce the capacity function. Some of the basic properties of the capacity function are established. In particular, we prove theorems 1.1 and 1.2 in section 2 assuming two important properties of the capacity function. These two properties are established in sections 3 and 4. In section 3, we show that the capacity function has a continuous extension to the degenerated spherical triangles by relating it to the Lobachevsky function. In section 4, we study the behavior of the derivative of the capacity function at the degenerated spherical triangles.

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2. Spherical Triangles and Proofs of Theorems 1.1 and 1.2

We prove theorems 1.1 and 1.2 assuming several technical properties on spherical triangles in this section. For simplicity, we assume that the indices i, j, k are pairwise distinct in this section.

2.1. Given a spherical, Euclidean or hyperbolic triangle with inner angles x_1, x_2, x_3, let y_1, y_2, y_3 be the edge lengths so that y_i-th edge is facing the angle x_i. Let $\lambda = 0, -1, 1$ be the curvature of the underlying space, i.e., $\lambda = 1$ for spherical triangles, $\lambda = -1$ for hyperbolic triangles and $\lambda = 0$ for Euclidean triangles. The cosine law states that,

$$\cos(\sqrt{\lambda} y_i) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}, \tag{2.1}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Furthermore, the partial derivatives of y_i as a function of $x = (x_1, x_2, x_3)$ are given by the following lemma.

Lemma 2.1. *For any spherical or hyperbolic triangle of inner angles x_i, x_j, x_k and the corresponding edge lengths y_i, y_j, y_k, where $\{i, j, k\} = \{1, 2, 3\}$, the following hold.*

(a) $\partial y_i/\partial x_i = \sin(x_i)/A_{ijk}$ *where* $A_{ijk} = \sin(\sqrt{\lambda} y_i) \sin x_j \sin x_k/\sqrt{\lambda}$ *satisfies* $A_{ijk} = A_{jki}$,

(b) $\partial y_i/\partial x_j = \partial y_i/\partial x_i \cos y_k$.

The proof is a simple exercise in calculus, see for instance [**Lu1**].

The space of all spherical triangles parametrized by its inner angles x_1, x_2, x_3, denoted by M_3, is the open tetrahedron $\{x = (x_1, x_2, x_3) \in (0, \pi)^3 | x_i^* > 0, \sum_{i=1}^3 x_i > \pi\}$ where $x_i^* = 1/2(\pi + x_i - x_j - x_k)$, $\{i, j, k\} = \{1, 2, 3\}$. To see that these inequalities are necessary, we first note that the sum of inner angles of a spherical triangle is larger than π. To see $x_1^* > 0$, we note that if x_1, x_2, x_3 are the inner angles of a spherical triangle A, then $x_1, \pi - x_2, \pi - x_3$ also form the inner angles of a spherical triangle B so that $A \cup B$ forms a region bounded by two great circles intersecting at an angle x_1. It follows that the sum $x_1 + \pi - x_2 + \pi - x_3 > \pi$. This shows $x_1^* > 0$ is necessary. It is not difficult to show that these four inequalities are also sufficient.

Corollary 2.1.

(a) *The differential* 1*-form* $w = \sum_{i=1}^3 \ln\tan(y_i/2)dx_i$ *is closed in the open set* M_3*.*

(b) *The function* $\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^x w$ *is well defined on* M_3 *and is strictly convex.*

(c) *The differential 1-form* $\tilde{w} = \sum_{i=1}^3 \ln\tanh(y_i/2)dx_i$ *is closed in the set* $\mathcal{H}_3 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_i > 0, \quad x_1 + x_2 + x_3 < \pi\}$ *of all hyperbolic triangles.*

(d) *The function* $\tilde{\theta}(x) = \int_{(0,0,0)}^x \tilde{w}$ *is a well defined smooth function on* $\mathcal{H}_3$*.*

Proof. To show part (a), it suffices to prove $\partial(\ln\tan(y_i/2))/\partial x_j$ is symmetric in i, j. By Lemma 2.1, the partial derivative is found to be

$$1/\sin(y_i)\partial y_i/\partial x_j = \cos(y_k)[\sin(x_i)/\sin(y_i)]/A_{ijk}. \tag{2.2}$$

By the sine law, one sees clearly that the partial derivative is symmetric in i, j. Note also that

$$\partial(\ln\tan(y_i/2))/\partial x_i = [\sin(x_i)/\sin(y_i)]/A_{ijk}. \tag{2.3}$$

Since the space M_3 is simply connected, we see that the function $\theta(x)$ is well defined on M_3. To show that the function θ is strictly convex, let us calculate its Hessian matrix $H = [h_{rs}]_{3\times 3}$. By definition, we have $h_{rs} = \partial(\ln\tan(y_r/2))/\partial x_s$. By (2.2) and (2.3), we have $h_{ij} = h_{ii}\cos y_k$ and $h_{11} = h_{22} = h_{33} > 0$ by the sine law. Thus the matrix H is a positive multiplication of the matrix $[a_{rs}]$ where $a_{ij} = \cos y_k$ and $a_{ii} = 1$. For a spherical triangle of edge lengths y_1, y_2, y_3, the matrix $[a_{rs}]$ is always positive definite. Indeed, let v_1, v_2, v_3 be the three unit vectors in the 3-space forming the vertices of the spherical triange; then by definition, a_{rs} is the inner product of v_r with v_s. Thus the matrix $[a_{rs}]$ is positive definite since it is the Gram matrix of three independent vectors.

The verifications of parts (c) and (d) are similar and will be omitted. q.e.d.

2.2. The closure of M_3 in $\mathbf{R}^3$ is given by $\bar{M}_3 = \{x \in [0,\pi]^3 | x_i^* \geq 0, x_1 + x_2 + x_3 \geq \pi\}$. In sections 3 and 4, we will establish the following two properties concerning the function θ. Recall that the Lobachevsky function $\Lambda(t) = -\int_0^t \ln|2\sin u|du$. The function is continuous on the real line $\mathbf{R}$ and is an odd periodic function of period π. See Milnor [**Mi**] for more details.

Proposition 3.1. *The capacity function*

$$\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^{x} \sum_{i=1}^{3} \ln \tan(y_i/2) dx_i$$

is given by the following,

$$\theta(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda((\pi + x_1 + x_2 + x_3)/2) + 4\Lambda(\pi/4),$$

and the capacity function $\tilde{\theta}(x) = \int_{(0,0,0)}^{x} \sum_{i=1}^{3} \ln \tanh(y_i/2) dx_i$ *is given by*

$$\tilde{\theta}(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda((\pi + x_1 + x_2 + x_3)/2).$$

In particular, both θ and $\tilde{\theta}$ have continuous extensions to the closure $\bar{M}_3$ of the moduli space of spherical triangles $M_3 = \{(x_1, x_2, x_3) \in (0,\pi)^3 | x_1 + x_2 + x_3 > \pi$ and $x_i^* > 0$, $i = 1, 2, 3\}$ and the closure of $\{(x_1, x_2, x_3) \in (0,\pi)^3 | x_1 + x_2 + x_3 < \pi\}$. Geometrically, $16\Lambda(\pi/4) - 4\theta(x_1, x_2, x_3)$ is the volume of the hyperbolic ideal octahedron whose vertices are the intersection points of the three circles bounding the spherical triangle (x_1, x_2, x_3).

Proposition 4.1. *For any point $a \in \bar{M}_3 - M_3$ and a point $p \in M_3$, let $f(t)$ be the function $\theta((1-t)a + tp)$ where $t \in (0, 1)$. If a is not one of $(0, 0, \pi)$, $(0, \pi, 0)$, $(\pi, 0, 0)$, (π, π, π), then*

$$\lim_{t \to 0^+} f'(t) = -\infty.$$

If $a \in \{(0, 0, \pi), (0, \pi, 0), (\pi, 0, 0), (\pi, \pi, \pi)\}$, then the limit $\lim_{t\to 0^+} f'(t)$ exists and is a finite number.

In the rest of the section, we prove theorems 1.1 and 1.2 assuming propositions 3.1 and 4.1.

2.3. Given a spherical triangle of inner angles x_1, x_2, x_3, we define its *capacity* to be $\theta(x_1, x_2, x_3)$ where θ is the function introduced in Corollary 2.1. For a spherical angle structure, we define its capacity to be the sum of the capacities of its spherical triangles. To write down the capacity function explicitly, let us fix some notations. First, let us label the set of all corners in (S, T) by integers $\{1, \dots, n\}$. If three corners labeled by a, b, c are of the form $(e_1, f), (e_2, f), (e_3, f)$, we denote it by $\{a, b, c\} \in \Delta$ and call $\{a, b, c\}$ forms a triangle. For a spherical angle structure $x : C(S, T) \to (0, \pi)$, we use x_r to denote the value of x at the r-th corner and consider $x = (x_1, \dots, x_n)$ as a vector in $\mathbf{R}^n$. Under this identification, the space of all spherical angle structures $AS(S,T) = \{x \in (0,\pi)^n |$ whenever r,s,t form a triangle, $(x_r, x_s, x_t) \in M_3\}$ becomes an open convex polyhedron of dimension n. The capacity of the spherical angle structure x, denoted by $\Theta(x)$, is given by

$$\Theta(x) = \sum_{\{r,s,t\} \in \Delta} \theta(x_r, x_s, x_t).$$

Since $\theta(x_1, x_2, x_3)$ is strictly convex, we have

Lemma 2.2. *The capacity function Θ defined on $AS(S,T)$ is a strictly convex function.*

2.4. Given any map $D : E \to (0,\infty)$, we denote $AS(S,T;D)$ the subspace of all spherical angle structures with edge invariant equal to D.

Lemma 2.3. *If $AS(S,T;D)$ is non-empty, then the critical points of $\Theta|_{AS(S,T;D)}$ are exactly those spherical angle structures derived from spherical polyhedral metrics.*

Proof. For simplicity, let us set $G = \Theta|_{AS(S,T;D)}$. Applying the Lagrangian multipliers to Θ on $AS(S,T)$ subject to the set of linear constraints $D_x(e) = D(e)$ for $e \in E$, we see that at a critical point of G, there is a map $C : E \to \mathbf{R}$ (the multipliers) so that, for all indices i,

$$\partial\Theta/\partial x_i = C_e \tag{2.4}$$

where the i-th corner is of the form (e,f), i.e., the i-th corner is facing the edge e. Let the three corners of the triangle f be labeled by i,j,k. Then $\partial\Theta/\partial x_i = \ln\tan(y_i/2)$ where y_i is given by the cosine law (2.1). This shows, by (2.4), that the edge length of e in the spherical triangle of inner angles x_i, x_j, x_k depends only on C_e. In particular, if f' is the second triangle in T having e as an edge, then the length of e calculated in f' in the spherical angle structure is the same as the length of e calculated using f. In summary, we see that there is a well defined assignment of edge lengths $l : E \to (0,\pi)$ so that the assignment on the three edges of each triangle forms the lengths of a spherical triangle and the inner angles induced by l is x.

To see the result in the other direction, suppose we have a point in $AS(S,T;D)$ which is induced from a spherical polyhedral metric $l : E \to (0,\infty)$. We want to show that the point is a critical point of G. Since the constraints $D_x = D$ are linear, the critical points p of G on $AS(S,T;D)$ are the same as those points $q \in AS(S,T;D)$ so that there is a map $C : E \to \mathbf{R}$ satisfying (2.4) at q. Evidently at a spherical angle structure derived from a spherical polyhedral metric $l : E \to \mathbf{R}_{>0}$, we define C_e to be $\ln\tan(l(e)/2)$. Then (2.4) follows. q.e.d.

It is well known that for a smooth strictly convex function f defined on a convex open W set in $\mathbf{R}^n$, the gradient of f is a diffeomorphim from W to an open set in $\mathbf{R}^n$. As a consequence of Lemma 2.2 and Lemma 2.3, we see Theorem 1.1 follows.

2.5. To prove Theorem 1.2, by Proposition 3.1, the function Θ on the space of all spherical angle structure $AS(S,T;D)$ has a continuous extension to the closure $\bar{A}S(S,T;D)$ of $AS(S,T;D)$ in $\mathbf{R}^n$. The closure is evidently compact since it is contained in $[0,\pi]^n$. Take a minimal point a of Θ in the closure $\bar{A}S(S,T;D)$. If the point a is in $AS(S,T;D)$, we are done. We claim that $a \in \partial AS(S,T;D)$ is impossible. Suppose otherwise, there is a triple of indices $\{u',v',w'\}$ so that $(a_{u'},a_{v'},a_{w'})$ is in the boundary of M_3. Take a point $p \in AS(S,T;D)$ and consider the smooth path $\gamma(t) = (1-t)a + tp$ for $t \in (0,1]$ in $AS(S,T;D)$. Let $g(t) = \Theta(\gamma(t))$. We have $g(t) \geq g(0)$ for all $t > 0$ by the choice of the point a. Thus, $\liminf_{t\to 0^+} dg/dt \geq 0$. But, by proposition 4.1, we have

$$\lim_{t\to 0^+} dg/dt = -\infty. \tag{2.5}$$

This produces a contradiction. Here is the more detailed argument to see (2.5).

Let Δ_1 be the set of all triples of indices $\{\{u,v,w\}|$ so that $\{u,v,w\} \in \Delta$ and $(a_u,a_v,a_w) \in \partial M_3\}$ and $\Delta_2 = \Delta - \Delta_1$. Then the function g can be written as

$$g(t) = \sum_{\{u,v,w\}\in\Delta_1} \theta(x_u(t),x_v(t),x_w(t)) + \sum_{\{u,v,w\}\in\Delta_2} \theta(x_u(t),x_v(t),x_w(t))$$

where $x(t) = x(\gamma(t))$. The derivative $g'(t)$ can be expressed as

$$\begin{aligned} g'(t) = &\sum_{\{u,v,w\}\in\Delta_1} d/dt[\theta(x_u(t),x_v(t),x_w(t))] \\ &+ \sum_{\{u,v,w\}\in\Delta_2} d/dt[\theta(x_u(t),x_v(t),x_w(t))]. \end{aligned}$$

Note that since the edge invariant D is assumed to be strictly less than π, if $\{u,v,w\}$ is in Δ_1, then the triple (a_u,a_v,a_w) is in $\partial M_3 - \{(0,\ 0,\ \pi),\ (0,\ \pi,\ 0),\ (\pi,\ 0,\ 0),\ (\pi,\ \pi,\ \pi)\}$. Thus by Proposition 4.1, as t tends to 0, each term in the first sum tends to $-\infty$. Each term in the second sum tends to a finite number as t tends to 0. Thus we see (2.5) holds.

2.6. The above proof in fact shows the following stronger result. *A cycle* in the triangulated surface (S,T) is an ordered collection of edges and triangles $\{e_1, f_1, e_2, f_2, \ldots, e_n, f_n\}$ so that e_i and e_{i+1} are edges in f_i and e_1, e_n are edges of f_n. An edge invariant assignment D is said to contain a $\{0,0,\pi\}$-cycle if there is a cycle of edges and a point $a \in \partial AS(S,T;D)$ so that $D_a(e_i) = \pi$ and the inner angles of each f_i in a are $0, 0, \pi$.

Theorem 2.1. *Given any triangulated surface and any edge invariant function $D : E \to (0, \pi]$ which contains no $\{0, 0, \pi\}$-cycles, if there is a linear spherical structure having D as the edge invariant, then there exists a spherical polyhedral metric having D as the edge invariant function.*

The proof is evident.

3. Continuous Extension of the Capacity Function

We show that the capacity of spherical triangles extends continuously to the degenerated triangles. For the rest of the section, we take a spherical or hyperbolic triangle of inner angles x_1, x_2, x_3 and edge lengths y_1, y_2, y_3 so that y_i-th edge is facing the x_i-th inner angle. We use $x = (x_1, x_2, x_3)$ and $x_i^* = 1/2(\pi + x_i - x_j - x_k)$. As a convention, we assume the indices $\{i, j, k\} = \{1, 2, 3\}$. The main result of the section is the following.

Proposition 3.1. *The capacity function*

$$\theta(x) = \int_{(\pi/2,\pi/2,\pi/2)}^{x} \sum_{i=1}^{3} \ln \tan(y_i/2) dx_i$$

is given by the following:

$$\theta(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda((\pi + x_1 + x_2 + x_3)/2) + 4\Lambda(\pi/4), \tag{3.1}$$

and the capacity function $\tilde{\theta}(x) = \int_{(0,0,0)}^{x} \sum_{i=1}^{3} \ln \tanh(y_i/2) dx_i$ is given by

$$\tilde{\theta}(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda((\pi + x_1 + x_2 + x_3)/2). \tag{3.2}$$

In particular, both θ and $\tilde{\theta}$ have continuous extensions to the closure $\bar{M}_3$ of the moduli space of spherical triangles $M_3 = \{(x_1, x_2, x_3) \in (0, \pi)^3 | x_1 + x_2 + x_3 > \pi$ and $x_i^* > 0,\ i = 1, 2, 3\}$ and the closure of $\{(x_1, x_2, x_3) \in (0, \pi)^3 | x_1 + x_2 + x_3 < \pi\}$. Geometrically, $16\Lambda(\pi/4) - 4\theta(x_1, x_2, x_3)$ is the volume of the hyperbolic ideal octahedron whose vertices are the intersection points of the three circles bounding the spherical triangle (x_1, x_2, x_3).

Proof. The proof is a straight forward computation using the cosine law. Recall that the cosine law (2.1) says

$$\cos y_i = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}.$$

Using the summation formulas for cosine function that

$$\begin{aligned}\cos(a+b) &= \cos a\cos b - \sin a\sin b,\\ \cos(a-b) &= \cos a\cos b + \sin a\sin b,\\ \cos a + \cos b &= 2\cos((a+b)/2)\cos((a-b)/2),\\ \cos a - \cos b &= 2\sin((a+b)/2)\sin((b-a)/2),\end{aligned}$$

we can rewrite the cosine law as one of the following:

$$\cos y_i - 1 = 2\frac{\sin x_i^* \cos((x_i+x_j+x_k)/2)}{\sin x_j \sin x_k}, \tag{3.3}$$

and

$$\cos y_i + 1 = 2\frac{\sin x_j^* \sin x_k^*}{\sin x_j \sin x_k}. \tag{3.4}$$

In particular,

$$\frac{1-\cos y_i}{1+\cos y_i} = -\frac{\sin x_i^* \cos((x_i+x_j+x_k)/2)}{\sin x_j^* \sin x_k^*}. \tag{3.5}$$

However, we also have the trigonometric identity,

$$\tan^2(u/2) = \frac{1-\cos u}{1+\cos u}.$$

This shows that the cosine law for spherical triangles can be written as

$$\tan^2(y_i/2) = -\frac{\sin x_i^* \cos((x_i+x_j+x_k)/2)}{\sin x_j^* \sin x_k^*}. \tag{3.6}$$

By the same calculation and using $\tanh^2(u/2) = (\cosh u - 1)/(\cosh u + 1)$, we obtain the cosine law for hyperbolic triangles as

$$\tanh^2(y_i/2) = \frac{\sin x_i^* \cos((x_i+x_j+x_k)/2)}{\sin x_j^* \sin x_k^*}. \tag{3.7}$$

Since by definition $\partial\theta/\partial x_i = \ln\tan(y_i/2)$, by (3.6), we have

$$\begin{aligned}\partial\theta/\partial x_i = 1/2[&\ln\sin x_i^* - \ln\sin x_j^* - \ln\sin x_k^*\\ &+ \ln(|\sin((x_1+x_2+x_3+\pi)/2)|)].\end{aligned} \tag{3.8}$$

Since the function $F(x_1, x_2, x_3)$ given by the right hand side of (3.1) has the partial derivative

$$\begin{aligned}\partial F/\partial x_i &= 1/2[\ln(2\sin x_i^*) - \ln(2\sin x_j^*) - \ln(2\sin x_k^*)\\ &\qquad + \ln(2|\sin((x_1+x_2+x_3+\pi)/2)|)]\\ &= 1/2[\ln\sin x_i^* - \ln\sin x_j^* - \ln\sin x_k^*\\ &\qquad + \ln(|\sin((x_1+x_2+x_3+\pi)/2)|)],\end{aligned}$$

we see that $\partial F/\partial x_i = \partial\theta/\partial x_i$. In particular, these two functions differ by a constant on M_3. Since $\theta(\pi/2, \pi/2, \pi/2) = 0 = F(\pi/2, \pi/2, \pi/2)$, the result follows. In particular, we see that θ has a continuous extension to the 3-space $\mathbf{R}^3$. The same calculation using (3.7) verifies (3.2).

Since three great circles bounding a spherical triangle decompose the 2-sphere into eight spherical triangles, it follows that the convex hull of the six intersection points of three circles is the union of eight hyperbolic tetrahedra. Each of them has three vertices at the sphere at infinity and one vertex the Euclidean center. By (3.1) and known formula for volume of hyperbolic tetrahedra with three vertices at the sphere at infinity [**Vi**], i.e., (3.10) below, we see that $16\Lambda(\pi/4) - 4\theta(x_1, x_2, x_3)$ is the volume of the hyperbolic octahedron which is the convex hull of the six points. q.e.d.

3.1. Remarks. Proposition 3.1 shows that the functions $\theta(x_1, x_2, x_3)$ and $\tilde{\theta}(x_1, x_2, x_3)$ are essentially $W(x)$ where

$$W(x_1, x_2, x_3) = -\sum_{i=1}^{3} \Lambda(x_i^*) - \Lambda((\pi + x_1 + x_2 + x_3)/2). \tag{3.9}$$

This function $W(x)$ is closely related to

$$V(x_1, x_2, x_3) = \sum_{i=1}^{3} (\Lambda(x_i) + \Lambda(x_i^*)) - \Lambda((\pi + x_1 + x_2 + x_3)/2). \tag{3.10}$$

For a spherical triangle x, the function $V(x)/2$ is known to be the hyperbolic volume of a hyperbolic tetrahedron with three vertices at the sphere at infinity so that the link at the finite vertex is the spherical x (see [**Vi**], also [**Le**]). For a hyperbolic triangle (x_1, x_2, x_3), $V(x)$ is the volume of the convex hull of the intersection points of circles bounding the triangle. This is the function used by Leibon as the capacity. For a Euclidean triangle x, $V(x)/2$ is the volume of the hyperbolic ideal tetrahedron with dihedral angles $x_1, x_1, x_2, x_2, x_3, x_3$. Peter Doyle [**Le**] noticed that $V(x)$ is not concave on M_3 and took $V(x)$ as a different capacity for spherical triangles. He observed that the critical point of this capacity for spherical angle structures with prescribed Delaunay invariant are the spherical cone metrics. On the other hand, $V(x)$ is concave in the set $\{(x_1, x_2, x_3) \in [0, \pi]^3 | x_1 + x_2 + x_3 \leq \pi\}$ ([**Le**]). For a spherical triangle x, $-4W(x)$ is the volume of the ideal hyperbolic octahedron whose vertices are the intersection points of the circles bounding the triangle. For a Euclidean triangle x, we have $W(x) = -V(x)/2$. We do not know the geometric meaning of $W(x)$ for a hyperbolic triangle x. The other related works are [**CV**] and [**BS**].

3.2. It can be shown that functions W and V in (3.9) and (3.10) are the only functions, up to scaling and adding of linear functions, with the required properties. To be more precise, if $F(x_1, x_2, x_3)$ is a smooth function of the inner angles (x_1, x_2, x_3) of a triangle so that $\partial F/\partial x_i$ is a universal function of the edge length y_i, then $F = c_1 W + c_2(x_1 + x_2 + x_3) + c_3$ for some constants c_1, c_2 and c_3. Similarly, if $F(x_1, x_2, x_3)$ is a smooth function so that $\partial F/\partial x_i^*$ is a universal function of y_i, then $F = c_1 V + c_2(x_1 + x_2 + x_3) + c_3$ for some constants c_1, c_2 and c_3. This shows that if one intends to find the constant curvature cone metrics in $AS(S, T; D)$, $AS(S, T; \mathcal{D})$, $AH(S, T; D)$ or $AH(S, T; \mathcal{D})$ by a variational method so that the energy is contructed locally by summing up the energies of the triangles, then all the possible candidates of the energies are $c_1 V + c_2(x_1 + x_2 + x_3) + c_3$ and $c_1 W + c_2(x_1 + x_2 + x_3) + c_3$.

4. Degeneration of Spherical Triangles

The goal of this section is to understand how a sequence of spherical triangles degenerates and to understand the behavior of the derivatives of the capacity on the sequence of degenerated spherical triangles. Recall that the moduli space M_3 of spherical triangles is an open regular tetrahedron in the 3-space. The closure $\bar{M}_3$ of M_3 is the closed tetrahedron. We call a point in the boundary $\partial M_3 = \bar{M}_3 - M_3$ a *degenerated* spherical triangle (with respect to inner angles). The goal of the section is to prove,

Proposition 4.1. *For any point $a \in \bar{M}_3 - M_3$ and a point $p \in M_3$, let $f(t) = \theta((1-t)a + tp)$ where $t \in [0, 1]$. If a is not equal to any of the points $(0, 0, \pi), (0, \pi, 0)$, $(\pi, 0, 0)$, (π, π, π), then*

$$\lim_{t \to 0^+} f'(t) = -\infty. \tag{4.1}$$

If $a \in \{(0, 0, \pi), (0, \pi, 0), (\pi, 0, 0), (\pi, \pi, \pi)\}$, then the limit $\lim_{t \to 0^+} f'(t)$ exists and is a finite number.

4.1. The moduli space M_3 of spherical triangles is given by $\{x \in (0, \pi)^3 | x_i^* > 0, x_1 + x_2 + x_3 > \pi\}$ which is the open regular tetrahedron inscribed in the standard cube $[0, \pi]^3$. The four vertices of the tetrahedron are $v_1 = (\pi, 0, 0), v_2 = (0, \pi, 0), v_3 = (0, 0, \pi)$ and $v_4 = (\pi, \pi, \pi)$, and its four triangular faces lie in the planes given by the linear equations $x_i^* = 0$, i=1,2,3, and $x_1 + x_2 + x_3 = \pi$ respectively. We now decompose the boundary ∂M_3 into a disjoint union of six parts, denoted by I, II, III, IV, V and VI, as follows. Here I is the open triangle $\Delta v_1 v_2 v_3$. Part II is the union of the three open triangles $\Delta v_4 v_i v_j$ where $\{i, j\} \subset \{1, 2, 3\}$. Part III is the union of three open edges of the triangle I, i.e., III is the union of open intervals $v_i v_j$ where $\{i, j\} \subset \{1, 2, 3\}$. Part IV is the union of the three open intervals $v_4 v_i$. Part V is $\{(\pi, \pi, \pi)\}$. Part VI is $\{(0, 0, \pi), (0, \pi, 0), (\pi, 0, 0)\}$. The algebraic description of them is as follows.

$$\begin{aligned}
\mathrm{I} &= \{a \in (0,\pi)^3 | a_1 + a_2 + a_3 = \pi, a_i^* \in (0,\pi)\},\\
\mathrm{II} &= \cup_{i=1}^3 \{a \in (0,\pi)^3 | a_i^* = 0, a_j^*, a_k^* \in (0,\pi), a_1 + a_2 + a_3 > \pi\},\\
\mathrm{III} &= \cup_{i=1}^3 \{a \in [0,\pi)^3 | a_i^* = 0, a_j^*, a_k^* \in (0,\pi), a_1 + a_2 + a_3 = \pi\},\\
\mathrm{IV} &= \cup_{i=1}^3 \{a \in (0,\pi]^3 | a_i = \pi, a_j^* = a_k^* = 0, a_i^* \in (0,\pi), a_1 + a_2 + a_3 > \pi\},\\
\mathrm{V} &= \{a \in [0,\pi]^3 | a_i^* = 0, i = 1,2,3, a_1 + a_2 + a_3 = 3\pi\},\\
\mathrm{VI} &= \cup_{i=1}^3 | a_j^* = a_k^* = 0, a_i^* = 2\pi, a_1 + a_2 + a_3 = \pi\}.
\end{aligned}$$

As usual, we have used the convention that $\{i,j,k\} = \{1,2,3\}$ above.

4.2. We now prove Proposition 4.1 by considering the limit $\lim_{t\to 0^+} f'(t)$ according to the type of the degenerated spherical triangle a. Let $a = (a_1,a_2,a_3)$, $p = (p_1,p_2,p_3)$ and let $x_i = x_i(t) = (1-t)a_i + tp_i$. We use $y_i = y_i(t)$ to denote the corresponding edge lengths of the triangle $x = (x_1,x_2,x_3)$. Note that $x_i \to a_i$ and $x_i^* \to a_i^*$ as time t tends to 0, also $dx_i/dt = p_i - a_i$. By definition,

$$f'(t) = \sum_{i=1}^{3} \ln\tan(y_i(t)/2)(p_i - a_i). \tag{4.2}$$

By (3.8), we write,

$$\ln\tan(y_i/2) = S(x_i^*) - S(x_j^*) - S(x_k^*) + C(x) \tag{4.3}$$

where $S(u) = 1/2\ln\sin(u)$ and $C(x) = 1/2\ln|\cos((x_1+x_2+x_3)/2)|$. Assume in the following computation that (i,j,k) is a cyclic permutation of $(1,2,3)$, or, more precisely, we take $j = i+1, k = i+2$ where indices are counted modulo 3. Substitute (4.3) into (4.2), and we obtain,

$$\begin{aligned}
f'(t) &= \sum_{i=1}^{3} (S(x_i^*) - S(x_{i+1}^*) - S(x_{i+2}^*) + C(x))(p_i - a_i) \\
&= \sum_{i=1}^{3} (S(x_i^*) - S(x_{i+1}^*) - S(x_{i+2}^*))(p_i - a_i) \\
&\quad + C(x)\left(\sum_{i=1}^{3} p_i - \sum_{i=1}^{3} a_i\right) \\
&= 2\sum_{i=1}^{3} S(x_i^*)(p_i^* - a_i^*) + C(x)\left(\sum_{i=1}^{3} p_i - \sum_{i=1}^{3} a_i\right).
\end{aligned} \tag{4.4}$$

We now discuss the limit of $f'(t)$ as t tends to 0 according to the type of the degenerated triangle a.

4.3. Case 1, the triangle a has type I, i.e., $a_1 + a_2 + a_3 = \pi$ and $a_i, a_i^* \in (0, \pi)$. In particular, $\lim_{t\to 0^+} S(x_i^*) = S(a_i^*)$ exists in $\mathbf{R}$. Thus the unbounded term in (4.4) is the last term $C(x)(\sum_{i=1}^3 p_i - \sum_{i=1}^3 a_i)$ which tends to $-\infty$ due to $a_1 + a_2 + a_3 = \pi$, $p_1 + p_2 + p_3 > \pi$ and $\lim_{t\to 0^+} C(x) = -\infty$. This shows the proposition for case 1.

4.4. Case 2, the triangle a has type II. For simplicity, we may assume that $\pi + a_1 = a_2 + a_3$, i.e., $a_1^* = 0$, $a_i, a_2^*, a_3^* \in (0, \pi)$, and $a_1 + a_2 + a_3 \in (\pi, 3\pi)$. Then the unbounded term in (4.4) is $2S(x_1^*)(p_1^* - a_1^*)$. All other terms are bounded since the $\lim_{t\to 0^+} S(x_i^*(t)) = S(a_i^*)$ is finite for $i = 2, 3$ and $\lim_{t\to 0^+} C(x) = 1/2 \ln|\cos(a_1 + a_2 + a_3)/2)|$ is also finite. On the other hand, $p_1^* > 0$, $a_1^* = 0$ and $\lim_{t\to 0^+} S(x_1^*) = -\infty$, and we see that $\lim_{t\to 0^+} f'(t) = -\infty$.

4.5. Cases 3,4, the triangle a has type III or IV. In these cases, exactly two of the four equations $a_1^* = 0, a_2^* = 0, a_3^* = 0$, or $a_1 + a_2 + a_3 = \pi$ hold. To be more precise, in the case III, we may assume without loss of generality that $a_1^* = 0$, $\sum_{i=1}^3 a_i = \pi$, $a_2^*, a_3^* \in (0, \pi)$. Thus, in (4.4), exactly two terms, $2S(x_1^*)(p_1^* - a_1^*)$ and $C(x)(\sum_{i=1}^3 p_i - \pi)$ tend to $-\infty$ as t approaches 0. The other two terms remain bounded. Thus the result follows.

In the case IV, we may assume for simplicity that $a_1^* = a_2^* = 0$ and $\sum_{i=1}^3 a_i > \pi$ and $a_3^* > 0$. Then due to $0 < \sum_{i=1}^3 a_i^* = 3\pi - \sum_{i=1}^3 a_i$, we have $\sum_{i=1}^3 a_i < 3\pi$. This shows that $\lim_{t\to 0} C(x) = C(a)$ is finite. Thus in (4.4), there are again exactly two terms, namely $2S(x_1^*)(p_1^* - a_1^*)$ and $2S(x_2^*)(p_2^* - a_2^*)$ tend to $-\infty$ as t approaches 0. The other two terms remain bounded. Thus the result follows again.

4.6. Case 5, the triangle a is an equator (π, π, π). In this case $a_i^* = 0$ and $a_1 + a_2 + a_3 = 3\pi$. Using (4.2) and (4.3), we have,

$$
\begin{aligned}
(4.5) \qquad f'(t) &= \sum_{i=1}^3 (S(x_i^*) - S(x_j^*) - S(x_k^*) + C(x))(p_i - \pi) \\
&= \sum_{i=1}^3 [(S(x_i^*) - S(x_j^*)) + (C(x) - S(x_k^*))](p_i - \pi).
\end{aligned}
$$

We note that both limits $\lim_{t\to 0^+}(S(x_i^*) - S(x_j^*))$ and $\lim_{t\to 0^+}(C(x) - S(x_k^*))$ exist in $\mathbf{R}$. Indeed, by definition,

$$
\begin{aligned}
x_i^* &= 1/2[\pi + x_i - x_j - x_k] \\
&= 1/2[\pi + (1-t)(a_i - a_j - a_k) + t(p_i - p_j - p_k)] \\
&= 1/2[\pi + (1-t)(-\pi) + t(p_i - p_j - p_k) \\
&= 1/2(t(p_i - p_j - p_k + \pi)) = tp_i^*,
\end{aligned}
$$

$$\begin{aligned} x_1 + x_2 + x_3 &= t(p_1 + p_2 + p_3) + (1 - t)3\pi \\ &= 3\pi + t(p_1 + p_2 + p_3 - 3\pi). \end{aligned}$$

Thus, $S(x_i^*) - S(x_j^*) = 1/2(\ln \sin(tp_i^*) - \ln \sin(tp_j^*))$ which tends to $1/2(\ln \sin p_i^* - \ln \sin p_j^*)$ as t tends to 0. Similarly, $C(x) - S(x_k^*)$ tends to the finite number $1/2(\ln |\sin((p_1 + p_2 + p_3 - 3\pi)/2)| - \ln(\sin(p_k^*)))$.

4.7. Case 6, the triangle a is of type VI. For simplicity, we assume that $a = (\pi, 0, 0)$. Thus $a_1 + a_2 + a_3 = \pi$, $a_1^* = 2\pi$, $a_2^* = a_3^* = 0$. We use (4.5) to calculate the limit $\lim_{t\to 0} f'(t)$. The calculation is exactly the same as that of case 5. Indeed, each of the four terms $S(x_i^*)$ and $C(x)$ tends to $-\infty$ as t approaches zero. On the other hand, by the same argument as in 4.6, both of the limits $\lim_{t\to 0^+} S(x_i^*)/S(x_k^*)$ and $\lim_{t\to 0^+} S(x_i^*)/C(x)$ are finite. Thus the result follows.

This ends the proof of Proposition 4.1.

4.8. Remark. We give a geometric interpretation of the stratification I, II, ..., VI of the degenerated triangles. The type I boundary point $x \in \{x \in (0, \pi)^3 | x_1 + x_2 + x_3 = \pi\}$ corresponds to the "Euclidean triangle". Geometrically, it represents a point which is the limit of spherical triangles shrinking to a point so that its inner angles tend to three numbers in $(0, \pi)$. In particular, if one defines the edge length $y_i = 0$ for these triangle, the cosine law (2.1) still makes sense in terms of taking limit. The type II points in $\{x \in (0, \pi)^3 | x_1 + x_2 + x_3 > \pi, x_i^* = 0, x_j^* > 0, x_k^* > 0\}$ correspond to the other codimension-1 faces. They represent the "exceptional Euclidean triangles". Geometrically, it is the limit of sequence of spherical triangles expanding to a union of two geodesics from a point to its antipodal point so that the inner angles tend to three numbers in $(0, \pi)$. In particular, the edge lengths are $y_i = 0, y_j = y_k = \pi$ and a type II triangle has two vertices. Note that the edge length function y_i extends continuously on the set $M_3 \cup I \cup II$. There are two types of codimension-2 faces. The first type, denoted by III, consists of three open edges of the form $\{x = (x_1, x_2, x_k) \in [0, \pi)^3 | x_i = 0, x_j, x_k > 0 \text{ and } x_j + x_k = \pi\}$. This is a further degeneration of "Euclidean triangles". The second type of codimension-2 face, denoted by IV, consists of the three open edges of the form $\{x = (x_1, x_2, x_3) \in (0, \pi]^3 | x_i = \pi, x_j = x_k \in (0, \pi)\}$. Geometrically, it corresponds to a degenerated spherical triangle so that two of its three distinct vertices are antipodal points. Due to the location of the third vertex (of inner angle π), the length function y_r does not extend continuously from M_3 to $M_3 \cup IV$. Finally, there are two types of vertices. The first type, denoted by V, is the point (π, π, π) corresponding to the equator, and the second type, denoted by VI, consists of $(0, 0, \pi), (0, \pi, 0), (\pi, 0, 0)$ corresponding to a degenerated triangle whose three distinct vertices lie in a great circular arc of length at most π.

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Department of Mathematics
Rutgers University
Piscataway, NJ 08854
E-mail address: fluo@math.rutgers.edu