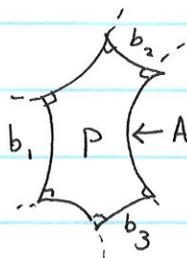
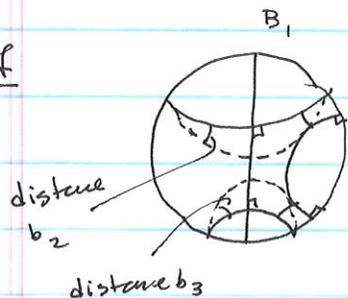


### Lecture 11. Fenchel-Nielsen Coordinate

Key lemma  $\forall b_1, b_2, b_3 > 0, \exists!$  right-angled hyperbolic hexagon  $P$  whose three non-pairwise adjacent edges have lengths  $b_1, b_2, b_3$ .

Pf



the existence

the uniqueness

$P'$

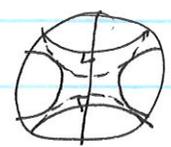
↓  
due the uniqueness of the geodesic  $A_i$  of distance  $b_2, b_3$

Uniqueness If  $l_1, l_2$  are two disjoint geodesics of positive distance apart, then  $\exists$  exactly two geodesics  $A_i$  s.t.  $d(A_i, l_i) = b_i \quad i=2,3$ .

Pf



transferring common perpendicular



symmetric

□

Now Add Gauss-Bonnet Here

Lecture 11 Fenchel-Wielsen Coordinate

Corollary (a) The sum of the inner angles of a hyperbolic  $n$ -gon  $< (n-2)\pi$

In fact its area =  $(n-2)\pi - \sum_{i=1}^n \alpha_i$

(b) If  $\Sigma_g$  is a closed genus  $g$  hyperbolic surface, then its area is

$$-2\pi \chi(\Sigma_g) = \int_{\Sigma_g} -dA \quad (K_g = -1) \quad (\text{Gauss-Bonnet})$$

Pf (a) Decompose the polygon into triangles, total of  $(n-2)$  of them



so

$$\text{Area} = \sum_{\Delta_i} (\pi - \alpha_i - \beta_i - \gamma_i) = (n-2)\pi - \sum (\text{inner angles})$$



(b) triangulated the surface  $(\Sigma, d)$  into hyperbolic

triangles, say there are  $V, E, F$  many vertices, edges and triangles

let  $\alpha_i$ 's be the set of all inner angles

$\beta_i$   
 $\gamma_i$

$\Delta_1, \dots, \Delta_F$



$$\text{Area}(\Sigma) = \sum_{\Delta_i} \text{Area}(\Delta_i) = \sum_{\Delta_i} (\pi - \alpha_i - \beta_i - \gamma_i)$$

$$= F \cdot \pi - \sum (\alpha_i + \beta_i + \gamma_i)$$

$$= F \cdot \pi - \sum_{\text{vertices } v} \left( \sum_{\alpha_i \text{ at } v} \alpha_i \right)$$

$$= F \cdot \pi - 2\pi \cdot V$$

$$= (-2\pi) \left( V - \frac{F}{2} \right) = (-2\pi) \left( V - \frac{3F}{2} + F \right) \quad \text{But } E = \frac{3F}{2}$$

$$= -2\pi \chi(\Sigma)$$



$\subset \mathbb{H}$

□

Ex Between any two geodesics  $l_1, l_2$  there exists at most one geodesic  $g$  perpendicular to both

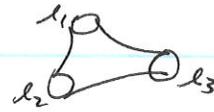


$\Rightarrow$  if not  $\Rightarrow \exists$  right angled quadrilateral, or



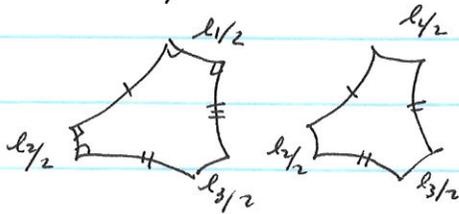
triangles of two inner angles  $\frac{\pi}{2}, \frac{\pi}{2}$ .

Thus  $\forall l_1, l_2, l_3 > 0 \exists!$  hyperbolic 3-holed sphere w/ geodesic boundary of lengths  $l_1, l_2, l_3$ .

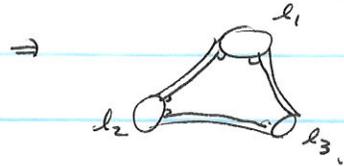


Pf Existence, take two right-angled

hexagons of edge lengths  $l_1/2, l_2/2, l_3/2$



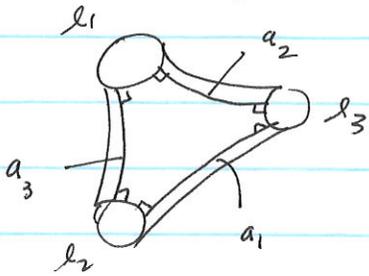
isometrically glue them along the other three edges



(Metric double)

Uniqueness Suppose  $P$  is such a hyperbolic 3-holed sphere

let  $a_1, a_2, a_3$  be the shortest paths between pairs of boundaries

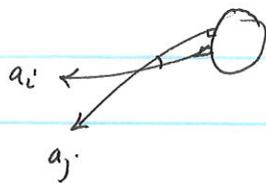


claim (1)  $a_i \perp l_j$ 's (shortest)

(2)  $a_i$  simple, no self intersection  $\neq \emptyset$  (shortest)

(3)  $a_i \cap a_j = \emptyset$  : if  $a_i \cap a_j \neq \emptyset$

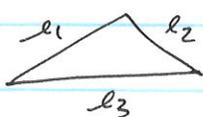
$\Rightarrow \exists$  a hyperbolic triangle  $\Delta$  of interior angles  $\frac{\pi}{2}, \frac{\pi}{2}, \alpha$ .



$\Rightarrow$  cut  $P$  open along  $a_1, a_2, a_3$  we obtain two right-angled hexagons  $H_+, H_-$ .  $H_+$  is isometric to  $H_-$  : they have the same edges lengths at three non-adjacent edges

$\Rightarrow P$  is a metric double of  $H_+$   $\square$

BM. This is very similar to,  $\forall l_1, l_2, l_3 \geq 0 \quad l_1 + l_2 > l_3 \Rightarrow \exists$  triangle of edge lengths  $l_1, l_2, l_3$ .



Lecture 12 Bers's Constant.

Recall  $\text{Area}(B_r(p)) = 4\pi \sinh^2(\frac{r}{2})$

Thm (Bers) There exists a constant  $C_g$  s.t.  $\forall$  hyperbolic metric  $d$  on  $\Sigma_g$  surface of genus  $g$  has a pants decomposition so that each decomposition curve has length  $\leq C_g$ .

Notation Given an essential loop  $d$ ,  $d^*$  is the closed geodesic  $d^* \simeq d$ .

It is not known the best constant  $C_g$  ( $C_g \leq g \ln g$ )

*w/ or without boundary*

Def  $(M, d)$  Riemannian mfd  $p \in M, r > 0$  is called an injectivity radius if the exponential map  $\text{Exp}: T_p M \rightarrow M$  is 1-1 on  $\overset{p}{B}_r(0)$

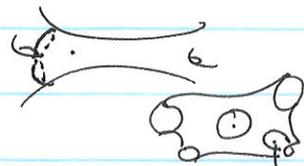
$\Rightarrow B_r(p)$  is diffeomorphic (homeomorphic) to  $B_r(0)$

Eg: Cylinder

$S^1 \times \mathbb{R}$



$r = \pi \cdot \max$

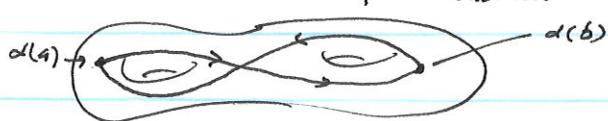


2-corned

Lemma 1. If  $d: S^1 \rightarrow \mathbb{H}/\Gamma = \Sigma_g$  is a geodesic ~~loop~~ <sup>i.e.</sup>  $\exists a \neq b \in S^1$  s.t.

$d|_{S^1 - \{a, b\}}$  is geodesic; then  $d \ncong pt$  on  $\mathbb{H}/\Gamma$   
cuts of two distinct

Eg



May have two corners:

Pf If not, then  $d \cong pt \Rightarrow \exists$  an extension  $F: \mathbb{D} \rightarrow \mathbb{H}/\Gamma$  of  $d$  ( $\ncong pt$ )

By the lifting thm,  $F$  lifts to  $\tilde{F}: \mathbb{D} \rightarrow \mathbb{H}$  s.t.  $\tilde{F}|_{\partial \mathbb{D} - \{a, b\}}$  is a geodesic:

$\Leftrightarrow \exists$  two geodesics in  $\mathbb{H}$  intersecting at two points

But that is impossible!



□

Corollary A geodesic loop  $d: [0, 1] \rightarrow \Sigma_g$  s.t.,  $d(0) = d(1)$  +  $d|_{(0, 1)}$  geodesic  
 (May not be a closed geodesic).  $\Rightarrow d \ncong pt$ .

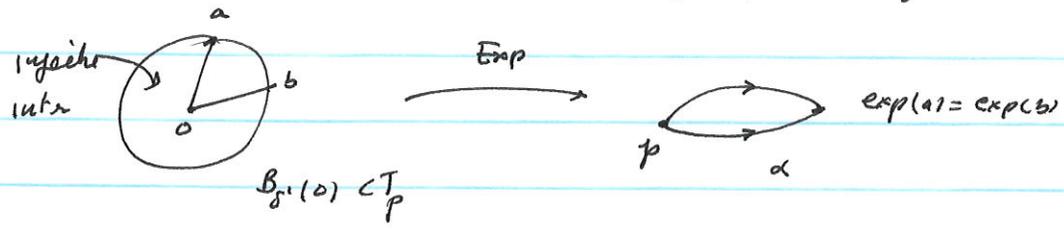


lecture 12

Lemma 2. For  $(\Sigma_g, d)$ , let  $L =$  the length of the shortest geodesic in  $\Sigma_g$   $\delta = L/2$   
(Why does it exist?). Then  $\delta$  is an injectivity radius for  $\forall p \in \Sigma_g$ .

pf

If not,  $\exists p \in \Sigma_g$  and  $\delta' \leq \delta$  s.t.  $\text{Exp}: T_p \Sigma_g \rightarrow \Sigma_g$  is not injective on  $B_{\delta'}(0)$ .



Thus, the large loop  $\alpha$  has length  $\leq L$  is essential

$\Rightarrow \alpha \simeq \alpha^*$  closed geodesic s.t.  
 $l(\alpha^*) < l(\alpha) \leq 2\delta' \leq 2\delta = L$

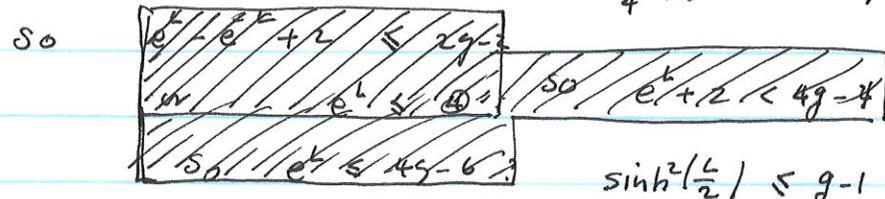
Contradicting the choice of  $L$ .

Corollary 3 The length of the shortest geodesic in  $(\Sigma_g, d) \leq 2 \sinh^{-1}(\sqrt{g-1})$   
 $L \leq 2 \ln(4g-2) = \ln(-2\chi(\Sigma) - 2)$

pf If otherwise,  $L \geq 2 \ln(4g-2)$

then  $\text{Area}(\Sigma_g, d) = 4\pi(g-1) \geq \text{Area}(B_{L/2}(p))$

$\leq 4\pi \sinh^2(L/2)$   
 $= \frac{4\pi}{4} (e^L + e^{-L} + 2) = (e^L + e^{-L} + 2)\pi$



$\frac{e^L + e^{-L} + 2}{4} \leq g-1$

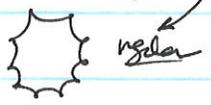
so  $e^L \leq 4g-2$

$L \sim \ln(4g)$

In general

(embedded ball)

[RM] This is sharp.



Corollary If  $r > 0$  is injectivity radius of  $p$  in a cpt surface  $X$  w/ geodesic  $\partial X$   
 $\forall B_r(p) \subset X$   $-2\pi\chi(X) \geq 4\pi \sinh^2(r)$   $\frac{e^r + e^{-r} + 2}{4} \leq -\frac{\chi(X)}{2}$

$\Rightarrow e^{\frac{r}{2}} \leq -2\chi(X) + 2$  i.e.  $r \leq \frac{1}{2} \ln(-2\chi(X) + 2) = R$

Lemma  $(X, d)$  opt hyperbolic,  $\partial X$  geodesic,  $X \neq \Sigma_{0,3} = \delta$  s.t each boundary component of  $X$  has length  $\leq c$ . Then  $\exists$  a closed geodesic  $\alpha$  in  $X$  s.t

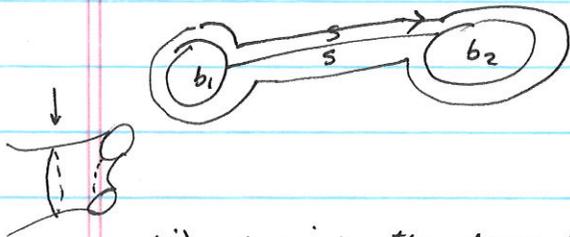
- (1)  $L(\alpha) \leq 2c + 2 \ln(-2\chi(X) + 2) = 2c + 2R$   $R = \frac{1}{2} \ln(-2\chi(X) + 2)$
- (2)  $\alpha$  is not in  $\partial X$

Pf. If Not.  $\forall$  closed geod  $\beta \subset X - \partial X$  has length  $> 2c + 2R$ .

Claim If  $s$  is a geodesic arc from  $\partial X$  to  $\partial X \Rightarrow \text{length}(s) \geq 2R$

Pf. If Not  $\exists$  such  $s$  of length  $< 2R$

- (i)  $s$  joins different boundary comp. produce a new curve  $a$  as shown.



$\text{length}(a) \leq 2c + 2\text{length}(s) \leq 2c + 2R$

$\Rightarrow \text{length}(a^*) < 2c + 2R$

$\Rightarrow a^* \subset \partial X \Rightarrow X = \Sigma_{0,3}$  impossible

- (ii)  $s$  joins the same boundary



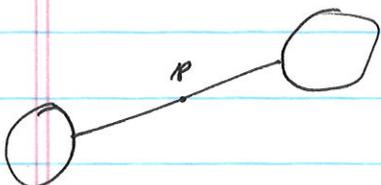
One of  $a_1$  or  $a_2$  in  $X$  is not boundary

component of  $X$

s.t  $\text{length}(a_i) \leq c + \text{length}(s)$ .  $\Rightarrow$  contradiction.

Now, let  $S$  be the shortest arc from  $\partial X$  to  $\partial X$  (perpendicular to  $\partial X$ )

+  $p$  be the mid point of  $S$

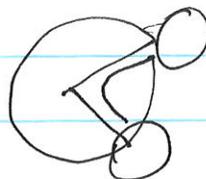


We claim the ball  $B_{\frac{R}{2}}(p)$  is embedded.

$\Rightarrow \text{Area}(X) \geq \text{area}(B_{\frac{R}{2}}(p)) \Rightarrow \frac{\pi}{2} < \frac{1}{2} \ln(-2\chi(X) + 2)$

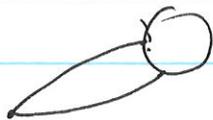
$R < \ln(-2\chi(X) + 2)$  contradiction.

To see it: (i)  $B_{\frac{R}{2}}(p) \cap \partial X = \emptyset$ .



$\exists$  a path  $s$  joining  $\partial X$  to  $\partial X$  of length  $< R$ !

(ii')



$\Rightarrow$  area argument  $\Rightarrow \text{Exp}|_{T_p X} \rightarrow -$  is  $\perp$  again  $\Rightarrow$

Pf of Bonnet's thm  $\Rightarrow$  Easy

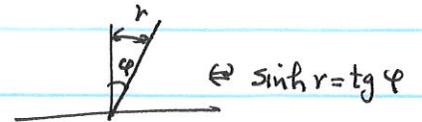
## Lecture 11. Basic Facts about Hyperbolic Surfaces

Recall right-angled hexagon, Gauss-Bonnet, existence of hyperbolic parts.

This gives a way to describe hyperbolic metrics on surfaces  $\Sigma_g$  of genus  $g \geq 2$

Corollary Each hyperbolic metric on  $\Sigma_g$  is an isometric gluing of hyperbolic parts.

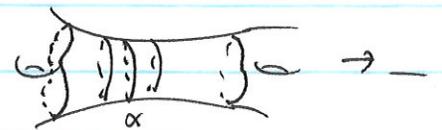
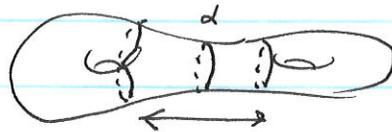
The collar lemma and Poincaré constant



Collar lemma Suppose  $\alpha$  is a simple closed geodesic in a hyperbolic surface  $\Sigma = \mathbb{H}/\Gamma$  of length  $l$ . Let  $\delta$  be ~~the neighborhood~~. Then  $\sinh(2\delta) \sinh(\frac{l}{2}) = 1$ .

$N_\delta(\alpha) = \{ z \in \mathbb{H}/\Gamma \mid d(z, \alpha) \leq \delta \}$  is an embedded annulus

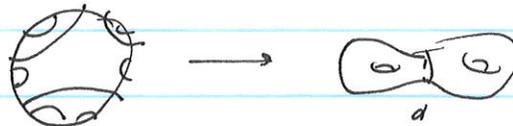
isometric to  $\{ z \in \mathbb{H} \mid d(z, \gamma\text{-axis}) \leq \delta \} / z \sim e^l z$



Topologically  $N_\delta(\alpha)$   $t > 0$  small, embedded

Proof:  $\pi: \mathbb{H} \rightarrow \mathbb{H}/\Gamma$  quotient map  $\pi^{-1}(\alpha)$  disjoint union of geodesics since  $\alpha$  is simple.

$\Gamma$ : action

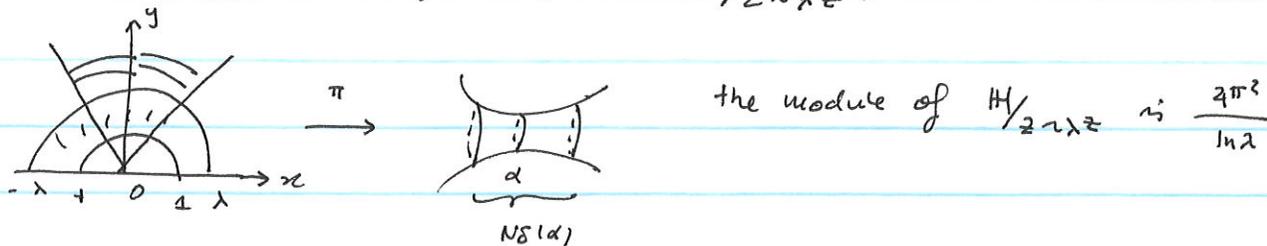


The condition  $\Leftrightarrow$  for any two  $\tilde{\alpha}_1 \neq \tilde{\alpha}_2$  components of  $\pi^{-1}(\alpha)$   $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 = \emptyset$   
then  $N_\delta(\tilde{\alpha}_1) \cap N_\delta(\tilde{\alpha}_2) = \emptyset$

Let  $\hat{\delta}$  be the largest  $\delta$  s.t.  $N_{\hat{\delta}}(\alpha)$  embedded annulus

### Lecture 13 The collar lemma

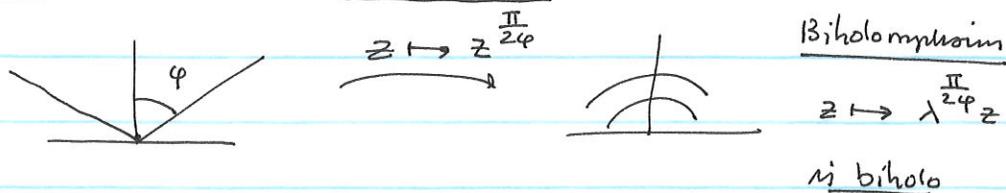
Eg  $\alpha$  a simple closed geodesic in  $\mathbb{H}/z \sim \lambda z$ .  $length(\alpha) = \ln \lambda$   $z > 1$



The collar of  $\alpha$  is  $N_\delta(\alpha) = \{ z \mid d(z, \alpha) \leq \delta \} / z \sim \lambda z$  :

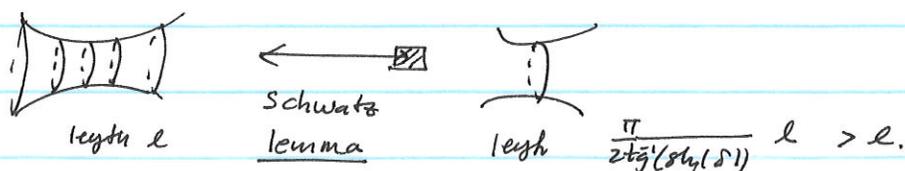
Note the module of  $N_\delta(\alpha)$  +  $\mathbb{H}/z \sim \lambda z$  are related by the formula:

$$\boxed{\operatorname{tg} \varphi = \sinh \delta}$$



Thus  $N_\delta(\alpha) \cong_{\text{biholo}} \mathbb{H} / z \sim \lambda^{\frac{\pi}{2\varphi}} z$

$$\text{Thus the length of } \alpha = \frac{\pi}{2\varphi} \cdot \ln \lambda = \frac{\pi}{2 \operatorname{tg}^{-1}(\sinh(\delta))} (\ln \lambda)$$



Collar Thm Suppose  $\alpha$  is a simple closed geodesic in  $\mathbb{H}/\Gamma$ , complete hyperbolic surface, of length  $l$ . Let  $\delta$  be:  $\sinh(2\delta) \sinh(\frac{l}{2}) = 1$ . Then the neighborhood  $N_\delta(\alpha) = \{ z \in \mathbb{H}/\Gamma \mid d(z, \alpha) < \delta \}$  is isometric to

$$\left\{ z \in \frac{\mathbb{H}}{z \sim e^l z} \mid d(z, \text{y-axis}) < \delta \right\} \quad (\text{Marquies tube})$$

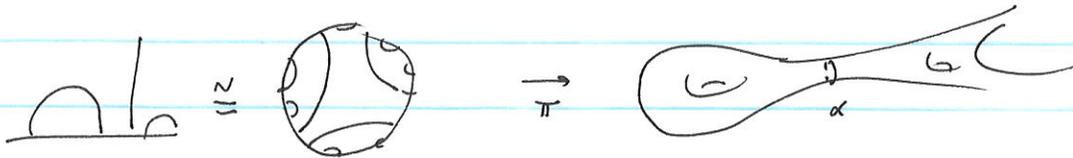
Homework  $\exists c > 0$  The moduli of  $\text{mod } N_\delta(\alpha) \geq c$  for all  $l, \delta$ .

$$\left( \begin{array}{l} \text{Now for } \delta = \frac{1}{2} \sinh^{-1} \left( \frac{1}{\sinh(\frac{l}{2})} \right) \quad \sinh(\delta) = \sinh \left( \frac{1}{2} \sinh^{-1} \left( \frac{1}{\sinh(\frac{l}{2})} \right) \right) \\ \text{For } l \gg 1. \quad \sinh^{-1} \left( \frac{1}{e^{l/2}} \right) = \sinh^{-1} \left( e^{-l/2} \right) \sim e^{-l/2} \quad \delta = e^{-l/2} \\ \sinh(\delta) \end{array} \right)$$

What happens when  $l \rightarrow \infty$ :  $\text{length}(N_\delta(\alpha)) > \text{length}(\alpha) = l \rightarrow \infty$  So no way.

Let  $\delta$  be the largest number st  $N_\delta(\alpha)$  embedded.

Proof. Let  $\pi: \mathbb{H} \rightarrow \mathbb{H}/\Gamma$  be the quotient map. Since  $\alpha$  simple,  $\pi^{-1}(\alpha)$  consists of disjoint geodesics in  $\mathbb{H}$ .



The condition  $\emptyset \neq \forall \tilde{\alpha}_1 \neq \tilde{\alpha}_2$  geod in  $\pi^{-1}(\alpha)$   $N_\delta(\tilde{\alpha}_1) \cap N_\delta(\tilde{\alpha}_2) = \emptyset$ .

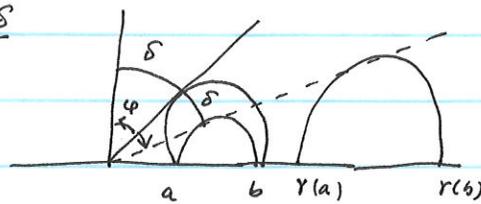
Notation:  $[a \neq b]$   $a \neq b$  in  $\mathbb{R} \cup \{\infty\}$ ,  $[a, b]$  geodesic from  $a$  to  $b$ .

After a conjugation, we may assume:

(1)  $[0, \infty] \subset \pi^{-1}(\alpha)$  and  $\gamma(z) = \lambda z \in \Gamma, \lambda = e^l$

(2) if  $[a, b] \subset \pi^{-1}(\alpha)$   $0 < a < b \Rightarrow \delta[a, b] \cap [a, b] = \emptyset$ .

dist $[0, \infty], [a, b] = 2\delta$



$\Rightarrow \lambda a > b$

so  $\lambda > (b/a)$

(3)  $N_\delta[a, b]$  is tangent to  $N_\delta[0, \infty]$

good:  $\sinh(2\delta) \sinh(l/2) \geq 1$

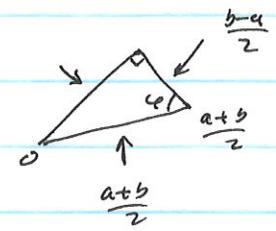
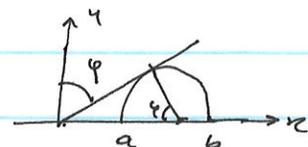
Now by the assumpti. of tangency

$2\delta = \text{dist}[0, \infty], [a, b]$

so  $\sinh(2\delta) = \text{tg}(\varphi) = \frac{\sqrt{(\frac{a+b}{2})^2 - (\frac{b-a}{2})^2}}{(\frac{b-a}{2})}$

(distance angle formula)  $= \frac{2\sqrt{ab}}{(b-a)}$

$= 2 \frac{\sqrt{b/a}}{(b/a - 1)} \geq 2 \frac{\sqrt{\lambda}}{(\lambda - 1)} = \frac{2}{(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})} = \frac{2}{e^{\frac{l}{2}} - e^{-\frac{l}{2}}} = \frac{1}{\sinh(l/2)}$



i.e.  $\sinh(2\delta) \sinh(l/2) \geq 1$

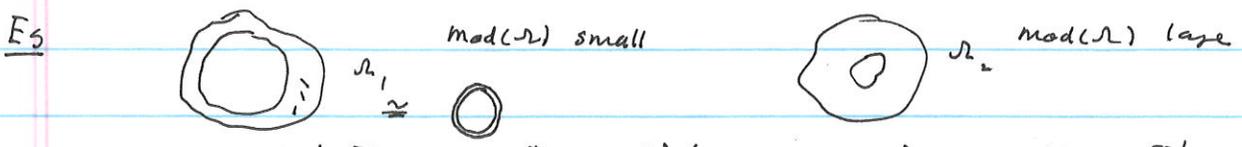
□

$\text{length}(\text{shortest geod}) = \frac{4\pi^2}{\text{mod}(\Omega)}$   
 -13.3-

Lecture 13. Collar lemma

Def The module of the ring  $\Omega = \{z \mid r_1 < |z| < r_2\}$ ,  $\text{mod}(\Omega) = \log \frac{r_2}{r_1}$ . By uniformization theorem, each ring  $\Omega$  (i.e. Riemann surface homeo to  $\mathbb{C}^*$ ) is biholomorphic to  $\Omega_{r_1, r_2}$ .

We define  $\text{mod}(\Omega) = \text{mod}(\Omega_{r_1, r_2})$



Eg.  $\Omega = \{z \mid 0 < \text{Im} z < h\} / z \sim z + w$  has  $\text{mod}(\Omega) = \frac{2\pi h}{w}$ . (HW)

Corollary  $\forall g \geq 2, \exists M_g > 0$  s.t. any closed Riemann surf  $\Sigma_g$  of genus  $g$  contains an essential ring domain  $\Omega \subset \Sigma_g$  st  $\text{mod}(\Omega) \geq M_g$  (essential  $\Leftrightarrow$  homotopically non-trivial)

Pr. Let  $\alpha$  be the shortest geod in  $\Sigma_g$   $\delta = \frac{1}{2} \sinh^{-1} \left( \frac{1}{\sinh(\frac{l(\alpha)}{2})} \right)$ ,  $\Omega = N_\delta(\alpha)$

Known  $l(\alpha) \leq \log(4g-2) \Rightarrow \text{mod}(\Omega) \geq m_g$  a lower bound  
 $\Rightarrow \delta \geq \delta_0(g)$  (  $\lim_{l(\alpha) \rightarrow 0} \text{mod}(\Omega) \rightarrow \infty$  ). HW.  $\square$

There were not too much study of max module ring in Riemann surfaces. This is a very interesting subject.

Def  $\Sigma$  Riemann surface.  $\text{Maxmod}(\Sigma) = \max \{ \text{mod}(\Omega) \mid \Omega \text{ essential ring in } \Sigma \}$ .  
 ( $\Leftrightarrow$  counterpart of the shortest geodesics). ~~A maximum on extremal surfaces is a~~ An extremal

surface is a Riemann surface  $\Sigma$  st  $\text{Maxmod}(\Sigma) \leq \text{Maxmod}(\Sigma')$  for all other Riemann surface  $\Sigma' \cong_{\text{homeo}} \Sigma$ . What can you say about these surfaces?

Eg. ~~maximal torus~~ is  $\mathbb{C} / z + iz$ .  $\text{maxmod}(\mathbb{C} / z + iz) = 2\pi$   
 The extremal torus is  $\mathbb{C} / z + iz$ .

Schwarz lemma infinitesimal version.  $(\Sigma_i, d_i)$  hyperbolic surfaces  $\varphi: \Sigma_1 \rightarrow \Sigma_2$  holomorphic map. Then  $\forall v \in T_p \Sigma_1 - \{0\}$ ,  $\|D\varphi(v)\|_{d_2} \leq \|v\|_{d_1}$ , s.t. equality holds for one  $v$  iff  $\varphi$  is an isometry. In particular  $\forall$  loop  $\alpha \subset \Sigma_1$ ,  $l(\alpha) \geq l(\varphi(\alpha))$ .

Proof let  $\pi_i: \mathbb{D} \rightarrow \Sigma_i$  be the covering map st  $\pi_1(0) = p$ ,  $\pi_2(0) = \varphi(p)$ .

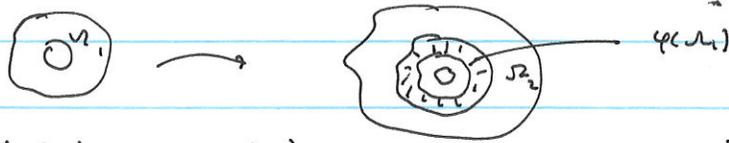
~~Proof~~ By covering space theory  $\exists$  holomorphic lifting  $\tilde{\varphi}: (\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$  st  $\pi_2 \tilde{\varphi} = \varphi \circ \pi_1$ .

Now Schwarz lemma applied to  $\tilde{\varphi}$ :  $\|D\tilde{\varphi}(v)\|_0 \leq \|v\|_0 \Rightarrow$  done.

Corollary

# Lecture 13 Collar lemma

Corollary If  $\varphi: \Omega_1 \rightarrow \Omega_2$  is a holomorphic embedding of a riig  $\Omega_1$  to  $\Omega_2$  s.t  $\varphi$  is a homotopy equivalence (i.e  $\varphi \neq pt$ )



Then  $mod(\Omega_2) \geq mod(\Omega_1)$  s.t equality holds iff  $\varphi$  biholo.

Pf Let  $d_1, d_2$  be the pome metric +  $d_1, d_2$  be shortest geodesics in  $(\Omega_1, d_1) (\Omega_2, d_2)$ . Then  $l(d_2) \leq l(\varphi(d_1)) \leq l(d_1) \Rightarrow$  result

$l(d_2) \leq l(\varphi(d_1)) \leq l(d_1) \Rightarrow$  result  
 $\uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow$   
 shortest \qquad \qquad \qquad Schwarz \qquad \qquad \qquad  $(\varphi(d_1) \perp d_2)$   
 $l(d) = \frac{4\pi^2}{mod(\Omega)} \Rightarrow$  done □

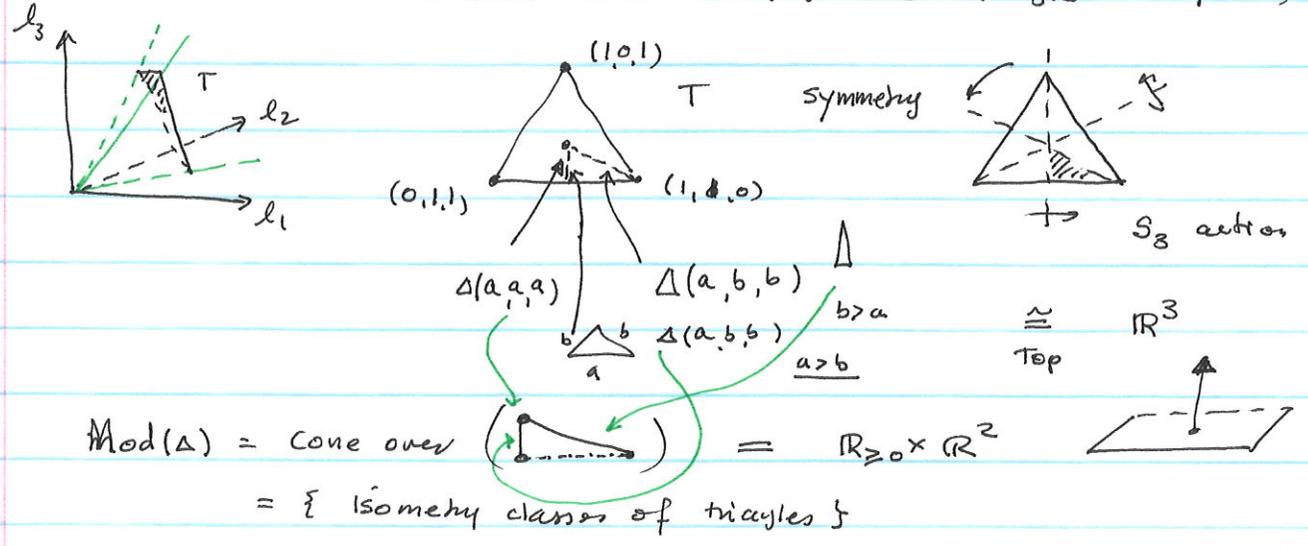
Eg For a flat torus  $T = \mathbb{C} / \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  if  $\alpha$  is a simple loop  $\subset T$  then  $mod(Riig(\alpha)) = \frac{2\pi Area(T)}{l^2(\alpha)}$

Thus, we are looking for torus of the (largest) minimal width.

Conclusion It must be  $\mathbb{C} / \mathbb{Z} + \mathbb{Z}i$ .

Lecture 14: Moduli space + Teichmüller space

Eg 1  $\text{Teich}(\Delta) = \{(l_1, l_2, l_3) \in \mathbb{R}_{>0}^3 \mid l_i + l_j > l_k\}$   
 = Cone from 0 to an equilateral triangle  $T$  (open  $T$ )



$\text{Mod}(\Delta) = \text{Cone over } (\text{triangle}) = \mathbb{R}_{\geq 0} \times \mathbb{R}^2$   
 = { Isometry classes of triangles }

Riemann surface

$S$  topological surface

$$\text{Mod}(S) = \{ [(\Sigma, \phi)]_{\mathbb{R}} \mid \Sigma \cong S \text{ homeo}, (\Sigma, \phi) \cong_{\text{biholo}} (\Sigma', \phi') \}$$

isometry biholomorphism classes of  $\mathbb{R}$ -surf homeo to  $S$ .

The Teichmüller space of  $S$ ,

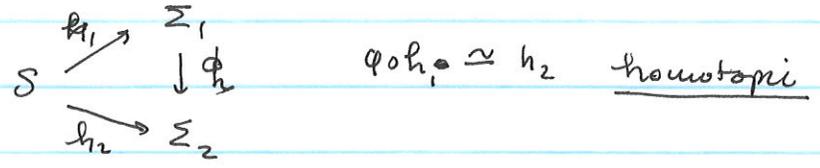
Easy definition  $\text{Teich}(S) = \{ [(S, \phi)]_{\mathbb{R}} \mid h: (S, \phi) \cong (S, \phi') \text{ biholo. st. } h \cong \text{id} \}$

Problem when the underlying  $S$  changes, how to define  $\text{Teich}(S)$ ?

Def A marked Riemann surf (of type  $S$ ) is  $(\Sigma, \phi, h)$  where  
 $h: S \rightarrow \Sigma$  homeo. (o.p.)  $(\Sigma, \phi)$  complex structure

$$(\Sigma_1, \phi_1, h_1) \cong_{\text{Teich}} (\Sigma_2, \phi_2, h_2) \iff \exists \text{ biholo } \psi: (\Sigma_1, \phi_1) \rightarrow (\Sigma_2, \phi_2)$$

so that



Eg  $\Sigma = S$   $h = \text{id} \implies \phi \cong \text{id}: S \rightarrow S!$

- Rm 1. We may replace  $h: S \rightarrow \Sigma$  by the homotopy class of o.p. homeo
- 2. We may even ~~assume~~  $\phi$  by  $\hat{\phi}: \pi_1(S) \rightarrow \pi_1(\Sigma)$  isomorphism  
 replace  $h$  by  $h^*$

Eg Mod(S'xS') and Teich(S'xS').

Claim Mod(S'xS')  $\cong$  H/PSL(2,Z) + Teich(S'xS')  $\cong$  H.

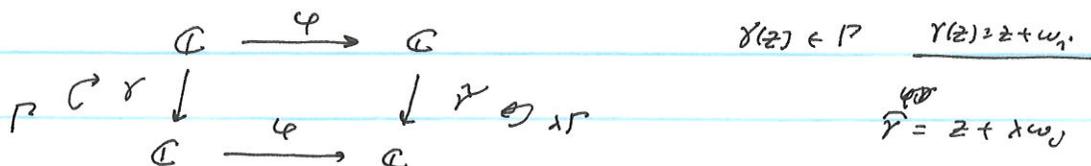
Uniformization  $\Rightarrow$  Each complex torus  $\cong \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$   $\omega_1/\omega_2 \notin \mathbb{R}$   $\mathbb{H}$   
 $\omega_2/\omega_2 \in \mathbb{R}$

Note 1 if  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$  Then  $a\omega_1 + b\omega_2, c\omega_1 + d\omega_2$  New basis

$$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(c\omega_1 + d\omega_2)$$

② The isomorphism  $\varphi(z) = \lambda z \quad \lambda \neq 0: \mathbb{C} \rightarrow \mathbb{C}$

conjugate  $\rho$  to  $\lambda\rho = \mathbb{Z}(\lambda\omega_1) + \mathbb{Z}(\lambda\omega_2)$



$\Rightarrow \varphi$  induces iso  $\mathbb{C}/\rho \rightarrow \mathbb{C}/\lambda\rho$



$$\Rightarrow \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}(\omega_1/\omega_2)$$

$$\cong \mathbb{C}/\mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(c\omega_1 + d\omega_2) \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right)$$

Conclusion if  $z \in \mathbb{H}$  and  $z' = \frac{az+b}{cz+d} \Rightarrow \mathbb{C}/\mathbb{Z} + \mathbb{Z}z \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}z'$

On the other hand, if  $\varphi: \mathbb{C}/\mathbb{Z} + \mathbb{Z}z \rightarrow \mathbb{C}/\mathbb{Z} + \mathbb{Z}z'$   $z, z' \in \mathbb{R}$

biho, we claim  $z' = \frac{az+b}{cz+d} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$ .

Proof. First.  $\forall \alpha \in \mathbb{C} \quad [z] \mapsto [z+\alpha]$  biho of  $\mathbb{C}/\mathbb{Z} + \mathbb{Z}z' \hookrightarrow$

So we may assume that  $\varphi([0]) = [0]$

Now by covering lifting thm.  $\Rightarrow \exists$  biholomphi  $\tilde{\varphi}: \mathbb{C} \rightarrow \mathbb{C} \quad \tilde{\varphi}(0) = 0$

$$\begin{array}{ccc}
 \tilde{\varphi}: \mathbb{C} & \rightarrow & \mathbb{C} \\
 \downarrow & & \downarrow \\
 \mathbb{C}/\rho & \xrightarrow{\varphi} & \mathbb{C}/\rho'
 \end{array}
 \Rightarrow \tilde{\varphi}(z) = \lambda z \quad \lambda \in \mathbb{C}^\times$$

Furthermore  $\tilde{\varphi}(\Gamma) = \Gamma'$  i.e.  $\Gamma' = \lambda \Gamma$   $\{z, 1\}$   $\{ \lambda z, \lambda \}$  basis

$$\Rightarrow \begin{aligned} \lambda z &= az' + b \\ \lambda \cdot 1 &= cz' + d \end{aligned} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{C}) \quad (\det A = \pm 1)$$

But  $z, z' \in \mathbb{H} \Rightarrow ad - bc = 1 \quad \det(A) = 1 \quad \underline{\text{Done}}$

Prop The Teichmüller space  $T(S'KS') = \mathbb{H}$ .

Key topological fact  $h_1, h_2: S'KS' \rightarrow S'KS'$  homeo  $\mathbb{D}$  s.t.  $h_1 \cong h_2$   
 $\iff (h_1)_* = (h_2)_*: \pi_1(S'KS') \cong \pi_1(S'KS')$

Thus a marked complex torus  $(X, \varphi, h) \Leftrightarrow (X, \varphi, \{\alpha, \beta\})$   
 $\alpha, \beta$  two generators of  $\pi_1(X)$ , (oriented generators) ( $\alpha \rightarrow \beta$  defines orientation)

$\Leftrightarrow$  Each marked torus =  $(\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, (\omega_1, \omega_2))$   $\omega_1/\omega_2 \in \mathbb{H}$

Now  $(\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, (\omega_1, \omega_2)) \underset{\text{Teich}}{\cong} (\mathbb{C}/\mathbb{Z}\omega'_1 + \mathbb{Z}\omega'_2, (\omega'_1, \omega'_2))$

$\iff \omega_1/\omega_2 = \omega'_1/\omega'_2$   $\varphi$   $\varphi$  holomorph  $\tilde{\varphi}(0) = 0$   $\tilde{\varphi}(\omega_i) = \lambda \omega'_i$

The same proof as above  $\Rightarrow$  done.

Question What about  $\omega_1/\omega_2 \in -\mathbb{H} \Leftrightarrow$  orientation issue to o.p.  $\Leftrightarrow$  -

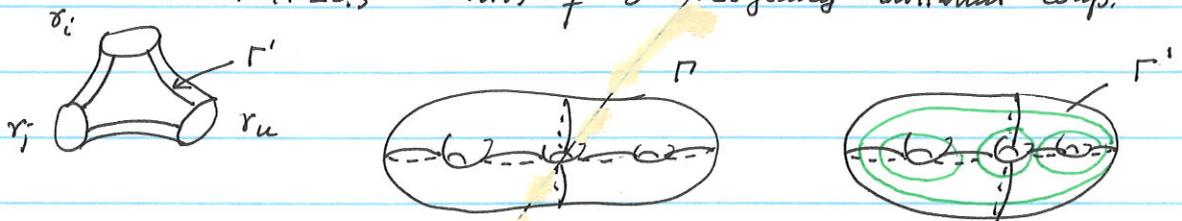
# lecture 14 The Fenchel-Nielsen Coordinate

$S$  a closed surface of genus  $g \geq 2$

$\text{Teich}(S) = \{ [(X, d, \varphi)] \mid d \text{ hyperbolic metric } \varphi: S \rightarrow X, \text{ "homotopy class" of orientation preserving homeo } \}$

Now, fix a 3-holed sphere decoup  $P = \{ r_1, \dots, r_{3g-3} \}$  of  $S$ , let  $\Gamma'$  be a set of disjoint s.c.c called "seams" s.t for each 3-holed sphere  $\Sigma_{0,3} \subset S - \Gamma'$

$\Gamma' \cap \Sigma_{0,3}$  consists of 3 arcs joining different comp.



with  $(S, P, \Gamma')$  we can associate the FN coordinate,

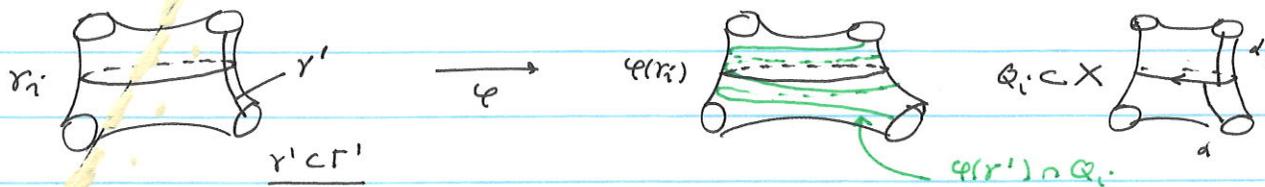
$$\text{FN: } \text{Teich}(S) \rightarrow (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} : [(X, d, \varphi)] \mapsto (l_i, t_i)$$

$(l_1, \dots, l_{3g-3}) \in \mathbb{R}_{>0}^{3g-3}$  length coord       $(t_1, \dots, t_{3g-3})$  twist-coord

The length..  $\forall r_i$  let  $\varphi(r_i)^*$  be the geod  $\simeq \varphi(r_i)$  in  $(X, d)$   $l_i = \text{length}(\varphi(r_i)^*)$

We may assume, for  $(X, d)$  that  $\varphi(r_i) = \varphi(r_i)^*$  after a homotopy

For each  $r_i$ , let  $Q_i =$  union of two hyperbolic 3-holed spheres in  $X$  adjacent to  $\varphi(r_i)^*$  "  $P_i' \cup P_i''$  "

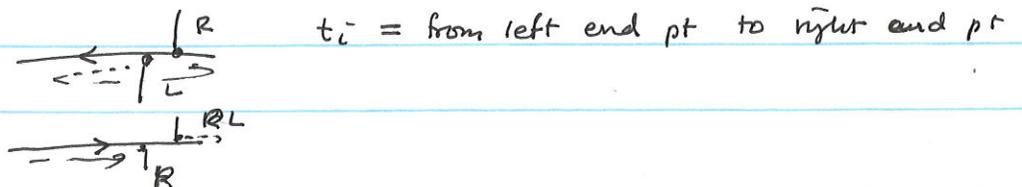


Now  $\varphi(r_i) \cap Q_i \simeq d_1 \times d_2 \times d_3 \text{ rel}(\partial Q_i)$   $\partial Q_i =$  invariant.

where  $d_1, d_3$  are the shortest geod in  $P_i' \cup P_i''$ ,  $d_2 \subset \varphi(r_i)$

Define  $t_i =$  the signed distance of  $d_2 \subset \mathbb{R}$ . (length)

Signed distance Fix any orientation on  $\varphi(r_i)$ , use analatn of  $X \Rightarrow$  left + right side of  $\varphi(r_i)$ .



Thm (Fenchel-Nielsen) Fix  $(S, \Gamma, \Gamma')$  the map

(sketch) FN:  $\text{Teich}(S) \longrightarrow \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3}$  is 1-1 onto

PF: Well defined o.k. (it is independent of the choice of two  $\gamma_i, \gamma_i'$ ).

Onto clearly from the construction.

Given  $(l, x) \rightarrow$  produce hyperbolic pairs of given lengths

Use  $t$  to glue them isometrically.  $\Rightarrow (X, d)$ . Use  $t_i \rightarrow$  map

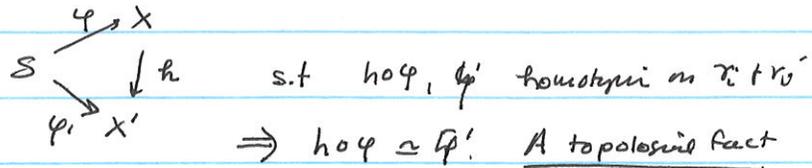
$h: S \rightarrow X$ . sending  $h(\gamma_i')$  to a curve homotopic to  $d_1 * d_2 * d_3$ .

1-1: If  $(X, d, \varphi), (X', d', \varphi')$  two marked hyperbolic surfaces of the same FN coordinates  $\Rightarrow (X, d, \varphi) \cong (X', d', \varphi')$

By the construction  $l_i + t_i \Rightarrow \exists$  an isometry  $h: X \rightarrow X'$

sending geodesics  $\varphi(\gamma_i)$  to  $\varphi'(\gamma_i)$ . SAME twisting  $\Rightarrow$  glued nicely

The homotopy  $\Rightarrow$



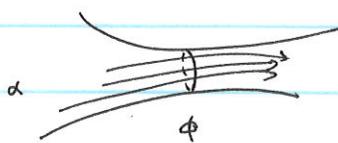
□

Lecture 15

Wolpert's formula cosine

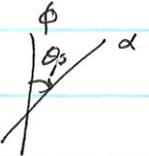
Kerckhoff's calculation.

Thus (Wolpert) If  $d_t$  is a family of hyperbolic metrics obtained by  $t$ -twisting along a simple closed geodesic  $\phi$  in  $d_0$ , and  $\alpha(t)$  is the closed geodesic in  $d_t$  homotopic to geodesic  $\alpha(0)$ ,



then

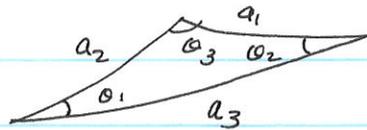
$$\frac{d}{dt} \Big|_{t=0} l(\alpha(t)) = \sum_{p \in d_0 \cap \phi_0} \cos \theta_p$$



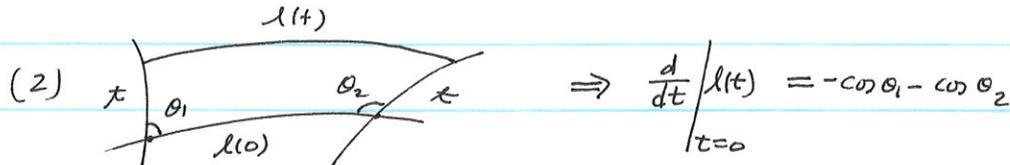
$\theta_p$  from  $\phi$  to  $\alpha$ .

Proof It is basically the cosine law for triangles

lemma (1) Cosine law

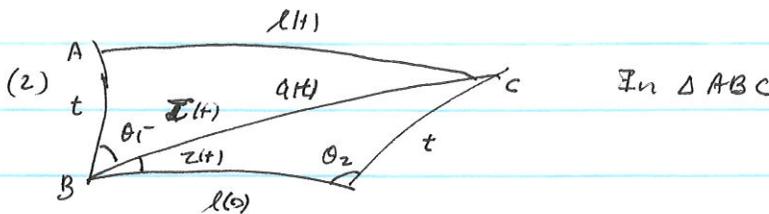


$$\cosh(a_3) = \cosh(a_1) \cosh(a_2) - \sinh(a_1) \sinh(a_2) \cos \theta_3$$



$$\Rightarrow \frac{d}{dt} \Big|_{t=0} l(t) = -\cos \theta_1 - \cos \theta_2$$

pf (1) standard, I take it as homework.



$$\cosh(l(t)) = \cosh t \cosh(a(t)) - \sinh t \sinh a(t) \cos(\theta_1 - z(t)) \quad , \quad z(0) = 0$$

so  $\frac{d}{dt} \Big|_{t=0} \Rightarrow$

$$\sinh(l(0)) l'(0) = \cosh(a(0)) a'(0) (\sinh a(0)) a'(0) - \sinh a(0) \cos(\theta_1)$$

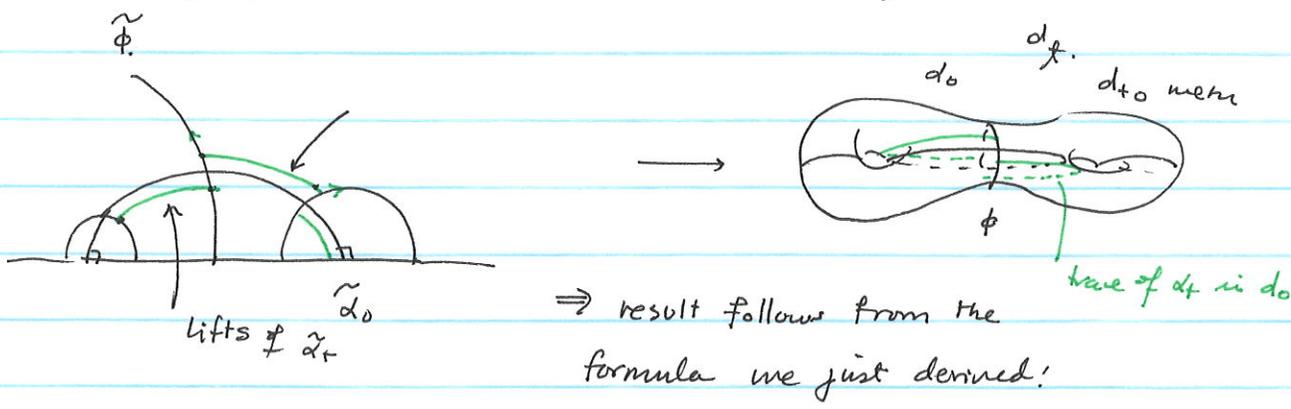
$$\Rightarrow l'(0) = a'(0) - \cos(\theta_1)$$

Now  $a(t)$ :  $\cosh(a(t)) = \cosh t \cosh(l(0)) - \sinh t \sinh(l(0)) \cos \theta_2$

derivative  $\Rightarrow (\sinh l(0)) a'(0) = \cancel{\cosh l(0)} - \sinh l(0) \cos \theta_2$

$$\Rightarrow a'(0) = -\cos \theta_2 \quad \square$$

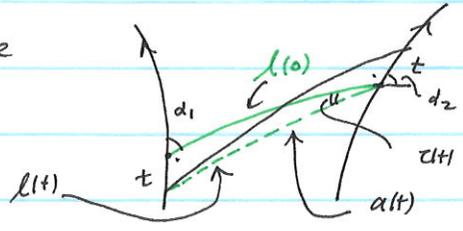
Proof lift everything to the universal cover and write things down in  $d_0$  metric.



\$\Rightarrow\$ result follows from the formula we just derived!

□

In reality, it should be



that is the actual situation.

In this case:

$$\begin{aligned} \cosh l(t) &= \cosh t \cosh a(t) - \sinh t \sinh a(t) \cos(\pi - d_2 + \tau(t)) \\ &= \cosh a(t) - t \cdot \sinh(a(t)) \cos(\pi - d_2) + o(t^2) \end{aligned}$$

$$\Rightarrow l'(0) = a'(0) + \sinh(l(0)) \cos(d_2)$$

Next

$$\begin{aligned} \cosh a(t) &= \cosh t \cosh l(0) - \sinh t \sinh l(0) \cos(\pi - d_1) \\ &= \cosh l(0) + t \sinh l(0) \cos(d_1) \end{aligned}$$

$$\Rightarrow a'(0) = \cancel{\sinh l(0)} \cos d_1 \quad \text{Done}$$

□

(Wolpert)

Corollary The symplectic form  $\sum d l_i \wedge d x_i$  on  $T(\Sigma_g)$  is invariant under the action of the MCG.