Lecture 3. A Brief Introduction to Hyperbolic Geometry

§3.1. The Hyperboloid Model of Hyperbolic Geometry

Recall that the unit sphere $\mathbb{S}^n = \{ x \in \mathbb{E}^{(n+1)} | (x,x) = 1\}$ is defined by the standard inner product (,) in the Euclidean space $\mathbb{E}^{n+1}$. The hyperbolic n-dimensional space is defined in the similar way using the **Minkowski Space** $\mathbb{E}^{n,1}$ which is $\mathbb{R}^{n+1}$ with inner product $<,>$ where

$$< X, Y >= \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

with $X = (x_1, ..., x_{n+1}), Y = (y_1, ..., y_{n+1})$. By definition, the n-dimensional (hyperboloid model) of the **hyperbolic Space** $\mathbb{H}^n$ is $\{x \in \mathbb{E}^{n,1} \mid \ < x, x >= -1, \ x_{n+1} > 0\}$; the **de Sitter Space** $\mathbb{S}^n_1$ is defined to be $\{x \in \mathbb{E}^{n,1} \mid \ < x, x >= 1\}$; and the **light Cone** is defined to be $\{x \in \mathbb{E}^{n,1} \mid \ < x, x >= 0\}$.

Example 1. If n = 2, and $X = (x, y, z)$, then $< X, X >= x^2 + y^2 - z^2$. The hyperbolic plane $\mathbb{H}^2$ and the de-Sitter space are shown in figure 3.1.

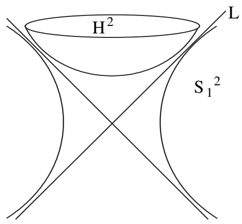


Figure 3.1

Lemma 3.1. The restriction of the bilinear form $<,>$ to the tangent space $T_p\mathbb{H}^n$ is positive defined.

Proof. Since $< p, p >= -1$, the tangent space at p, $T_p\mathbb{H}^2 = \{x \in \mathbb{R}^{n+1} | < x, p >= 0\}$. Now suppose otherwise that there exists $x \in T_p\mathbb{H}^2$, such that $< x, x >< 0$. By the choice of x and p, we have

(1)$< x, x >< 0$, i.e., $\sum_1^n x_i^2 < x_{n+1}^2$,

(2)$< p, p >= -1$, i.e., $\sum_1^n p_i^2 + 1 = p_{n+1}^2$, and

(3)$< x, p >= 0$, i.e., $\sum_1^n x_i p_i = x_{n+1} p_{n+1}$.
Therefore,

$$(\sum_1^n x_i^2)(1 + \sum_1^n p_i^2) \leq x_{n+1}^2 p_{n+1}^2 = (\sum_1^n x_i p_i)^2 \leq (\sum_1^n x_i^2)(\sum_1^n p_i^2).$$

This is impossible.$\square$

By this lemma, the restriction $<,> \mid$ produces a Riemannian metric d on $\mathbb{H}^n$. This Riemannian metric d is called *the hyperbolic metric* on $\mathbb{H}^n$.

Definition 3.1. Let $\mathbb{O}(m,1)$ be the subgroup of $\mathbb{GL}(m+1,\mathbb{R})$ preserving $<,>$ in $\mathbb{E}^{m,1}$. Let $\mathbb{O}_+(m,1)$ be subgroup of $\mathbb{O}(m,1)$ preserving $\mathbb{H}^n$. Let $\mathbb{O}(m)$ be the orthogonal group of $\mathbb{R}^m$ preserving the standard inner product.

Lemma 3.2. The group $\mathbb{O}_+(n,1)$ acts on $\mathbb{H}^n$ preserving the Riemannian metric $<,> \mid$. Furthermore, the $\mathbb{O}_+(n,1)$-action is transitive on $\mathbb{H}^n$.
Proof. By the definition of $\mathbb{O}_+(n,1)$, we see that it acts isometrically on $\mathbb{H}^n$. To see part (b), let $e_{n+1} = (0,...,0,1) \in \mathbb{H}^n$, and $e_i = (0,...,0,1,0,...,0)$ be the standard basis for $\mathbb{R}^{n+1}$. For any $p \in \mathbb{H}^n$, let $p^\perp = \{x \in \mathbb{R}^{n+1} \mid < x, p >= 0\} = T_p\mathbb{H}^n$. Then $e_{n+1}^\perp = \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$. By lemma 3.1, there are orthogonal basis $v_1,...,v_n$ in $p^\perp$ with respect to $<,> |_{p^\perp}$ so that $< v_i, v_j >= \delta_{ij} =< e_i, e_j >$ for $i,j \leq n$. Define a linear map $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, by $\phi(v_i) = e_i,\ i = 1,2,...,n$ and $\phi(p) = e_{n+1}$, then $\phi \in \mathbb{O}_+(n,1)$. By the construction $\phi(p) = e_{n+1}$, thus (b) follows. $\square$

The Isotropy group of $\mathbb{O}_+(n,1)$ at e_{n+1} is

$$\{g \in \mathbb{O}_+(n,1) \mid g(e_{n+1}) = e_{n+1}\} = \mathbb{O}(n) \oplus Id = \{[\begin{array}{cc} A & 0 \\ 0 & 1 \end{array}] | A \in \mathbb{O}(n)\}.$$

Since $\mathbb{O}(n)$ acts transitively on the set of all 2-dimensional subspaces of $\mathbb{E}^n$. It follows that $Isotropy(\mathbb{O}_+(n,1), e_{n+1})$ acts transitively on 2-dimensional linear subspaces of $T_{e_{n+1}}\mathbb{H}^n$. Together with lemma 3.2, this shows that the hyperbolic space $(\mathbb{H}^n, d)$ has a constant sectional curvature.

Definition 3.2. (Totally Geodesic Submanifolds)
If (M,g) is Riemannian manifold and $S \subset M$ is a k-dimensional smooth submanifold, we say S is totally geodesic in (M,g) if for all geodesics γ tangent to S at one point, $\gamma \subset S$.

Example 2. 1-dim totally geodesic submanifolds are the geodesics.
Example 3. Affine planes $\subset \mathbb{R}^n$ are totally geodesic in $\mathbb{E}^n$.

Lemma 3.3. If Φ is an isometry of (M,g) and $S = \{x \mid \Phi(x) = x\}$, then S is totally geodesic.
Proof. Say γ as above is a geodesic tangent to S at point p, then $\Phi \circ \gamma(t)$ is another through p and tangent to γ at p. Since geodesic through a point with given tangent vector is unique, it follows that $\Phi(\gamma(t)) = \gamma(t)$ for all t. Then $\gamma \subset S$.

By considering the isometry $F(x_1, x_2, x_3, ..., x_{n+1}) = (x_1, x_2, -x_3, ..., -x_{n+1})$ and its fixed points, we find one geodesic $\mathbb{H}^n \cap (\mathbb{R}^2 \times 0)$ in $\mathbb{H}^n$. Using lemma 3.2, we have,

Corollary 3.4. All geodesics in $\mathbb{H}^n$ are of the form $\mathbb{H}^n \cap (2-dim\ linear\ spaces)$. All totally geodesic submanifolds in $\mathbb{H}^n$ are of the form $\mathbb{H}^n \cap (linear\ subspaces)$.

Lemma 3.5. (a) If $u, v \in \mathbb{H}^n$, their hyperbolic distance d satisfies $\cosh(d) = -< u, v >$.
(b) If $u, v \in \mathbb{S}^n_1$(de Sitter Space), then the dihedral angle θ between $u^\perp = \{x \in \mathbb{H}^n \mid < x, u >= 0\}$ and $v^\perp$ satisfies $\cos\theta = -< u, v >$.
Proof. We will prove part (a) only. Part (b) follows by the same argument. Using $\mathbb{O}_+(n+1, 1)$, we may assume $u = e_{n+1}, v \in (0, ..., 0, x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}^2$ Thus, we may assume that n=1. In this case, $u = (0, 1), v = (\sinh d, \cosh d)$. Let $\gamma(t) = (\sinh t, \cosh t), t \in [0, d]$ be the geodesic. We have $\gamma'(t) = (\cosh t, \sinh t)$, and $< \gamma'(t), \gamma'(t) >= \cosh^2 t - \sinh^2 t = 1$. Thus $\gamma(t)$ is arc-length parameterized. It follows that

$$d(u, v) = \int_0^d \sqrt{< \gamma'(t), \gamma'(t) >} dt = d.$$

This shows $\cosh(d) = -< u, v >$ due to $< u, v >= -\cosh(d)$. $\square$

As a consequence, we see that $(\mathbb{H}^n, <,> |)$ is a complete Riemannian manifold since each geodesic can be extended to infinity.

Exercises.
(1) Show that the medias of a hyperbolic triangle intersect in one point.
(2) Can you find a regular dodecahedron(or icosahedron) in $\mathbb{H}^3$?

§3.2. The Klein Model of $\mathbb{H}^n$

Consider the open unit disc $\mathbb{D}^n \subset \mathbb{R}^n \times 1 \subset \mathbb{E}^{n+1,1}$. Define a bijective map $\pi : \mathbb{H}^n \to \mathbb{D}^n$, by sending a point x to $\lambda x \in \mathbb{D}^n$ where $\lambda \in \mathbb{R}_{>0}$. The map π is a radius projection sending geodesics to line segments, totally geodesic submanifolds to affine subspaces.

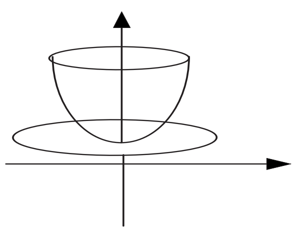


Figure 3.2

As a consequence, in the Klein model $\mathbb{D}^n$ of $\mathbb{H}^n$, all geodesics are Euclidean line segments inside the open unit disk.

§3.3. The Upper Half Space Model of $\mathbb{H}^n$

Let us begin with the 2-dimensional case. The upper half plane is $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) >$

$0\}$, where $z = x + iy$. Define a Riemannian metric on it by $ds = \frac{dx^2+dy^2}{y^2} = \frac{-4dzd\overline{z}}{(z-\overline{z})^2}$. The area form of the metric is $\frac{dxdy}{y^2}$, and the length of a curve γ is $L(\gamma) = \int_a^b \frac{|\gamma'(t)|}{Im(\gamma(t))}dt$. We can list all orientation preserving isometries of the metric as follows. First, from the definition, we see easily that $f(z) = \lambda z$ where $\lambda \in \mathbb{R} > 0$ and $g(z) = z + a$ where $a \in \mathbb{R}$ are isometries. Next we claim that $h(z) = -\frac{1}{z}$ preserves the Riemannian metric. Indeed, let $w = h(z)$ and

$$h^*(ds) = -\frac{4dwd\overline{w}}{(w-\overline{w})^2} = -\frac{4d(\frac{1}{z})d(\frac{1}{\overline{z}})}{(\frac{1}{z}-\frac{1}{\overline{z}})^2} = -4\frac{\frac{1}{z^2\overline{z}^2}dzd\overline{z}}{\frac{(\overline{z}-z)^2}{z^2\overline{z}^2}} = -4\frac{dzd\overline{z}}{(z-\overline{z})^2}.$$

It is well known from complex analysis that each Möbius transformation $F(z) = \frac{az+b}{cz+d}, \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in \mathbb{SL}(2,\mathbb{R})$ of the upper half plane is a compositions of f, g and h above. It follows that $SL(2,\mathbb{R})$ acts as isometries on $\mathbb{H}^2$. We leave it as an exercise to the reader to verify that all orientation preserving isometries of $\mathbb{H}^2$ are Möbius transformations. (Hint: shows that $SL(2,\mathbb{R})$ acts transitively on the unit tangent vectors.)

Our next task is to find all geodesics in the upper half plane model. Since we know a lot of isometries, it suffices to find one geodesic and then use isometries to find others. Here is a simple calculation.

Example 4. The positive y-axis is a geodesic in $\mathbb{H}^2$.

Let $\gamma(t)$ be a path from ia to ib, $t \in [0,1]$, in $\mathbb{H}^2$ where $b > a > 0$. Write $\gamma(t) = (x(t), y(t))$, $y(t) > 0$. Then $\gamma'(t) = (x'(t), y'(t))$ and its length is

$$L(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}dt \geq \int_0^1 \frac{\sqrt{y'(t)^2}}{y(t)}dt \geq |\int_0^1 \frac{y'(t)}{y(t)}dt| = \ln\frac{b}{a}.$$

Note that equalities hold if and only if $x(t) = 0$ and $y'(t) \geq 0$. That is the same as $\gamma(t)$ is monotonic lying in the positive y-axis. This shows that the positive y-axis is a geodesic and the distance is

$$d_{\mathbb{H}^2}(ia, ib) = \ln\frac{b}{a}.$$

Definition 3.3. The cross ration of four complex numbers a, b, c, d is defined to be $(a, b, c, d) \equiv \frac{a-c}{a-d} : \frac{b-c}{b-d}$.

Using the cross ratio, we obtain,

$$d_{\mathbb{H}^2}(ia, ib) = \ln(ia, ib, \infty, 0).$$

Since Möbius transformations preserve cross ratio and the set of all circles and line, it follows that the hyperbolic distance between two points $z, w \in \mathbb{H}^2$ is

$$d_{\mathbb{H}^2}(z, w) = \ln(z, w, \widetilde{w}, \widetilde{z}),\ z, w \in \mathbb{H}^2$$

where $\widetilde{z}, \widetilde{w} \in \mathbb{R}$ and $z, w, \widetilde{w}, \widetilde{z}$ are in a circle perpendicular to $\mathbb{R}$. See figure below.
By using the isometrics $z \to \frac{az+b}{cz+d}$, we obtain,

Corollary 3.6. All geodesics in $\mathbb{H}^2$ are (portions of) vertical lines or circles perpendicular to the x-axis.

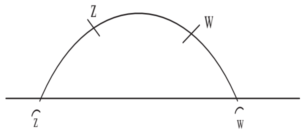


Figure 3.3

All of the above computations can be carried out without too much changes to any dimension. The outcome is the upper half space model of of $\mathbb{H}^n$. Here are the definitions. Let the upper half space in $\mathbb{R}^n$ be $\mathbb{H}^n = \{x = (x_1, ..., x_{n-1}, t) \in \mathbb{R}^n \mid t > 0\}$. Define a Riemannian metric on it by

$$ds^2 = (dx_1^2 + ... + dx_{n-1}^2 + dt^2)/t^2 = (d_E s)^2/t^2$$

Note that this Riemannian metric is obtained from the Euclidean metric $d_E s$ by multiplying by a positive function. Thus the notion of angles in both metrics are the same. This is termed *conformal* in Riemannian geometry and the upper half space model is conformal to the Euclidean geometry. The group of isometries of the Riemannian metric ds can be found as follows. The following maps can be seen from the definition that they preserve the Riemannian metric:

(1) $f(x) = \lambda x$ where λ is a positive real number,
(2) $g(z, t) = (Az, t)$ where $z = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}_{>0}$ and $A \in \mathbb{O}(n-1)$,
(3) $h(z, t) = (z + a, t),\ a \in \mathbb{R}^{n-1}$.

The inversion I about the unit sphere is an isometry of $\mathbb{H}^n$ where
(4) $I(x) = \frac{x}{\|x\|^2} = \frac{x}{(x,x)}$.

Lemma 3.7. The inversion I preserves the Riemannian metric ds in $\mathbb{H}^n$ and in particular preserve angles in $\mathbb{E}^n$.

The verification of lemma 3.7 is a generalization of the calculation that we performed above for n=2. We omit the details.

Among the properties of the inversion, the most interesting one is that the inversion I preserves the set of all (codimension-1) spheres and planes. Here is a proof. The equation of a codimension-1 sphere or a plane in $\mathbb{R}^n$ is given by

$$Ax \cdot x + Ba \cdot x + C = 0$$

where $A, B, C \in \mathbb{R}$, $a \in \mathbb{R}^n$, and $(x \cdot a)$ is the standard Euclidean inner product. Now replace x by $\frac{x}{(x \cdot x)}$. The above equation becomes, after a simple substitution,

$$A + Ba \cdot x + Cx \cdot x = 0.$$

It is an equation of a sphere or plane again.

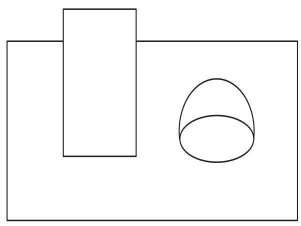
Remark 1. It can be shown that all isometries of $\mathbb{H}^n$ are compositions of (1),(2),(3) and (4).

Remark 2. A *Möbius Transformation* of $\mathbb{R}^n$ is defined to be a composition of the inversion I with $Ax + b,\ A \in \mathbb{O}(n),\ b \in \mathbb{R}^n$. Note that a Möbius transformation is a self bijective map of $\mathbb{R}^n \cup \{\infty\}$.

Remark 3. The inversion F about a sphere of radius r centered at c is the bijective map of $\mathbb{R}^n \cup \{\infty\}$ sending x to $r\frac{x-c}{||x-c||^2} + c$. Since F and I are conjugate under a linear transformation of the form $Ax + b$ where $A \in (O)(n)$, all properties of I still hold for F.

Remark 4. A **theorem of Liouville** says if $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ is open and connected, then for any angle preserving smooth embedding $f : \Omega \to \mathbb{R}^n$, there exists a Möbius transformation F of $\mathbb{R}^n$ so that $F|_\Omega = f$. It shows the drastic difference between dimension-2 and dimension at least 3. For in dimension-2, any injective holomorphic map is a smooth angle preserving embedding.

By exactly the same argument, we can now find all geodesics in the upper half space model $\mathbb{H}^n$. First using the same argument as in example 4, we prove that the positive t-axis $0 \times \mathbb{R}_{>0}$ is a geodesic in $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$. Then using isometries and the basic property of Möbius transformations, we conclude that all geodesics in $\mathbb{H}^n$ are either vertical lines or semi-circles perpendicular to $\mathbb{R}^{n-1} \times 0$. All totally geodesic submanifolds in $\mathbb{H}^n$ are hemi-spheres, planes perpendicular to $\mathbb{R}^{n-1} \times 0$.



In the 3-dimensional upper half space model of $\mathbb{H}^3$, a totally geodesic plane in $\mathbb{H}^3$ corresponds to a circle in the complex plane. Thus a convex hyperbolic polytope in $\mathbb{H}^3$ is given by a circle pattern in $\overline{\mathbb{C}}$. This is the starting point of Thurston's work relating circle packing with the 3-dimensional hyperbolic geometry in 1978.

§3.4. The Poincare Disc Model $\mathbb{B}^n$ of $\mathbb{H}^2$

Let $\sum = \{x \in \mathbb{R}^n \mid \ \| x - (0, ..., 0, -1) \| = \sqrt{2}\}$ be the sphere of radius $\sqrt{2}$ centered at $c = (0, .., 0, -1)$. The sphere $\sum$ intersects both the unit sphere $\partial\mathbb{B}^n = \{x| \, ||x|| = 1\}$ and the horizontal plane $\mathbb{R}^{n-1} \times 0$ at an angle $\pi/4$. An easily calculation shows that the inversion I_1 about $\sum$,

$$I_1(x) = \sqrt{2}\frac{x-c}{||x-c||^2} + c$$

sends the upper half space $\mathbb{H}^n$ bijectively onto $\mathbb{B}^n = \{x| \, ||x|| < 1\}$. Define a Riemannian metric on $\mathbb{B}^n$ by pulling back $(I_1^{-1})^*(ds)$. One can show that

$$(I_1^{-1})^*(ds) = \frac{4\sum_{i=1}^{n} dx_i^2}{(1 - \sum_{i=1}^{n} x_i^2)^2}.$$

It is highly recommended that the reader to carry out the calculation for $n = 2$. In this case, one can use the Möbius transformation $-\frac{z-i}{z+i}$ to replace I_1. The pair $(\mathbb{B}^n, (I_1^{-1})^*(ds))$ is called the Poincare disk model of the hyperbolic space. By using the basic properties of the Möbius transformation, we see that all geodesics in $\mathbb{B}^n$ are circular arcs and line segments perpendicular to the unit sphere $\partial\mathbb{B}^n$. All totally geodesic submanifolds are (portions of) spheres and planes perpendicular $\perp \partial\mathbb{B}^n$.

The advantage of using the Poincaré model is due to its symmetry. For instance, it is obvious from the expression of the Riemannian metric that a hyperbolic ball centered at the origin has to be a Euclidean ball. Using Möbius transformations, one then concludes that all hyperbolic balls are Euclidean balls (even though they may have different centers and radii).

Finally, we should say a word about the equivalences of all these different models. That the upper half space model and the Poincare models are isometric due to the construction. So are the Klein model and the hyperboloid model. The isometry from the hyperboloid model and the upper half space model can be established using Cartan's theorem in Riemannian geometry. Cartan's theorem says that any two simply connected, complete Riemannian metrics of constant curvature -1 are isometric. Evidently, both models are simply connected and complete (due to the infinite extendability of the geodesics). Finally, due to the knowledge of the isometry groups, we see that both have constant sectional curvature. It remains to verify that the constant curvature is -1. This can be done by calculating the curvature for $n = 2$ at single point. We leave it to the reader to verify this.