

Thur SFS Finite Cover

Conf Thurston finite cover. In particular, \exists surface group.

The Gromov Norm

refine: Thurston's note, chapter 6
Gosha: Gromov, bounded cohomology

§1 X topological space

$$S_n(X, \mathbb{Z}) = \text{singular } n\text{-chains} . \left\{ \sum_{i=1}^m x_i \sigma_i \mid x_i \in \mathbb{Z} \quad \sigma_i: \sigma^i \rightarrow X \right\}$$

$$S_n(X, \mathbb{R}) = \left\{ \mid x_i \in \mathbb{R} \right\}$$

$$c = \sum x_i \sigma_i \quad \text{its norm} \quad \|c\| = \sum |x_i|.$$

$$H_n(X, K) = \{ [c] \mid c \text{ } n\text{-chain w/ } K \text{ coefficient}, c \sim c' \text{ iff } c - c' = \partial d \}$$

Def (Gromov) The pseudonorm $\|[c]\| = \inf \{ \|c\| \mid c' \sim c \}$.

Eg. $H_1(S, \mathbb{Z}) \cong \mathbb{Z}$ generator $d = [a]$ $a: \overset{\sigma'}{\underset{\sigma}{[a_1, a_2]}} \rightarrow S'$ $a(t) = e^{2\pi i t}$

$$\begin{aligned} \|\alpha\| &= 0. \quad a \approx \frac{1}{n} a_n = c \quad a_n: [a_1, a_2] \rightarrow S' \\ &\Rightarrow \|\alpha\| \leq \inf \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_{>0} \right\}. \end{aligned}$$

Prop. 1 Basic property. $\alpha, \beta \in H_n(X, \mathbb{R})$

$$(1) \quad \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

$$(2) \quad \|\lambda \alpha\| = |\lambda| \|\alpha\|$$

$$(3) f: X \rightarrow Y \text{ continuous} \quad \|f_{*}(\alpha)\| \leq \|\alpha\|.$$

Proof Trivial

$\|\cdot\|$ Gromov pseudo-norm on $H_n(X, \mathbb{R})$

Question Is $\|\alpha\| \neq 0$ for some α ?

§2. M^n closed orientable manifold. $\Rightarrow H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. Its generator d_M is called the fundamental class of M .

Eg. 1 $d_{S^1} = [a]$

Eg. 2. $d_{S^1 \times S^1} = [a_1 + a_2]$: $a_1 \xrightarrow{S^1} \boxed{S^1}$ $a_2 \xrightarrow{S^1}$

Fundamental class = n -cycle which covers each pt of M^n algebraically exactly once.

Ex. 3 d_{S^n} : $a: S^n \rightarrow S^n$

$a/\!: \partial S^n \rightarrow N$ north pole.

$a: \partial S^n \rightarrow S^n \setminus \{N\}$ homeomorphism

Homework Prove this. $H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}[[d_{\text{S}^n}]]$.

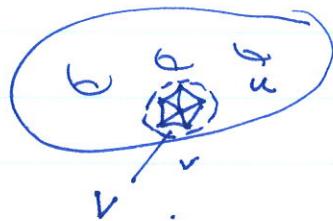
Ex. Σ_g orientable surface, with a triangulation. T :

$$\chi_{\Sigma} = [\sum_{\substack{\sigma_i \in T^{(1)} \\ \text{oriented}}} \sigma_i^2]$$



Proof. Take $v \in T^{(0)}$ a vertex $U = \Sigma - \{v\}$ $V = \text{open ball neighborhood of } v$

cancelling $\text{st}(v)$



By M-V

U, V

$U \cap V \cong S^1$ circle around v

(\mathbb{Z} coeff)

$$0 \rightarrow H_2(U \cap V) \xrightarrow{\partial} H_1(U \cap V) \xrightarrow{a} H_1(U) \oplus H_1(V) \xrightarrow{\text{II}} (i_*(\beta), -i_*(\beta))$$

Claim $\beta = 0$, $a: S^1 \hookrightarrow U \cap V$ inclusion

$$\underline{h([a]) = 0} \quad (a = \partial(\text{the surface}))$$

Or you see it from the π_1)

Thus $H_2(U \cap V) \xrightarrow[\partial]{\cong} H_1(U \cap V)$

Question Find a cycle whose boundary goes to $[a]$.

$$\sigma_u = \sum \sigma_i^2$$

$$\sigma_v = \sum \sigma_j^2$$

$$0 \rightarrow S_2(U \cap V) \rightarrow S_2(U) \oplus S_2(V) \rightarrow S_2(U \cap V) \rightarrow 0$$

$$c \mapsto (i_*(c), -i_*(c))$$

$$\begin{matrix} & s \\ & \downarrow \\ (\sigma_u, +\sigma_v) & \end{matrix}$$

$$\downarrow \sigma_u + \sigma_v$$

$$S_2(U \cap V) \rightarrow (\partial \sigma_u, -\partial \sigma_v)$$

$$(a) \rightarrow$$

done

if

Eg. $\sigma_{\Sigma_2} = \sum_{i=1}^6 \sigma_i^2$ with indicated



Def The Gromov norm of an orientable n -mfld, closed $\Rightarrow \|\sigma_m\|$.
 $\|M\| \stackrel{\Delta}{=} \|\sigma_m\|$

Eg. $\|S^1\| = 0$.

Degree, $F: M \rightarrow N$, $\deg(F) = d$: $f_*: H_n(M, \mathbb{Z}) \rightarrow H_n(N, \mathbb{Z})$ $f_*(d_m) = dd_n$

Prop 2 Lemma. $[\deg(f)] \cdot \|\sigma_{fN}\| \leq \|\sigma_M\|$. $[\deg(f)] \cdot \|N\| \leq \|M\|$

Proof

$$\|f_*(\sigma_m)\| \leq \|\sigma_{fN}\| = \|N\|$$

$$\|d \sigma_N\| = |d| \|N\|$$

Corollary If M^n admits a self map $f: M^n \rightarrow M^n$ of $[\deg(f)] > 1 \Rightarrow \|M\| = 0$

Eg. $\|S^n\| = 0$, $\|S^1 \times S^1\| = 0$, $\|\sum_g x_g S^1\| = 0$ □

Main Question What is $\|\Sigma_2\|$? Σ_g = closed orientable surface gen g.

Fix a hyperbolic metric d on Σ_2 .

Any regular 2-simplex $a: \sigma^2 \rightarrow \Sigma_2$ can be lifted

$$a: \sigma^2 \xrightarrow{\quad} \mathbb{H}^2 \downarrow \pi \xrightarrow{\quad} \Sigma_2$$



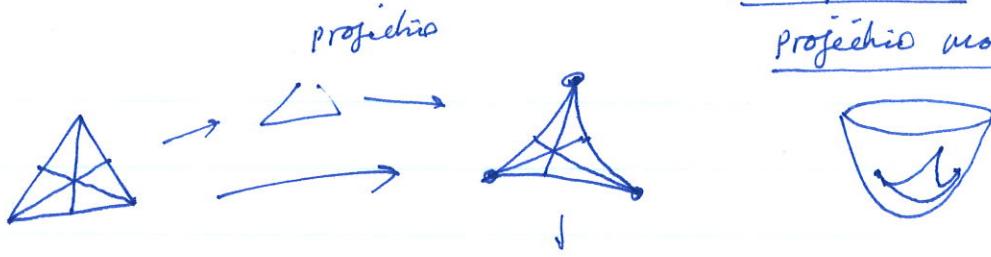
The straightening of a :

$$\hat{a}: \sigma^2 \rightarrow \mathbb{H}^2 : \hat{a}(v_i) = v_i^1 \quad v_i \text{ standard basis}$$

Then $\hat{a}(x_1, x_2, x_3) =$ canon hyperbolic triangle with
verts v_1^1, v_2^1, v_3^1

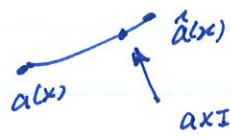
The simplest
projection model

-4-



Note $\alpha + \hat{\alpha}$ are canonically homotopic by

$(\alpha \times I)(x, t) =$ joint $\alpha(x)$ to $\hat{\alpha}(x)$ by geodesic segment



These construction is canonical sit.

① For $c = \sum x_i q_i \in S_d(\Sigma, d)$ cycle

$\tilde{c} = \sum x_i \tilde{q}_i$ after a 2-cycle homologous

$$(\partial \tilde{c}) = \alpha - \hat{\alpha} + \partial(c).$$

Conclusion Def A geometric chain: $c = \sum x_i \sigma_i$ $\sigma_i: \mathbb{P}^2 \rightarrow (\Sigma_g, \alpha)$
is a straighten chain.

straight

Cleary $\|c\| \geq \|\tilde{c}\|$ The straightness.

Now, back to Σ_2 (or in fact any closed hyperbolic metric)

If $\varepsilon > 0$, find straighten $\sum x_i q_i$ s.t. $\|\Sigma_2\| \geq \sum |x_i| - \varepsilon$

$$\begin{aligned} \text{Now } \text{Area}(\Sigma_2, d) &= \int_{\Sigma_2} dA = \sum_{q_i \text{ faces}} x_i \int_{q_i} dA \leq \sum |x_i| \cdot \pi = \pi \|\Sigma_2\| + \varepsilon \\ 2\pi/\chi(\Sigma_2) &= (4g-g)\pi \end{aligned}$$

$$\Rightarrow \underline{\|\Sigma_2\| \geq 4}$$

Gromov Norm

Let $v_n = \sup \{ \text{vol}(\text{hyperbolic } n\text{-simplex in } \mathbb{H}^n) \}$

Prop 1 $v_n < +\infty$

Thm (Gromov) 2. If (M^n, d) is a closed orientable hyperbolic n -mfld d.

then $\|M\| = \frac{\text{vol}(M^n, d)}{v_n}$.

(The most interesting application of this thm is in $\dim = 3$.

where 99% of 3-mflds are hyperbolic.

(For $n \geq 4$, \forall volume $K > 0 \exists$ only finitely many hyperboloids
closed mfld M^n $\text{vol}(M^n) < K$ X2. Wang)

Ex 1. $v_2 = \pi$

2. $v_3 = 3\Lambda(\frac{\pi}{3})$

$$\Lambda(x) = - \int_0^x \ln|2 \sin t| dt$$

Lobachevsky function

3.

Thm 3 (Haagerup-Munkholm) The regular ideal hyperbolic n -simplex has volume v_n . (= the maximum volume)

Acta Math. (47, 1981, no 1/-11) (My recent work: volume.)

I may give a proof of it for $n=3$

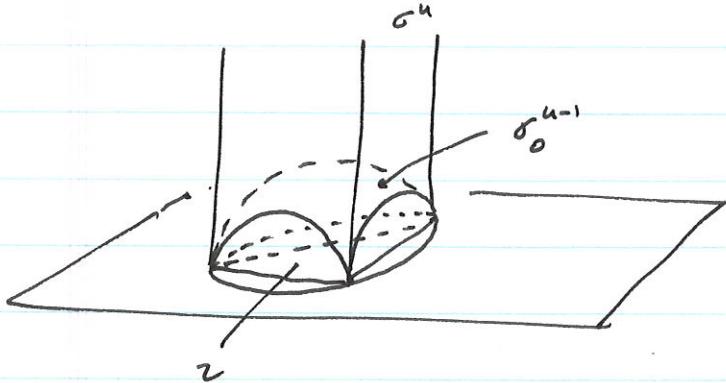
Prop 4. $\forall n \quad v_n < +\infty$ (All simplices are hyperbolic below)

Proof Every cpt n -simplex $\tau^n \subset \mathbb{H}^n \quad \tau^n \subset \sigma^n$ so idea, i.e all vertices are at $\partial \mathbb{H}^n$:



Thus, we may aim $\partial \sigma^n$. Put one vertex v_{12} of σ^n to do in

the upper half space model : $\mathbb{H}^n \rightarrow$



\mathbb{H}^n over a Euclidean
triangle $T^{n-1} \subset \mathbb{R}^{n-1} \times 0$

$$\subset \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{H}^n$$

We may come after
an isometry that

$$T^{n-1} \subset S^{n-1} \subset \mathbb{R}^n$$

unit sphere + vertex of $T^{n-1} \subset S^{n-1}$

$$dV_{\mathbb{H}} = \frac{dx_1 \cdots dx_{n-1} dx_n}{x_n^n} \quad (\text{metric } \frac{\sum dx_i^2}{x_n^2})$$

$$\text{vol}(\mathbb{H}^n) = \int_{\mathbb{H}} \int_{h(x_1 \cdots x_{n-1})}^{\infty} dV_{\mathbb{H}} = \int_{\mathbb{H}} \int_{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}}^{\infty} \left(\frac{dx_n}{x_n^n} \right) dx_1 \cdots dx_{n-1}$$

$$= \frac{1}{n-1} \int_{\mathbb{H}} \left(\frac{dx_1 \cdots dx_{n-1}}{\left(\sqrt{1-x_1^2-\cdots-x_{n-1}^2} \right)^{n-1}} \right)^{n-1} = \frac{1}{n-1} \int_{\mathbb{H}} \frac{dx_1 \cdots dx_{n-1}}{\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{\frac{n-1}{2}}}$$

Now, in the hyperbolic $(n-1)$ -space $S_+^{n-1} = \{x \in \mathbb{R}^n \mid |x|=1, x_n > 0\}$

the induced hyperbolic volume form, viewed on the disk

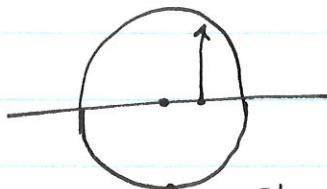
$$\mathbb{D}^{n-1} = \{x_1, \dots, x_{n-1} \in \mathbb{R}^{n-1} \mid \sum_{i=1}^{n-1} x_i^2 < 1\} \quad \text{is} \quad q(u) = (u, \sqrt{1-u^2})$$

Lemma 5.

$$\frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{\frac{n-1}{2}}} \quad \text{--- } q_f \uparrow$$

$$\leq \frac{1}{n-1} \int_{\mathbb{D}} \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{\frac{n-1}{2}}} = \binom{n-1}{n-1} \text{vol}(\mathbb{D}^{n-1}) \leq \left(\frac{2\pi}{n-1} \right) V_{n-1}$$

lemmas (Homework)



P: stereographic

projection from $(0, \dots, 0, -1)$ of S^{n-1} to \mathbb{R}^{n-1}

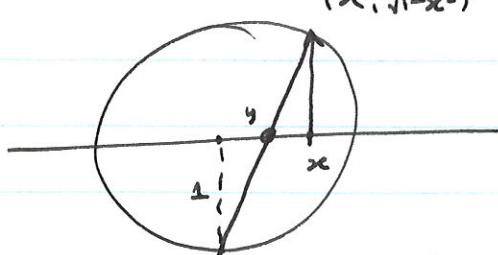
sending $\boxed{\mathbb{D}^n}$ S_+^{n-1} to \mathbb{D}^{n-1} . Consider the map $\phi(x) = (x, \sqrt{1-x^2})$

$\mathbb{D}^{n-1} \rightarrow S_+^{n-1}$ followed by p^{-1} : $F = p^{-1} \circ \phi: \mathbb{D}^{n-1} \rightarrow \mathbb{D}^{n-1}$. $y = F(x)$

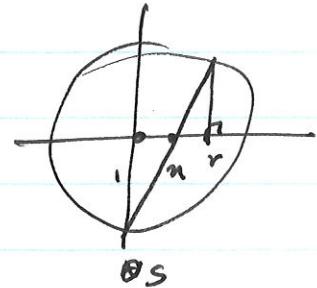
Then the

$$F^* \left(\frac{2^{n-1} dx_1 \dots dx_{n-1}}{\underbrace{\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{n-1}}_{\text{volume } (\mathbb{D}^{n-1}, d)}} \right) = \frac{dx_1 \dots dx_{n-1}}{\underbrace{\left(1 - \sum_{i=1}^{n-1} x_i^2 \right)^{\frac{n+1}{2}}}_{\text{hyperbolic.}}}$$

$n=1$



$$y = \frac{x}{1 + \sqrt{1-x^2}}$$



$$\Rightarrow y^1 = \frac{1}{\sqrt{1-x^2} (1 + \sqrt{1-x^2})}$$

Using polar coordinate: r radius $\theta \in S^{n-2}$ direction)

$$F(r, \theta) = \left(\frac{r}{1 + \sqrt{1-r^2}}, \theta \right) \quad u = \frac{r}{1 + \sqrt{1-r^2}} \quad u' = \frac{1}{\sqrt{1-r^2}(\sqrt{1-r^2} + 1)}$$

$$\text{volume form on } (\mathbb{D}^{n-1}, dr) \quad \frac{\alpha^{n-1} r^{n-2} dr d\theta}{(1-r^2)^{n-1}}$$

$$1-u^2 = \left(\frac{2\sqrt{1-r^2}}{1 + \sqrt{1-r^2}} \right)$$

$$\text{So } F^* \left[\frac{2^{n-1} u^{n-2} du d\theta}{(1-u^2)^{n-1}} \right] = 2^{n-1} \cdot \frac{r^{n-2}}{(1+\sqrt{1-r^2})^{n-2}} \frac{dr d\theta}{(1-r^2)(1+\sqrt{1-r^2})} \frac{2^{n-1} (\sqrt{1-r^2})^{n-1}}{(1+\sqrt{1-r^2})^{n-1}}$$

$$= \boxed{\frac{r^{n-2} dr d\alpha}{(1-r^2)^{n+2}}}$$

$$\underline{dx_1 \wedge \dots \wedge dx_{n-1} = r^{n-2} dr d\alpha}$$

Read Milnor: to see what for $n=3$

the volume of an ideal hyperbolic tetrahedron of dihedral angles a, b, c

$$\text{is } \Lambda(a) + \Lambda(b) + \Lambda(c), \quad a, b, c > 0 \quad a+b+c \leq \pi$$

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Straight simplex in hyperbolic manifold (M, d)

vertices v_1, \dots, v_{n+1}

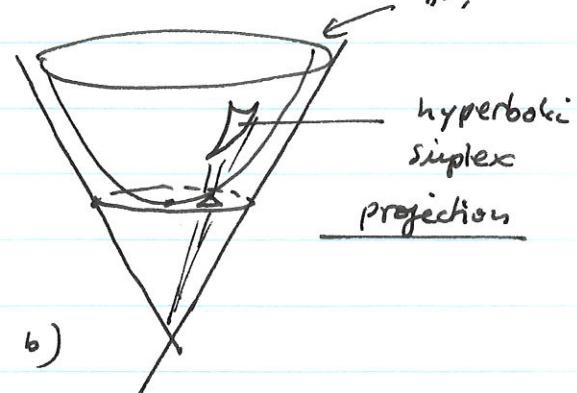
Recall $\sigma^n \subset \mathbb{R}^n$ Euclidean standard simplex (affine)

If $\tilde{\sigma}^n \subset \mathbb{H}^n$ a hyperbolic n -simplex, with vertices v_1, \dots, v_{n+1} , there exists a canonical map (may be collinear)

$$\Phi: \sigma^n \longrightarrow \tilde{\sigma}^n$$

Called the canonical parametrization of $\tilde{\sigma}^n$: by \mathbb{H}^n ,

taking the projective model.



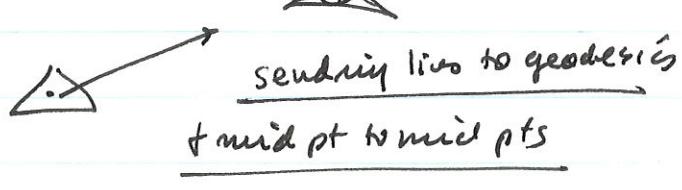
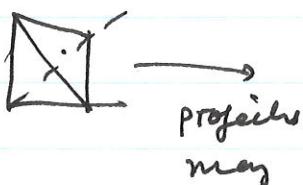
$$\sigma^n \rightarrow \tilde{\sigma}^n \subset D^n \times 1 \subset \mathbb{R}^{n+1}$$

projection

$$\text{The barycenter } \rightarrow (\tilde{\sigma}^n, \varphi) \xrightarrow[\text{projection map}]{} (\sigma^n, b)$$

The map is canonical

$$\boxed{\varphi(\partial_i \sigma^n) = \partial_i \tilde{\sigma}^n}$$



similar

For each n -simplex $f: \sigma^n \rightarrow (M^n, d)$, its straight part

$$\tilde{f}: \sigma^n \rightarrow (M^n, d)$$

is obtained as follows. Let $f: \sigma^n \rightarrow M$ to $\tilde{g}: \boxed{\sigma^n} \rightarrow H^n$

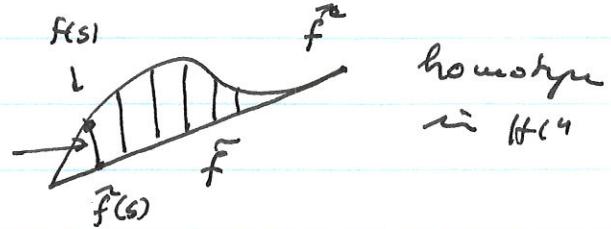
Let $\tilde{g}: \sigma^n \rightarrow H^n$ be a geometric straight simplex w/ the same

set of vertices as that of g . Let $\tilde{f} = \pi \circ \tilde{g}$. $\pi: H^n \rightarrow M^n$ be the

universal cover. Then, $f \simeq \tilde{f}$ canonical homotopy by going

the geodesic  segments.

geodesic
seg of
length



Next

\Rightarrow For each ~~simp~~ simplex n -chain $c = \sum x_i \sigma_i, i \in I^n$

its straight part is $\tilde{c} = \sum x_i \tilde{\sigma}_i$ is homologous to c

$$\tilde{c} - c = \underline{\partial c}$$

Thus, we may use straight chains to estimate Gromovian

Proof $\|M\| \otimes \geq \frac{\text{vol}(M)}{v_n}$

Proof as before

$$\forall \varepsilon > 0, \exists \text{ straight } c = \sum x_i \sigma_i, \sum |x_i| \leq \|M\| + \varepsilon \Rightarrow \frac{1}{v_n} \int_c \text{dvol} \geq \frac{1}{v_n} \int_M \text{dvol} =$$

$$\|M\| + \varepsilon \geq \sum |x_i| \geq \frac{1}{v_n} (\sum |x_i| v_n) \geq \frac{1}{v_n} \int_M \text{dvol} =$$

Gromov Norm

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Thuz. We see that, the same

Prop 3. If there is an upper bound for the volume of all n -simplices in H^1 , say U_n , then.

$$\|M\| \geq \frac{\text{vol}(M, \cdot)}{U_n}.$$

By exactly the same argument

The same proof $\Rightarrow \| \Sigma_g \| \geq 4g-4$.

Thuz 4 $\| \Sigma_2 \| = 4 \quad (\Rightarrow \| \Sigma_g \| = 4g-4)$

Proof \exists a $(g-1)$ -fold covering map

$$f: \Sigma_g \rightarrow \Sigma_2. \quad \underline{\text{see it}}$$

$$\underline{\text{Also}}, \quad \| \Sigma_g \| \leq 4g+4$$

$$\Rightarrow (g-1) \| \Sigma_2 \| \leq \| \Sigma_g \| \leq 4g+4$$

Let $g \rightarrow \infty$

$\Rightarrow \| \Sigma_g \|$ never realized by ?

done.

#

Thuz 5. If $f: M \rightarrow N$ d -fold covering of oriented subl. then

$$\|M\| = d \|N\|$$

Proof Known $\|M\| \geq d \|N\|$

To show the other way, $\forall \varepsilon > 0$. find $c = \sum_{i \in I} x_i \in \sigma_n$ st $\sum |x_i| \geq \|N\| + \varepsilon$.

Now, σ_i lifts to $\sigma_{i1}, \dots, \sigma_{id} \in M$

$\tilde{c} = \sum x_i (\sigma_{i1} + \dots + \sigma_{id})$ a fundamental cycle d_M

($f_*^{*}(c) = d \sigma$, $f^*[\sigma] = f_*$ injective).

$$\|M\| \leq \sum |x_i| \cdot d \leq d \|N\| + d \varepsilon.$$

Rm If M supports some geometric structure with $\sup \{ \text{Vol}(\sigma^n) \}$
 σ^n geometric simplex } $< +\infty \Rightarrow \|M\| > 0.$

" \Leftarrow " (Kuiper): $\|M^n\| \leq \frac{\text{Vol}(M)}{v_n}.$

Consider $M = \mathbb{H}^n/\Gamma$ $\Gamma = \pi_1(M)$ as the deck transformation group.

Let $p \in \mathbb{H}^n$, U a compact fundamental domain for Γ
 $d = \text{diam}(U)$

Ex. $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{H}^n = \mathbb{R}^2$, $p = (\frac{1}{2}, \frac{1}{2})$

Take $D > d$. a D -regular n -simplex in \mathbb{H}^n : regular simplex with
all edge length D .

$$\begin{aligned} \text{Vol}(M) &= \int_{\sum x_i \sigma_i} d\text{vol} = \sum x_i \int_{\sigma_i} d\text{vol} = \sum x_i \text{Vol}(\sigma_i) \\ &\leq \sum |x_i| v_n = v_n (\|M\| + \varepsilon) \end{aligned}$$

To make it close almost eq.

Need a chain, $c = \sum x_i \sigma_i$ s.t. $\begin{cases} c \subset d_M \\ x_i \geq 0 \\ \text{Vol}(\sigma_i) \sim v_n \end{cases}$

Thm of (Haagerup-Munkholm) $v_n = \text{Vol}(\text{regular ideal } n\text{-simplex})$

Easily fact: ①. $\text{Vol}(D\text{-regular } n\text{-simplex}) \xrightarrow{=} v_n \text{ as } D \rightarrow +\infty$

②. $\forall d > 0, \exists D' \text{ s.t. if } D > D', \text{ and}$
 $\lim_{D \rightarrow +\infty} \text{Vol}(\sigma^n) = v_n.$

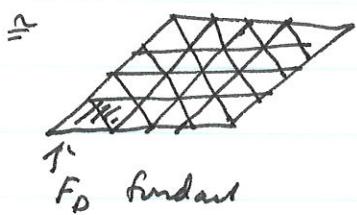
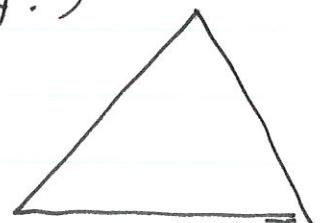
σ^n is a n -simplex where lengths $l_{n,1}$ are with $[D-d, D+d]$, then

$$|\text{Vol}(\sigma^n) - v_n| < \varepsilon. \quad (\text{Almost } D\text{-regular})$$

Thus, Kuiper was able to construct it explicitly.

(Roughly: cover M^n by regular n -simplices evenly!)

Torus. $S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z} + i\mathbb{Z}$ covered by



$$\mathbb{R}^2 = \mathbb{C} / \mathbb{Z}(1) \oplus \mathbb{Z} e^{(2\pi i / 6)}$$



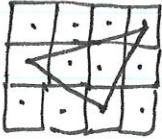
$$p \in U,$$

(1) An oriented good simplex: σ^n = hyperbolic simplex w/ vertex y_1, \dots, y_{n+1} $y_i \in P$

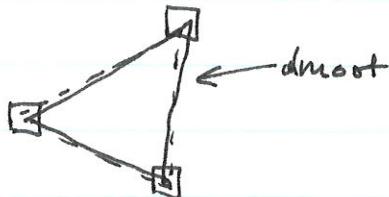
(2) Coeff $m(\sigma^n) =$ Haar measure { regular D -simplex τ^n with one vertex in each $y_i U$, } ^{simply}

Fix D

Eg.



σ^n



Haar measure: $S_D = \{ \text{all } D\text{-regular } n\text{-simplices in } H^n \} \xleftrightarrow{\text{onto}} \prod_{i=1}^n \text{Iso}(H^n)$

$$g \tau_0 \xrightarrow{\psi} g$$

Fix a τ_0 regular D -simplex

$\text{Iso}(H^n) =$ Orthonormal frames on H^n has a Lebesgue invariant under the left action (= right action) (up to multiplication)

Homework

$$\text{Iso}^+(H) = S^1 \times \mathbb{C} \quad (z + \mathbb{C} e^{i\theta} + b) \mapsto (e^{i\phi}, b)$$

The product measure on $S' \times \mathbb{C}^*$ is the Haar measure.

Define (infinite chain)

$$\tilde{F}_D = \sum_{\sigma_i \text{ good}} m(\sigma_i) \sigma_i \quad (\text{as a formal sum})$$

Countable terms

\exists D -regular simplex $\Rightarrow \tilde{F}_D$ exists

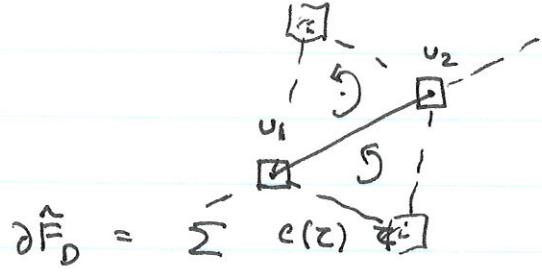
$$\textcircled{1} \quad \gamma \in P \quad \gamma_* \tilde{F}_D = \tilde{F}_D$$

(Haar measure invariant under action

$$\gamma \sigma_i = \gamma(r_1 p, \dots, r_{n+1} p) = (\gamma r_1 p, \dots, \gamma r_{n+1} p)$$

$$\textcircled{2} \quad \partial \tilde{F}_D = 0.$$

Eg 3 2-Dim.



$$\partial \tilde{F}_D = \sum c(z) \quad z \text{ segment from } r_1 p \rightarrow r_2 p$$

What are its coefficients

$$c(z) = m(D\text{-triangles with 2 vertices in } v_1, v_2) \text{ induced action}$$

- so that

- ~~$m(D\text{-tri})$~~ on z is from $r_1 p \rightarrow r_2 p$)

- $m(D\text{-triangles w/ 2 vertices in } v_1, v_2)$. So the induced orientation on z is from $r_1 p \rightarrow r_2 p$)

$\equiv 0$ by reflection on the line through z

The same \Rightarrow result.

Let $\sigma_1, \dots, \sigma_N$ be the set of all ~~regular~~^{good} simplexes s.t.

$$(1) \quad \sigma_i \neq r\sigma_j \quad \forall i \neq j$$

$$(2) \quad \forall \text{ good simplex } \exists \quad r \quad \text{s.t.} \quad \sigma = r\sigma_i$$

(a coset) $\underline{\text{def}} \quad F_D = \sum_{r \in r} r \left(\sum_{i=1}^N m(\sigma_i) \sigma_i \right)$

Defn. $F_D = \sum_{i=1}^N m(\sigma_i) (\pi_0 \sigma_i) \in S_n(M, \mathbb{R})$

Then F_D is a ~~cycle~~ by (2)

$$\begin{aligned} \|F_D\| &= \sum_{i=1}^N m(\sigma_i) = m(\text{Span of all D-regular n-simplexes}) \\ &= m(\text{Iso}^+(H^n)/P) \quad \text{definition of Haar.} \\ &= \text{vol}(M) = \end{aligned}$$

Finally!

$\partial F_D = 0$ so we see that

$$+ H_n(M, \mathbb{R}) = \mathbb{R} \quad F_D \in \partial K_D d_M, \quad k_D \in \mathbb{R}_{>0}$$

$$K_D \int_{F_D} R \, d\text{vol} = \int_{\partial K_D d_M} \Phi \, d\text{vol} = k_D \text{vol}(M)$$

$$\begin{aligned} \sum_{i=1}^N m(\sigma_i) \frac{\text{vol}(\sigma_i)}{| } &\cong \sum_{i=1}^N m(\sigma_i) V_n(D) \\ &= V_n(D) \text{vol}(M) \\ \text{almost D-regular D} \gg 1 \end{aligned}$$

So $\frac{K_D}{V_n(D)} \rightarrow 1$ as $D \rightarrow \infty$

~~$K_D \rightarrow V_n$~~ as $D \rightarrow \infty$

Now, we define the cycle to be

$$C_D = \frac{F_D}{V_n(D)} = \sum_{i=1}^N \left(\frac{m_D(\sigma_i)}{K_D} \right) \pi \cdot \sigma_i \in d_M$$

Let us see

$$\|M\| \leq \|C_D\| = \frac{1}{K_D} \sum_{i=1}^N m(\sigma_i) \xrightarrow[D \rightarrow \infty]{} \frac{\text{Vol}(M)}{V_n}$$

i.e.

$$\|M\| \leq \frac{\text{Vol}(M)}{V_n}$$

#

R.M Remark if one defines a modified Gromov norm

using cubes I^n instead of simplices σ^n . We have a "new"

$$\|M\|' = \inf \left\{ \sum |x_i| \mid \sum_i c_i \in d_M, c_i: I^n \rightarrow M \right\}$$

Same Proof

$$\begin{aligned} \|M\|' &= \frac{\text{vol}(M)}{\sup \{ \text{vol}(I^n) \mid I^n \text{ hyperbolic } n\text{-cube} \}} \\ &= \frac{\text{vol}(M)}{\text{vol}(n\text{-regular ideal } n\text{-cube})} \end{aligned}$$

Question ① Is it true that

$$\|M \times N\|' = \|M\|' \|N\|'$$



cylinder

clearly \exists constant c_n

$$\frac{1}{c_n} \|M\|$$

$$\leq \|M'\|' \leq c_n \|M\|$$

$$\Delta \leftarrow \square \text{ colby cause}$$



① Gromov Proved: $\exists c_{n>0}$ st

$$\frac{1}{c_n} \|M_n\| \times \|N^n\| \leq \|M^n \times N^n\| \leq c_n \|M^n\| \|N^n\|$$

② Is it true that $\|M \times N\|' = \|M\|' \times \|N\|'$ (my question)

One way is easy: $\|M \times N'\| \leq \|M'\| \times \|N'\|$

Indeed, $\forall \varepsilon > 0$, $\exists \sum x_i d_i \in \alpha_M$ $\sum |x_i| \leq \|M\| + \varepsilon$
 $\sum y_j d_j \in \alpha_N$ $\sum |y_j| \leq \|N\| + \varepsilon$

$$\Rightarrow \sum_{i,j} x_i y_j \cdot c_i \times d_j \in \alpha_{M \times N}$$

$$\begin{aligned} \|M \times N\|' &\leq \sum (|x_i| |y_j|) = (\sum |x_i|)(\sum |y_j|) \\ &\leq (\|M\| + \varepsilon)(\|N\| + \varepsilon) \quad \text{let } \varepsilon \rightarrow 0 \end{aligned}$$

Main Question

What is $\|\sum_2 \times \sum_2\|' = ?$

Recent paper in Topology: 2005

Google: Gromov norm of product.