

Ergodicity of Geodesic flow on hyperbolic surface.

Thm 1. $\Gamma \subset PSL(2, \mathbb{R})$ discrete s.t $m(\rho \setminus PSL(2, \mathbb{R})) < +\infty$. Then the geodesic flow on $T^1(\rho \setminus \mathbb{H}) = \rho \setminus PSL(2, \mathbb{R})$ is ergodic

Pf. If $B \subset T^1\Sigma$ ($\Sigma = \rho \setminus \mathbb{H}$) is an invariant measurable set. \Leftrightarrow let

$f(x) = \chi_B(x)$ be the characteristic function of B , so

$$f(x \cdot g) = f(x) \quad \forall g \in A = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a > 0 \right\} \quad (1)$$

Goal $f = 0$ (or 1) a.e.

Known. $PSL(2, \mathbb{R})$ acts from right transitively on $\rho \setminus PSL(2, \mathbb{R}) = X$

It suffices to show $\forall g \in PSL(2, \mathbb{R}) \quad f(x \cdot g) = f(x)$ a.e.

But $PSL(2, \mathbb{R})$ generated by A . $A^+ = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \right\} \cup A^- = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}$

It suffices to show $\forall h \in A^\pm$

$$f(x \cdot h) = f(x) \quad \text{a.e.}$$

Now take $h \in A^-$ (say) $g = \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \quad a > 1 \quad n \gg 1 \in A$

Then

$$\int_X |f(x \cdot h) - f(x)| dm_x = \int_X |f(y \cdot g^n \cdot h \cdot g^{-n}) - f(y)| dm_y$$

$$= \int_X |f(y \cdot g^n \cdot h \cdot g^{-n}) - f(y)| dm_y$$

Now we know: $\underbrace{g^n h \cdot g^{-n}}_{\gamma_n} \rightarrow id \quad \text{As } n \rightarrow \infty$

Lemma $\forall \alpha \in L^1(X)$

$$\int_X |\alpha(y \cdot \gamma_n) - \alpha(y)| dm_y \rightarrow 0$$

Pf Step 1 $\alpha \in L^1(X)$ & continuous w.r.t. ~~weak topology~~.

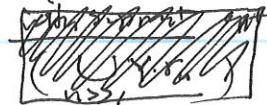
Then (i) $\alpha(y \cdot \gamma_n) - \alpha(y) \rightarrow 0$ pointwise

(ii) $|\alpha(y \cdot \gamma_n) - \alpha(y)| \leq 2M \quad M = \max(|\alpha|)$

2M

So $|\alpha(y \cdot \gamma_n) - \alpha(y)|$ is dominated by a L^1 function

\Rightarrow Lebesgue dominated convergence \Rightarrow done



Ergodicity of Geodesic Flow

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(2) Now continuous ~~approx~~^{are} function are dense in $L^1(X)$ (general fact).

So $\forall \alpha \in L^1(X)$, $\forall \varepsilon > 0$. $\exists \beta$, continuous ~~approx~~^{continuous} s.t. $\|\alpha - \beta\|_{L^1} < \varepsilon/3$.

For β . $\exists N$ s.t. $n \geq N$ $\int_X |\beta(x \cdot r_n) - \beta(x)| dm < \varepsilon/3$

Now

$$\begin{aligned} \int_X |\alpha(x \cdot r_n) - \alpha(x)| dm &\leq \int_X |\alpha(x \cdot r_n) - \beta(x \cdot r_n)| dm \\ &\quad + \int_X |\beta(x \cdot r_n) - \beta(x)| dm + \int_X |\beta(x) - \alpha(x)| dm \end{aligned}$$

$$y = x \cdot r_n$$

$$\overline{dm_y} = dm_x, \quad \int_X |\alpha(y) - \beta(y)| dm_y + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

\Rightarrow the result. □

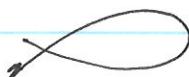
Corollary: The set of all simple geodesics in $T^* \Sigma$ has zero measure.

Pf $W = \{v \mid \text{geodesic through } v \text{ not self intersect}\}$

Clearly $g_t W = W$ invariant. Why is W measurable?

Reason: W^c is open! Open sets are measurable.

Indeed if $v \in W^c$, then for a $w' \in$ small neighborhood of v , geodesic $g(w')$



self intersects

$$\Rightarrow m(W^c) > 0 \Rightarrow m(W) = 0$$

Eg Back to the proof of ergodicity of $\mathbb{Z}(r)$ action on S^1 $r(z) = z \cdot e^{2\pi i \varphi}$. $\varphi \in \mathbb{R} \cup \{\infty\}$

The same as above: $f = \chi_B$ Find $n_i \rightarrow \infty$ s.t. $r^{n_i} \rightarrow \infty \in \mathbb{C}^\times$

density

$$\|f(x) - f(x \cdot r^{n_i})\| \rightarrow 0 \quad \forall \varphi$$

$$= \lim_{n_i \rightarrow \infty} \|f(x) - f(x \cdot r^{n_i})\| = 0$$

Several More basic ergodic theorem (general).

Let (X, μ) probability space; $\varphi: X \rightarrow X$ μ preserving bijection. Thus φ defines an action of \mathbb{Z} on X .

$\forall f \in L^1 \rightarrow \varphi$ measurable, define

$$T(f)(x) = f(\varphi(x))$$

Note if $f \in L^1(X)$ (or $L^2(X)$)

$$\|f\|_{L^1} = \|\tau(f)\|_{L^1} \quad \text{or} \quad \|f\|_{L^2} = \|T(f)\|_{L^2}$$

$$\|\tau(f)\|_{L^1} = \int_X |f(\varphi(x))| d\mu_x \quad \frac{y = \varphi(x)}{d\mu_y = d\mu_x} \quad \int_X |f(y)| d\mu_y = \|f\|_{L^1}$$

Eg Boltzmann's ergodic hypothesis:  $\varphi(x) = \text{time } t=1 \text{ position}$

$$f \in L^1(X) \quad \underset{\text{space avg}}{\int_X f d\mu} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) \quad \underset{\text{time avg}}{\curvearrowright} \quad \text{can be computed, obsrv!}$$

Thm (Von Neumann's ergodic theorem) Assume $\varphi: (X, \mu) \rightarrow (X, \mu)$ ergodic.

Then

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n+1} \sum_{i=0}^n T^i(f) - \left(\int_X f d\mu \right) \right\|_{L^2} = 0$$

$\forall f \in L^2(X)$.

$\mathcal{H} = L^2(X)$ Hilbert space; $T: \mathcal{H} \rightarrow \mathcal{H}: f \mapsto T(f)$ linear, isometry (unitary)

Thm (Abstract Ergodic theorem) Let \mathcal{H} be a Hilbert space and $T: \mathcal{H} \rightarrow \mathcal{H}$ be unitary. Let $\mathcal{H}_0 = \{x \in \mathcal{H} \mid Tx = x\}$ fixed space of T and $P: \mathcal{H} \rightarrow \mathcal{H}_0$ orthogonal projection. Then

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n+1} \sum_{i=0}^n T^i f - Pf \right\| = 0$$

Note ergodicity of $\varphi \Rightarrow Tf = f \Rightarrow f = \text{const function } P(f) = \left(\int_X f d\mu \right)_X$

Pf step 1. if $f \in \mathcal{H}_0$, then $P(f) = f + \frac{1}{n+1} \sum_{i=0}^n T^i(f) = f$ done

Step 2 It suffices to prove the result for $x \in \mathcal{H}_0^\perp$

$$\begin{aligned}
 (2.1) \quad & \text{if } y = Tx - x \quad x \in \mathcal{H} \quad (\text{Note } y \in \mathcal{H}_0^\perp \Leftrightarrow \langle y, v \rangle \\
 &= \langle Tx - x, v \rangle = \langle Tx, v \rangle - \langle x, v \rangle \\
 &= \langle x, T^*v \rangle - \langle x, v \rangle = \langle x, v \rangle - \langle x, v \rangle \\
 &= 0) \quad T^*T = \text{id}, \quad Tu = u
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } T^i y &= T^{i+1}x - T^i x \\
 \Rightarrow \frac{1}{n+1} \sum_{i=0}^n T^i y &= \frac{1}{n+1} [T^{n+1}(x) - x] \quad \text{bounded} \Rightarrow \underline{\text{done}}
 \end{aligned}$$

(2.2) The set $W = \{y = Tx - x \mid x \in \mathcal{H}\}$ is dense in \mathcal{H}_0^\perp , i.e. $W^\perp = \mathcal{H}_0$.

Indeed,

$$\begin{aligned}
 &\text{if } v \in \mathcal{H} \text{ s.t. } \langle v, Tx - x \rangle = 0 \quad \forall x \in \mathcal{H} \\
 \Leftrightarrow &\langle T^*v, x \rangle - \langle v, x \rangle = 0 \quad \forall x \\
 \Leftrightarrow &T^*v = v \quad \text{for all} \\
 \Leftrightarrow &TT^*v = T^*v \quad TT^* = \text{id unitary} \\
 \Leftrightarrow &v = Tu \quad \underline{\text{i.e.}} \quad v \in \mathcal{H}_0.
 \end{aligned}$$

□

Thus (Birkhoff's ergodicity) $\varphi: (X, \mu) \rightarrow (X, \mu)$ ergodic, (X, μ) probability space

$$\forall f \in L^1(X) \Rightarrow \text{a.e. } x \in X \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) = \int_X f d\mu$$

Pf It suffices to prove it for $0 \leq f \leq 1$.

$$\text{let } A_n(x) := \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) \quad \bar{A}(x) = \overline{\lim_{n \rightarrow \infty} A_n(x)}$$

Goal

$$\int_X \bar{A} d\mu \leq \int_X f d\mu$$

$$\boxed{\bar{A}(x) = \bar{A}(\varphi^N(x)) + N}$$

$$\begin{aligned}
 (\text{Indeed. Apply it to } 1-f \Rightarrow \int_X \bar{A} d\mu \geq \int_X f d\mu \Rightarrow \int_X (\bar{A} - \underline{A}) d\mu \leq 0)
 \end{aligned}$$

$$\Rightarrow \bar{A} = \underline{A} \quad \underline{\text{a.e.}}$$

Birkhoff Ergodic Thm

Birkhoff's ergodic thm $\varphi: (X, \mu) \rightarrow (X, \mu)$ measure preserving (X, μ)

probability space. Then $\forall f \in L^1(X)$, there \bar{f} a.e. $x \in X$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) = \bar{f}(x) \text{ exists}$$

$$\text{S.t. } \bar{f}(\varphi(x)) = \bar{f}(x) \text{ a.e. } \int_X \bar{f} d\mu = \int_X f d\mu$$

In particular, if φ ergodic $\Rightarrow \bar{f}(x) = \int_X f d\mu$.
(M. Keane)

PF. It suffices to show for $0 \leq f \leq 1$. Let

$$A_n(x) = \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) \quad \bar{A}(x) = \lim_{n \rightarrow \infty} A_n(x)$$

Goal

$$\int_X \bar{A}(x) d\mu \leq \int_X f d\mu \quad (1)$$

$$\text{Note } \bar{A}(\varphi^i(x)) = \bar{A}(x) \quad (2)$$

Indeed, if so, apply (1) to $1-f \Rightarrow \int_X \bar{A}(x) d\mu \geq \int_X f d\mu \quad (2)$

$$\Rightarrow \int_X (\bar{A}(x) - A(x)) d\mu = 0 \Rightarrow \bar{A}(x) = A(x) \text{ a.e.}$$

Proof of (1) $\forall \varepsilon > 0$, define

$$\tau(x): X \rightarrow \mathbb{Z}: x \mapsto \min \{ n \mid A_n(x) \geq \bar{A}(x) - \varepsilon \}$$

τ is measurable.

Case 1 τ is essentially bounded, i.e., $\exists M > 0$ s.t. for a.e. $x \in X$

$$\tau(x) \leq M \quad (3)$$

$$\Rightarrow \forall \text{ a.e. } x \in X \quad \tau(\varphi^i(x)) \leq M \quad \forall i.$$

i.e. \forall a.e. $y \in X$, $\exists i \leq M$ s.t

$$f(y) + f(\varphi(y)) + \dots + f(\varphi^i(y)) \geq (i+1)(\bar{A}(y) - \varepsilon) \quad (4)$$

(3) \Rightarrow for a.e. $x \in X$ $\tau(\varphi^i(x)) \leq M$.

Now for these x and $n \in \mathbb{Z}_{>0}$,

decompose the orbit $x, \varphi(x), \dots, \varphi^n(x)$ as

$$\begin{aligned}
 & \varphi^0(x), \dots, \varphi^{i_1}(x), \quad \text{s.t.} \quad f(x) + \dots + f(\varphi^{i_1}(x)) \geq (i_1 + 1) (\bar{A}(x) - \varepsilon) \\
 0 \leq i_1 \leq M \quad & + \\
 & \varphi^{i_1+1}(x), \dots, \varphi^{i_2}(x), \quad \text{s.t.} \quad f(\varphi^{i_1+1}(x)) + \dots + f(\varphi^{i_2}(x)) \geq (i_2 - i_1) (\bar{A}(\varphi^{i_1}(x)) - \varepsilon) \\
 i_2 - i_1 \leq M \quad & : \\
 & \vdots \\
 & \varphi^{i_{r-1}+1}(x), \dots, \varphi^{i_r}(x), \quad \text{s.t.} \quad f(\varphi^{i_{r-1}+1}(x)) + \dots + f(\varphi^{i_r}(x)) \geq (i_r - i_{r-1}) (\bar{A}(\varphi^{i_{r-1}}(x)) - \varepsilon) \\
 i_r - i_{r-1} \leq M \quad & \\
 & \varphi^{i_r+1}(x), \dots, \varphi^n(x) \quad (n - r \leq M)
 \end{aligned}$$

so let sum up all above + use $\bar{A}(\varphi^j(x)) = \bar{A}(x)$

$$\begin{aligned}
 \Rightarrow \sum_{i=0}^n f(\varphi^i(x)) & \geq \sum_{j=0}^r (i_j - i_{j-1}) (\bar{A}(x) - \varepsilon) \\
 & = i_r (\bar{A}(x) - \varepsilon) \\
 & \geq (n - M) (\bar{A}(x) - \varepsilon)
 \end{aligned}$$

divide by $n + 1$ + integrate

$$\begin{aligned}
 \int_X \frac{1}{n+1} \sum_{i=0}^n f(\varphi^i(x)) dx & \geq \left(\frac{n-M}{n+1} \right) \left(\int_X \bar{A}(x) dx - \varepsilon \right) \\
 \text{Let } n \rightarrow \infty \quad & \downarrow \text{Neumann} \\
 \int_X f d\mu & \geq \int_X \bar{A}(x) dx
 \end{aligned}$$

Case 2 ε is Not essentially bounded.

$$\exists M > 0 \text{ st } \mu(\{x | \varepsilon(x) > M\}) < \varepsilon \quad (\text{due to } \mu(x) = 1)$$

$$\text{let } f' \triangleq f + g \quad g = \chi_{\{x | \varepsilon(x) > M\}}$$

$$A'_n(x) = \sum_{i=0}^n f'(x) \geq A_n(x)$$

$$\underline{\text{Claim}} \quad \min \{ n | A'_n(x) \geq (\bar{A}(x) - \varepsilon) \} \leq M$$

Indeed, if $\tau(x) \leq M \Rightarrow A_n'(x) \geq A_n(x)$ done definition
 if $\tau(x) > M$, $A_0(x) = f(x) + g(x) \geq 1 \geq (1-\varepsilon)$.

By the same argument

$$(n+1) A_n'(x) \geq (n-M) (\bar{A}(x) - \varepsilon)$$

$$\text{Divide + integrate} \quad \int_X f' d\mu \geq \int_X \bar{A} d\mu - \varepsilon$$

$$\int_X f d\mu + \varepsilon$$

$$\Rightarrow \int_X f d\mu \geq \int_X \bar{A} d\mu - 2\varepsilon. \quad \square$$