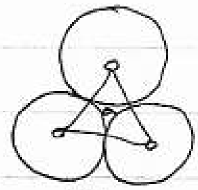
Extra Lectures ~~notes~~ 11-12 7-10-2012

Thurston's proof of uniqueness (Marden-Rodin)

If $\gamma : V \to \mathbb{R}_{>0}$ and $\tilde{\gamma} : V \to \mathbb{R}_{>0}$ be two normalized CP metrics

$\gamma(v_i) = \tilde{\gamma}(v_i) = 1 \quad i = 1, 2, 3$



all inside

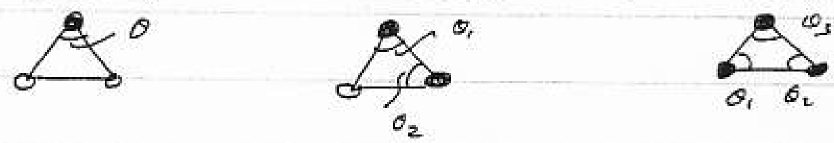
Suppose $\gamma \neq \tilde{\gamma}$. let $V_+ = \{ v \in V \mid \gamma(v) > \tilde{\gamma}(v) \}$, $V_- = \{ v \in V \mid \gamma(v) \leq \tilde{\gamma}(v) \}$

Claim. θ, $\tilde{\theta}$ the corresponding angles in γ and $\tilde{\gamma}$.

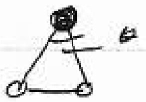
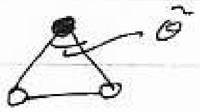
$$\sum_{\substack{v \in V_+ \\ \theta \text{ at } v}} \theta \;\underset{=}{\neq}\; \sum_{\substack{v \in V_+ \\ \tilde{\theta} \text{ at } v}} \tilde{\theta} = 2\pi \#(V_+). \quad \underline{\text{But}}$$

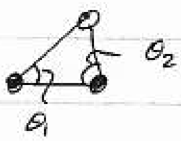
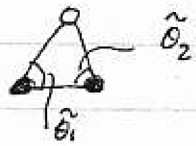
$\bullet \in V_+ \quad \circ \in V_-$

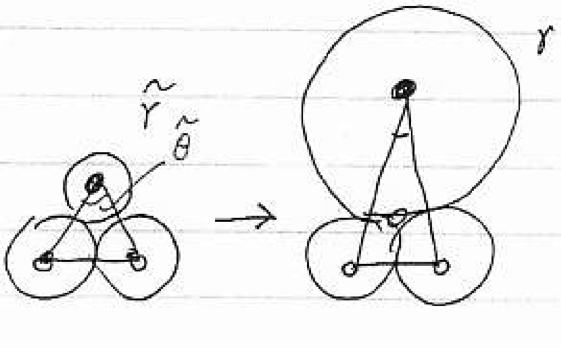
The sum $= \sum \theta + \sum (\theta_1 + \theta_2) + \sum (\theta_1 + \theta_2 + \theta_3)$



$=$

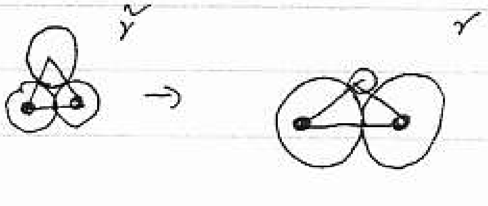
Lemma (1). In  , $\theta < \tilde{\theta}$ 

(2). In  , $\theta_1 + \theta_2 < \tilde{\theta}_1 + \tilde{\theta}_2$ 



$\underline{\tilde{\theta} > \theta}$

$\boxed{\text{done}}$

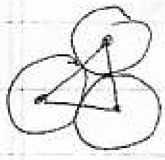


$\tilde{\theta}_1 + \tilde{\theta}_2 > \theta$

de Verdier on circle packing

<u>Thm</u> Let Δ be a Euclidean triangle of edge lengths $r_i + r_j = l_k$ and angles $\theta_i\ \theta_j\ \theta_k$

<u>then</u> $r_j \dfrac{\partial \theta_i}{\partial r_j} = r_i \dfrac{\partial \theta_j}{\partial r_i} > 0$ i.e. $\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial \theta_j}{\partial u_i} > 0 \quad i \neq j \quad u_i = \log r_i$



~~Thm~~ Furthermore, the matrix $-\left[\dfrac{\partial \theta_i}{\partial u_j}\right]_{3\times 3}$ is positive semi definite $\forall\ (u_1, u_2, u_3) \in \mathbb{R}^3$.

<u>Corollary.</u> $\exists$ a function $F(u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}$ s.t $\dfrac{\partial F}{\partial u_i} = \theta_i$ and

(1). F is concave in u

(2). $F(u + \lambda(1,1,1)) = F(u) + \lambda\pi$

(3). F is strictly concave on $u_1 + u_2 + u_3 = 0$, <u>or</u> if $p - q \notin \lambda(1,1,1)$, then

$g(t) = F(tp + (1-t)q)$ is strictly concave in t.

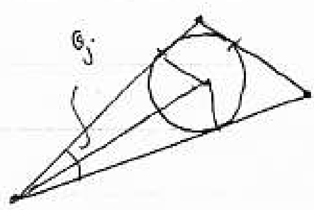
<u>Proof</u> Check symmetry

$$r_j \frac{\partial \theta_i}{\partial r_j} = r_j \left[\frac{\partial \theta_i}{\partial l_i}\frac{\partial l_i}{\partial r_j} + \frac{\partial \theta_i}{\partial l_j}\frac{\partial l_j}{\partial r_j} + \frac{\partial \theta_i}{\partial l_k}\frac{\partial l_k}{\partial r_j}\right] \qquad l_k = r_i + r_j$$

$$= r_j \left(\frac{\partial \theta_i}{\partial l_i} + \frac{\partial \theta_i}{\partial l_k}\right)$$

$$= r_j \frac{\partial \theta_i}{\partial l_i}\left(1 - \cos(\theta_j)\right) \qquad \text{(Lemma)}$$

$$= r_j \frac{l_i}{2A} \cdot 2 \cdot \sin^2\left(\frac{\theta_j}{2}\right) \qquad \text{(sine law)}$$



$$= \frac{r_j}{B} \sin(\theta_i) \sin^2\left(\frac{\theta_j}{2}\right)$$

$$= \frac{1}{2B}\, \underbrace{r_j \left[\frac{\sin(\theta_j/2)}{\cos(\theta_j/2)}\right]} \cdot \boxed{\cancel{\sin^2(\tfrac{\theta_i}{2}) \sin^2(\tfrac{\theta_j}{2}) = \tfrac{2}{B} R}} \cdot \left[\sin(\theta_i)\sin(\theta_j)\right]$$

$$= \frac{1}{2B} \cdot (\text{inscribed circle radius}) \ \sin(\theta_i) \sin(\theta_j) > 0$$

+ symmetric.

Now, the sum of each row $\dfrac{\partial \theta_1}{\partial r_i} + \dfrac{\partial \theta_2}{\partial r_i} + \dfrac{\partial \theta_3}{\partial r_i} = 0 \Rightarrow$ result.

<u>Lemma</u> $a_{ij} = a_{ji} > 0 \quad i \neq j \qquad a_{ij} + a_{ii} + a_{in} = 0 \quad \Rightarrow \quad -[a_{ij}]$ semipositive definite of rank 2. where null eigenvector is $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, two other eigenvalues > 0

<u>RM</u> $a, b, c > 0$ $\begin{bmatrix} a+b & -b & -a \\ -b & b+c & -c \\ -a & -c & c+a \end{bmatrix}$ <u>det = 0</u> $a+b > 0$

$\begin{vmatrix} a+b & -b \\ -b & b+c \end{vmatrix} \geq 0 \Rightarrow$ <u>done</u>.

$\Rightarrow \quad v \cdot A \cdot v \geq 0 \quad$ w/ $\quad v \cdot A \cdot v > 0$ if $v \neq (\lambda, \lambda, \lambda)$.

<u>Proof of the corollary</u>

<u>Calculus</u> $F(x_1, \cdots, x_n)$ smooth on $\mathbb{R}^n$ $\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] \geq 0 \Rightarrow F$ convex

$$F(\lambda p + (1-\lambda) q) \leq \lambda F(p) + (1-\lambda) F(q) \quad 0 \leq \lambda \leq 1$$

$u + \lambda(1,1,1)$, u $\Rightarrow$ similar triples $\theta_1\ \theta_2\ \theta_3$ the same!

$$F(u) = \int_0^u \sum_{i=1}^3 \theta_i \, du_i$$

Finally, $g'(t) = \nabla F \cdot (p-q) \qquad g''(t) = (p-q) \cdot H(f) \cdot (p-q) \overset{+}{>} 0.$ $\square$

Lecture 5. <u>Proof of AKT thm</u>

<u>Proof</u>. Given any $r: \mathbb{R}^V \longrightarrow \mathbb{R}_{>0}$ or $u: \mathbb{R}^V \to \mathbb{R}$, $(u(v) = \lg r(v))$

Construct $W(u): \mathbb{R}^V \to \mathbb{R}$ $\qquad u_i = u(v_i)$

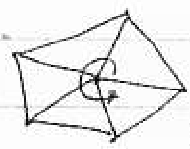
$$: \sum_{\triangle v_i v_j v_k} F(u_i, u_j, u_k)$$

Then W is concave in u, since it is the sum of concave function

From $u \Rightarrow$ polyhedral metric on Δ:

<u>Lemma</u>. $\dfrac{\partial W}{\partial u_i}$ = sum of angles at v_i

<u>Definition</u>.



<u>Now, we prove uniqueness</u>: suppose r', r'' two CP with isomorphic

nerves normalized s.t $r_1' = r_2' = r_3' = r_1'' = r_2'' = r_3'' = 1$. Then their cycle such u', u''

Are the same $\Rightarrow$ $\nabla W(u') = \nabla W(u'') = \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 2\pi, \cdots, 2\pi\right)$

Consider $g(t) = \boxed{} - W(tu' + (1-t)u'') \qquad 0 \le t \le 1$.

<u>First</u> (1) $g(t)$ convex in $[0,1]$.

(2) $g(t)$ smooth in t

(3) $g(t)$ is Not linear in t <u>Since</u> $u'' - u' \ne \lambda(1, \cdots, 1)$

<u>To see (3)</u>. $F(t u_i' + (1-t) u_i'', \cdots)$ is not linear in t

<u>Unless</u> they are propositional (Move again) due to $u_i' = u_i'' = 1$ $(i = 1, 2)$

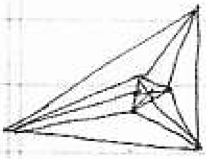
$\Rightarrow u' = u''$

$\Rightarrow \quad g''(t) \ge 0$ and $g''(t_0) > 0 \Rightarrow g'(0) < g'(1)$

But $g'(0) = g'(1) = \nabla W(u') \cdot (u' - u'')$

C.P

<u>Existence</u>. All such G is isomorphic to 1-skeleton of a <u>Geometric</u> $\mathcal{T}$ tryulation of Δ



Let α_i's be the inner angles of all triangles in $\mathcal{T}$.

(1) $\alpha_i + \alpha_j + \alpha_k = \pi$ if i, j, k inside one type $\sigma \in$

(2) $\displaystyle\sum_{\text{at } v} \alpha_i = \begin{cases} 2\pi & v \text{ interior} \\ \frac{\pi}{3} & v \text{ boundary} \end{cases}$

(Angle struct

<u>Lemma</u> (Verdierer) Suppose $a_1, a_2, a_3 \geq 0$ $a_1 + a_2 + a_3 = \pi$, Then the function $\hat{F}(u_1, u_2, u_3) = F(u_1, u_2, u_3) - \sum_{i=1}^{3} a_i u_i : \underbrace{\{u_1 + u_2 + u_3 = 0\}}_{P} \to \mathbb{R}$

has a unique minimal point $\hat{u}$ so that $\lim_{\substack{u \to \infty \\ u \in P}} \hat{F}(u) = -\infty$ $\max |u_i - u_j| \to +\infty$

<u>Proof</u> Let $(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in P \rightsquigarrow$ the Euclidean triangle of inner angs $a_1, a_2, a_3 \Rightarrow$

$$\frac{\partial \hat{F}}{\partial u_i}(\hat{u}) = 0 \quad \text{by definition}$$

$\Rightarrow$ $\hat{F}$ has a critical pt at $\hat{u}$.

But $\hat{F}$ is concave $\Rightarrow$ $\hat{u}$ is a ~~minimal~~ local max pt of $\hat{F}$.

But $\hat{F}$ is strictly concave in P. $\Rightarrow$ $\hat{u}$ is the unique ~~minimal~~ max

Furthermore $\hat{F}$ concave on P $\Rightarrow$ $\lim_{\substack{u \to \infty \\ u \in P}} \hat{F}(u) = -\infty$ $\square$

Now, consider the function $\hat{W} : \mathbb{R}^V \to \mathbb{R}$ restricted to $\Pi = \{u \in \mathbb{R}^V \mid \sum_v u_v$

$$\hat{W}(u) = W(u) - \sum_{v \neq v_1, v_2, v_3} 2\pi u - \frac{\pi}{3} \sum_{i=1}^{3} u_i$$

$\hat{W}$ is still concave and can be written as

$$= \sum_{\triangle v_i v_j v_k} \hat{F}(u_i, u_j, u_k) \quad \text{by the choice of angs } a_i$$

<u>Claim</u> $\hat{W}$ has a local max point in Π.

<u>Proof</u> Take a sequence $u^{(n)} \in \Pi$ s.t. $|u^{(n)}| \to +\infty$.

Then $\exists$ a triangle $\Delta v_i v_j v_k$ s.t. $\in P$ $\Downarrow \Uparrow$ $\max |u_i^{(n)} - u_j^{(n)}| \to$

$$\left|(u_i, u_j, u_k) - \frac{u_i+u_j+u_k}{3}(1,1,1)\right| \to \infty \quad \Leftarrow \Leftrightarrow \quad \max_{j \Delta_k^i} |u_i^{(n)} - u_j^{(n)}| \to \infty$$

~~[illegible]~~ <u>Not</u>

$\Downarrow$

the term $\geq \nearrow \infty$ each term $\geq$ bounded

$$\Rightarrow \lim_{u^{(n)} \to \infty} \hat{W}(u^{(n)}) = -\infty .$$

Therefore there exists a local max $\overset{p}{\text{for}}$ $\hat{w}$: $\frac{\partial \hat{w}}{\partial u_i}(p) = 0$

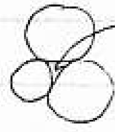
$$\Rightarrow \text{Cone angle} < \begin{matrix} \pi/3 & \text{at } v_1 v_2 v_3 \\ 2\pi & \text{at } v \in V - \{v_1 v_2 v_3\} \end{matrix}$$

This is the OP metric. $\square$

Lectures 13-14. Infinite Circle Packing

He-Schramm's Rigidity of Infinite Circle Packing

Thm (He-Schramm) Let C_0 be the regular hexagonal packing on $\mathbb{E}^2$ and C' be a circle packing of $\mathbb{E}^2$ whose nerve is isomorphic to $N(C_0)$. Then C' is regular, ie. all radii are the same.



Their theorem works in more general setting

$$C' < \begin{array}{l}\text{Each circle tangent to 6 others} \\ \mathbb{E}^2 = \cup \text{ circle discs} \cup \text{ interstices}\end{array} \Rightarrow \text{All circles have the same radii}$$

Proof

Part 1 fixed point index. All maps are piecewise smooth (and continuous)

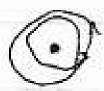
$\gamma \quad z \mapsto \frac{z}{|z|} \quad S^1$ $\boxed{\text{degree}}$ on $\, l: \gamma \to \mathbb{C}-\{0\}$

Def (winding number) $\gamma: S^1 \to \mathbb{C}-\{0\}$ oriented loop, then its winding number

$$\text{wind}(\gamma) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z} \qquad \left(= \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt = \frac{1}{2\pi i} \ln(\gamma(t))\Big|_0^1\right)$$

Eg 1 $\gamma: S^1 \to \{c\} \in \mathbb{C}-\{0\}$ $\quad \text{wind}(\gamma) = 0$

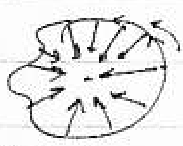
$\gamma = id: S^1 \hookrightarrow S^1 \subset \mathbb{C}-0 \quad \text{ind}(\gamma) = 1 \quad \gamma: S^1 \to c \in \mathbb{C}-\{0\}$



Fact: if $\gamma_1 \simeq \gamma_2: S^1 \to \mathbb{C}-\{0\}$, $\text{wind}(\gamma_1) = \text{wind}(\gamma_2)$ $\quad H$

Eg 2 Let us assume all Jordan curves are counterclockwise oriented: $\gamma: S^1 \to J \subset \mathbb{C}$ positively oriented $\quad \gamma(S^1) = J$

$$\text{wind}(\gamma) = \begin{cases} 1 & 0 \in \mathring{\Omega} \\ 0 & \notin \overline{\Omega}\end{cases}$$



$$= \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z} = \begin{cases} 0 & \frac{1}{z} \text{ analytic in } \Omega \\ 1 & \text{residue thm}\end{cases}$$

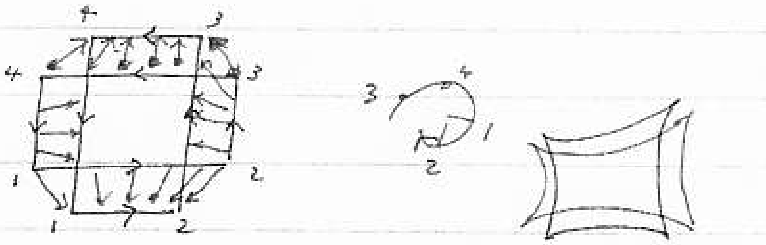
[**Prop.** Let $\Omega \subset \mathbb{C}$ be a Jordan domain $F: \overline{\Omega} \to \mathbb{C}$ a continuous map s.t $F|\Omega$ is analytic]

Def $\gamma \subset \mathbb{C}$ Jordan curve $f: \gamma \to \mathbb{C}$ p.w. smooth s.t $f(x) \neq x \ \forall x \in J$ (fixed point free). Then $\text{ind}(f) = \text{wind}(f(z) - z|_{z\in\gamma})$ $\quad \gamma = \gamma(t)$

$$= \frac{1}{2\pi i}\oint_{f-\gamma} \frac{dz}{z} = \frac{1}{2\pi i}\oint_0^{2\pi} \frac{f'(\gamma(t))\gamma'(t) - \gamma'(t)}{f(\gamma(t)) - \gamma(t)}\,dt$$

Lemma 1. If $H: \gamma \times I \to \mathbb{C}$ cont. s.t $H(x,t) \neq x$ $\forall x \in \gamma, t \in I$, then $\text{ind}(H(x,0)) = \text{ind}(H(x,1))$. (Homotopy invariance

Eg 3. The index of $h = -1$



Prop 2. Suppose $\Omega \subset \mathbb{C}$ Jordan domain $F: \overline{\Omega} \to \mathbb{C}$ cont. s.t., $F|_{\Omega}$ analytic and $f = F|_{\partial\Omega}$ is f.p.f (fixed point free), then $\text{ind}(f) \geq 0$

Proof In fact $\text{Ind}(f) = \sum_{\substack{p \in \Omega \\ f(p)=p}} \text{ind}(F, p)$

Let $g(z) = F(z) - z$

$$\text{ind}(f) = \frac{1}{2\pi i}\int_{g(\partial\Omega)} \frac{dz}{z} \overset{?}{=} \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{g'(z)}{g(z)}\, dz$$ (definition + change of variables)

$$\overset{\text{Residue}}{=} \sum_{\substack{p \in \Omega \\ g(p)=0}} \text{Res}(g(p)) = \boxed{\sum_{\substack{p \in \Omega \\ f(p)=p}} \text{ind}(F, p)} \geq 0$$ □

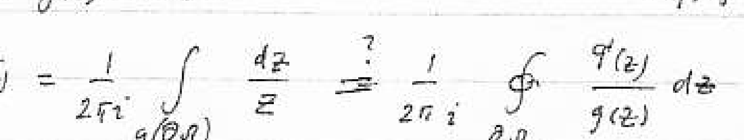
Definition?

homeomorphism

Corollary 1. Suppose S_1, S_2 are two oriented (positively oriented) $f: S_1 \to S_2$ f.p.f map $\Rightarrow \text{ind}(f)$.

2. Suppose J_1, J_2 are two positively oriented Jordan curves s.t a_1^1, a_2^1, a_3^1 are three points in J_1 cyclically oriented s.t $a_j^2 \cdot a_j^1 \neq a_j^1$. Then $\exists$ a f.p.f $f: J_1 \to J_2$ $f(a_i^1) = a_i^2$ $i = 1, 2, 3$ s.t $\text{ind}(f) \geq 0$

Pf (1)



We will deal w/ $S_1 \cap S_2$ at two pts.

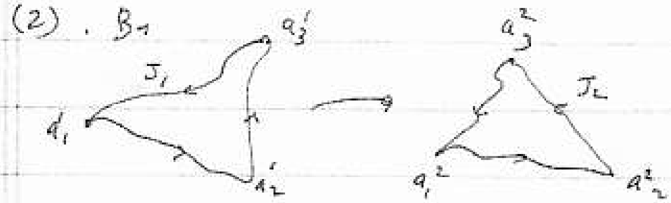
Construct a Möbius transf $\varphi: S_1 \to S_2$ extending to the disk s.t $\varphi(A) = f(A)$, $\varphi(B) = f(B)$. $\varphi: D_1 \to D_2$ where D_i disks $\partial D_i = S_i$

Now $f \simeq \varphi: S_1 \to S_2$ by a homotopy without fixed pts.

extending along ... $h(x,t) = t f(x) + (1-t)\varphi(x)$

Therefore $\operatorname{ind}(f) = \operatorname{ind}(\varphi) \geq 0$ since φ is analytic

(2).



By the Riemann mapping theorem & Caratheodory extension.

□

Proof of the thm

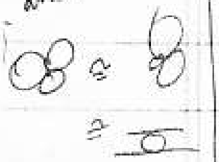
Suppose otherwise, ∃ a non-regular hexagonal packing C_1 of $\mathbb{E}^2$.
"ABD are circles appeared in both C & C'.

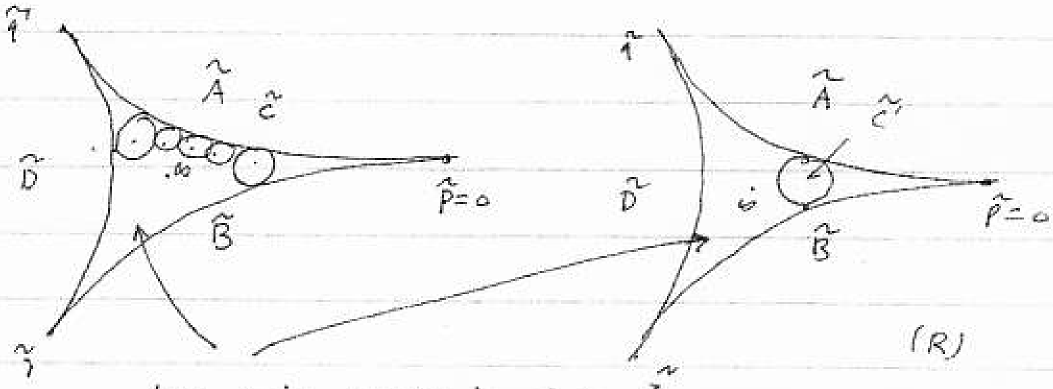
May assume C_1 $\xleftrightarrow{\text{corresponding}}$ C_0

$C \neq C'$

Consider the triangle $Apqr$ in $\hat{\mathbb{C}}$. after a further inversion:

Any two triple tangent circles are Möbius equi

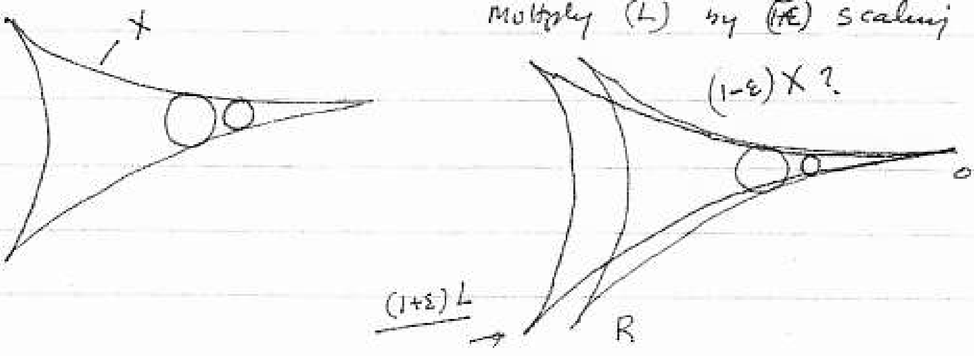




(L) two circle packings of the same combinatorics

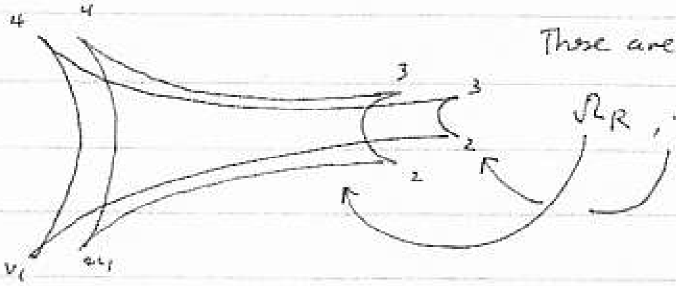
Say $\tilde{C}'$ has larger diameter

Befor Multply (L) by $\cancel{(1+\varepsilon)}$ $(1-\varepsilon)$ scaling



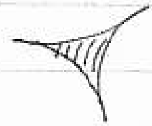
He-Schramm

<u>expand</u>:



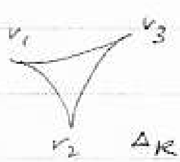
These are two Jordan domains Ω_R, Ω_L

Each is tilled by the circles and their intersects

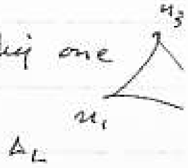


Define a P.L smooth, continuous map $f: \overline{\Omega}_R \to \overline{\Omega}_L$ as follows.

(1) for each oriented intersect



in $\overline{\Omega}_R$ find its corresponding one



$\overline{\Omega}_L$, produce the Riemann mapping $\varphi: \Delta_R \to \Delta_L$ $\varphi(v_i) = u_i$

(2) for each circles

extend φ to a map to the circle.

$\Rightarrow$ the required map. extend it to ∞.

Call the map $F: \overline{\Omega}_R \to \overline{\Omega}_L$

By example 3. $Ind(F|\partial\Omega_R) = -1$.

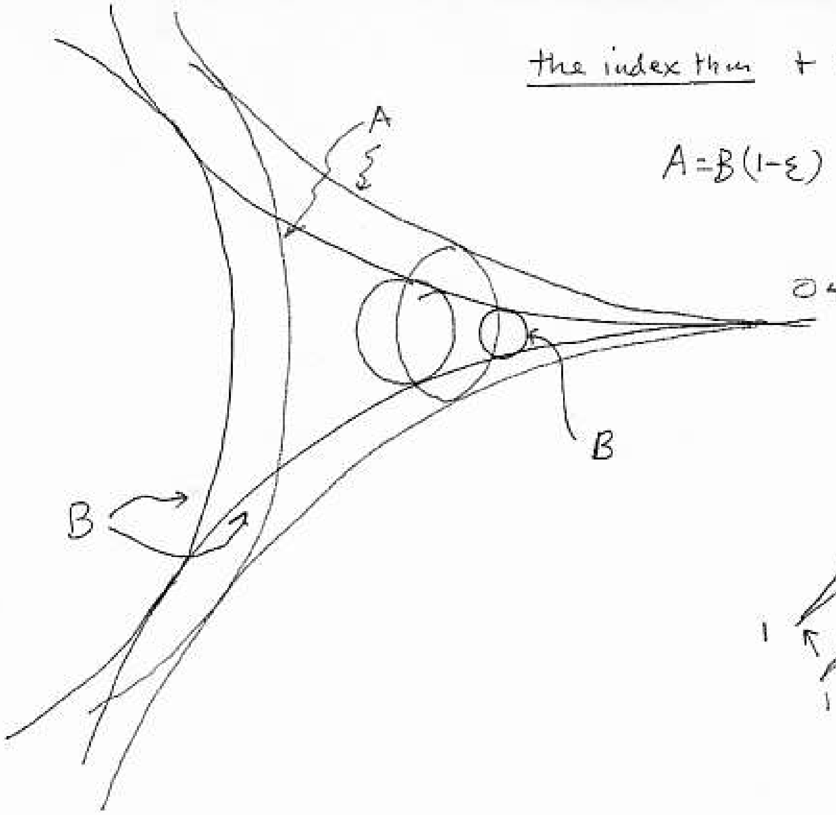
<u>Now</u>

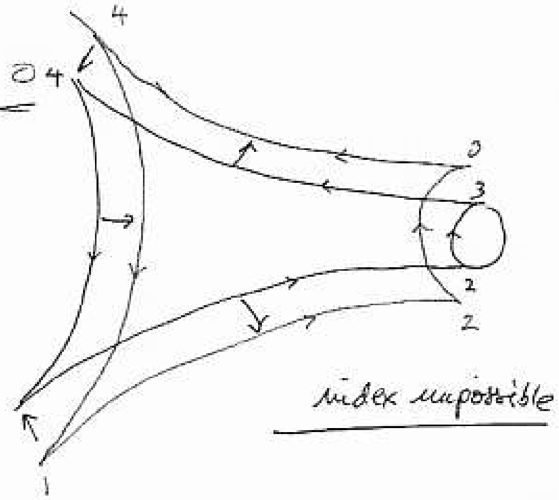
$$\text{the index}(F/\partial\Omega_R) = \sum_{O} ind(f|_O) + \sum_{\nabla} ind(f|_\Delta)$$

by the additivity of the integrals ≥ 0

the index thm + index calculation

$A = B(1-\varepsilon)$ $\quad$ ε - small





index impossible

A rederivation of $\gamma_j \frac{\partial \theta_i}{\partial \gamma_j}$ symmetric in i, j

$$= \gamma_j \left(\frac{\partial \theta_i}{\partial l_i} + \frac{\partial \theta_i}{\partial l_K} \right) = \gamma_j \frac{\sin\theta_i}{B} (1 - \cos\theta_j) \cancel{= \frac{4\gamma_j}{B} \sin\frac{\theta_i}{2} \cos\frac{\theta_i}{2} \sin^2\frac{\theta_j}{2}}$$

$$\cancel{= \frac{4}{B} \left(\gamma_j \,\mathrm{tg}\frac{\theta_j}{2} \right) \cdot \left[\sin\frac{\theta_i}{2} \cos\frac{\theta_i}{2} \sin\frac{\theta_j}{2} \cos\frac{\theta_j}{2} \right] = \frac{4\gamma}{B} \cdot (\text{symmetric in } i, j)}$$

Where r_j = radius of the inscribed circle.

$$= \frac{\gamma_j}{B} \sin\theta_i \cdot 2\sin^2\frac{\theta_j}{2} = \frac{1}{B} \left[\gamma_j \tan\left(\frac{\theta_j}{2}\right) \right] \sin\theta_i \sin\theta_j$$

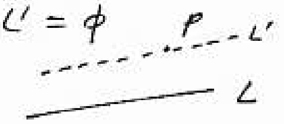
$$= \frac{r}{B} \sin\theta_i \sin\theta_j$$

Lectures 15.16 Hyperbolic Geometry

Geometry X consists of points, and objects $\mathcal{L} = \{\text{lines } L \text{ in } X\}$

Euclid: 5 axioms

1. $\forall\ p \neq q \in X.\ \exists!\ L \in \mathcal{L}$ s.t $p, q \in L$
2. $\forall\ L \in \mathcal{L}$ can be extended to ∞ in both directions ($\exists$ metric)
3. $\forall\ p \in X$ and $\forall\ r > 0,\ \exists!$ circle of radius r centered at p ($\exists$ metric)
4. All right angles are congruent (isometry group) ($\exists$ angle mesurement
5. $\forall\ L \in \mathcal{L},\ \forall\ p \notin L,\ \exists!\ L' \in \mathcal{L}$ s.t $p \in L'$ and $L \cap L' = \phi$



The major problem in geometry: $(1)+(2)+(3)+(4) \overset{?}{\Rightarrow} (5)$

F. Bolyai $\rightsquigarrow$ J. Bolyai (1802–1860), Gauss, Lobachevsky, Rigorous, ~~Decart~~ Descartes

The non-Euclidean geometry (= hyperbolic Geometry)

Def Cross ratio: $a, b, c, d \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ distinct then $(a,b,c,d) = \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d} \in \mathbb{C}-\{\ \}$

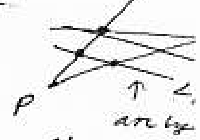
Möbius transf $f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$: $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ 1-1, onto

$GL(2,\mathbb{C}) \to$ Möb: $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto f$ onto homo.

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$ $f(: \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ 1-1 onto

Basic properties

Eg Projection map



Prop 1 (a) $(f(a), f(b), f(c), f(d)) = (a, b, c, d)$

Pf f is a composition of $g_1(z) = \lambda z$, $g_2(z) = z + \alpha$, $g_3(z) = \frac{1}{z}$. Check $\square$

Eg If b, c, d distinct, then $f(z) = (z, b, c, d)$ is a Möbius transf

s.t f: $b \mapsto 1$
$c \mapsto 0$
$d \mapsto \infty$

Corollary (a) $\forall$ four dist $a, b, c, d \ \underset{\text{Möbius}}{\cong}\ ((a,b,c,d), 1, 0, \infty)$

(b) Given $\boxed{a,b,c,d}$ distinct quadriples a_1, a_2, a_3, a_4 & b_1, b_2, b_3, b_4

there exists Möbius f s.t $f(a_i) = b_i \Leftrightarrow (a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4)$

Prop 3 Let $\lambda = (a, b, c, d)$

(a) $(a, b, c, d) = (b, a, d, c) = (c, d, a, b)$

(b) $(a, d, b, c) = (\lambda - 1)/\lambda$ and $(a, c, d, b) = \frac{1}{1-\lambda}$, $(b, a, c, d) = \frac{1}{\lambda}$

(c) $\lambda \in \mathbb{R} \Leftrightarrow a, b, c, d$ lie in a circle or line

(d) $(a, b, x, z) = (a, b, x, y)(a, b, y, z)$: $\frac{a-x}{a-z} \Big/ \frac{b-x}{b-z} = \left(\frac{a-x}{a-y} \Big/ \frac{b-y}{b-z}\right)\left(\frac{a-y}{a-z} \Big/ \frac{b-y}{b-z}\right)$

Pf We need a lemma which says M

Lemma 4 Möbius f preserves the set of lines and circles + the angles

Pf It suffices to prove it for $h(z) = \frac{1}{z} = \frac{z}{z\bar{z}}$ inversion

Lemma 5. In $\mathbb{E}^n$ the inversion $h(x) = \frac{x}{|x|^2}$ preserves the set of all $(n-1)$-spheres + $(n-1)$-planes

Pf The equations for them are $A\langle x, x\rangle + \langle B, x\rangle + C = 0$ $B \in \mathbb{R}^n$

Replace $x = \frac{y}{\langle y, y\rangle}$ $\Rightarrow$ $\frac{A\langle y, y\rangle}{|y|^2} + \frac{\langle y, B\rangle}{|y|^2} + c = 0$

So is $A + \langle B, y\rangle + C\langle y, y\rangle = 0$.

Angle preserving due to complex analyticity of $f(z)$. □

Thus $(a, b, c, d) \in \mathbb{R} \Leftrightarrow (a, b, c, d), 0, 1, \infty \in \mathbb{R} \cup \{\infty\} \Leftrightarrow a, b, c, d$ image of $\mathbb{R} \cup \{\infty\}$ under Möbius $\Rightarrow$ Done.

(b) Say $(a, d, b, c) = \beta(\lambda, \infty, 1, 0) = \frac{\lambda - 1}{\lambda - 0} = \frac{\lambda - 1}{\lambda}$ done.

$d \to \infty$, $b \to 1$, $c \to 0$

Later in Poincaré model

Eg (Hilbert Geometry)

$\Omega \subset \mathbb{R}^2$ open unit disk, $\mathcal{L} = \{$all lines in $\mathbb{R}^2 \cap \Omega\}$,

$$d(p, q) = |\lg|(p, q, a, b)|| = \lg\left|\left(\frac{p-a}{p-b}\right)\Big/\left(\frac{q-a}{q-b}\right)\right|$$

$q \to a$ it tends to ∞
$p \to b$ it tends to ∞

False argument

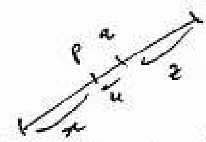
~~The set of all Möbius transf preserving disk $D = \{z \mapsto e^{i\theta}\frac{z-a}{\bar{a}z-1} \mid |a| < 1\}$~~

~~this group acting on D preserving distance $d(p, q)$ (Möbius inv)~~

~~acts transitively on D. Thus a ball, May assume centered at $q = 0$~~

~~Projective maps $\varphi: \mathbb{D} \to \mathbb{D}$~~

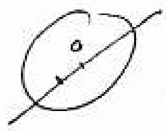
$$\{ p \mid d(p,o) = r \} = \{ p \mid \lg\left(\left|\frac{p-a}{p-b}\right| \Big/ \left|\frac{a}{b}\right| \right) = r \}$$



$d = \lg\left[(1+\frac{y}{x})(1+\frac{y}{z})\right]$

$$= \{ p \mid \frac{|p-a|}{|p-b|} = e^r \}$$

$|p-a| =$

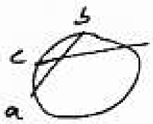


$$= \{ p \mid \frac{|p|+1}{1-|p|} = e^r \}$$

$$= \{ p \mid |p| = \frac{e^r-1}{e^r+1} \} = \{ p \mid |p| = \coth(\tfrac{r}{2}) \}$$

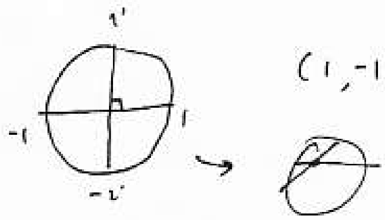
$\sinh(t) = \frac{e^t - e^{-t}}{2}$, $\cosh(t) = \frac{e^t + e^{-t}}{2}$, $\coth(t) = \sinh(t)/\cosh(t)$. Is a circle

<u>Def</u> Two lines $[a,b]$, $[c,d]$ are <u>perpendicular</u>



if $(a,b,c,d) = -1$

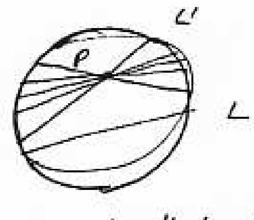
<u>Eg</u>



$$(1,-1,i,-i) = \frac{1-i}{1+i} \Big/ \frac{-1-i}{-1+i} = \left(\frac{1-i}{1+i}\right)\left(\frac{1-i}{1+i}\right)$$

$$= \frac{-2i}{2i} = -1$$

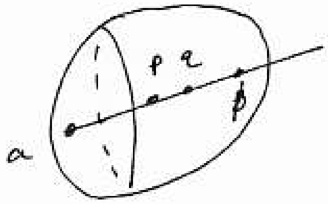
Thus, we obtain a model of a 2-dim geometry in which the fifth axiom does not hold:



This is a model of Hyperbolic geometry (called Klein model)

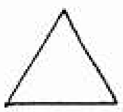
<u>Above is an example of Hilbert Geometry</u>:

$\Omega \subset \mathbb{R}^n$ open bounded convex set



<u>dist the same</u>

<u>lines are shortest distance paths</u>



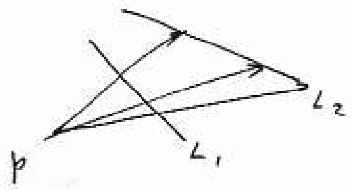
Ω open triangle Is already very interesting in itself

<u>Key thm</u> The distance so defined satisfies $d(p,q) + d(q,r) \geq d(p,r)$

<u>Let us prove it</u>:

Proof Lemma Projection from a point preserves cross ratio

pf If L_1, L_2 two lines with arc length parametrizations s, t and $p \in \mathbb{C}$



then $\pi_p : L_1 \to L_2$ the projection is a fractional linear $\pi_p(s) = \dfrac{\alpha t+\beta}{\gamma t+\delta}$ $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

$\alpha\delta - \beta\gamma \neq 0$

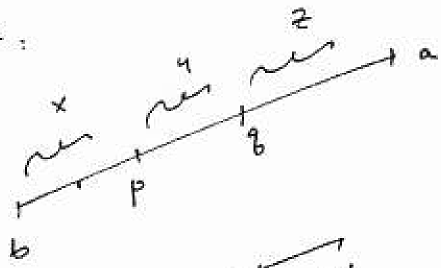
pf After a translation, we may assume $p = 0$

points in L_1, L_2 are $(as+b,\ cs+d)$ and $(a't+b',\ c't+d')$

the map preserves the slope: $\dfrac{as+b}{cs+d} = \dfrac{a't+b'}{c't+d'}$ $\Rightarrow$ done

□

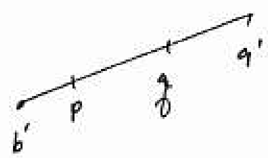
The distance:



$$d(p,q) = \ln \left| \frac{p-a}{p-b} \Big/ \frac{q-a}{q-b} \right|$$

$$= \ln \left(\frac{y+z}{x}\right) \Big/ \left(\frac{z}{x+y}\right) = \ln\left[\left(1+\frac{y}{z}\right)\left(1+\frac{y}{x}\right)\right]$$

Thus



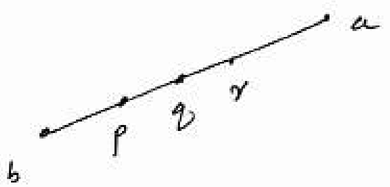
if $\Omega' \subset \Omega$ convex $\Rightarrow d_\Omega(p,q) \leq d_{\Omega'}(p,q)$

Now to show triangle inequality: $p, q, r \in \Omega$

step 1 p, q, r colinear. Then $d(p,q) + d(q,r) = d(p,r)$

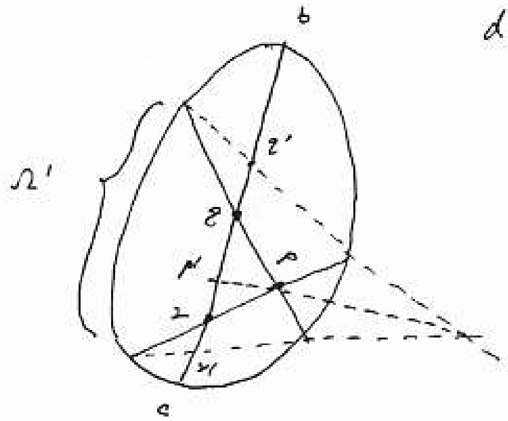
$$\Updownarrow$$

$$|(p,q,a,b)| \cdot |(q,r,a,b)| = |(p,r,a,b)|$$



ie $(a,b,p,q)(a,b,q,r) = (a,b,p,r)$ Yes

Step 2 $\quad p, q, r$ not colinear. We want



$$d(q,r) \le d(q,p) + d(p,r)$$

But $d(q,r) \le d_{\Omega'}(q,r) = \ln |(q,r,r',q')|$

$$\overset{\text{step 1}}{=} d_{\Omega'}(q,p') + d_{\Omega'}(p',r)$$

$$= \ln|(q,p',r',q')| + \ln|(p',r,r',q')|$$

$$\underset{\text{inv}}{\overset{\text{projective}}{=}} d(q,p) + d(p,r)$$

$\square$

Major Problems: What is geometry?

Gauss, Riemann quadratic pos. def forms on tangent space $\longrightarrow$ on mflds

$\longleftrightarrow$ general relativity.

Quantum world: discrete space time?

Riemann's Paper: introduced $\dfrac{\Sigma\, dx_i^2}{(1+k\,\Sigma x_i^2)^2}$ constant curvature metrics

RM The Hilbert distance is projective invariant $\varphi: \mathbb{R}^n \supset \cdot \to \mathbb{R}^n$ projective

$$|(x_1, x_2, x_3, x_4)| = |(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))|$$

when x_1, x_2, x_3, x_4 colinear!

What is a projective map φ on $\mathbb{R}^n$:

Take $A \in GL(n+1, \mathbb{R})$ $\qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\varphi(x) = \pi_1\left(A\begin{bmatrix} x \\ 1 \end{bmatrix} \Big/ \text{last column} \right)$$

Eg $n=1$ $\quad a, b, c, d \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) \qquad x \mapsto \pi_1 \begin{bmatrix} ax+b \\ cx+d \end{bmatrix} \Big/ cx+d = \frac{ax+b}{cx+d}$

$n=2$ $\quad (x_1, x_2) \longmapsto \left(a_{11}x_1 + a_{12}x_2 + a_{13},\ a_{21}x_1 + a_{22}x_2 + a_{23}\right) \Big/ (a_{31}x_1 + a_{32}x_2 + a_{33})$

$\det[a_{ij}] \ne 0$

RM $\quad A, B, C, D \in \mathbb{R}^n$ distinct, cross ratio

$$(A,B,C,D) = \frac{|A-C|}{|A-D|} : \frac{|B-C|}{|B-D|}.$$

It is invariant under projective map. if A, B, C, D colinear!

Lecture ~~16~~ 17–18 Hyperbolic Geometry 7-30-2012

Why hyperbolic geometry: uniformization theorem . Geometry. (X, G, Θ) bsept grp, object

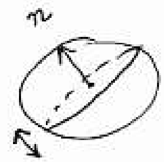
Schwartz Picks lemma. Euclidean 5 axioms

We begin with the spherical geometry : $(\mathbb{E}^3, \langle , \rangle)$ the standard inner product

$\mathbb{S}^2 = \{ x \in \mathbb{E}^3 \mid (x, x) = 1 \}$ $\quad Iso(\mathbb{S}^2) \underset{\supset}{\overset{\text{why}}{\equiv}} O(3) = \{ A : \mathbb{R}^3 \to \mathbb{R}^3 \mid (Ax, Ax) = 1$

Metrics. $\forall\, p \in \mathbb{S}^2$ $T_p\mathbb{S}^2$ induced $\langle , \rangle$.

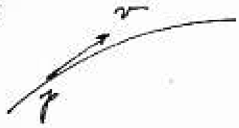
Geodesics $\Leftrightarrow$ $s = P \cap \mathbb{S}^2$ $\quad P$ 2-dim plane $\subset \mathbb{R}^3$.



Why is s shortest: $\quad$ dist $(u, v) = \cos^{-1}(u, v)$.

Lemma!: (Σ^2, g) Riemannian surf $\quad \varphi : (\Sigma^2, g) \to (\Sigma^2, g)$ isometry s.t Fix(φ) = a smooth curve s. Then s is a geodesic (or (M^n, g) $\varphi \in Iso$ Fix(φ) = smooth curve γ $\Rightarrow \gamma$ geodesic!)

Pf : Take $p \in s$ + $v \in T_p\Sigma$ tangent to s



Then v determines a geodesic $\gamma(t)$

Now $\quad \varphi(\gamma(t)), \gamma(t)$ are both geodesics from p w/ the same tangent vect

uniqueness of geodesic $\Rightarrow$ $\quad \varphi(\gamma(t)) = \gamma(t) \Rightarrow \quad \gamma(t) \subset s \Rightarrow$ done

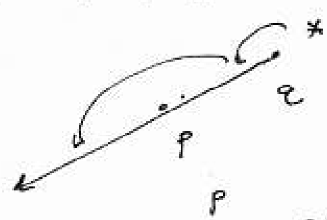
Eg $\quad s = P \cap \mathbb{S}^2$ is a geodesic since the reflection $\quad x \mapsto x - 2(n, x)\cdot n$

n unit normal vector to P $\quad$ is an isometry w/ fixed pt P. $\square$

Eg $\quad$ Half turn about $p \in \mathbb{E}^2$ $\quad$ ($x \mapsto -x : \mathbb{R}^2 \to \mathbb{R}^2$ orientation preserving involution)

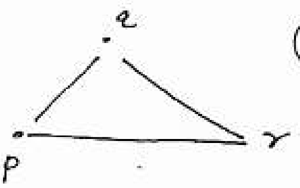
$\varphi_p(x) = 2p - x \qquad \varphi_p(p) = p. \qquad \varphi_p(\varphi_p(x)) = 2p - (2p - x) = x$

$\varphi_p \circ \varphi_q(x) = 2p - (2q - x) = x + 2(p - q)$ $\quad$ translation

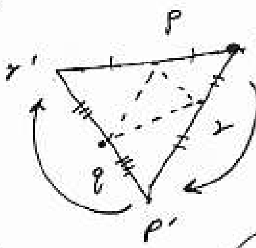


What is $\varphi_p \circ \varphi_q \circ \varphi_r$

$q' \mapsto q'$



(?)



$Iso^+(\mathbb{E}^2) = \langle \varphi_p \mid p \in \mathbb{E}$

$\gamma = \varphi_{p_1}\varphi_{p_2}$ or $\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}$

These are in $Isot(\mathbb{E}^2)$

Q: $\quad$ The π-rotation about a point $p \in \mathbb{S}^2$

$\varphi_p(x) = 2(p, x)\cdot p - x \qquad p.$

Hw $\quad$ The group $\langle \varphi_p, \varphi_q, \varphi_r \rangle$ is discrete in $Iso^+(\mathbb{R}^2)$

Eg (Gauss) $\quad SO(3) = Iso^+(\mathbb{E}^3)$ is generated by φ_p, $p \in \mathbb{S}^2$, + is also:

$\forall\, \varphi \in SO(3) \qquad \varphi = \varphi_{p_1} \circ \varphi_{p_2}$

$A = Iso^+(\mathbb{E}^3) = SO(3) \qquad A = B \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} B^T \quad B \in SO(3)$

@ When is $\langle \varphi_p, \varphi_q, \varphi_r \rangle$ a discrete (= finite) subgroup of $SO(3)$?

The hyperbolic geometry $\mathbb{H}^n$

Riemann wrote down the metric $\dfrac{\sum dx_i^2}{(1+\kappa \sum x_i^2)^2}$. $-1<\kappa<1$.

Gauss, Lobachevsky, Bolyai independently discovered $n=2$

Minkowski space $\mathbb{E}^{n,1} = (\mathbb{R}^{n+1}, \langle , \rangle)$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

$O(n+1,1) \triangleq \{ f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \text{ linear} \mid \langle f(x), f(y) \rangle = \langle x, y \rangle \}$ orthogonal group

Eg 1. If $\langle m, m \rangle = 1$, then $\varphi_m(x) = x - 2\langle m, x \rangle m$ (reflection) $\in O(n+1,1)$

$$\langle \varphi_m(x), \varphi_m(y) \rangle = \langle x - 2\langle m, x \rangle m, y - 2\langle m, y \rangle m \rangle$$

$$= \langle x, y \rangle + 4\langle m, x \rangle \langle m, y \rangle - 2\langle m, x \rangle \langle m, y \rangle - 2\langle m, y \rangle \langle m, x \rangle = \langle x, y \rangle.$$

$(x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_n, -x_{n+1})$

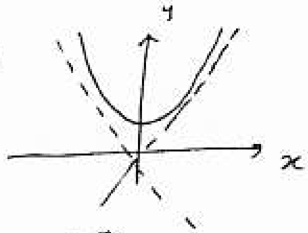
The hyperbolic space

$$\mathbb{H}^n = \{ x \in \mathbb{E}^{n,1} \mid \langle x, x \rangle = -1, \; x_{n+1} > 0 \}$$

the light-cone $\{ x \in \mathbb{E}^{n,1} \mid \langle x, x \rangle = 0 \}$

de Sitter space $\{ x \in \mathbb{E}^{n,1} \mid \langle x, x \rangle = 1 \}$

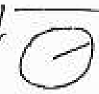
Eg 2. $n=1$ $\mathbb{H}^1$:



$x^2 - y^2 = -1$ $y^2 = x^2 + 1$ $y > 0$

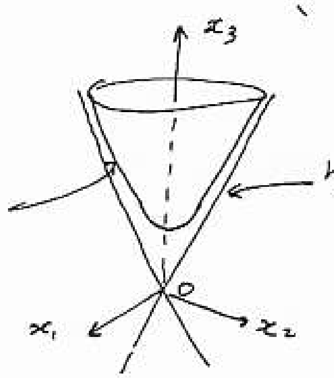
$y = \sqrt{1+x^2}$

parameter ~~$x = \cosh$~~ $\begin{cases} x = \sinh(t) \\ y = \cosh(t) \end{cases}$

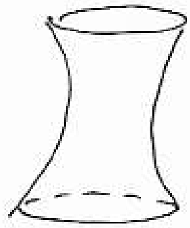


$x = \cos(t)$
$y = \sin(t)$

$n=2$ $\mathbb{H}^2$

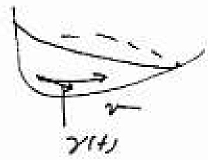


de Sitter



$x_3 = \sqrt{1 + x_1^2 + x_2^2}$

We identify $T_p \mathbb{H}^n = \{ v \in \mathbb{R}^{n+1} \mid \langle v, p \rangle = 0 \}$



indeed tangent $v = \dot\gamma(t)\big|_{t=0}$ $\quad \gamma(t) \in \mathbb{H}^n \quad \gamma(0) = p$

$$\Rightarrow \quad \langle \gamma(t), \gamma(t) \rangle = -1 \qquad \text{Take } \frac{d}{dt}\Big|_{t=0} \Rightarrow$$

$$2 \langle \dot\gamma(0), \gamma(0) \rangle = 0 \quad \Rightarrow \quad \langle v, p \rangle = 0$$

Conversely, each $v \in T_p \mathbb{H}^n$ is $\dot\gamma(0)$

<u>Lemma 1</u> The restriction of $\langle , \rangle$ to $T_p \mathbb{H}^n$ is positive definite

<u>pf.</u> If not, say $v = (x_1, \cdots, x_{n+1}) \in T_p \mathbb{H}^n$ $\quad p = (p_1, \cdots, p_n, p_{n+1})$ s.t

$$\langle v, v \rangle = -1 . \qquad \langle v, p \rangle = 0 \qquad \langle p, p \rangle = -1$$

$$\Leftrightarrow \quad \sum_1^n x_i^2 = -1 + x_{n+1}^2 \qquad \langle x, x \rangle = -1$$

$$\sum_1^n p_i^2 = -1 + p_{n+1}^2 \qquad \langle p, p \rangle = -1$$

$$\sum_1^n x_i p_i = x_{n+1} p_{n+1} \qquad \langle v, p \rangle = 0$$

<u>Now</u>

$$\Big(1 + \sum_1^n x_i^2\Big)\Big(1 + \sum_1^n p_i^2\Big) = x_{n+1}^2 p_{n+1}^2 = \Big(\sum_1^n x_i p_i\Big)^2 \underset{\text{Cauchy}}{\leq} \Big(\sum_1^n x_i^2\Big)\Big(\sum_1^n p_i^2\Big)$$

$\square$

<u>RM 1.</u> This shows if $p, q \in \mathbb{H}^n$ $\quad \langle p, q \rangle < 0$ if $\langle q, q \rangle \leq 0$, $q_{n+1} > 0$.

<u>RM 2</u> $O(n+1)$ acts transitively on $F(S^n)$ orthonormal frame bundle

$\Leftrightarrow$ (v_1, v_2, p) orthonormal basis of $\mathbb{E}^{n+1}$

related to $(e_1, \cdots, e_{n+1})$ by $\varphi \in O(n+1)$ <u>done</u>

Exactly the same proof works for $\mathbb{H}^n$

<u>Lemma 2</u> Let $F(\mathbb{H}^n)$ be the orthonormal frame bundle over $\mathbb{H}^n$. Then

$O^+(n+1, 1) = \{ \varphi \in O(n+1, 1) \mid \varphi(H^n) = H^n \ \ (\varphi(H^n) = -H^n) \}$ acts transitively

on $F(\mathbb{H}^n)$

<u>Pf</u> $p \in H^n$ $(v_1, \cdots, v_n)$ orthonormal frame $\Rightarrow$ $(v_1, \cdots, v_n, p)$ orthonormal basis of

$\mathbb{E}^{n+1,1}$ $\Rightarrow \exists\ \varphi \in O(n+1, 1)$ $\quad \varphi(e_i) = v_i \quad i = 1, 2, \cdots, n. \quad \varphi(e_{n+1}) = p$

For this φ $\quad \varphi(H^n) = H^n$ since both e_{n+1} + $p \in H^n$.

$\square$

<u>Corollary 3.</u> $Iso(\mathbb{H}^n) = O^+(n+1,1)$, $Isotropy(e_{n+1}) = O(n)$

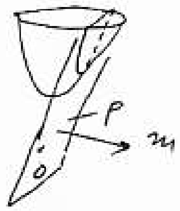
<u>pf</u> By the construction we have "$\supset$"

To see "$\subset$" $\varphi \in Iso(\mathbb{H}^n) \Rightarrow \varphi(e_1, \dots, e_{n+1}) = \Psi(e_1, \dots, e_{n+1})$ for some $\Psi \in O^+(n+1,1)$ by the lemma $\Rightarrow$ May assume $\varphi(e_i) = e_i \Rightarrow \varphi = id$ (isometry)

□

<u>Lemma 4.</u> All geodesics $L \subset \mathbb{H}^n$ are $P \cap \mathbb{H}^n$ P 2-dim plane $\ni 0$.

<u>pf</u> How do you do that for $\mathbb{S}^n$.



Show that $P \cap \mathbb{H}^n$ are fixed pts of an isometry

<u>n=2</u> the reflection $\varphi_P : x \mapsto x - 2\langle m, x\rangle m$ where

m is the normal vector to $P := \{ y \in \mathbb{R}^{n+1} \mid \langle y, m\rangle = 0\}$

<u>Note</u> $\langle m, m\rangle > 0$! since otherwise $P \cap \mathbb{H}^n = \phi$. $m \in$ <u>de Sitter space</u>

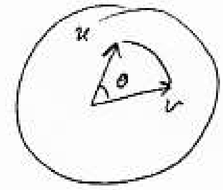
<u>n=3.</u> Find it: φ_P: "rotation" about the axis.

<u>Eg</u> $\forall$ Line $L \subset \mathbb{E}^n$, $\exists$ isometry $\varphi \in Iso(\mathbb{E}^n)$ s.t. $Fix(\varphi) = L$.

Exactly the same works for $\mathbb{S}^n$, $\mathbb{H}^n$.

□

<u>Distance Formula:</u> if $u, v \in \mathbb{S}^n$ $d_{\mathbb{S}^n}(u,v) = \cos^{-1}(\langle u, v\rangle)$

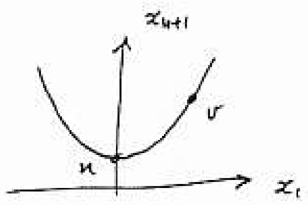


$d_{\mathbb{S}^n}(u,v) = \theta$ $\cos(\theta) = \langle u, v\rangle$

<u>Prop 5.</u> If $u, v \in \mathbb{H}^n$, then $d_{\mathbb{H}^n}(u,v) = \cosh^{-1}(-\langle u, v\rangle)$

<u>Proof.</u> Using iso φ may assume $\varphi(u) = e_{n+1}$ and $v = (\sinh(t), 0, \dots, \cosh(t))$

(In the plane (x_1, x_{n+1})-plane) <u>why</u>?)



$d(u,v) =$ length of the H^1 from (u,v) $0 \le s \le t$

$\gamma(s) = (\sinh(s), \cosh(s))$

$\gamma'(s) = (\cosh(s), \sinh(s))$, $\|\gamma'(s)\| = \cosh^2(s) - \sinh^2(t) = 1$

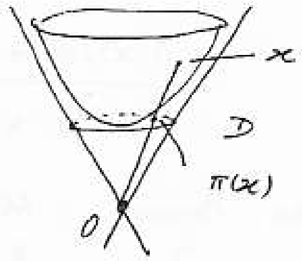
So <u>length</u> $d(u,v) = \int_0^t ds = t$ $-\langle u, v\rangle = -\langle (0,1), (\sinh(t), \cosh(t))\rangle = \cosh(t)$

<u>done</u> □

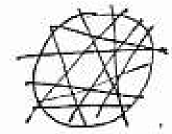
The Klein Model of $\mathbb{H}^n$

Let $D = \{ (x_1, \cdots, x_n, 1) \in \mathbb{R}^{n+1} \mid \sum x_i^2 < 1 \}$

$\pi: \mathbb{H}^n \longrightarrow \mathbb{D}$. $\quad \pi(x_1, \cdots, x_{n+1}) = \pi\left(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}, 1\right) = \frac{1}{x_{n+1}} \cdot x$



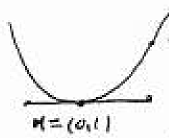
π is 1-1 onto continuous, geodesics in D: lines



Prop The distance on D, pull backed from $\mathbb{H}^n$ by π, is the Hilbert distance. ($2d_{Hilbert}$)

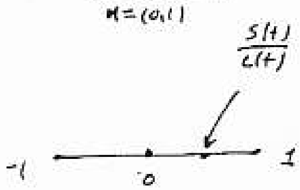
Pf. step 1 Only need to check for $n=2$.

Step 2 For the case of $n=1$



$d(u,v) = \cosh^{-1}(-\langle\ \rangle) = t$!

projection to D'



$$d_{Hilbert}(\pi(u), \pi(v)) = \lg\left(\frac{s(t)}{c(t)}, 0, -1, 1\right)$$

$$= \lg\left(\frac{\frac{s(t)}{c(t)}+1}{\frac{s(t)}{c(t)}-1} : \frac{-1}{+1}\right) = \lg\left[\frac{s(t)+c(t)}{c(t)-s(t)}\right] = \lg\left(\frac{e^t}{e^{-t}}\right) = 2t$$

step 3. This holds for all $p, q \in \mathbb{D}$, i.e.: $2d_{Hilbert}(p,q) = d_{\mathbb{H}^n}(\pi^{-1}(p), \pi^{-1}(q))$

Let $p' = \pi^{-1}(p)$ $q' = \pi^{-1}(q)$. Find $\varphi \in \mathrm{Iso}(\mathbb{H}^n)$ $\varphi(p') = u$, $\varphi(q') = v$ as in step 2

Note φ induces a projective map $\tilde{\varphi}: D^n \to D^n$!

Indeed $\varphi(z) = (\varphi_1(z), \cdots, \varphi_{n+1}(z))$ φ_i linear.

Now $\tilde{\varphi}: D^n \longrightarrow D^n$

$$\begin{array}{l} (x,1) \\ \quad \searrow^{\pi^{-1}} \\ \qquad \lambda(x,1) \xrightarrow{\varphi} \lambda\varphi(x,1) \end{array}$$

$\tilde{\varphi}(x,1) = (\varphi_1(x,1), \cdots, \varphi_{n+1}(x,1)) / \varphi_{n+1}(x,1)$

fractional linear

$\Rightarrow$ preserves Hilbert distance!

$$\Rightarrow 2d_{Hil}(p,q) \overset{\tilde{\varphi}}{=} 2d_{Hil}(\pi(u), \pi(v)) \overset{\text{step 2}}{=} d(u,v) \overset{\varphi\ \text{iso}}{=\!=} d(p', q') \triangleq d_{\mathbb{H}^n}(\pi^{-1}(p), \pi^{-1}(q)). \quad \square$$

So Hilbert metric on disk is Riemannian.

→ BACK

$\underline{n=2}$ Klein model

$$s = (x, y) \qquad x^2+y^2<1$$

metric
$$\frac{(1-x^2)\,dx^2 + xy\,dx\,dy + (1-y^2)\,dy^2}{(1-x^2-y^2)^2}$$

Relationship between Poincaré & Klein.

$$\begin{array}{ccc} u & & s \end{array}$$

$$u \longmapsto s \qquad s = \frac{2u}{1+|u|^2}$$

Klein $\rightarrow$ Poincaré

$$s \longmapsto u = \frac{\left(1-\sqrt{1-|s|^2}\right)}{|s|^2}\cdot s$$

Excellent Calculation

Lecture 19, 20. 3D Hyperbolic Geometry

8-6-2012

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Quick Introduction to Hyperbolic 3-Space

Lemma Let $f(x) = \frac{x}{|x|^2} : \mathbb{R}^4 \cup \{\infty\} \circlearrowleft$. Then

$$Df(a) \cdot v = \frac{1}{|a|^2} \cdot \left(v - 2\left(v, \frac{a}{|a|}\right) \frac{a}{|a|} \right)$$

Scaling of a reflection about $a^\perp$.

Proof, t small

$$f(a+tv) - f(a) = \frac{a+tv}{(a+tv, a+tv)} - \frac{a}{(a,a)} = \frac{a+tv}{(a,a)+2t(a,v)} - \frac{a}{(a,a)} + o(t^2)$$

$$= \frac{a}{|a|^2\left(1+2t\left(v, \frac{a}{|a|^2}\right)\right)} + \frac{v}{(a,a)} - \frac{a}{(a,a)} + o(t^2)$$

$$= \frac{a}{|a|^2}\left(1 - 2t\left(v, \frac{a}{|a|^2}\right)\right) + \frac{v}{(a,a)} - \frac{a}{(a,a)} + o(t^2)$$

$$= \frac{1}{|a|^2}\left(v - 2t\left(v, \frac{a}{|a|}\right)\frac{a}{|a|}\right) + o(t^2)$$ $\square$

Corollary The hyperbolic metric $ds^2 = \frac{|dx^2|}{x_n^2}$ in $U_n = \{x \in \mathbb{R}^4 \,/\, x_n > 0\}$

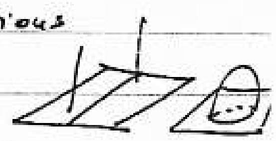
is invariant under $f(x) = \frac{x}{|x|^2}$

Pf Let $y = \frac{x}{|x|^4}$ $\quad y_n = x_n / |x|^2$ $\quad \boxed{dx^2 = \sum dx_i^2}$

so $|dy|^2 = \frac{1}{|x|^4}|dx|^2 \Rightarrow \frac{|dy^2|}{y_n^2} = \frac{\frac{1}{|x|^4}|dx^2|}{x_n^2/|x|^4} = \frac{dx^2}{x_n^2}$ $\square$

Conclusion: $x \mapsto \lambda x$ $\lambda > 0$, all reflections about plane (hyper-plane) $P \perp \partial$ + f are isometries of U_n. They generate $Iso(U_n) = Iso(\mathbb{H}^4)$ where restriction to $\mathbb{R}^{n-1} \times 0$ are Möbius transformations

$n=3$: $\mathbb{C} \to \mathbb{C}$ are $z \mapsto \frac{az+b}{cz+d}$ or $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$.



Easy Fact $\varphi: (M^n, g) \circlearrowleft$ isometries s.t $N = Fix(\varphi)$ smooth submfd then N is totally geodesic. if γ is a geodesic tangent to N at one point $\Rightarrow \gamma \subset N$

$\Rightarrow$ All vertical planes + 2-sphere $\perp \mathbb{C} \times 0$ are totally geod.

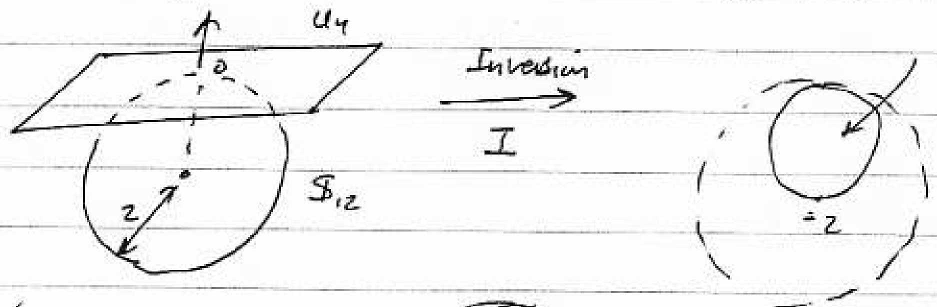
$\Rightarrow$ All ... lines + circles $\perp$ $\mathbb{C} \times 0$ are geodesics

Since totally geodesic submanifolds intersecting in a totally geodesic submanifold. We obtain: geodesics in $U_n = \{x \in \mathbb{R}^n \,/\, x_n > 0\}$ are circles + lines $\perp \mathbb{R}^{n-1} \times 0$.

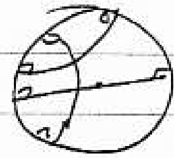
Also $Iso(\mathbb{H}^n)$ contains all compositions of reflections and inversions about codim-1 ~~submani~~ spheres and planes $\perp \mathbb{R}^{n-1} \times 0$

Eg Another way to find geodesics in U_n: show directly that $\gamma(t): [0,1] \to U_n$, $\gamma(0) = (0, \dots, 0, a)$, $\gamma(1) = (0, 0, \dots, 0, b)$ is a geodesic iff $\gamma(t)$ is in the x_n-axis. Then use isometry of U_n to find other geodesics

Eg. The ball model (Poincaré disk model) of hyperbolic n-space is obtained from $U_n = \{x \in \mathbb{R}^n \,/\, x_n > 0\}$ by inversion about the sphere of radius 2 centered at $(0, 0, \dots, 0, -2)$

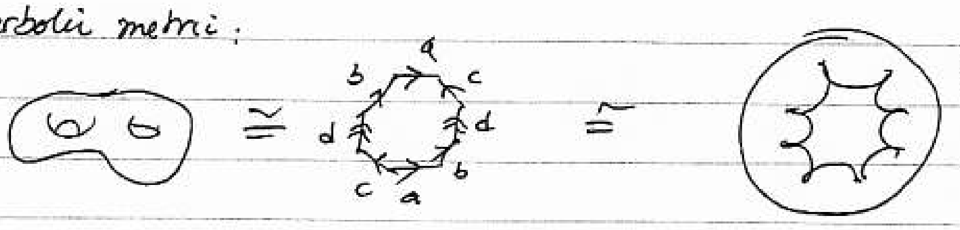


geodesics in $I(U_n)$ are :



$n=2$

Use this model: One can show every closed surface of genus $g \geq 2$ has a hyperbolic metric:



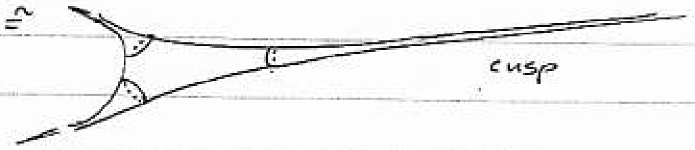
regular hyperbolic $4g$-gon of inner angle $\frac{2\pi}{4g}$.

Thurston's deep insight:
- 2-d: Almost all surfaces are hyperbolic (i.e. $\exists$ complete hyperbolic metric)
- 3-D, the SAME!

Can you find ONE hyperbolic manifold in dim ≥ 3?

Eg 2d. First: $\mathbb{H}/\Gamma(2)$ $\quad \Gamma(2) = \{A \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \mid A \equiv I \bmod 2\}$,

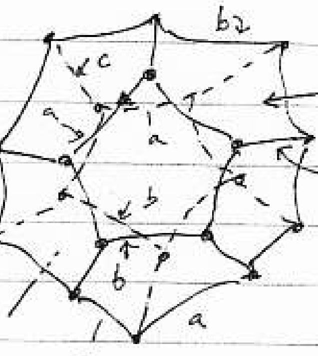
$\cong \mathbb{C} - \{0,1\} \cong$



Eg 3d. Poincaré homology sphere: Poincaré conjecture (original) M^3 closed 3-mfd

$H_1(M^3) = 1 = H_2(M^3) = 1 \quad H_3(M^3) = \mathbb{Z} \Rightarrow M^3 \cong S^3$

Poincaré counter-example, regular dodecahedron symmetric w.r.t $x \mapsto -x$



Each $F \rightsquigarrow$ $\frac{4\pi}{5}$ RH Rotation $\sim (x \sim -x)$

$\boxed{\Rightarrow \deg = 3}$

$\exists$ spherical regular deca of dihedral $2\pi/3$

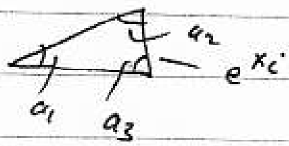
Seifert-Weber: the SAME but use $\frac{2\pi}{5}$ rotation followed by $x \mapsto -x$

the quotient is a compact Hyperbolic 3-mfd since there exists a regular hyperbolic dodecahedron of dihedral angle $2\pi/5$. (The same as 2-dim

Thm (Milnor, Lobachevsky) ideal tetra in $\mathbb{H}^3$ of angs a, b, c has volume $\Lambda(a)+\Lambda(b)+\Lambda(c)$

where $\Lambda(x) = -\int_0^x \ln|2\sin(t)|\,dt$ $\quad$ ($C^0(\mathbb{R})$ periodic π) concave

Cohen-Kenyon-Propp:



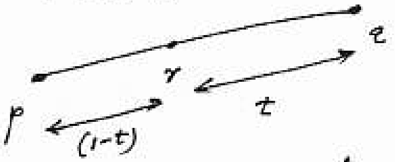
$\exists$ convex $F(x)$ $\quad \frac{\partial F}{\partial x_i} = \theta_i$

F is legendre dual to Λ $\quad \frac{\partial V}{\partial \theta_i} = -\ln|\sin(\theta_i)| = -x_i + c.$

Def $X \subset \mathbb{R}^n$ ($\neq \phi$) convex if $\forall\, p, q \in X$ and $t \in [0,1]$ $\quad tp + (1-t)q \in X$, $\quad tp+(1-t)q = r$

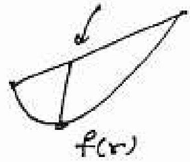
$$r - p = (t-1)p + (1-t)q = (1-t)(q-p)$$

$$r - q = t(p-q)$$



$f: X \to \mathbb{R}$ convex if $\forall\, p, q \in X$ $\quad t \in [0,1]$

$$f(tp + (1-t)q) \leq t f(p) + (1-t) f(q)$$



Def $p_1, \dots, p_K \in X$ $\quad t_1, \dots, t_K \geq 0$ $\quad \sum t_i = 1$ $\quad$ then $\sum_{i=1}^{K} t_i p_i$

is a convex combination of $p_1, \dots, p_K$

Easy Fact

(1) $\{X_\alpha \mid \alpha \in A\}$ collection of convex sets in $\mathbb{R}^n$ $\Rightarrow$ $\bigcap_\alpha X_\alpha$ is convex

(2) $\{f_\alpha \mid f_\alpha : X \to \mathbb{R} \;\; \alpha \in A\}$ ... functions $\Rightarrow$ $\sup_\alpha \{f_\alpha(x)\} : X \to \mathbb{R} \cup \{\infty\}$ convex

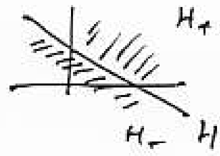
(3) X convex $\Rightarrow$ closure $\bar{X}$ of X convex

RM Convexity well defined for $f: X \to \mathbb{R} \cup \{+\infty\}$: $+\infty + a = +\infty$

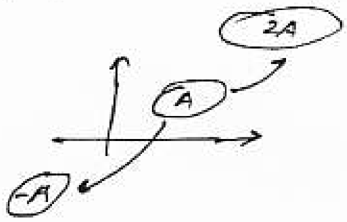
Eg 1. Hyper-plane $\quad H = \{x \in \mathbb{R}^n \mid x \cdot a = b\}$ $\quad a \in \mathbb{R}^n - \{0\},\; b \in \mathbb{R}$

half-space $\quad H_+ = \{x \mid x \cdot a \geq b\}$ $\quad H_- = \{x \mid x \cdot a \leq b\}$

Are convex



Eg 2. A, B convex in $\mathbb{R}^n$ then



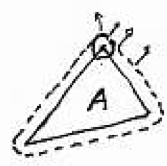
(1) $\lambda A = \{\lambda \cdot x \mid x \in A\}$ is convex

(2) $A + B = \{x \mid x = a + b \;\; a \in A, b \in B\}$

(3) $A - B = \{x \mid x = a - b \;\; — \;\;\}$ $\quad$ are convex

(4) $B = \{|x| \leq \varepsilon\}$ $\quad$ ε-ball

then $\quad A + B = \{x \in \mathbb{R}^n \mid d(x, A) \leq \varepsilon\}$

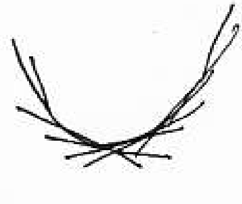


C^1 But Not C^2

Note: $\partial(A+B)$ is SMOOTH.

(5) Epigraph of f convex

Eg 3. We will see: X convex + closed $\Rightarrow X = \bigcap_{X \subset H_+}$ half space H_+



f convex l. semi. cont. $\quad \Rightarrow \quad f = \sup\{\text{Affine maps}\}$

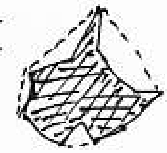
Def $A \neq \phi \subset \mathbb{R}^n$ the convex hull $\mathrm{Conv}(A) = \bigcap_{\substack{X \text{ convex} \\ A \subset X}} X$.

Eg $\mathrm{Conv}(A) = \{$ all convex combinations of $p_1, \cdots, p_k \in A \} = C(A)$

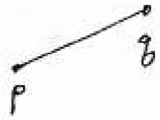
$C(A) \subset \mathrm{Conv}(A)$ clear X contains it $\forall X$

"$\supset$" since $C(A)$ is convex.

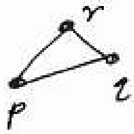
Geometry

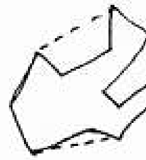


Eg

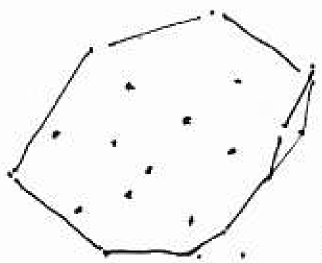


$\mathrm{Conv}\{p, q\} = \{ tp + (1-t) q \mid t \in [0,1] \}$





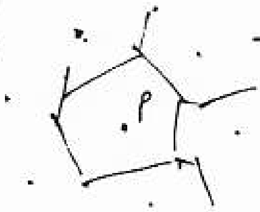
Eg The Voronoi Construction: Given a finite set of points $P \subset \mathbb{R}^2$ one produces the "best" decomposition of $\mathrm{Conv}(P)$. $\mathbb{R}^2$

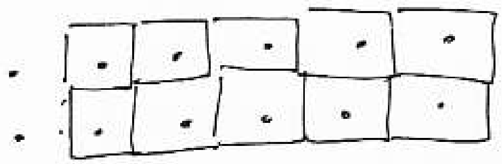


$\forall \; p \in P$ consider

$\Omega(p) = \{ x \in \mathbb{R}^2 \mid \quad d(x,p) \leq d(x,q) \quad \forall \; q \in P - \{p\} \}$

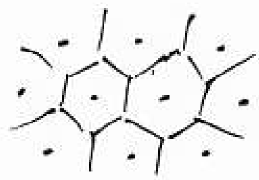
$\Omega(p)$ is convex





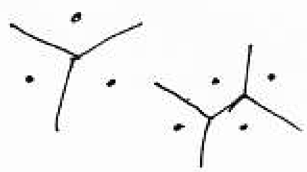
$\mathbb{Z} \oplus \mathbb{Z}$

0-dim $\Rightarrow$ 1-dim + 2-dim



$\mathbb{Z} \oplus e^{\pi i/3} \mathbb{Z}$

RM This construction works in ANY metric spaces (X, d).



Caratheodory Thm If $A \subset \mathbb{R}^n$, then $\forall \; p \in \mathrm{Conv}(A) \Rightarrow p = \sum_{i=1}^{k} x_i p_i \quad p_i \in A$, $k \leq n+1$

i.e., $\mathrm{Conv}(A)$ is a union of n-dim simplices

Pf By definition $p = \sum_{i=1}^{k} x_i p_i$. $x_i > 0$, $\sum_1^k x_i = 1$, $p_i \in A$

(We may assume that k is the smallest) claim if $k \geq n+2$, we may reduce k. Rewrite $p = \sum_i^{k'} \tilde{x}_i \tilde{p}_i$ $\boxed{k' < k}$

Indeed $k \geq n+2 \Rightarrow k-1 \geq n+1$ ie $p_k - p_1, \cdots, p_2 - p_1$ linear dep

$\Rightarrow$ numbers $\sum_{i=1}^{k-1} b_i (p_i - p_1) = 0$ or

$$\sum_{i=1}^{k} b_i p_i = 0 \qquad \left(b_k = -\sum_1^{k-1} b_i \right) \text{ ie } \sum_1^k b_i = 0$$

Not all b_i's are zero

Consider $p = \sum_{i=1}^{k} (x_i - t b_i) p_i$, $\sum (x_i - t b_i) = 1$

$\exists\, t$ s.t $x_i - t b_i \geq 0 \;\; \forall i$ and One of them is zero.

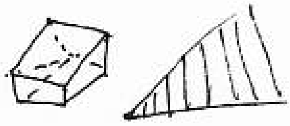
($t=0$ fine all ≥ 0, increase t till one of them $= 0$). $\square$

Corollary (Hw) If A is compact $\Rightarrow$ $Conv(A)$ is compact.

Def A compact convex polytope = Convex hull of a finite set.

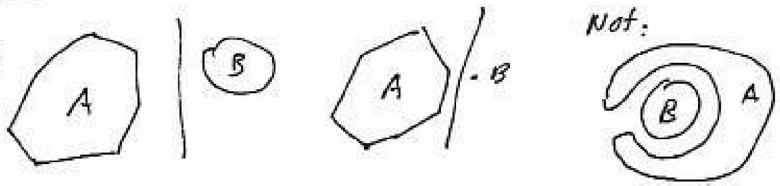
It is also the intersection of finite set of half spaces

(How do you prove that ??) . Convex polytope $= \bigcap_{i=1}^{n}$ Half spaces



The most important property of convex sets separation them

ie A, B disjoint convex sets $\Rightarrow$



(separation)

Thm Suppose X, Y non-empty convex sets in $\mathbb{R}^n$ s.t $X \cap Y = \phi$. Then

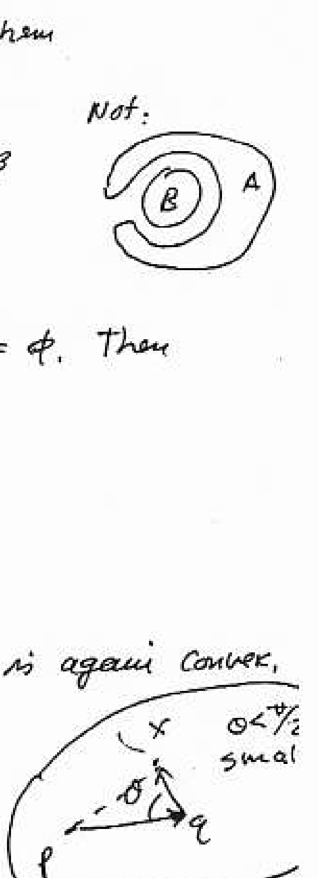
$\exists \;\; v \in \mathbb{R}^n - \{0\}$ and $b \in \mathbb{R}$ s.t

$\forall x \in X \qquad x \cdot v \geq b,$

$\forall y \in Y \qquad y \cdot v \leq b.$

Pf

Case 1 $Y = \{p\}$ ONE point. Let $\bar{X}$ = closure of X which is again convex,



step 1 $p \notin \bar{X}$, let $q \in \bar{X}$ of the shortest distance to p

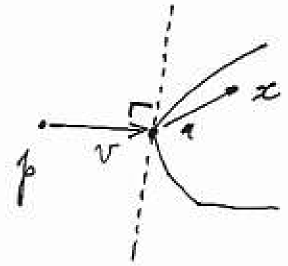
$\Rightarrow \;\; \forall \; x \in \bar{X}$ (shortest distance

$(x-q) \cdot v \geq 0 \qquad v = q - p$

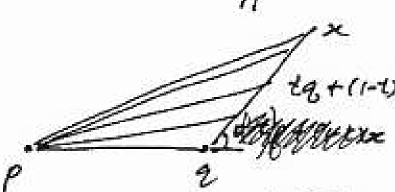
$\Uparrow \qquad (p-q) \cdot v < 0$

$\Rightarrow \quad x \cdot v \geq q \cdot v$

$p \cdot v < q \cdot v \qquad b = q \cdot v$



Geometric reason

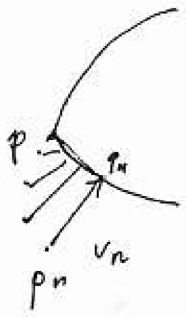


$(\theta < \frac{\pi}{2})$ (q is unique !)

Take $\frac{1}{n}$ / $t \to 1$

pause: X convex closed $X = \bigcap$ half

Step 2: If $p \in \bar{X}$, find $p_n \notin \bar{X}$, $p_n \to p$ and $q_n \in \bar{X}$ of shortest distance to p_n $\quad (\text{dist}(p_n, q_n) \Rightarrow 0)$



Let $v_n = \dfrac{q_n - p_n}{|q_n - p_n|}$ unit vector

After a choice of subsequence, may assume

$$q_n \to q = p \qquad \left(\begin{array}{l} d(p_n, p) \to 0 \\ d(p_n, p) \geq d(p_n, q_n) \\ \downarrow \\ 0 \end{array} \right)$$

$$v_n \to v$$

Now $\forall\, x \in \bar{X}$ $\quad (x - q_n) \cdot v_n \geq 0$

$$(p_n - q_n) \cdot v_n \leq 0$$

Take the limit $\Rightarrow$ $(x - q) \cdot v \leq 0$ $\Rightarrow$ done $\quad (p = q)$

$$\Rightarrow \boxed{x \cdot v \leq q \cdot v = p \cdot v}$$

Case 2: General Y.

Consider $C = X - Y = \{ c = x - y \mid x \in X,\ y \in Y \}$ it is still convex

Now $X \cap Y = \phi \Rightarrow 0 \notin C$

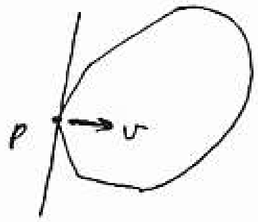
$\exists\, v \in \mathbb{R}^n - \{0\}$ s.t. $\forall\, c = x - y \in C$

$$c \cdot v \leq 0 \cdot v = 0$$

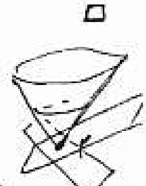
i.e. $\quad x \cdot v \leq y \cdot v \qquad \underline{b = \sup\{x \cdot v \mid x \in X\}}$

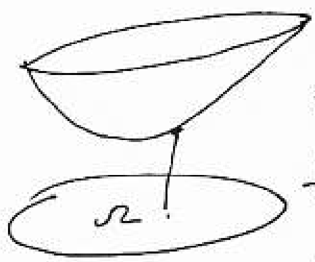
$\square$

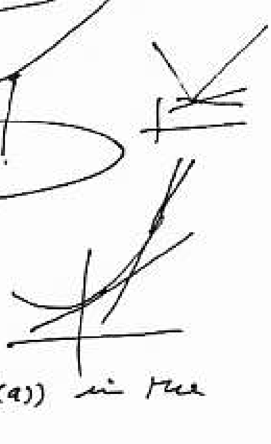
Corollary: $\forall\, p \in \partial X = \bar{X} - X$, $\exists\, v \neq 0$ s.t. $\quad p \cdot v \geq x \cdot v \quad \forall\, x \in X$



these are the supporting hyper-planes of X at boundary points.







Eg: $f: \Omega \to \mathbb{R}$ convex $\quad \Omega \subset \mathbb{R}^n$ open convex

then $\forall\, a \in \Omega\ \exists\ v \in \mathbb{R}^n$ and s.t.

$$f(x) \geq f(a) + v \cdot (x - a)$$

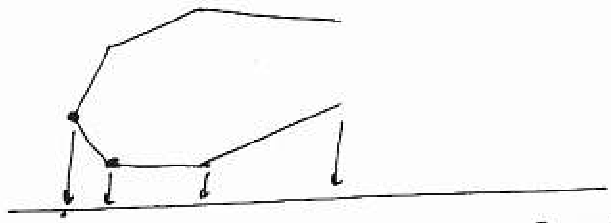
Indeed $X = \{ (x, t) \in \Omega \times \mathbb{R} \mid t \geq f(x) \}$ is convex w/ $(a, f(a))$ in the boundary $\Rightarrow$ done

v — sub-derivative of a function f.

Thm Notations $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, write $x\geq 0 \Leftrightarrow x_i\geq 0\ \forall i$; $x>0 \Leftrightarrow \forall i$ x

Linear programming: $\max\{f(x) \mid x\in P\}$ P convex polytope $f(x)=a.x$ linear

ANS: solution always appears at the vertex of P



Key issue: How to algorithmically find the solution? G. Dantzig, Kantorovich. (1951.)

Farkas lemma Suppose $A:\mathbb{R}^n\to\mathbb{R}^m$ linear. Then

$$\{Ax=b,\ x\geq 0\}=\phi \Leftrightarrow \exists\ h\in\mathbb{R}^m \text{ s.t } A^t(h)\geq 0,\ h.b<0$$

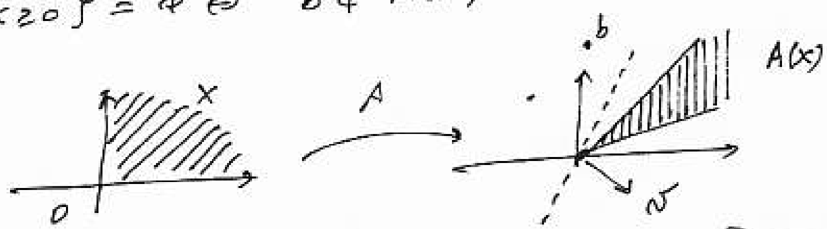
All problem $\begin{matrix}Ax=b\\ Bx\geq c\end{matrix} \Rightarrow \begin{matrix}F(x)=b\\ x\geq 0\end{matrix}$ introduce $y=Bx-c$

Pf. "$\Leftarrow$" clear Since if so

$$0\overset{>}{<}\langle h,b\rangle=\langle h, Ax\rangle=\langle A^t h, x\rangle\geq 0$$

"$\Rightarrow$" Consider the convex set $X=\mathbb{R}^n_{\geq 0}=\{x\in\mathbb{R}^n \mid x\geq 0\}$

Then $\{Ax=b,\ x\geq 0\}=\phi \Leftrightarrow b\notin A(X)$ — convex



seperation thm $\exists$ $h\in\mathbb{R}^n$

$b.h<0$

$y.h\geq 0$ $\forall\ y\in A(X)$

$b.h<K$
$y.h\geq K$ $y=0$
$K\leq 0$ $\Rightarrow$ $K\leq 0$
$y.h_1=\mu<0 \Rightarrow$

i.e. $\forall\ x\geq 0$ $\langle Ax, h\rangle\geq 0$

$\Updownarrow$

$\langle x, A^t h\rangle\geq 0$ $\forall\ x\geq 0$

$\Updownarrow$

$A^t h\geq 0$ $\square$

Recall Thurston's Circle Packing, Curvature set $\{K\}=\{z\in\mathbb{R}^V \mid \sum_{v\in V} z(v)=2\pi\chi(\Sigma)$,

$\forall\ I\subsetneq V$ $\sum_{v\in I} z(v)<2\pi|I|-\pi|F_I|\}$ $\underset{\text{Farkas}}{\Leftrightarrow}$ Angle structures of de Verdiere

$\chi(\Sigma)=0$

Legendre-Fenchel Duality

$f: \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ convex ($\Rightarrow$ $\{x \mid f(x)<+\infty\}$ is convex in $\mathbb{R}^n$)

Its dual $f^*(y) = \sup\{x\cdot y - f(x) \mid x\in\mathbb{R}^n\}$ is convex. $\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$

Eg $f:\mathbb{R}\to\mathbb{R}$ $f(x) = x$, $f^*(y) = \begin{cases} 0 & y=1 \\ \infty & y=+\infty, \end{cases}$ $f^{**} = f$

(Duality)

Thm. $f: \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ convex s.t (1) $\exists x_0$ $f(x_0) < +\infty$ (2) $\underline{\lim}_{x\to a} f(x) \ge f(a)$

(the epigraph $\{(x,y)\in\mathbb{R}^n\times\mathbb{R} \mid y \ge f(x)\}$ closed convex), Then $f^{**} = f$.

Conversely it is also true

pf First $f(x) + f^*(y) \ge x\cdot y$ $\forall x,y$ ($f^*(y) \triangleq \sup \ge x\cdot y - f(x)$),

$\Rightarrow$ $f(x) \ge x\cdot y - f^*(y)$ $\forall y \Rightarrow f(x) \ge f^{**}(x) = \sup\{x\cdot y - f^*(y)\}$

Next. To show $f \le f^{**}$

$\forall a$, then graph f has a supporting plane at $(a, f(a))$

$\Rightarrow$ $\exists b$ s.t ~~$f(a) \le f(x) - (x-a)\cdot b$ $\forall x$~~

$\Rightarrow$ $f(x) \ge f(a) + (x-a)\cdot b$ $\forall x$

$\Rightarrow$ $a\cdot b - f(a) \ge x\cdot b - f(x)$ $\forall x$

$\Rightarrow$ $a\cdot b - f(a) \ge f^*(b)$

$\Rightarrow$ $a\cdot b - f^*(b) \ge f(a)$ —— (1)

$\Rightarrow$ $f^{**}(a) \ge a\cdot b - f^*(b) \ge f(a)$

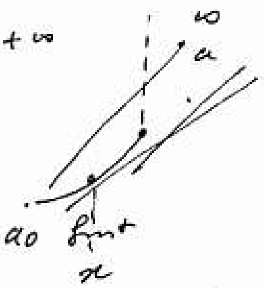
Q1 What if $f(a) = +\infty$? $\boxed{\text{HW}}$ show that this holds if f has closed epigraph & $f \not\equiv \infty$.

Q2. How do you show that $\exists c$, $f^*(c) < +\infty$

Q2 can be answered easily: Since $\exists x_0$ $f(x_0) < +\infty$ f closed $\Rightarrow$

$\exists$ supporting plane at $(x_0, f(x_0))$ for the epi-graph $\Rightarrow$ (1) holds

Q1. if $f(a) = +\infty$



$\sup\{a\cdot y - f^*(y)\} = +\infty$ why? $\forall N\ \exists y$

$a\cdot y - f^*(y) \ge N$

$a\cdot y \ge N + f^*(y) \ge N + x\cdot y - f(x)$ $\forall x, \forall y$

($\forall N$.)

$\Rightarrow$ $(a-x)\cdot y \ge N - f(x)$

or $f(x) \ge N + (x-a)\cdot y$ $\rightarrow$

Another sep

Legendre Dual

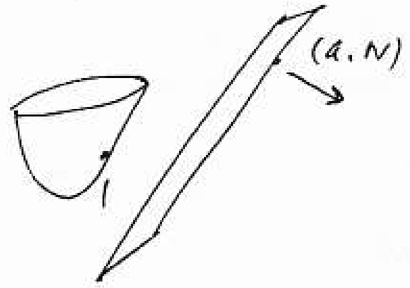
Q1. To show: $\forall N, \exists a\ b$ s.t $\forall x$

$$f(x) \geq N + (x-a)\cdot b$$

Proof Done if $f(x) = +\infty$

if $f(x) < +\infty$, Then $(a, N) \notin \{(\tilde{x}, f(\tilde{x})) \mid f(\tilde{x}) < +\infty\}$

Separation



$\forall x \in C$

$\exists\ b$ st $\quad (a, N)\cdot(-b, 1) \leq (x, f(x))(-b, 1)$

$$-ab + N \leq -x\cdot b + f(x)$$

$\square$

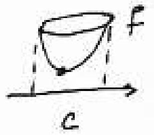
Lecture 21, 22, Main theorem in Convex Optimization

8-8-2017

Notation $X=(x_1,\cdots,x_n)\in\mathbb{R}^n$. $X\geq 0 \Leftrightarrow x_i\geq 0\ \forall i$; $\quad X>0 \Leftrightarrow x_i>0\ \forall i$.

Suppose X is convex $\subset\mathbb{R}^n$ $\quad f, g_1,\cdots, g_k: X\to\mathbb{R}$ convex, $h:\mathbb{R}^n\to\mathbb{R}^m$ affine

Main Problem $\quad \min\{f(x)\mid x\in X,\ g(x)\leq 0,\ h(x)=0\}\quad g=(g_1,\cdots,g_k)$ - convex



Lemma 1. Suppose g convex, h affine as above. If $\{x\in X\mid g(x)<0,\ h(x)=0\}=\phi$, then $\exists\ (p,q)\neq 0\in\mathbb{R}^k\times\mathbb{R}^m$ with $p\geq 0$ s.t

$$(p, g(x))+(q, h(x))\geq 0 \quad .\quad \forall\ x\in X$$

Pf Let $W=\{(y,z)\in\mathbb{R}^k\times\mathbb{R}^m\mid \exists\ x\in X \text{ s.t } z=h(x) \text{ and } y>g(x)\}$.

Then W is convex: $\quad (y_i, z_i)\in W\ \ i=1,2\ $ for x_i. For $t\in[0,1]$

$$ty_1+(1-t)y_2 > tg(x_1)+(1-t)g(x_2)\geq g(tx_1+(1-t)x_2). \text{ —— done}$$

Now $(0,0)\notin W \Rightarrow$ by separation thm $\exists\ (p,q)\neq 0$ s.t

$$p\cdot y+q\cdot z\geq 0 \qquad \forall\ (y,z)\in W$$

(1) $p\geq 0$, if one $p_i<0$, then by taking y st $y_i\gg 1$ $\quad p\cdot y+q\cdot z<0$

(2) $\forall\ \varepsilon>0 \quad (g(x)+(\varepsilon,\cdots,\varepsilon), h(x))\in W \Rightarrow$

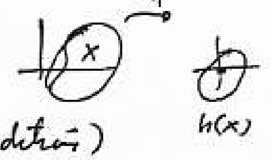
$$p\cdot g+(\varepsilon\cdots\varepsilon)\cdot p+q\cdot h(x)\geq 0 \qquad \forall\ \varepsilon.\ \text{let } \varepsilon\to 0.$$

□

Def A pair (g,h) is "good" if

(1) $\{u\in\mathbb{R}^m\mid u\cdot h(x)\geq 0 \quad \forall x\in X\}=\{0\}$

(2) $\exists\ x_0\in X$ s.t $g(x_0)<0$ and $h(x_0)=0$ (Slater Condition)



Main Thm in Convex Optimization

Suppose (g,h) a good pair and x_0 is a solution to CP i.e $\quad x_0\ g(x_0)\leq 0\ \ h(x_0)=$

$$f(x_0)=\min\{f(x)\mid x\in X,\ g(x)\leq 0,\ h(x)=0\}$$

Then $\exists\ (\lambda,\mu)\in\mathbb{R}^k\times\mathbb{R}^m,\ \lambda\geq 0$ s.t

$$\forall\ x\in X \qquad f(x)+\lambda\cdot g(x)+\mu\cdot h(x)\geq f(x_0)\geq f(x_0)+\lambda\cdot g(x_0)+\mu\cdot h(x_0)$$

Pf Consider $(f(x)-f(x_0), g(x)): X\to\mathbb{R}^{k+1}$, and $h:\mathbb{R}^m\to\mathbb{R}^m$.

X_0 a solution $\Rightarrow \{x\in X\mid f(x)<f(x_0),\ g(x)<0,\ h(x)=0\}=\phi$. By the lemma

$\exists\ (\delta,\lambda,\mu)\in \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^m-\{0\}$ s.t $(\delta,\lambda)\geq 0$, $\forall\ x\in X$

$$\delta(f(x)-f(x_0))+\lambda\cdot g(x)+\mu\cdot h(x)\geq 0$$

$\delta>0$, if otherwise $\quad \lambda\cdot g(x)+\mu\cdot h(x)\geq 0,\ \forall x\in X$. Let $x=x_0 \Rightarrow \lambda=0 \Rightarrow \mu\cdot h(x)\equiv 0 \Rightarrow$

$\mu=0 \Rightarrow$ impossible

□

Lecture Last Brunn-Minkowski

Brunn-Minkowski Inequality

Thm $A, B \subset \mathbb{R}^n$ compact sets. Then

$$\left(\mathrm{vol}(A+B)\right)^{\frac{1}{n}} \geq \left(\mathrm{vol}(A)\right)^{\frac{1}{n}} + \left(\mathrm{vol}(B)\right)^{\frac{1}{n}}$$

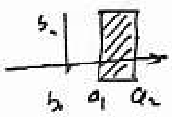
$$A+B = \{ a+b \mid a \in A, b \in B \}$$

---

pf A box in $\mathbb{R}^n$: $X = [a_1, b_1] \times \cdots \times [a_n, b_n]$ $\quad \mathrm{vol}(X) = \prod_i^n (b_i - a_i)$

step 1 If A, B are boxes say $A = \prod [a_i, b_i]$, $B = \prod [c_i, d_i]$

then $A+B = \prod [a_i + c_i, b_i + d_i]$



Thus let $x_i = b_i - a_i$, $y_i = d_i - c_i$, then

$$\mathrm{vol}(A+B) = \prod_i^n (x_i + y_i)$$

$$\mathrm{vol}(A) = \prod x_i$$

$$\mathrm{vol}(B) = \prod y_i$$

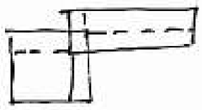
Now

$$\left[\prod\left(\frac{x_i}{x_i+y_i}\right)\right]^{\frac{1}{n}} + \left(\prod \frac{y_i}{x+y_i}\right)^{\frac{1}{n}} \leq \frac{1}{n}\left[\sum\left(\frac{x_i}{x_i+y_i}\right) + \sum\left(\frac{y_i}{x_i+y_i}\right)\right] = 1$$

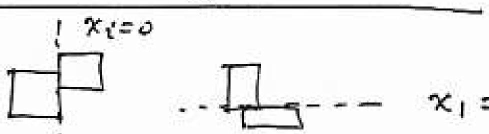
done

$$\boxed{\sqrt[n]{\alpha_1 \cdots \alpha_n} \leq \frac{1}{n}(\alpha_1 + \cdots + \alpha_n)}$$

step 2 Suppose A, B are union of finite boxes with disjoint interiors.



We use induction on the total number of boxes in A and B



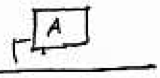
Say A contains at least two boxes.

then $\exists$ a coord x_i s.t $x_i \geq c$ & $x_i \leq c$ both contain boxes

We may replace A by $A + \alpha$ $\quad \alpha \in \mathbb{R}^n$ without chge the statement (Translation). Define

& let $x_i = 1$

$$A^+ = A \cap \{x \mid x_1 \geq 0\} \qquad A^- = A \cap \{x \mid x_1 \leq 0\}$$



B^+, B^- similarly

We may assume $\exists$ two boxes in A separated by $\{x \mid x_1 = 0\}$

After a translation of A (and chge of coordinate x)

ie A^+ & A^- contain boxes of A

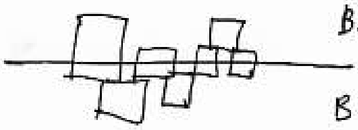
Translate B st

$$\frac{vol(A^+)}{vol(A)} = \frac{vol(B^+)}{vol(B)}$$

Suppose A, B contain m_A, m_B many boxes

Now. B^+, B^- each has ~~at most~~ is a union of at most m_B boxes where B is a union of k boxes

& A^+, A^- ~~has~~ contains fewer than $m_A - 1$ boxes.



$l = \#$ boxes in A

By induction hypothesis, B-M ineq holds for (A^+, B^+) & (A^-, B^-)

Also $(A^+ + B^+) \cap (A^- + B^-)$ has zero measure $\subset \{x_1 = 0\}$. $\boxed{A = A^- \cup A^+}$

Now:

& $A+B) = (A^+ + B^+) \cup (A^- + B^-)$ + some $\boxed{A^+ + B^-}$

$$vol(A+B) \geq vol(A^+ + B^+) + vol(A^- + B^-)$$

$$\overset{hyp}{\geq} \left[vol(A^+)^{\frac{1}{n}} + vol(B^+)^{\frac{1}{n}}\right]^n + \left[vol(A^-)^{\frac{1}{n}} + vol(B^-)^{\frac{1}{n}}\right]^n$$

$$= vol(A^+)\left[1 + \underbrace{\left(\frac{vol(B^+)}{vol(A^+)}\right)^{\frac{1}{n}}}\right]^n + vol(A^-)\left(1 + \underbrace{\left(\frac{vol(B^-)}{vol(A^-)}\right)^{\frac{1}{n}}}_{\text{equal}}\right)^n$$

$$= \left(vol(A^+) + vol(A^-)\right)\left(1 + \left(\frac{vol(B)}{vol(A)}\right)^{\frac{1}{n}}\right)^n$$

$$= vol(A)\left(1 + \left(\frac{vol(B)}{vol(A)}\right)^{\frac{1}{n}}\right)^n$$

$$= vol(A)^{\frac{1}{n}} + vol(B)^{\frac{1}{n}}.$$

Step 3   Every cpt set can be approximated by ~~50~~ finite union of boxes

Take limit.

□

Corollary   if $\lambda \in [0,1] \Rightarrow$

$$vol(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \underset{BM}{\geq} \left(vol(\lambda A)\right)^{\frac{1}{n}} + \left(vol((1-\lambda)B)\right)^{\frac{1}{n}}$$

$$= \lambda\, vol(A)^{\frac{1}{n}} + (1-\lambda)\, vol(B)^{\frac{1}{n}}$$

$vol(-)$ is convex w.r.t Minkowski sum