Lectures 1.2

5-31-2012

MSC Lectures: Topics in Geometry

Feng Luo, lectures at MSC 2012. Summer

Plane: cover a wide range of topics in geometry. (Not Riemannian)

(1) Start triangular billiard problem, integral geometry, a little dynamics system, ergodicity theorem, Poincaré recurrent

(2). Spherical, hyperbolic and Hilbert geometry, circle packing, rigidity

(3). Convex polytopes, Cauchy rigidity, Steinitz realization theorem

(4). Brunn-Minkowski, Hadwiger's theorem

(5) Optimal transportations & its applications

(6) Pick's Area theorem

(7) Hyperbolic geometries

(8) Basic convexity theory

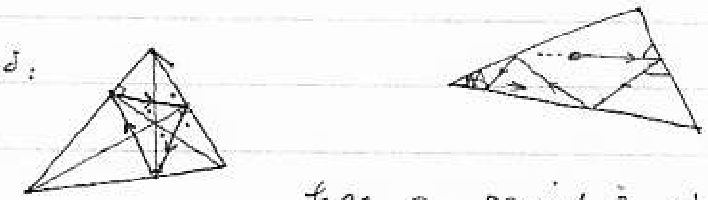
Lecture 1.2 The triangle Billiard Problem

There are many famous elementary number theory problems (p, $p+2$ primes?) whose solution will bring fame. Not too many such ones in elementary geometry.

$\mathbb{E}^2$ = Euclidean plane = $\mathbb{C}$. $\mathrm{Iso}^+(\mathbb{E}^2) = \{ z \mapsto e^{i\theta} z + a \}$

Conjecture: Each triangular billiard board in $\mathbb{E}^2$ admits a closed trajectory.

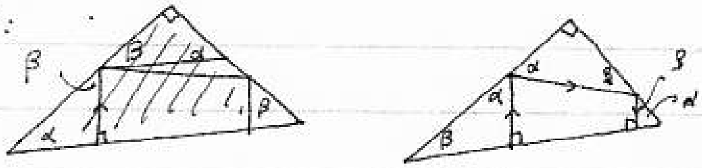
Eg.1. Δ acute angled:



has a period 3 closed trajectory.

S. Tabachnikov (Fagnano traj.

Eg 2. Δ right angled:



$\boxed{\text{period 6}}$

Best result: R. Schwartz: Conjecture holds if angle $\alpha < 100°$

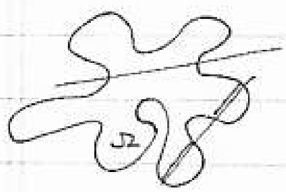
Thm 1 (Masur) If all angles of Δ are in $\mathbb{Q}\pi$, then Δ has a closed trajectory.

Some basic integral geometry ($\Leftrightarrow$ differential geometry, Gauss-Riemann)

Let $\mathcal{L}$ be the set of all lines in $\mathbb{E}^2$. The isometry group $\mathrm{Iso}(\mathbb{E}^2)$ ($z \mapsto e^{i\theta} z + \alpha$ or $z \mapsto e^{i\theta} \bar{z} + \alpha$) acts on $\mathcal{L}$.

Thm 2. There exists a measure m on $\mathcal{L}$ invariant under $\mathrm{Iso}(\mathbb{E}^2)$.

Crofton's Thm 3. If $\Omega \subset \mathbb{E}^2$ is a Jordan domain w/ smooth $\partial\Omega$, then

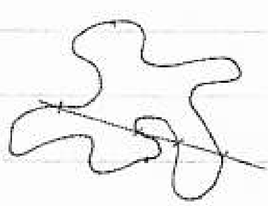


$$\pi \, \mathrm{Area}(\Omega) = \int_{\{L \in \mathcal{L}\}} \mathrm{Length}(L \cap \Omega) \, dm$$

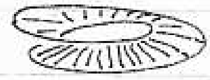
Lecture 1.2 Triangle Billiards + Integral Geometry

Crofton–Poincaré Thm 4 Ω smooth Jordan domain, then

$$2\,\text{length}(\partial\Omega) = \int_{\{L\in\mathcal{L}\}} \#\{L\cap\partial\Omega\}\,dm$$

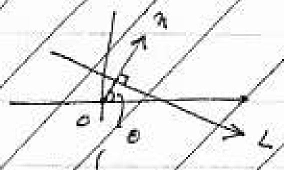


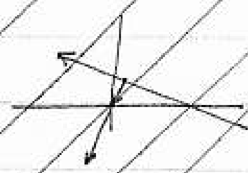
RM These theorems hold in any dimension $\mathbb{R}^N$.

Q (HW) Show that $\mathcal{L} \cong$ Möbius band  (open)

Let $\mathcal{L}^* = \{$set of all oriented lines in $\mathbb{E}^2\}$. $\mathcal{L} = \mathcal{L}^*/L\sim -L$.

Lemma $\mathcal{L}^*$ can be parametrized by $\mathbb{R} \times S^1 \ni (\rho, e^{i\theta})$, $0\le\theta<2\pi$





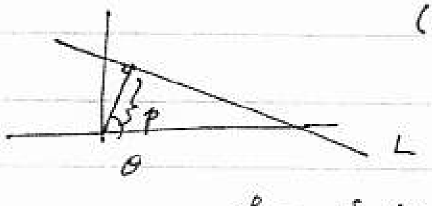
identification $(\rho, z) \sim (-\rho, -\frac{1}{z})$

(L, n) positive oriented

$\rho\ge 0$, $\vec{n} = \rho e^{i\theta}$

An unoriented line $L \subset \mathbb{E}^2$ is determined by $(\rho,\theta)\in\mathbb{R}_{\ge0}\times[0,2\pi)$

$(0,\theta)\sim(0,\theta+\pi)$ $\quad 0<\theta<\pi$



$L \Leftrightarrow (\cos\theta)x + (\sin\theta)y = \rho$

$\mathcal{L} = \{(\rho,\theta)\in\mathbb{R}_{\ge0}\times[0,2\pi)\}/(0,\theta)\sim(0,\theta+\pi)$.

Lemma 1. The measure $|d\rho\wedge d\theta|$ on $\mathcal{L}$ is invariant under $\mathrm{Iso}^+(\mathbb{E}^2)$ action

(Basic $dx\wedge dy$ in $\mathbb{E}^2$ invariant under $\mathrm{Iso}^+(\mathbb{E}^2)=\{z\mapsto e^{i\varphi}z+b\}$)

Pf. Each $\mathrm{Iso}^+(\mathbb{E}^2)$ is a composition $z\mapsto e^{i\varphi}z$ and $z\mapsto z+(a+ib)$

For $z\mapsto e^{i\varphi}z$: $\quad(\rho,\theta)\mapsto(\rho,\theta+\varphi) \Rightarrow d\rho\wedge d(\theta+\varphi) = d\rho\wedge d\theta$

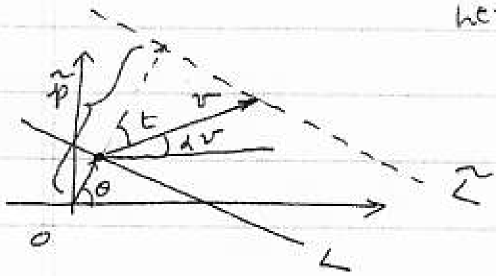
φ fixed $\qquad \boxed{d\varphi=0}$

Lecture 1.2 Integral Geometry

For $\varphi(z) = z + v$. $\quad v = re^{i\alpha}$

if L has coord. (p, θ) and $\varphi(L) = \tilde{L}$ has coord $(\tilde{p}, \tilde{\theta})$

then, $\quad \tilde{\theta} = \theta$.



Let $\tilde{p}$ be the shortest distance from $\tilde{L}$ to O

+ t be the angle between v + the normal vector n to $\tilde{L}$

Then

$$t + \alpha = \theta \quad \text{or} \quad t = \theta - \alpha, \; dt = d\theta$$

so $\quad \tilde{p} = p + |v| \cos t = p + r\cos(t)$

so $\quad d\tilde{p} = dp + (-r\sin(t))\, d\theta$

$$\Rightarrow \quad d\tilde{p} \wedge d\tilde{\theta} = (dp - r\sin(t)\, d\theta) \wedge d\theta = dp \wedge d\theta.$$

$\square$

Eg The space of all lines intersecting $\{|z| \le 1\}$ has ~~[illegible]~~ coordinate (p, θ) $\quad 0 \le p \le 1, \quad 0 \le \theta \le 2\pi \Rightarrow$

$$\text{area}(|z| \le 1) = \int_0^1 \int_0^{2\pi} dp\, d\theta = 2\pi.$$

Def A line element v in $\mathbb{R}^2$: $(z, \{u, -u\})$ $\quad |u| = 1$, coord. (z, φ) $\quad 0 \le \varphi <$

The set of all line elements in $\mathbb{R}^2$

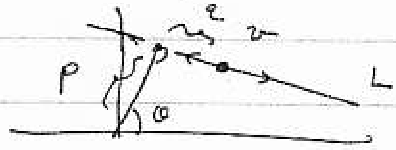
$$\mathcal{D} = \{(z, \varphi) \mid z \in \mathbb{R}^2 \quad 0 \le \varphi < \pi\} \quad \text{coord } (x, y, \varphi)$$

$\text{Iso}^+(\mathbb{E}^2)$ acts naturally on $\mathcal{D}$

$$\gamma(z) = e^{i\theta} z + b \qquad (\underbrace{x, y}_{z}, \varphi) \mapsto (\gamma(z), [\varphi + \theta])$$

$[\varphi] = \begin{cases} \varphi & 0 \le \varphi < \pi \\ \varphi - k\pi & 0 \le \varphi - k\pi < \pi, \; k \in \mathbb{Z} \end{cases}$

Given $v \in \mathcal{D}$, v determines a line $L \in \mathcal{L}$ + $q, q \in \mathbb{R}$

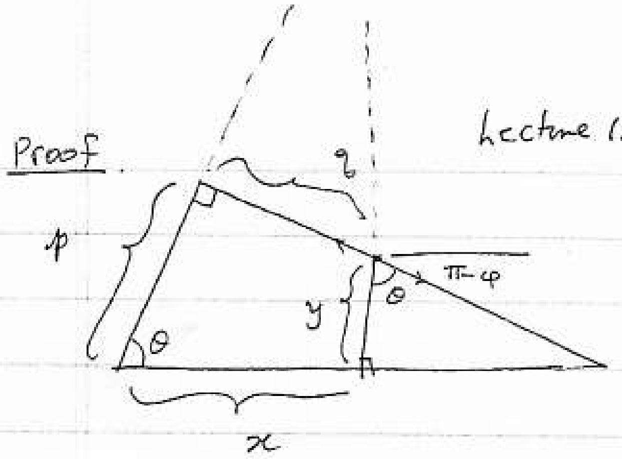


q : distance of base pt of v to end pts of p.

Lemma $\quad dx \wedge dy \wedge d\varphi = dp \wedge dq \wedge d\theta$

Lecture 1.2. Billiards + Crofton Theorem

<u>Proof</u>



$$v \overset{\text{coord}}{\Leftrightarrow} (x, y, \varphi)$$

$$\varphi = \frac{\pi}{2} - \theta$$

<u>Then</u> $\begin{cases} p = x\cos\theta + y\sin\theta \\ q = x\sin\theta - y\cos\theta \end{cases}$

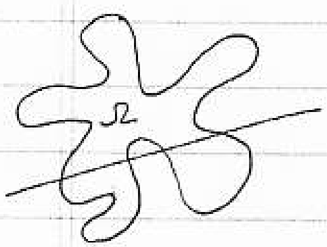
<u>Proof</u>. (1) $\dfrac{p}{\cos\theta} = x + y\,\mathrm{tg}(\theta)$

(2) $\dfrac{q}{\cos\theta} = x\,\mathrm{tg}\theta - y$

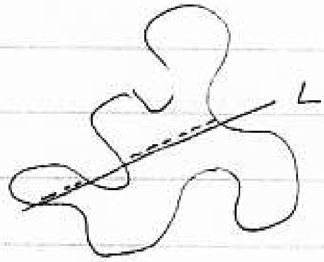
So $\quad dp\wedge dq = \det\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} dx\wedge dy + F\,dx\wedge d\theta + G\,dy\wedge d\theta$

<u>Therefore</u>, $dp\wedge dq\wedge d\theta = (\pm 1)\, dx\wedge dy\wedge d\theta$.

Now the proof of Crofton's formula (thm 3)



$$\int_{\{L\in\mathcal{L} \,|\, L\cap\Omega\neq\phi\}} \text{length}(L\cap\Omega)\, dp\wedge d\theta$$



Let $W = \{L\in\mathcal{L} \,|\, L\cap\Omega\neq\phi\}$, Let $W^* = \{v\in\mathcal{D} \,|\, \exists\, L\in\mathcal{L}$ s.t v is tangent to $L\cap\Omega\}$.

$$(L\cap\Omega) \to W^* \to W$$

Evidently $\quad \text{Length}(L\cap\Omega) = \displaystyle\int_{L\cap\Omega} dq,$

$$W^* \subset W\times\mathbb{R}$$
$$W^* = \Omega\times[0,\pi)$$

So, $\displaystyle\int_W \text{Length}(L\cap\Omega)\, dp\wedge d\theta = \int_{W^*}$

$$\int_W \text{length}(L\cap\Omega)\, dp\wedge d\theta = \int_W \Big(\int_{L\cap\Omega} dq\Big) dp\wedge d\theta \overset{\text{Fubini}}{=} \int_{W^*} \underset{\overset{\|}{dx\wedge dy\wedge d\varphi}}{dq\wedge dp\wedge d\theta} \overset{\text{lemma}}{=} \int_{W^*} \underline{d\varphi\, dx\, dy} = \pi \int_\Omega dx\,dy = \pi \text{ Area}$$

$$\int_W \Big(\int_{L\cap\Omega} dq\Big) dp\wedge d\theta \quad = \quad W^* \to \Omega\times[0,\pi)$$

→ Back

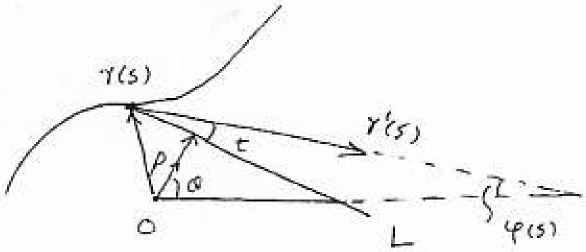
Lecture 3. Crofton-Poincaré & Haar Measure

<u>Thm</u> 1 (Crofton-Poincaré) Ω smooth Jordan domain in $\mathbb{R}^2$. Then

$$2\,\text{length}(\partial\Omega) = \int_{\{L\in\mathcal{L} \mid L\cap\Omega\neq\phi\}} \#\{L\cap\partial\Omega\}\, dp\wedge d\theta$$

<u>pf</u> Let $\gamma(s)$ be the arc length parameterization of $\partial\Omega$

$0\le s\le \text{length}(\partial\Omega)$



Fix $t\in[0,\pi)$, (s,t) determines a line $L\in\mathcal{L}$ $L\cap\partial\Omega\ni\gamma(s)$ s.t the angle between L & $\gamma'(s)$ at $\gamma(s) = t$

$0\le t\le\pi$

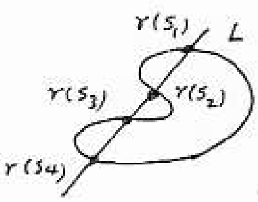
<u>Q</u> what is the coordinate (p,θ) of L ?

<u>Lemma</u> 2. $\quad dp\wedge d\theta = -\sin t\, ds\wedge dt$

<u>Now</u>,

$$\{(\gamma(s),t) \mid s\in[0,\text{length}(\partial\Omega)]\ t\in[0,\pi)\} \xrightarrow[\text{onto}]{1-1} \{L\in\mathcal{L} \mid L\cap\partial\Omega\neq\phi\}$$

each L is counted w/ multiplicity $\#\{L\cap\partial\Omega\}$



$L \;—\; (\gamma(s_i), t_i)\quad i=1,2,3,4$

<u>Thus</u>

$$\int_{\{L\in\mathcal{L} \mid L\cap\Omega\neq\phi\}} \#\{L\cap\partial\Omega\}\,dp\wedge d\theta \overset{\text{Lemma 2}}{=\!=} \int_0^{\text{Length}(\partial\Omega)} \left(\int_0^{\pi} \sin t\, dt\right) ds$$

$$= 2\,\text{length}(\partial\Omega).$$

<u>pf of lemma 2</u> Let $n = n(s,t)$ be the unit normal vector to $L = L(s,t)$. Say $n = e^{i\theta(s,t)}$ $\quad dn = i\cdot e^{i\theta}\cdot d\theta = i\,n\cdot d\theta$

<u>Now</u> $\quad p = \gamma(s)\cdot n\quad$ <u>also</u> $\quad \frac{\pi}{2}-\theta = t+\varphi(s),\quad \varphi(s) = \angle\,\gamma'(s),\ x\text{-axis}$

so $\quad dp = \gamma'(s)ds\cdot n + \gamma(s)\cdot dn = (\gamma'(s)\cdot n)\,ds + i(\gamma(s)\cdot n)\,d\theta$

$\quad d\theta = -dt + \varphi'(s)\,ds$

so $\quad dp\wedge d\theta \underset{d\theta\wedge d\theta=0}{=\!=} (\gamma'(s)\cdot n)\,ds\wedge d\theta = -\gamma'(s)\cdot n\, ds\wedge dt$

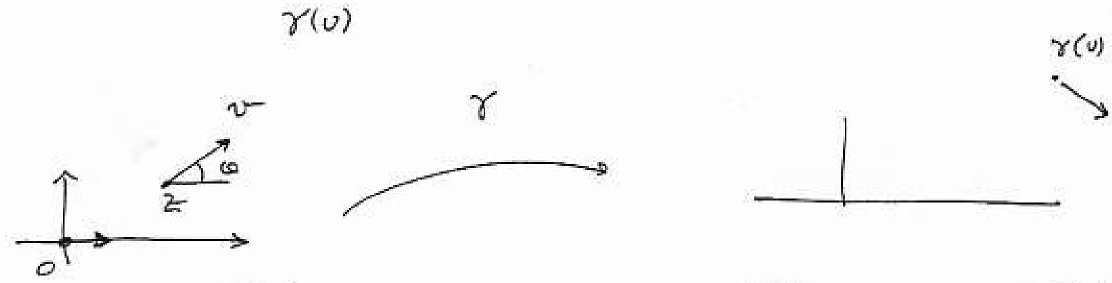
$\quad = -\cos(\frac{\pi}{2}-t)\,ds\wedge dt = -\sin t\, ds\wedge dt$

$\square$

Lecture 3

One more remark : + Haar measure

space of all unit tangent vectors to $\mathbb{E}^2$ $\underset{\mathcal{U}}{\underset{\shortparallel}{UTE^2}} = \{(z, \underset{v}{\underset{\shortparallel}{e^{i\theta}}}) \mid z \in \mathbb{C}$, $\theta \in \mathbb{R}\}$. $Iso^+(\mathbb{E}^2)$ acts $\mathcal{U}$



Fix $v_0 = \frac{\partial}{\partial x}\big|_0$. $\Phi : Iso^+(\mathbb{E}^2) \longrightarrow \mathcal{U} : \gamma \mapsto \gamma(\frac{\partial}{\partial x}\big|_0)$

Φ is 1-1 onto

(1) 1-1 if $\Phi(\gamma_1) = \Phi(\gamma_2) \iff \gamma_2^{-1}\gamma_1(\frac{\partial}{\partial x}\big|_0) = \frac{\partial}{\partial x}\big|_0$

$\gamma_2^{-1}\gamma_1(z)$ sends $0 \to 0$ + Also $\frac{\partial}{\partial x}\big|_0 \mapsto \frac{\partial}{\partial x}\big|_0$

Due to orientation preserving $\Rightarrow$ $\gamma_2^{-1}\gamma_1(z) = e^{i\varphi}z$ $\varphi = 0$

(2) Onto $v = (z, e^{i\theta})$ $\gamma(\omega) = e^{i\theta}\omega + z$ $\omega \mapsto -$

$0 \longmapsto z$ $\frac{\partial}{\partial x}\big|_0 \mapsto e^{i\theta}$

ie Write $\gamma \in Iso^+(\mathbb{E}^2)$ $\underbrace{\gamma(z) = e^{i\theta}z + b \Leftrightarrow (b, e^{i\theta}) \in \mathcal{U}}_{\text{1-1 onto}}$

Prop The measure $\eta = dx \wedge dy \wedge d\theta$ on $\mathcal{U}$ $z = x+iy$ is invariant under $Iso^+(\mathbb{E}^2)$ action. $(Iso^+(\mathbb{E}^2) = \underset{SO(2)}{\underset{\shortparallel}{S^1}} \ltimes \mathbb{C}$ (HW))

Pf. $\gamma(z) = e^{i\varphi}z$: $(x, y, \theta) \longmapsto ((\cos\varphi)x + \sin\varphi y, (\cos\varphi)x - \sin\varphi y, \theta + \varphi)$

check invariance

$\gamma(z) = z + b$ $(x, y, \theta) \longmapsto (x+b_1, y+b_2, \theta)$ inv

(x, y) θ

$\boxed{dx \wedge dy \wedge d\theta}$

This is the Haar measure on $Iso^+(\mathbb{E}^2)$

The space of all line elements $\{(z, \{u, -u\}) \mid |u| = 1\} = \mathcal{U}/_{(z,\theta) \sim (z, \theta+\pi)}$

What are the Haar measures?

Given a Lie group G^n (of dim $= n$) (e.g. $\mathbb{R}^n$, S^1, $(S^1)^n$, $Iso^+(\mathbb{R}^2)$, $SO(3)$, $SL(2,\mathbb{R})$)

there exists an differential n-form ω invariant under left multiplications

Eg 1. $G = \mathbb{R}^1$ $\quad O = \underset{0}{dx} \quad \underset{a}{dx}$ at $a \longrightarrow$ $\quad d(x+a) = dx$

$x \longmapsto x+a$

$0 \longmapsto a$

dx inv under translation

$\mathbb{R}^n$ $\quad dx_1 \wedge \cdots \wedge dx_n$ Lebesgue

Eg 2. $Iso^+(\mathbb{E}^2) \cong \mathbb{E}^2 \rtimes S^1 \cong \mathbb{E}^2 \times S^1$ $\quad dx \wedge dy \wedge d\theta$ $\quad e^{i\theta}$

Eg 3. $S^1 \ni e^{i\theta}$ $\quad d\theta$ $\quad$ What is it ? $\quad$ (HW, Show that $d\theta$ inv)

Eg 4. All those invariant measure on $\mathcal{L}$ are derived from the Haar measure on $Iso^+(\mathbb{E}^2)$

Eg 5. $X = \hat{\mathbb{R}} \times \hat{\mathbb{R}} - \{(x,x)\}$ $\quad \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ acted by $SL(2,\mathbb{R})$

$SL(2,\mathbb{R})$ acts on $\hat{\mathbb{R}}$. $\quad \gamma(x) = \dfrac{ax+b}{cx+d}$ $\quad a,b,c,d \in \mathbb{R}$ $\quad ad-bc = 1$

$\underset{\gamma}{SL(2,\mathbb{R})} \ni$ acts on $\gamma(x,y) = (\gamma(x), \gamma(y))$

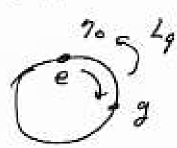
Hw $\quad \dfrac{dx \wedge dy}{(x-y)^2}$ is invariant under the group action

Solution. $\gamma(x) = \lambda x$ clear, $\quad \gamma(x) = x+a$ clear

$$\gamma(x) = \frac{1}{x} \qquad \frac{d(\frac{1}{x}) \wedge d(\frac{1}{y})}{(\frac{1}{x} - \frac{1}{y})^2} = \frac{\frac{1}{x^2}\frac{1}{y^2}\, dx \wedge dy}{\frac{1}{x^2 y^2}\,(y-x)^2} \quad \checkmark.$$

Eg 6 $\quad$ Construction of Haar measure, take n-form η_0 at $T_e G$

Let $L_g : G \to G$ $\quad x \mapsto g.x$ left multiplication.

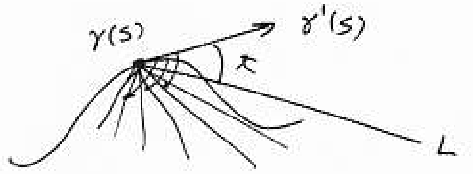


Define $\quad \eta_g = D(L_{g^{-1}})^* (\eta_0)$.

Eg 7. What is the Haar measure on $SL(2,\mathbb{R})$? (Hyperbolic Geometry)

Lecture 4. Triangular Billiards

Recall: for the smooth curve $\partial\Omega$ w/ arc length parameterization



Line through $\partial\Omega$ has a coord.

(s,t) s.t

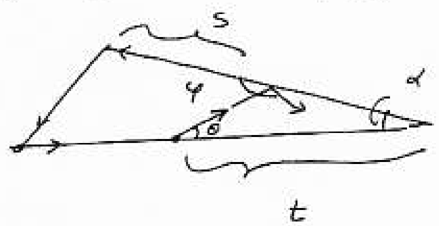
$\sin t \; ds \wedge dt \;=\; \underbrace{dp \wedge d\theta}_{\text{intrinsic Liouvill measure}}$

Now let us go back to triagle billiard problem

Let Δ be the triangle, $T = \partial\Delta - \{\text{vertices}\}$ $\quad\Delta$ oriented

Define the billiard map (Poincaré return map) $S^1 \sim e^{i\theta} \sim \theta \in [0, 2\pi)$

$F: \; T \times S^1 \to T \times S^1 \qquad T \times (0,\pi) \to T \times (0,\pi) \qquad \theta \in (0,\pi)$

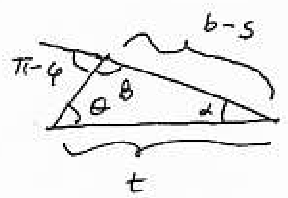


$(t, \theta) \longmapsto (s, \varphi)$. $\qquad T \in C^\infty$

θ: the angle from oriented $e \subset T$ to the trajectory.

Thm: $\quad |\sin\theta \, dt \wedge d\theta| = |\sin\varphi \, ds \wedge d\varphi|$ is invariant under F.

Pf Let α be the inner angle of Δ + b be the edge length of Δ which contains the (s, φ) pt.



$\boxed{\beta =}$ $\quad \beta = \varphi \qquad \pi - \varphi = \theta + \alpha$

so $\quad \beta = \pi - (\theta + \alpha) \qquad \varphi = \pi - (\theta + \alpha), \; d\varphi = -d\theta$

By the sine law: $\quad \dfrac{t}{\sin\beta} = \dfrac{b-s}{\sin\theta}$ or

$$\sin\theta \cdot t = (\sin\varphi)(b-s) \qquad (1)$$

Take d to (1) and wedge product w/ $d\theta$:

$$d(\sin\theta \cdot t) \wedge d\theta = d(\sin(\varphi)(b-s)) \wedge d\theta = d(\sin\varphi)(b-s)) \wedge (-d\varphi)$$

$$\|$$

$$\left(d(\sin\theta)\, t + \sin\theta \, \overset{d}{\,} t\right) \wedge d\theta = (-1)\left(\sin\varphi \; d(b-s)\right) \wedge d\varphi$$

$$\sin\theta \; dt \wedge d\theta = \sin\varphi \; ds \wedge d\varphi$$

$\square$

Lecture 4.

Rational Billiards

<u>Thm</u> (Masur) If Δ is a triangle whose angles are in $\mathbb{Q}\pi$, then Δ has a closed trajectory.

<u>Proof</u> Suppose all inner angles of Δ are in $\frac{2\pi}{N}\mathbb{Z}$, $N \in 4\mathbb{Z}_{>0}$

Fix this N and let

$$M = T \times \left\{ \frac{2\pi k}{N} \;\middle|\; k = 0, 1, \cdots, N \right\}. \quad \text{Note } \frac{\pi}{2} = \frac{2\pi}{4}.$$

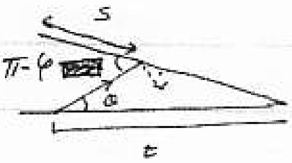
The return map $F: M \to M$ since the angles $\varphi \mapsto \theta + \left(\frac{2\pi k}{N}\right)$.

$\uparrow$ inner angle

<u>Lemma</u> Suppose e is an edge of Δ and $\underset{(s,\varphi)}{e \times \{\varphi\}} \subset M$. Then

$$|\sin(\varphi)\, ds|$$

is invariant under F.



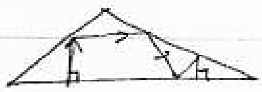
$$(b - s) = \frac{\sin\theta}{\sin(\varphi)}\, t$$

θ, φ constant <u>Now</u>: $\in \frac{2\pi}{N}\mathbb{Z}$

so $\quad -\sin\varphi\, ds = \sin\theta\, dt$

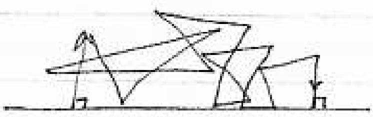
$\square$

Now let m be the measure on M evidently $m(M) < +\infty$



Let $E = e \times \{\frac{\pi}{2}\} \subset M \quad m(E) > 0$

By Poincare recurrent $\Rightarrow \exists\ v \in E$ s.t $F^n(v) \in E$. $\underline{n>0}$

$\Rightarrow$  double it $\Rightarrow$ trajectory

$\square$

Poincaré Recurrent

<u>Thm</u> (X, μ) probability $\phi: X \to X$ measure preserving, $E \subset X$ $\mu(E) > 0$. Then $\forall$ a.e $x \in E$, $\exists\ n_i \to \infty$ st $\phi^{n_i}(x) \in E$.

<u>Proof</u> Suppose Not. $\exists\ A \subset E$ $\mu(A) > 0$ s.t $\forall\ x \in A$. $\exists\ N$ $n \geq N$

$$\phi^n(x) \notin E \quad (\Rightarrow \phi^n(x) \notin A).$$

Replace E by A, we may assume: $m(E) > 0$, $m(E) < \infty$

$$\forall\ x \in E.\ \exists\ N \ \text{s.t}\ n \geq N \quad \phi^n(x) \notin E$$

Now let $F_n = \{ x \in E \,/\, i \geq n,\ \phi^i(x) \notin E \}. \subset F_{n+1}$

Then $E = \bigcup_{n=1}^{\infty} F_n$ $\quad \phi^{i}(x) \notin E\ \forall\ i \geq n \Rightarrow \underline{\phi^j(x) \notin E\ \forall\ j \geq n+1}$

But $\phi(F_n) \subset F_{n+1}$ by definition.

Since ϕ is measure preserving, we have $+$ $m(F_n) < +\infty$

$$\phi(F_n) = F_n \quad \text{a.e.}$$

$$\Rightarrow \quad \phi(E) = \bigcup_{n=1}^{\infty} \phi(F_n) \overset{\text{a.e}}{=} \bigcup_{n=1}^{\infty} F_n = E$$

which contradicts the assumption:

<u>i.e.,</u> $\forall$ a.e $x \in E$, $\phi(x) \in E \Rightarrow m(F_n) = 0$

$$\Rightarrow m(E) = 0.$$

$\square$

<u>Q</u> where do you use $m(X) < +\infty$

HighTeck: Koro.

Let $A_n = \bigcup_{k \geq n} \phi^{-k}(E)$. $E \subset A_0$ $A_i \subset A_j$ $i \leq j$. $A_i = \phi^{j-i}(A_j)$. $\Rightarrow$

$$m(A_i) = m(A_j)$$

For $n > 0$ $\quad E - A_n \subset A_0 - A_n$

$$\mu(E - A_n) \leq \mu(A_0) - \mu(A_n) = 0$$

$$\Rightarrow \quad \mu(E - A_n) = 0 \quad \forall n \quad \Rightarrow$$

$$\mu\Big(E - \bigcap_{n=1}^{\infty} A_n\Big) = \mu\Big(\bigcup_{n \geq 1} (E - A_n)\Big) = 0.$$

But $E - \bigcap_{n=1}^{\infty} A_n$ is precisely $x \in E$, $\exists\ K$ st $n \geq K$ $\phi^n(x) \in E$

$\quad$ $\|$ measure zero.

$\square$

Lecture 4 Poincaré Recurrent theorem

Thm (Poincare) Suppose $T: (X, m) \to (X, m)$ is a measure preserving bijection on a probability space $m(X) = 1$. If $E \subset X$ is a measurable set w/ $m(E) > 0$, then $\forall$ a.e $x \in E$, $\exists\, n_i \to +\infty$

$$\text{s.t} \quad T^{n_i}(x) \in E \qquad \text{(All sets are measurable)}$$

pf. If not, $\exists\ A \subset E$ w/ $m(A) > 0$ s.t.

$$\forall\ x \in A \quad \exists\ N = N_x \quad \text{s.t} \quad T^n(x) \notin E,\ \forall\ n \geq N \Rightarrow T^n(x) \notin A$$

We may replace E by A. so

$\exists\ E$ w/ $m(E) > 0$ s.t $\forall\ x \in E$, $\exists\ N$ and for $n \geq N$ $T^n(x) \notin E$.

Define for n

$$A_n = \{ x \in E \mid i \geq n \quad T^i(x) \notin E \}$$

By assumption $\quad E = \bigcup_{n=1}^{\infty} A_n$

Now for A_n

(1) $\quad A_n \subset A_{n+1}$

Since $\quad x \in A_n \Leftrightarrow T^i(x) \notin E \quad \forall\ i \geq n$

$$\Rightarrow T^i(x) \notin E \quad \forall\ i \geq n+1 \quad \Rightarrow x \in A_{n+1}$$

(2) $\quad T(A_n) \subset A_n$

Since $\quad x \in A_n \Rightarrow T^i(x) \notin E \quad \forall\ i \geq n$

$$\Rightarrow T^i(T(x)) = T^{i+1}(x) \notin E \quad \forall\ i \geq n \Rightarrow T(x) \in A_n$$

~~Note $T(E) =$~~ $\quad T$ preserves measure $\Rightarrow T(A_n) = A_n$ a.e.

$$\Rightarrow \quad T(E) = \bigcup_{n=1}^{\infty} T(A_n) \overset{a.e}{=} \bigcup_{n=1}^{\infty} A_n = E$$

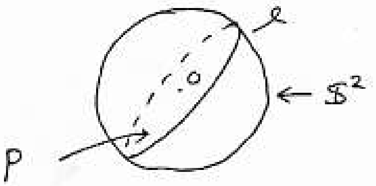
i.e $\forall$ a.e $x \in E$, $T(x) \in E \Longrightarrow$ a.e $x \in E$, $T^n(x) \in E\ \forall n$

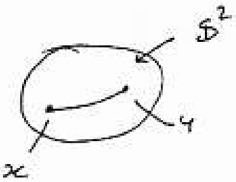
$\Rightarrow m(A_n) = 0 \Rightarrow m(E) = 0.$

Lecture 5. Basic Spherical Geometry

$\mathbb{S}^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$, $\mathrm{Iso}(\mathbb{S}^2) = O(3)$, Geodesics = lines $\ell = \mathbb{S}^2 \cap P$

where P = 2-dim linear subspace $\subset \mathbb{E}^3$

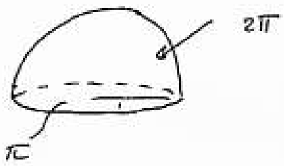




Fact $\forall\ x \neq y \in \mathbb{S}^2\ \exists$ geodesic ℓ containing x & y. ℓ is unique if $x \neq -y$

Q Why is great circles locally distance minimizing?

Q Are Crofton's thms still true?

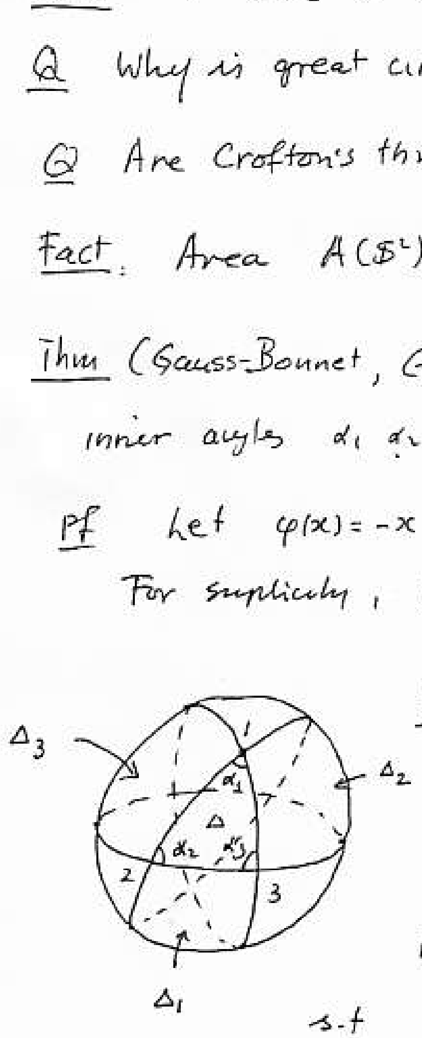


Fact: Area $A(\mathbb{S}^2)$ of $\mathbb{S}^2$ is 4π

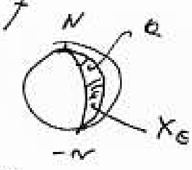
Thm (Gauss-Bonnet, Girard's) Let Δ be a spherical triangle of inner angles $\alpha_1\ \alpha_2\ \alpha_3$. Then its area $A(\Delta) = \alpha_1 + \alpha_2 + \alpha_3 - \pi$.

Pf Let $\varphi(x) = -x$ be the antipodal map $\varphi \in O(3)$ $\varphi \in \mathrm{Iso}(\mathbb{S}^2)$

For simplicity, $X \subset \mathbb{S}^2$ $-X \triangleq \varphi(X)$. so $A(-X) = A(X)$ = Area of X



Lemma 1 For $\theta \in (0, 2\pi)$, the lunes X_θ of bounded by two geodesics from N to $-N$ of angle θ has area $A(X_\theta) = 2\theta$.



Now let Δ_i be the spherical triangles as shown s.t $\Delta \cup \Delta_i$ forms X_{θ_i}

Then: $\mathbb{S}^2 = \Delta \sqcup -\Delta \sqcup \bigsqcup_{i=1}^{3} (\Delta_i \cup -\Delta_i)$. $\Rightarrow$

$$\begin{cases} A(\Delta) + A(\Delta_i) = 2\alpha_i \\ 2A(\Delta) + \sum_{i=1}^{3} A(\Delta_i) = 4\pi \quad \text{total area} \end{cases}$$

$\Rightarrow$ done

Consequence Sum of inner angles of a spherical triangle $> \pi$.

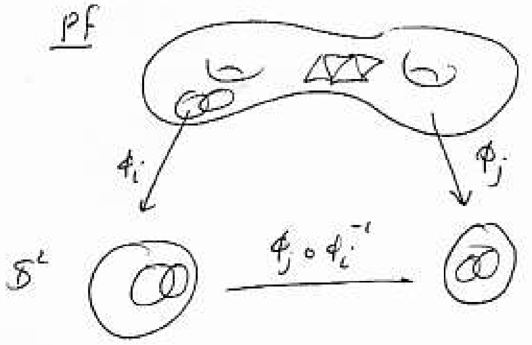
A spherical structure on a surface (Σ^2): $\mathbb{RP}^2 = \mathbb{S}^2/_{x\sim -x}$.

Corollary: If a closed surface Σ admits a spherical structure $\Rightarrow \chi(\Sigma) > 0$

ie $\Sigma = \mathbb{S}^2$ or $\mathbb{RP}^2$ (due to classification of closed surfaces)

Geometric structures on manifolds or surfaces

Pf



Produce a triangulation of Σ by geodesic triangles.

Now let us measure angles $\mathcal{T}^{(2)}$ all triangles

Girard's

$$\text{Area of } \Sigma = \sum_{\Delta_i \in \mathcal{T}^{(2)}} A(\Delta_i) = \sum_{\Delta_i \in \mathcal{T}^{(2)}} (\alpha_i + \beta_i + \gamma_i - \pi)$$

$$= \sum_{\alpha \text{ all angles}} \alpha - \pi \cdot \#(\mathcal{T}^{(2)})$$

F $\shortparallel$

$$= \sum_{v \text{ vertices}} \Big(\sum_{\text{angles } \alpha \text{ at } v} \alpha \Big) - \pi \#(\mathcal{T}^{(2)})$$

$V =$
$E =$
$F =$

$3E = 2F$

$$= 2\pi V - \pi F$$

$$= 2\pi\left(V - \frac{F}{2}\right) = 2\pi (V - E + F)$$

$$= 2\pi \chi(\Sigma) \quad \leftarrow \text{Gauss-Bonnet}$$

So $\chi(\Sigma) > 0$

My conjecture

Def A polynomial structure on Σ^2 = ~~trg~~ $\phi_i \circ \phi_j^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ polynomials

Conj If a closed surface Σ has a polynomial structure $\Rightarrow \chi(\Sigma) = 0$.

Kleinbottle.

Def (Chern Conjecture) An affine structure on M^4 is $\phi_i \circ \phi_j^{-1}$ $\mathbb{R}^4 \to \mathbb{R}^4$ Affine. $\chi(M) = 0$ for affine mfds.

True for $n=2$ (~~Benzecri~~ Benzecri thm 1959)

Corollary A flat surface Σ: charts w/ $\phi_i \circ \phi_j^{-1} \in \mathrm{Iso}(\mathbb{E}^2) \Rightarrow \chi(\Sigma) = 0$

$\alpha_i + \beta_i + \gamma_i = \pi$

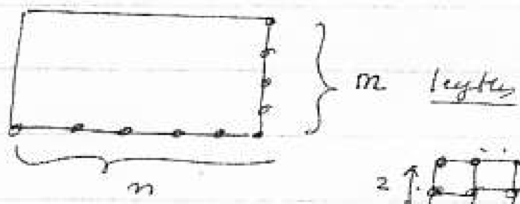
Lecture 5.

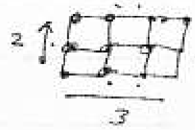
Pick's Area Theorem.

Thm. Let $P \subset \mathbb{R}^2$ be a polygon with vertices on lattice points on $L = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. Then
$$\text{Area}(P) = I + \frac{B}{2} - 1$$
where $B = \#(\partial P \cap \mathbb{Z}^2)$, $I = \#(\text{int}(P) \cap \mathbb{Z}^2)$.

Proof.

Eg.1. P rectangle parallel to x, y axis.





Then $\text{Area}(P) = mn$

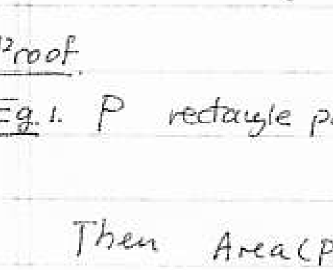
$B = 2(n-1) + 2(m-1) + 4 = 2n + 2m$

$I = (n-1)(m-1)$

Thus $I + \frac{B}{2} - 1 = nm - n - m + 1 - 1 + n + m = mn$

Eg2. P right-angled triangle w/ sides $\parallel$ x, y-axis, No lattice pts in hypotenuse $\Rightarrow$ True.

Lemma 1 Suppose P, Q are two lattice polygons s.t. $P \cap Q$ is an edge of P and Q. If Pick's theorem holds for P, Q, then it holds for $P \cup Q$. (Also Pick's thm holds for $P \cup Q$ & $P \Rightarrow$ holds for Q)



Proof Obviously, $\text{Area}(P \sqcup Q) = \text{Area}(P) + \text{Area}(Q)$

Thus, it suffices to show that the combinatorics area adds. Let I_P, B_P and I_Q, B_Q be the corresponding number and I, B be the numbers $\#((P \overset{\circ}{\cup} Q) \cap \mathbb{Z}^2)$ and $\#(\partial(P \cup Q) \cap \mathbb{Z}^2)$. Let E be $P \cap Q$. Then for $\alpha = \#(\overset{\circ}{E} \cap \mathbb{Z}^2)$

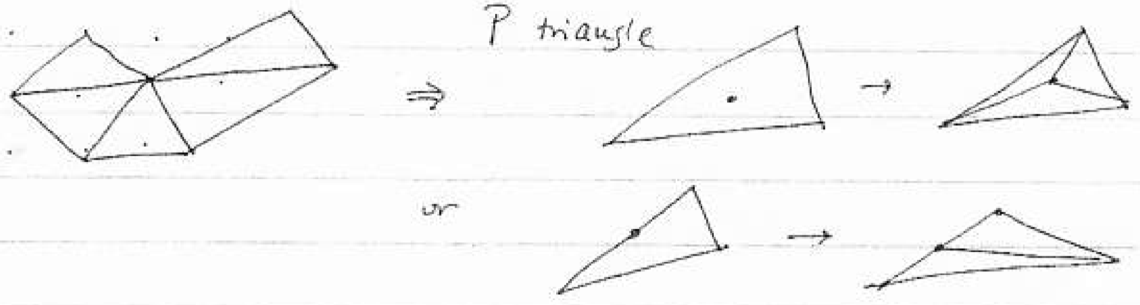
$$I = I_P + I_Q + \alpha$$
$$B = B_P + B_Q - (\alpha + 2 + \alpha)$$

Thus
$$I + \frac{B}{2} - 1 = I_P + I_Q + \alpha + \frac{B_P}{2} + \frac{B_Q}{2} - \alpha - 1 - 1$$
$$= \left(I_P + \frac{B_P}{2} - 1\right) + \left(I_Q + \frac{B_Q}{2} - 1\right)$$

Using the lemma, we conclude

Lemma 2. It suffices to prove it for lattice triangle P s.t. $\#(P \cap \mathbb{Z}^2) = 3$

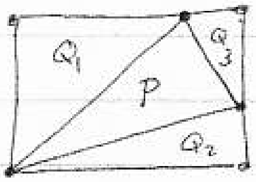
Pick's Area Theorem



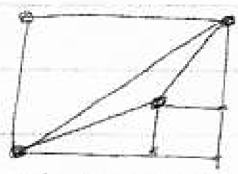
□

Lemma 3 It suffices to prove it for lattice rectangle P. + right angled triangle

Pf.



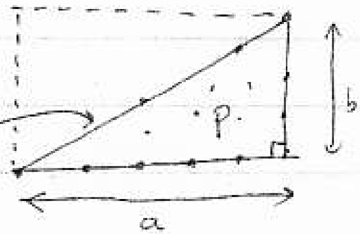
Area(P)
$= \text{Area}(R) - \sum \text{Area}(Q_i)$



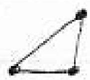
with no lattice pts in hypotenuse

Finally, it holds for right-angled triangle by lemma

No lattice



$\text{Area}(P) = \frac{1}{2}ab$, $I_p = \frac{1}{2}(a-1)(b-1)$



$B_p = a+b+1. = (a+1)+(b+1)-1.$

So $I_p + \frac{1}{2}B_p - 1 = \frac{1}{2}(ab - a - b + 1 + a + b - 1) - 1 = \frac{ab}{2}.$

R

π- rotation about a point $\underset{\shortparallel}{\overset{v_0}{}}(a,b)$ in $\mathbb{R}^2$ is

$(x, y) \longmapsto 2v_0 - (x, y)$ $\boxed{X \mapsto 2V - X}$

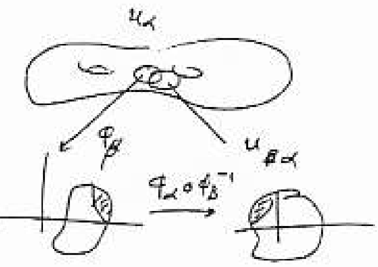
$\mathbb{Z}^2 \to \mathbb{Z}^2$ 1-1 onto

Lecture 5. Gauss-Bonnet

Recall a smooth mfd M^n : $\exists$ charts (U_α, ϕ_α) $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset$ open

s.t ① $\bigcup_\alpha U_\alpha = M^n$

② $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth



Def A smooth surface Σ^2 is said to have spherical structure or $(S^2, O(3))$ geometry if $\exists$ charts $\{(U_\alpha, \phi_\alpha) : \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset S^2$ open

st ① $\bigcup_\alpha U_\alpha = \Sigma^2$

② $\phi_\alpha \circ \phi_\beta^{-1}$ is a restriction of $g_{\alpha\beta} \in O(3)$ ($n+1$)

1. (Affine flat structure if $\phi_\alpha \circ \phi_\beta^{-1} = g_{\alpha\beta}|$ $\quad g_{\alpha\beta} \in \text{Affine}(\mathbb{R}^2)$
2. Euclidean flat $\quad g_{\alpha\beta}| \quad g_{\alpha\beta} \in \text{Iso}(\mathbb{E}^2)$ $g_{\alpha\beta}(x) = Ax + b$, $A \in GL(n,\mathbb{R})$
3. polynomial structure if $\phi_\alpha \circ \phi_\beta^{-1} = g_{\alpha\beta}$: $\quad g_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n$ polynomial

$f(x) = (f_1(x), \dots, f_n(x))$ each $f_i(x)$ polynomial)

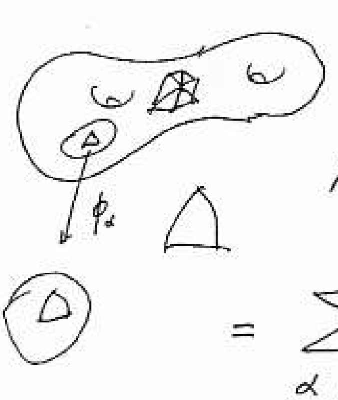
Corollary. Suppose Σ^2 closed surface w/ $(S^2, O(3))$ structure $\Rightarrow \Sigma^2 \cong S^2$ or $\mathbb{RP}^2 = S^2/_{x \sim -x}$

In fact $\text{area}(\Sigma^2) = 2\pi\, \chi(\Sigma)$

Pf. Σ^2 has $(S^2, O(3)) \Leftrightarrow$ geodesic, angle's area make sense!

Point: Every geometric property of S^2 invariant under $O(3)$ is defined on Σ^2. notions

Produce a triangulation $\mathcal{T}$ of Σ^2 by geodesics. Let V, E, F be the set of all vertices, edges + triangles in $\mathcal{T}$



Then

$$\text{Area}(\Sigma^2) = \sum_{\substack{\sigma \in T \\ \Delta}} \text{Area}(\Delta) \overset{\text{Girard}}{=\!=} \sum_{\substack{\sigma \in T \\ \text{inner angs}}} (\alpha_\sigma + \beta_\sigma + \gamma_\sigma - \pi)$$

$$= \sum_{\substack{\alpha \text{ inner} \\ \text{angs of } \mathcal{T}}} \alpha - \pi|F| = \sum_{v \in T} \Big(\sum_{\substack{\text{inner angs} \\ \alpha \text{ at } v}} \alpha \Big) - \pi|F|$$

$$= \sum_{v \in V} (2\pi) - \pi|F| = 2\pi|V| - \pi|F| = 2\pi\Big(|V| - \frac{|F|}{2}\Big)$$

$$= 2\pi\big(|V| - |E| + |F|\big) = 2\pi\chi(\Sigma). \qquad |E| = \frac{2}{3}|F|$$

Möbius Thm

Lecture 5 Gauss-Bonnet

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Similarly
<u>Corollary</u> Σ closed surface w/ $(\mathbb{E}^2, \cancel{O(3}\ \mathrm{Iso}(\mathbb{E}^2))$ structure $\Rightarrow \chi(\Sigma)=0$.

$\Rightarrow \Sigma = T^2$ or Kleinbottle.

<u>My conjecture</u> If closed surface Σ^2 has a polynomial structure $\Rightarrow \chi(\Sigma)=0$

Benzecri
<u>~~Benzicrez~~ thm</u> (1958) A closed affine surface $\Sigma \Rightarrow \chi(\Sigma)=0$

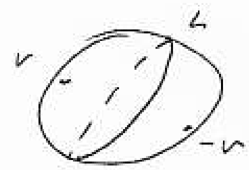
<u>Chern-Conjecture</u>: M^{2n} closed affine manifold $\Rightarrow \chi(M^4)=0$.

<u>open even for M^4!</u>

<u>Known</u> If M^n closed $|\pi_1(M)| < +\infty \Rightarrow M^n$ has no polynomial structure.

<u>RM</u> The Liouville measure on the space $\mathcal{L}$ of all geodesics in S^2

$$\mathcal{L} \ni L \leftrightarrow \{(v,-v) \mid v \in S^2\} \quad v \cdot L = 0$$

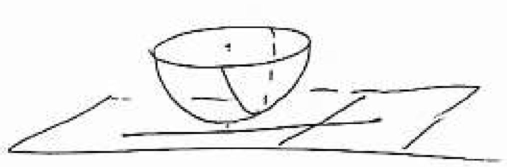


$$\Rightarrow \mathcal{L} \xrightarrow[\cong]{\pi} S^2/_{x\sim -x} \qquad \pi(gL) = g\cdot\pi(L)$$

$\mathrm{Iso} \circlearrowleft \mathcal{L}$, $\mathrm{Iso} \circlearrowleft S^2/_{x\sim -x}$

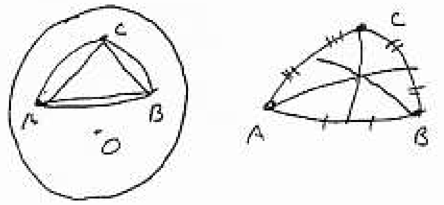
Thus, the space of all $\mathcal{L}$ = $\mathbb{RP}^2$ with the quotient Lebesgue measure,

<u>Jacobi Conjecture</u> $F: \mathbb{C}^n \to \mathbb{C}^n$ polynomial s.t. $\det DF(z) \neq 0 \ \forall z$
Then F is 1-1 onto w/ F^{-1} polynomial.



$$\mathcal{L}(\mathbb{R}^2) \xleftrightarrow[1-1]{} \mathcal{L}(S^2) - \{p\}$$

Eg. The three medians of a spherical triangle intersect at one point.

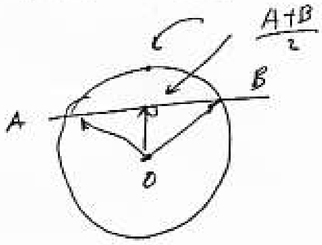


pf Given spherical ΔABC Let $\Delta_E ABC$ be the Euclidean triangle having the same set of vertices.

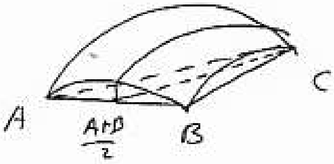
Let $\phi: \mathbb{R}^3 - 0 \to S^2$ $x \mapsto \frac{x}{\|x\|}$ the radial projection

Lemma If $A, B \in S^2$ + $\underline{A \neq -B}$ $L(A,B)$ the Euclidean line from A to B $\Rightarrow$ $\phi(L(A,B))$ spherical geodesic from A to B

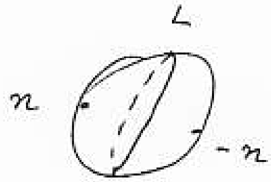
Furthermore $\phi(\frac{A+B}{2})$ = mid pt of A, B in S^2



If L is a line in $\mathbb{R}^3$ $0 \notin L$ $\phi(L)$ a geodesic in S^2.



—— $\Rightarrow$ ϕ(median of $\Delta_E ABC$) = median of ΔABC



Duality principle

The set of all lines $\mathcal{L}$ in S^2 = $S^2/_{x \sim -x}$

Since each line $L \subset S^2$ corresponds to its dual $\{n, -n\}$ $n \cdot L = 0$

Iso(S^2) acts on both + π is equivariant

$$\begin{array}{ccccc} \pi: \mathcal{L} & \longrightarrow & S^2/_{x\sim -x} & = & \mathbb{R}P^2 \\ \downarrow g & & & & \downarrow g \\ \mathcal{L} & \longrightarrow & S^2/_{x\sim -x} & = & \mathbb{R}P^2 \end{array}$$

Thus an invariant measure on $\mathcal{L}$ $\Leftrightarrow$ invariant measure on $\mathbb{R}P^2$ which is the quotient of the Lebesgue measure

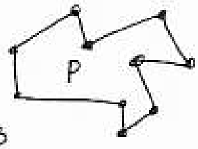
(BTW: Iso(S^2) acts on S^2 since $g(-x) = -g(x)$!) Total area of $\mathbb{R}P^2$ is 2π.

Q Is Crofton's thm still true ?

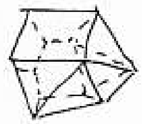
Lecture 6

Introduction to the Hilbert Third Problem.

$P \subset \mathbb{R}^n$ a compact polytope



$\mathcal{P}(\mathbb{R}^n)$ set of all compact polytopes



Def A decomposition of P is $\{P_1, \cdots, P_k\}$ s.t.,

(1) $P_i \in \mathcal{P}(\mathbb{R}^n)$

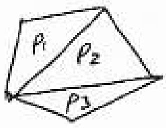
(2) $\text{int}(P_i) \cap \text{int}(P_j) = \phi$

(3) $P = \bigcup_{i=1}^{n} P_i$

equidissectible

$P \approx Q$ if they have the same decomps

Eg



clearly $\text{vol}(P) = \sum_i \text{vol}(P_i) = \text{vol}(Q)$

RM $\{P_i\}, \{R_j\}$ decomp of $P \Rightarrow \{P_i \cap R_j\}$ a decomposition. $\Rightarrow \sim$ equivalent relation.

Classical Thm 1. (Bolyai) 1832 If P, Q are two planar polygons of the same area, then they are equivalent.

Hilbert 3rd Problem Is it true that $\text{vol}(P) = \text{vol}(Q)$ for $P, Q \in \mathcal{P}(\mathbb{R}^n)$ $n \geq 3$ implies $P \approx Q$?

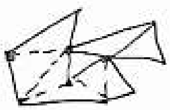
Gauss: it seems to be false for $n \geq 3$.

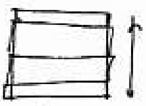
Thm 2 (Dehn). The regular tetra

$\not\approx$ Any box $[a,b] \times [c,d] \times [e,f]$.

Pf of thm 1 It suffices to show $P \approx [0,1] \times [0, \text{Area}(P)]$

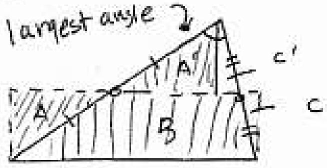
step 1. $\forall P \in \mathcal{P}(\mathbb{R}^2)$ can be decomposed into triangles clear

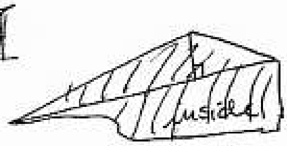




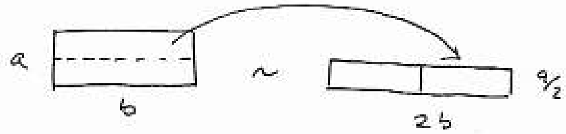
step 2. Each triangle $\Delta \approx [0,1] \times [0, \text{Area}(\Delta)]$

step 2.1. Each $\Delta \approx [a,b] \times [c,d] = A \cup B \cup C$

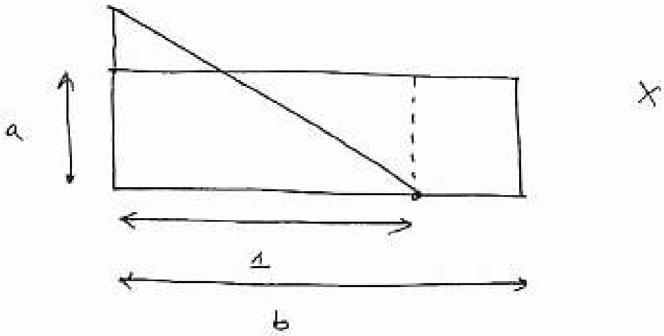


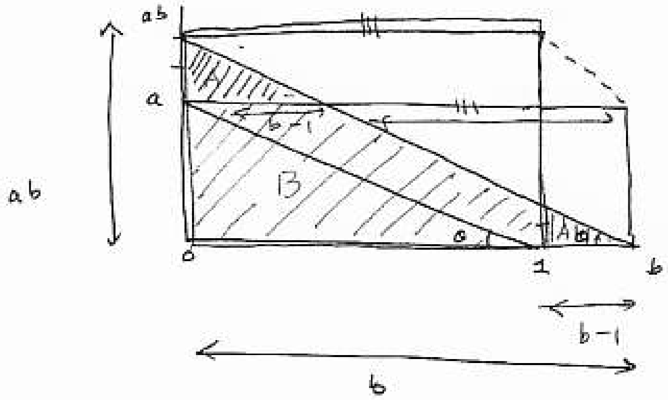


Step 2.2 $\qquad [0,a]\times[0,b] \approx [0,1]\times[0,ab]$



We may assume $b>1$



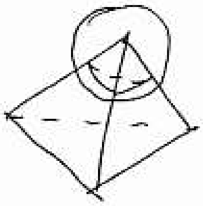
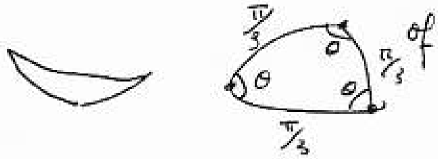


done

□

Pf of thm 2. (Dehn)

Step a A simple calculation show the dihedral angle θ of the regular tetra satisfies $\cos(\theta) = \frac{1}{3}$

 $\rightsquigarrow$ spherical triangle 

of edge lengths $\frac{\pi}{3}$

Now the cosine law for 

S^2 triangles

says $\quad \cos\theta_i = \dfrac{\cos l_i - \cos l_j \cos l_k}{\sin l_j \sin l_k} = \dfrac{\cos\frac{\pi}{6} - \left(\cos\frac{\pi}{6}\right)^2}{\left(\sin\frac{\pi}{6}\right)^2} =$

$$= \frac{\frac{1}{2}-\frac{1}{4}}{\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Step 3 $\forall\ n \in \mathbb{Z}_{>0}$ $\quad n\theta \neq \pi$ in fact $\cos(n\theta) = \frac{a_n}{3^n}$ $\quad 3 \nmid a_n \quad a_n \in \mathbb{Z}_{\neq 0}$

Let $\quad \alpha_n = n\theta \qquad \lambda_n = \cos(\alpha_n)$

Now use $\quad \cos(\alpha+\theta) + \cos(\alpha-\theta) = 2\cos(\alpha)\cos\theta$

$\alpha = n\theta$

$\Rightarrow$ $\boxed{\cancel{\cos((n+1)\theta)}}$

$$\cos((n+1)\theta) = 2\cos(n\theta)\cos(\theta) - \cos((n-1)\theta)$$

or $$\lambda_{n+1} = \frac{2}{3}\lambda_n - \lambda_{n-1}$$

Induction $\lambda_i = \frac{a_i}{3^i} \quad a_i \in \mathbb{Z} - \{0\}$

$$\Rightarrow \lambda_{n+1} = \frac{2}{3}\frac{a_n}{3^n} - \frac{a_{n-1}}{3^{n-1}} = \frac{(2a_n - 9a_{n-1})}{3^{n+1}}$$

$\boxed{\text{So}}$ $\quad a_{n+1} = 2a_n - 9a_{n-1} \in \mathbb{Z} \quad 3 \nmid a_{n+1}$.

Step 4 $\quad \theta$ and π are linearly independent over $\mathbb{Q}$ ($\mathbb{R}$ as $\mathbb{Q}$ vector space)

Take a basis of $\mathbb{R}/\mathbb{Q}$ s.t. θ and π are in the basis

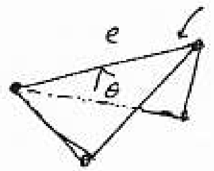
Define a $\mathbb{Q}$-linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$ st $f(\theta) = 1$, $f(\pi) = 0$

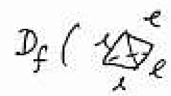
$$f(r_1 x_1 + \dots + r_n x_n) = \sum_i r_i f(x_i) \qquad r_i \in \mathbb{Q} \qquad \Downarrow \ f(\tfrac{\pi}{2}) = 0$$

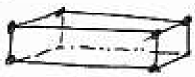
For each $P \in \mathcal{P}(\mathbb{R}^3)$ define Dehn's invariant of P

length(e) = $\ell(e)$

$$D_f(P) = \sum_{\text{edges } e \in P} \ell(e) f(\theta_e)$$



Eg $\quad D_f($  $) = 6 \cdot \ell(e) \cdot 1 = 6\ell \neq 0$

$D_f($  $) = \sum_{i=1}^{12} \ell_i f(\tfrac{\pi}{2}) = 0$

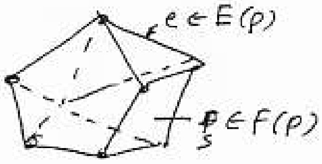
Lemma 3. If $\{P_1, \dots, P_k\}$ is a decomp of P, then

$$D_f(P) = \sum_{i=1}^{k} D_f(P_i)$$

$P \in \mathcal{P}(\mathbb{R}^3)$ $\quad E(P)$ the set of all edges in P. ∂P

$F(P)$ the set of all boundary faces of P

$V(P)$ the set of all vertices of P

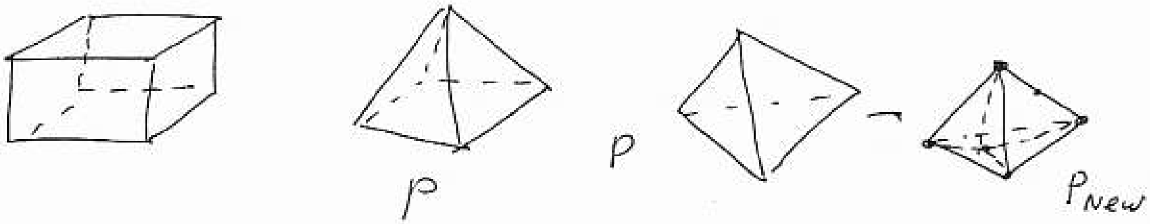


If $P_1, \cdots, P_n$ a decomp of P, then

$V = \bigcup V(P_i)$ gives a new decomposition of P

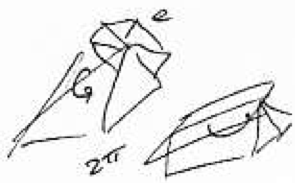
P is $\underline{P_{New}}$ (same set, difference cells) $\underline{E(P_{New}) = \bigcup E(P_i)}$

$\underline{\text{Now}}$



$$\sum_i D_f(P_i) = \sum_{e \in E(P_{New})} l(e) \sum_{\substack{\text{dihedral angles} \\ \theta \text{ in } P_i\text{'s at } e}} f(\theta)$$

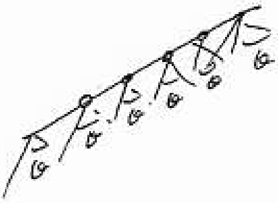
$$\underset{\text{linear}}{\overset{f\ \mathbb{Q}}{=\!=}} \sum_{e \in E(P_{New})} l(e)\, f\Big(\sum_{\theta \text{ in } P_i \text{ at } e} \theta \Big)$$



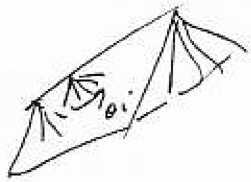
$$\underset{f(\pi)=0}{\overset{f(2\pi)=0}{=\!=}} \sum_{\substack{e \subset e' \in E(P) \\ \theta \text{ angle of } P \text{ at } e}} l(e)\, f(\theta)$$

$$\sum_{\substack{\theta \text{ in } P_i \text{ at} \\ e}} \theta = \begin{cases} 2\pi & e \in \operatorname{int}(P) \\ \pi & e \subset \operatorname{int}(S),\ S \in F(P) \\ \theta \text{ angle} & e \subset e' \end{cases}$$

$e' \in E(p)$



$$= \sum_{\substack{e' \in E(P) \\ \text{dihedral} \\ \text{angle } \theta}} l(e')\, f(\theta) = D_f(P).$$



$\boxed{\text{Open}}$ conjecture $\quad P \approx Q$ iff $\mathrm{Vol}(P) = \mathrm{Vol}(Q)$ and $D_f(P) = D_f(Q)$

for all $f : \mathbb{R} \to \mathbb{Q}$ $\quad f$ $\mathbb{Q}$ linear. w/ $f(\pi) = 0$

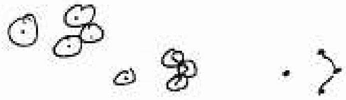
Algebraic K-theory

Lecture 7-8 Circle Packings (CP)

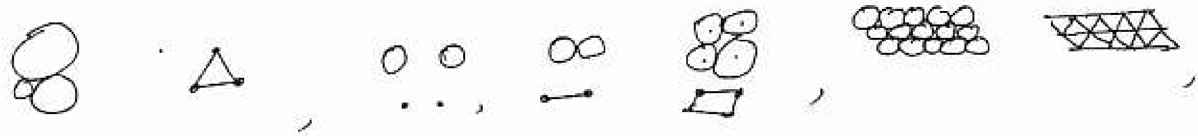
Remarks on Hilbert 3rd problem, the work of Sydler 1965 ANS is yes.

Now back to CP

Def A circle packing (CP) $\mathcal{C}$ is a collection of circles in $\mathbb{E}^2 = \mathbb{C}$ w/ disjoint interiors. The nerve (or combinatorics) $N(\mathcal{C})$ of $\mathcal{C}$ is the graph: whose vertices = centers of circles
whose edges = pairs of tangent circles



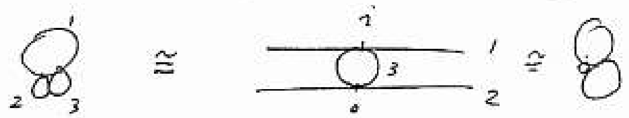
Eg 1



RM, We will focus only on connected nerve $N(\mathcal{C})$, connected graphs.

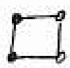
Def Two CP's $\mathcal{C}_1, \mathcal{C}_2$ are equivalent if $\exists$ a Möbius transformation $\gamma(z) = \frac{az+b}{cz+d}$ $a, b, c, d \in \mathbb{C}$, $ad-bc=1$ sending $\mathcal{C}_1$ to $\mathcal{C}_2$

Eg 2

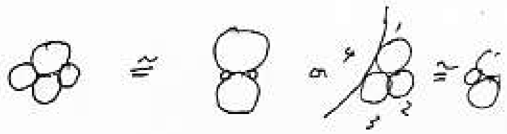


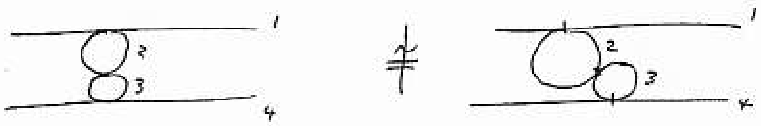
Thus, $\exists!$ CP whose nerve is

Eg 3. If $N(\mathcal{C}) \cong$

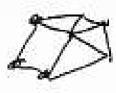


is the CP unique?





ANS. No

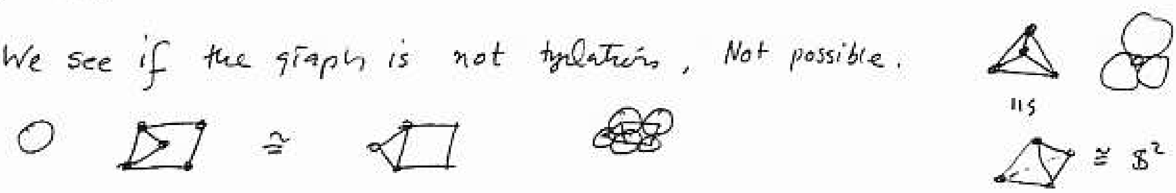


Basic Q1. (Existence) Given a connected planar graph $G \subset \mathbb{R}^2$, is there CP $\mathcal{C} \subset$ s.t $N(\mathcal{C}) \cong G$? (as planar graph)

(Uniqueness)

Q2 If $\mathcal{C}, \mathcal{C}'$ two CP s.t $N(\mathcal{C}) \cong N(\mathcal{C}')$, are $\mathcal{C}$ + $\mathcal{C}'$ equivalent?

Eg4 We see if the graph is not tylations, Not possible.



Uniqueness

Fundamental Thm relating geometry w/ Combinatorics

Thm 1.(AKT: Andreev-Koebe-Thurston) 1.

If G is the 1-skeleton of a finite triangulation of $S^2 = \mathbb{C} \cup \{\infty\}$, then $\exists !$ $CP_{\mathcal{G}}$ on $\mathbb{C}$ whose nerve $N(\mathcal{G})$ is isomorphic to G.

— A brief recall of triangulations of surfaces

Def A triangulated surface $(\Sigma, \mathcal{T})$:

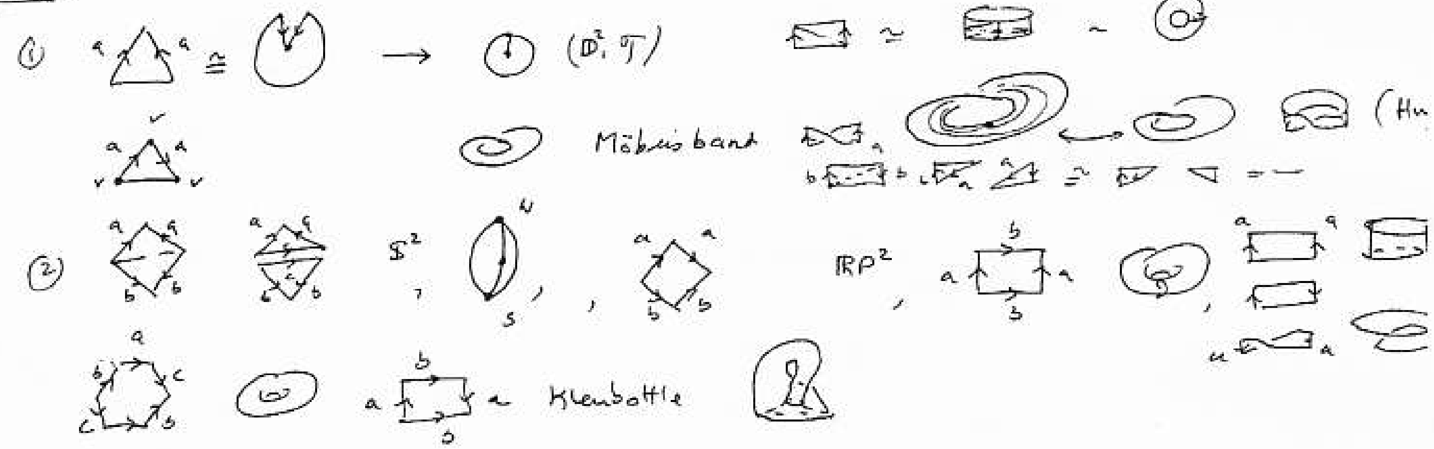
Take a finite collection of disjoint triangles $\Delta_1, \cdots, \Delta_n$. Identify pairs of all edges of $\Delta_1 \sqcup \cdots \sqcup \Delta_n$ by homeomorphisms.

$$\Sigma = \sqcup \Delta_i / \sim$$ a triangulated surface. $\mathcal{T}$ — comes from the cells in Δ_i.

V, E, T sets of all vertices, edges + triangles in $\mathcal{T}$

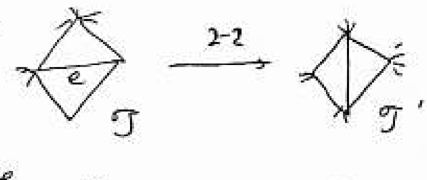
Σ is closed if each edge in $\Delta_1 \sqcup \cdots \sqcup \Delta_n$ is identified w/ some other edge

Eg.



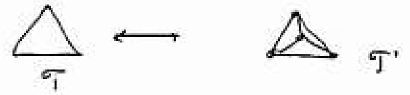
P_n - g n-sided polygon P_4 .

Def. Operations on $\mathcal{T}$. Given $(\Sigma, \mathcal{T})$, take $e \in E$ which is adjacent to 2-triangles



2-2 move on $\mathcal{T}$ or switch on $\mathcal{T}$

1-3 move



Added a vertex, 

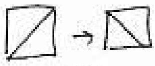
Thm 2.(Pachner)

(1) If $\mathcal{T}, \mathcal{T}'$ are two triangulations of a closed surface, then $\mathcal{T}, \mathcal{T}'$ are related by a seq. of 2-2 + 1-3 moves (Hw. prove it)

(2) if $\mathcal{T}, \mathcal{T}'$ are two triangulations of a cpt surface Σ with the same set of verties, then $\mathcal{T}, \mathcal{T}'$ are related by 2-2 moves.

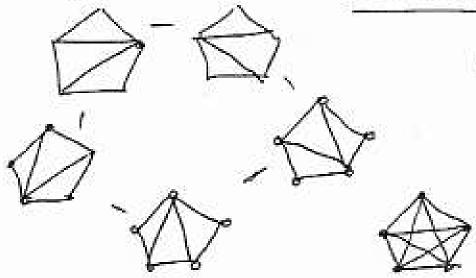
Eg <u>Triangulations of P_n</u> (Sleator-Tarjan-Thurston Conjecture) STTC.

two for P_4

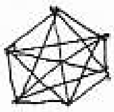


2→2

P_5



<u>2-2 moves</u> Any two trglation of P_5 (w/ vertis 5) are related by at most 2 2-2 moves



<u>Fact</u> (Hw). Any two trglations $\mathcal{T}, \mathcal{T}'$ of P_n w/ n vertis in ∂P_n are related by at most $2n-10$ moves.

<u>Conjecture</u>:

(STT): For $n \ge 12$, $\exists$ two trglations $\mathcal{T}_1, \mathcal{T}_2$ of P_n w/ n-verts in ∂P_n s.t they are Not related by any $2n-96$ <u>moves</u>.

<u>STT proved</u>: True for $n \gg 1$. using hyperbolic geometry ($\exists N$, s.t STT holds for $n \ge N$)

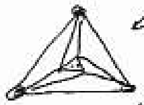
<u>Pf of thm 1</u>

<u>Uniqueness</u> (Marden-Rodin, Thurston).

Let $(S^2, \mathcal{T})$ be the triangulation where 1-skeleton is $\cong G$. & $\mathcal{C}, \tilde{\mathcal{C}}$ be the two circle packings with nerve $\cong G$.

We can always draw $\mathcal{T}$ as

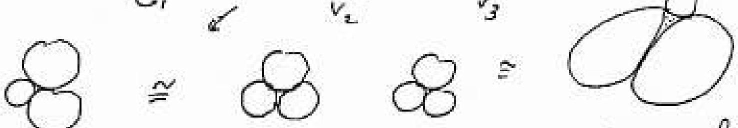
v_1 ↙ a triangle $\ni \infty$



v_2 v_3

triangulation of P_3

C_1 ↙ $\tilde{C}_2$



By using a Möbius transformation, we may assume Both of them are standard; for $\Delta 4$.

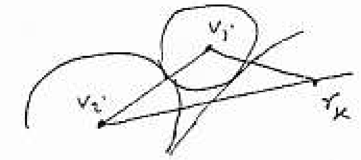
Let $V = \{v_1, v_2, v_3, v_4, \dots, v_n\}$ of $\mathcal{T}$.

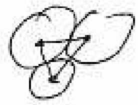
<u>Then</u> $\mathcal{C} \Rightarrow$ a radius function $r: V \longrightarrow \mathbb{R}_{\ge 0}$ $r(v_i) = 1$ $i = 1,2,3$

$\tilde{\mathcal{C}} \Rightarrow$ $\cdot\cdot$ $\cdot\cdot$ $\tilde{r}: V \longrightarrow \mathbb{R}_{\ge 0}$ $\tilde{r}(v_i) = 1$ $i = 1,2,3$.

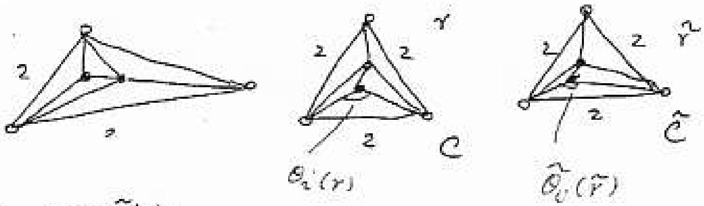
<u>We claim</u> $r = \tilde{r} \Rightarrow$ <u>$\mathcal{C} = \tilde{\mathcal{C}}$</u>

From r, we produce $l: E \longrightarrow \mathbb{R}_{>0}$, $l(v_i v_j) = r(v_i) + r(v_j)$





From l, we produce the <u>Euclidean metric</u>, i.e. the sum of inner angles at each v_i $(i \geq 4)$ is 2π.



Suppose $r \neq \tilde{r}$. say $\exists\ v$ s.t $r(v) > \tilde{r}(v)$.

Let $V_+ = \{ v \in V \mid r(v) > \tilde{r}(v) \}$, $V_- = \{ v \in V \mid r(v) \leq \tilde{r}(v) \}$ $(V_+ \neq \phi)$

Let $\theta, \tilde{\theta}$ be the corresponding angles in r + $\tilde{r}$ metric $(l, \tilde{l})$

<u>Claim</u>.

<u>Claim</u>: $2\pi|V_+| = \sum_{v \in V_+} \theta < \sum_{v \in V_+} \tilde{\theta} = 2\pi|V_+|$ (But both should be $2\pi|V_+|$) (curvature calcul[ation])

θ angle at v $\qquad$ $\tilde{\theta}$ angle at v

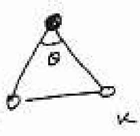
This shows $V_+ = \phi$. ~~Similarly $\{v \in V \mid r(v) < \tilde{r}(v)\}$~~ $r = \tilde{r}$.

<u>Pf of the claim</u> Let us paint $v \in V_+$ <u>black</u> + $v \in V_-$ white

The sum above breaks into portions

$$= \sum_{\text{one black vertex}} \theta + \sum_{\text{two black vertices}} \theta_1 + \theta_2 + \underbrace{\sum}_{\pi \# \{\sigma \in T \mid \text{all vertices of } \sigma \text{ are } V_+\}} (\theta_1 + \theta_2 + \theta_3)$$

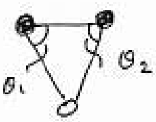
<u>Lemma 3.</u> (1) If



$r_i > \tilde{r}_i$
$r_j \leq \tilde{r}_j$
$r_k \leq \tilde{r}_k$

$\Rightarrow \quad \theta < \tilde{\theta}$

(2) If



$\Rightarrow \quad \theta_1 + \theta_2 < \tilde{\theta}_1 + \tilde{\theta}_2$

~~Lecture 9-10~~ Lecture 9-10
Circle Packings (Cont.)

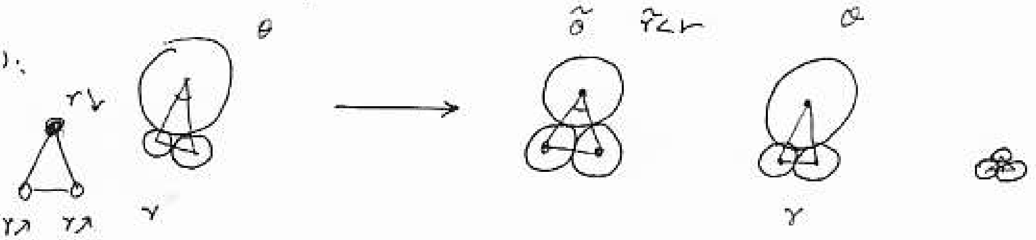
7-16-2012

~~28~~
~~27.57~~
-9.1-

Pf of lemma 3.

Let us draw a picture:

(1).

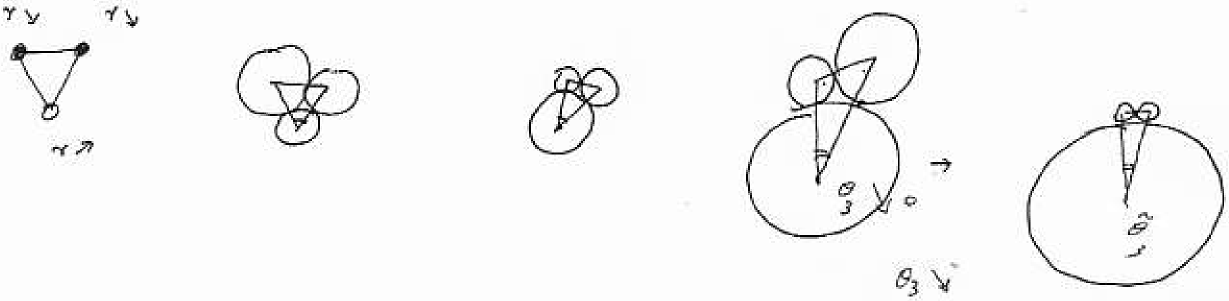


$\theta < \tilde\theta$

Greeks

$c > \tilde c \Rightarrow \theta > \tilde\theta$

(2).



$\theta_3 \searrow$

$\Leftrightarrow \theta_1 + \theta_2 \nearrow$

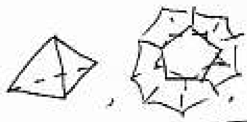
$\theta_1 + \theta_2 < \tilde\theta_1 + \tilde\theta_2$

$\square$

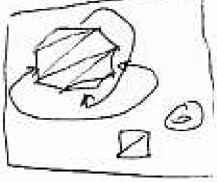
Done.

This shows $\mathcal{G} = \tilde{\mathcal{G}}$. there exists at most One circle packing realizing it.

It is good to ~~pose~~ pause + introduce polyhedral geometry.



Def Given a ~~polyhedral~~ triangulated surface (Σ, T), a Euclidean polyhedral metric on (Σ, T) is a map $l : E \longrightarrow \mathbb{R}_{>0}$ s.t if e_i, e_j, e_k form edges of a triangle $l(e_i) + l(e_j) > l(e_k)$



(Σ, T, l) — polyhedral metric. it is the metric gluing of Euclidean triangles.

The discrete curvature K of l: $V \to \mathbb{R}$

$$K(v) = 2\pi - \sum_{\substack{\text{angle } \theta \\ \text{at } v}} \theta \qquad v \in \operatorname{int}(\Sigma)$$

$$= \pi - \sum_{\substack{\text{angle } \theta \\ \text{at } v}} \theta \qquad v \in \partial\Sigma$$

Eg $\forall\ r : V \to \mathbb{R}_{>0}$ $\quad l : E \to \mathbb{R}_{>0}$ $\quad l(v_i v_j) = r(v_i) + r(v_j)$ is a polyhedral metric.

9-10

Lecture ~~8~~. Circle Packing

7-16-2012

— ~~9.2~~ — 9.2 —

Thm (Gauss-Bonnet) For any Euclidean polyhedral surface $(\Sigma, \mathcal{T}, l)$

$$\sum_{v\in V} K(v) = 2\pi\chi(\Sigma)$$

Pf. Let V_{in}, V_∂ be the sets of interior + boundary vertex set. $V = V_{in} \cup V_\partial$

$$\sum_{v\in V} K(v) = \sum_{v\in V_{in}} K(v) - \sum_{v\in V_\partial} K(v) = \sum_{v\in V_{in}}\Big(2\pi - \sum_{\theta \text{ at } v} \theta\Big) - \Big(\sum_{v\in V_\partial} \pi - \sum_{\theta \text{ at } v} \theta\Big)$$

$$= 2\pi\big(|V_{in}| + |V_\partial|/2\big) - \sum_{v\in V}\Big(\sum_{\theta \text{ at } v}\theta\Big)$$

$$= 2\pi\big(|V_{in}| + |V_\partial|/2\big) - \sum_{\theta \text{ ags in } \mathcal{T}} \theta$$

$$= 2\pi\big(|V_{in}| + |V_\partial|/2\big) - \sum_{\sigma\in\mathcal{T}}\Big(\sum_{\theta \text{ ags in } \sigma}\theta\Big)$$

$$= 2\pi\Big(|V_{in}| + \frac{|V_\partial|}{2}\Big) - \pi|F|$$

~~$= 2\pi\big(|V_{in}| + \frac{|V_\partial|}{2} - \frac{|F|}{2}\big) = 2\pi(|V| - |V_\partial| - |F|) = 2\pi\big(|V| - \frac{|V_\partial|}{2} + \frac{3|F|}{2} + |F|\big)$~~

~~Now $|E| = |V_\partial| + 2|F| = |E_\partial| + 2|F|$, $|F| = \frac{3F}{2}$, E_∂ boundary edges, $2|F|$: each triangle~~

$$= 2\pi\left(|V| - \left(\frac{|V_\partial|}{2} + \frac{3F}{2}\right) + F\right]$$

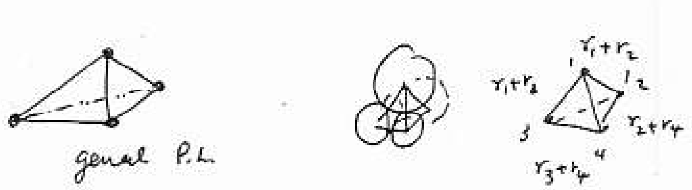
$$= 2\pi(|V| - |E| + |F|)$$

Now $3F = 2E - E_\partial \qquad |E_\partial| = |V_\partial|$. $\square$

$$= 2\pi\chi(\Sigma)$$

Basic problem in Geometry: relationships between metric + its curvature.

Eg. A CP metric on $(\Sigma, \mathcal{T})$: $r: V \to \mathbb{R}_{>0}$. Its curvature $K: V \to \mathbb{R}$



etc.

$\boxed{\text{Now the existence of } K=0}$

Thm (Thurston) $\forall (\Sigma, \mathcal{T})$ The curvature map $K: \mathbb{R}^V_{>0} \to \mathbb{R}^V$, $r \mapsto K_r$ is injective on $S = \{r \in \mathbb{R}^V_{>0} \mid \sum_{v\in V} r_i(v) = 1\}$, or K determines the metric up to scaling. Furthermore $K(\mathbb{R}^V_{>0})$ is a convex polytope in $\mathbb{R}^V$.

Pf. If $r_1, r_2 \in S$ ~~[illegible]~~ $K_{r_1}(r_1) = K_{r_2}(r_2)$, then $r_1 = r_2$

The proof is exactly the SAME as before! $V_+ = \{v \mid r_1(v) > r_2(v)\}$ etc.

Notation: X set. $\mathbb{R}^X = \{f: X \to \mathbb{R}\}$ vector space

AKT existence thm: $K \equiv 0$ is in the image $K(\mathbb{R}^V_{>0})^{-\{v_1 v_2 v_3\}}$ for $r_1 = r_2 = r_3 = 1$.

Circle Packings (Cont.)

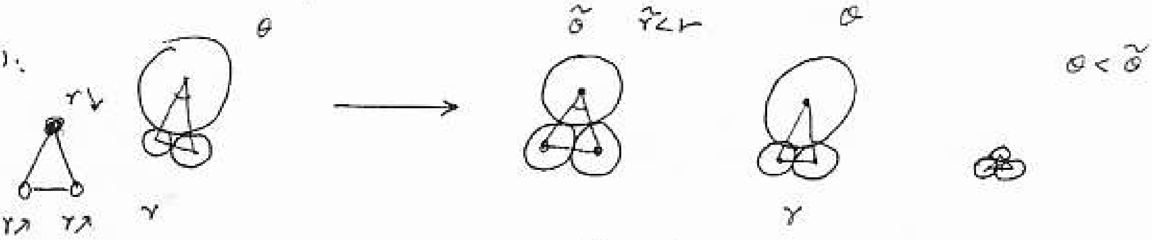
7-16-2012

~~28~~ ~~27.51~~ -9.1-

Pf of lemma 3.

Let us draw a picture:

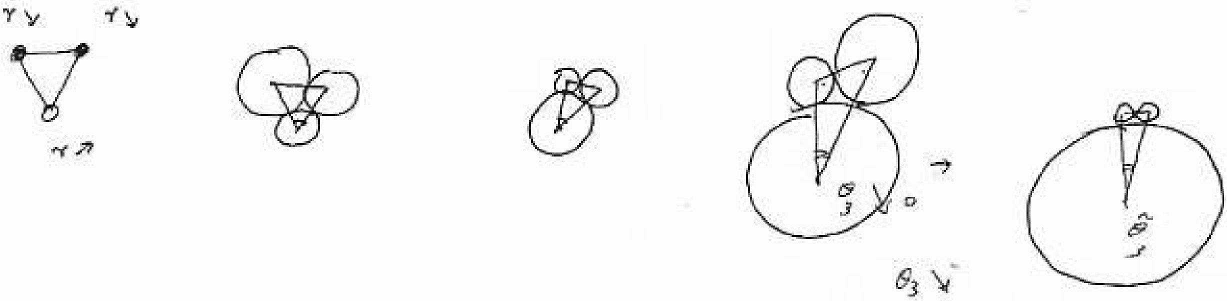
(1).



Greeks

$c \not> \tilde c \Rightarrow \theta > \tilde\theta$

(2).



$\theta_3 \searrow \Leftrightarrow \theta_1+\theta_2 \nearrow$

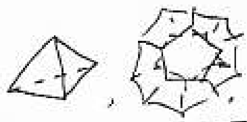
$\theta_1+\theta_2 < \tilde\theta_1+\tilde\theta_2$

$\square$

Done.

This shows $\theta = \bar\Theta$. there exists at most One circle packing realizing it.

It is good to pose pause & introduce polyhedral geometry.



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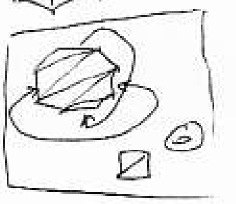
$$l(e_i) + l(e_j) > l(e_k)$$



(Σ, T, l) — polyhedral metric. it is the metric gluing of Euclidean triangles.

The discrete curvature K of l: $V \to \mathbb{R}$

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Eg $\forall\ r: V \to \mathbb{R}_{>0}$ $\quad l: E \to \mathbb{R}_{>0}$ $\quad l(v_iv_j) = r(v_i) + r(v_j)$ is a polyhedral metric.

Motivation, Prescribing curvature problem Given $\bar{K}: V \to \mathbb{R}$, Find r s.t. $K_r = \bar{K}$ 7-16-2012   L9-10

$\bar{K} \equiv 0 \Rightarrow$ CP existence   -9.3-

The set of curvatures : $P = K(\mathbb{R}^V_{>0}) = K(S)$

Consider $\Omega \subset \mathbb{R}^V$ to be the set : (Assume $\partial \Sigma = \phi$)

$$\Omega = \{ z \in \mathbb{R}^V \mid \sum_{v \in V} z(v) = 2\pi\chi(\Sigma), \ \forall\ I \subsetneq V$$

$$\sum_{v \in I} K(v) < 2\pi|I| - \pi|F_I| \}$$

$F_I = \{$ set of $\sigma \in T \mid$ vertices of σ are in $I \}$

<u>step 1</u> $K: S \to \Omega$, <u>i.e</u> $K(S) \subset \Omega$ $S = \{ x \in \mathbb{R}^V_{>0} \mid \sum x(v) = 1 \}$

Indeed, we use the same identity $I \subsetneq V$

$$\sum_{v \in I} K(v) = 2\pi|I| - \pi|F_I| - \sum \theta - \sum (\theta_1 + \theta_2) < 2\pi|I| - \pi|F_I|.$$

$\in I$, $\notin I$, $\notin I$ $\in I$, $\in I$, $\notin I$

+ Gauss-Bonnet

<u>step 2.</u> $\dim S = n-1 = \dim \Omega$ $K: S \to \Omega$ 1-1 $|V| = n$

Therefore, Brower invariance of domain ($f: \Omega \to \mathbb{R}^n$, Ω open in $\mathbb{R}^n$, f cont. 1-1 $\Rightarrow f(\Omega)$ open in $\mathbb{R}^n$)

$K(S) \subset \Omega$ is open

<u>step 3</u> Ω is connected, in fact Ω is convex

<u>step 4</u> $K(S) \subset \Omega$ is closed $\Rightarrow$ $K(S) = \Omega$ <u>Done</u>

Take $x_n \in S$ s.t $K_{x_n} \to w \in \Omega$ We <u>claim</u> that $w \in K(S)$

Indeed, we will show $\exists$ a subsequence $x_{n_i} \to a \in S \Rightarrow w = K(p)$

$\boxed{\text{If not,}}$ all coord of $x_n^{(v)} \in [0,1]$ ($x_n(v) \geq 0$, ≤ 1) $\in \mathbb{R}^V$

$\exists$ a subseq, say $x_n \to a$ $x_n(v) = a(v)$ $\forall v \in V$

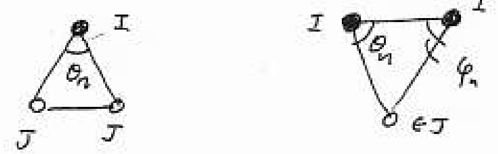
$\boxed{1 \geq a(v) \geq 0}$

<u>Now</u> $\sum_{v \in V} a(v) = 1$ clear due to $\sum_{v \in V} x_n(v) = 1$.

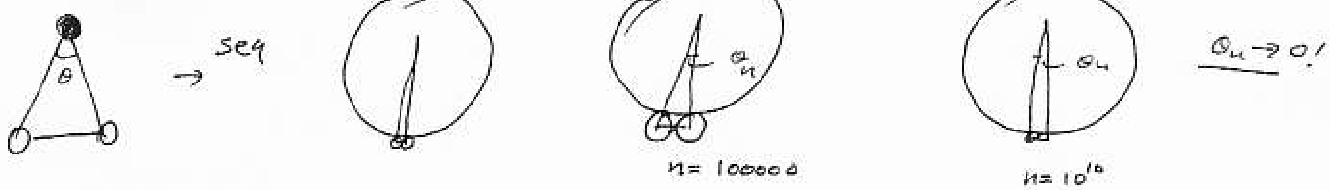
Claim <u>$a(v) > 0$!</u> $\forall v \in V \Rightarrow a \in S$

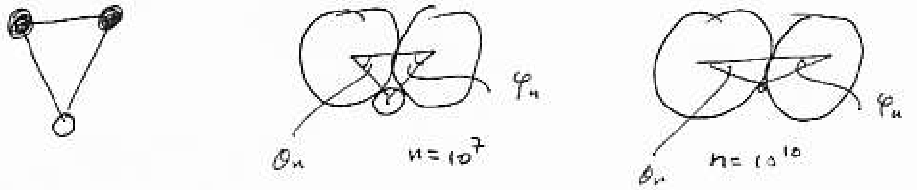
Let $I = \{ v \in V \mid a(v) > 0 \}$, $J = \{ v \in V \mid a(v) = 0 \}$ $\quad V = I \cup J$

$$\sum_{v \in I} K_{x_n}(v) = 2\pi |I| - \pi |F_I| - \sum \theta_n - \sum (\theta_n + \varphi_n)$$



Let us understand





$\theta_n + \varphi_n \to 0$

$$\xrightarrow{n \to \infty} 2\pi |I| - \pi |F_I|$$

i.e. For this specific choice of I, let $n \to \infty$, due to $K_{x_n} \to w$

$$\sum_{v \in I} w(v) = 2\pi |I| - \pi |F_I|$$

But we assumed $w \in \Omega$ i.e. $\sum_{v \in I} w(v) < 2\pi |I| - \pi |F_I|$

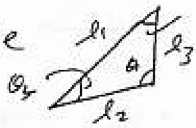
A contradiction!

Ω

Colin de Verdière's Variational Principle

Quantification of Thurston's proof

We begin with the quantification of the Greeks observation

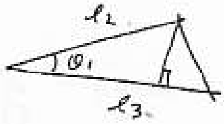
Given $\mathbb{E}^2$ triangle  of lengths l_1, l_2, l_3 and angles $\theta_1, \theta_2, \theta_3$

Now $\theta_i = \theta_i(l_1, l_2, l_3)$ by the cosine law

$$\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k} \qquad \{i,j,k\} = \{1,2,3\} \qquad \left[\theta_1(l_1, \underset{\text{fixed}}{l_2, l_3}) \nearrow l_1\right]$$

Prop. (1) $\dfrac{\partial\theta_i}{\partial l_i} = \dfrac{l_i}{2A} > 0 \qquad A = \text{area of } \Delta.$

(2) $\dfrac{\partial\theta_i}{\partial l_j} = -\dfrac{\partial\theta_i}{\partial l_i}\cos(\theta_k)$



pf.

(1) Take $\dfrac{\partial}{\partial l_i}$ to the cosine law

$$-\sin(\theta_i)\frac{\partial\theta_i}{\partial l_i} = -\frac{2 l_i}{2 l_j l_k} \Rightarrow \frac{\partial\theta_i}{\partial l_i} = \frac{l_i}{l_j l_k \sin(\theta_i)} = \frac{l_i}{2A}$$

(2). Take $\dfrac{\partial}{\partial l_j}$ to the cosine law

$$-\sin(\theta_i)\frac{\partial\theta_i}{\partial l_j} = \frac{1}{2l_k}\frac{\partial}{\partial l_j}\left[\frac{l_j^2 + l_k^2 - l_i^2}{l_j}\right]$$

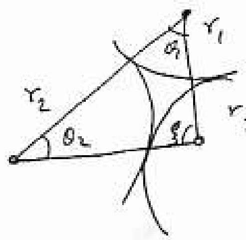
$$= \frac{1}{2l_k}\frac{1}{l_j^2}\left[2l_j \cdot l_j - (l_j^2 + l_k^2 - l_i^2)\right]$$

$$= \frac{1}{2 l_k l_j^2}\left(l_j^2 + l_i^2 - l_k^2\right) \Rightarrow$$

$$\frac{\partial\theta_i}{\partial l_j} = (-1)\left(\frac{l_j^2 + l_i^2 - l_k^2}{2 l_i l_j}\right)\left[\frac{l_i}{l_j l_k \sin(\theta_i)}\right] = (-1)\cos(\theta_k)\frac{\partial\theta_i}{\partial l_i}$$

Quantification of Thurston: $\dfrac{\partial\theta_i}{\partial r_i} < 0$, $\dfrac{\partial\theta_i}{\partial r_j} < 0$ Greeks $\dfrac{\partial\theta_i}{\partial l_i} > 0$.

Thm (de Verdiere) 1. Let Δ be a Euclidean triangle of edge lengths r_i+r_j, r_j+r_k and r_k+r_i and inner angles $\theta_1, \theta_2, \theta_3$.



Then $r_j \dfrac{\partial \theta_i}{\partial r_j} = r_i \dfrac{\partial \theta_j}{\partial r_i} > 0$ $\quad i \neq j \quad \Longleftrightarrow \quad \boxed{\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial \theta_j}{\partial u_i}}$

In particular $\dfrac{\partial \theta_i}{\partial r_i} < 0$ and $\dfrac{\partial \theta_i}{\partial r_j} < 0$. Let $u_i = \ln r_i$

Furthermore, $\exists$ a function $F(u_1, u_2, u_3)$ s.t. $\dfrac{\partial F}{\partial u_i} = \theta_i$ and

(1) F is concave in $(u_1, u_2, u_3) \in \mathbb{R}^3$

(2) F is strictly concave in the plane $P = \{u \in \mathbb{R}^3 \mid u_1+u_2+u_3 = 0\}$

(3) $F(u + (k,k,k)) = F(u)$

① $\alpha \neq \beta + k(1,1,1)$ $k \neq 0$ $\Rightarrow f(t\alpha + (1-t)\beta)$ is strictly concave in $t \in [0,1]$

Pf. $l_i = r_j + r_k \qquad \dfrac{\partial l_i}{\partial r_i} = 0, \quad \dfrac{\partial l_i}{\partial r_j} = 1 \qquad i \neq j$

So
$$r_j \frac{\partial \theta_i}{\partial r_j} = r_j \left[\frac{\partial \theta_i}{\partial l_i} \overbrace{\frac{\partial l_i}{\partial r_j}}^{=1} + \frac{\partial \theta_i}{\partial l_j} \overbrace{\frac{\partial l_j}{\partial r_j}}^{=0} + \frac{\partial \theta_i}{\partial l_k} \overbrace{\frac{\partial l_k}{\partial r_j}}^{=1} \right]$$

$$= r_j \left(\frac{\partial \theta_i}{\partial l_i} + \frac{\partial \theta_i}{\partial l_k} \right)$$

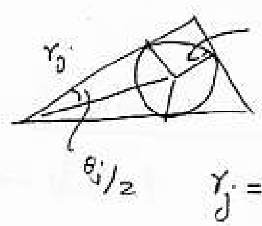
$$= r_j \left[\frac{\partial \theta_i}{\partial l_i} - \frac{\partial \theta_i}{\partial l_i} \cos(\theta_j) \right]$$

$$= r_j \frac{\sin(\theta_i)}{B} [1 - \cos(\theta_j)]$$

($B > 0$ indep of indices)

$$= \frac{r_j}{B} \sin(\theta_i) \cdot 2 \cdot \sin^2\left(\frac{\theta_j}{2}\right)$$

r = radius inscribed circle



$r_j = r \cdot \cot\left(\frac{\theta_j}{2}\right)$

$$= \frac{2r}{B} \cdot \cot\left(\frac{\theta_j}{2}\right) \sin(\theta_i) \sin^2\left(\frac{\theta_j}{2}\right)$$

$$= \frac{r}{B} \sin(\theta_i) \sin(\theta_j)$$

symmetric in i, j!

$\Rightarrow$ the differential 1-form $\displaystyle\sum_{i=1}^{3} \theta_i \, du_i$ is closed 1-form on $\mathbb{R}^3$

$\left(\dfrac{\partial \theta_i}{\partial u_j} = \dfrac{\partial \theta_j}{\partial u_i} \right) \Rightarrow$

Now. $\frac{\partial(\theta_1+\theta_2+\theta_3)}{\partial u_i} = 0 = \frac{\partial \pi}{\partial u_i}$

$\boxed{\text{So}}$ $\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_j}{\partial u_i} - \frac{\partial \theta_k}{\partial u_i} < 0$

$\Rightarrow$ the symmetric matrix

$$-\left[\frac{\partial \theta_i}{\partial u_j}\right]_{3\times 3} = \begin{bmatrix} a+b & -a & -b \\ -a & a+c & -c \\ -b & -c & b+c \end{bmatrix} = A \qquad a,b,c>0$$

This is a diagonally dominated matrix with well eigenvector (!!!)

It is well known A is semi-positive definite: $v^TAv \geq 0 \ \forall v$.
$+ \ v^TAv > 0 \quad v \in P - \{0\}$

Now $\exists \ F: \mathbb{R}^3 \to \mathbb{R}$ s.t., $F(u) = \int_0^u \sum_{i=1}^3 \theta_i du_i$ (Stokes thm)

Calculus $\mathbb{R}^3$
$w = p_1dx_1 + p_2dx_2 + p_3dx_3$
s.t $\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}$ $\boxed{dw}$
$\Leftrightarrow \exists f \ \frac{\partial f}{\partial x_i} = p_i$

$\frac{\partial F}{\partial u_i} = \theta_i \qquad \left[\frac{\partial^2 F}{\partial u_i \partial u_j}\right] = -A \leq 0$

Therefore F is concave in u. i.e (1) holds

(2) holds since $A|_P$ is positive definite $v^TAv > 0$ for $v \in P - \{0\}$.

(3) $F(u+(k,k,k)) = F(u) + k\pi$ since $\sum \theta_i d(u_i+k) = \sum \theta_i du_i$ $k =$ con

$\boxed{\frac{\partial F}{\partial u_i} = \theta_i \quad \sum\theta_i = \pi}$

□

The geometric meaning of $F \sim$ volume of some polytope in H^3.

Fact From calculus

① $f: (a,b) \to \mathbb{R}$ s.t. $f''(t) \geq 0 \Rightarrow f$ convex i.e

$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b) \quad \forall \lambda \in [0,1]$

② $f: \Omega \to \mathbb{R}$ s.t $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{N\times N} \geq 0$ semi-positive define $\Rightarrow f$ convex in Ω.
Ω open convex in $\mathbb{R}^N$

③ $f: \Omega \to \mathbb{R}$ s.t $H(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{N\times N}$ strictly positive define $\Rightarrow f$ strictly conv
Ω open convex in $\mathbb{R}^N$

$+ \ \nabla f: \Omega \to \mathbb{R}^N \quad x \mapsto \nabla f$ is 1-1

7-17-2012
10.4

de Verdier's proof of Thurston's rigidity,

For any dual $(\Sigma, \mathcal{T})$, the curvature map $\gamma: \mathbb{R}^V_{>0} \longrightarrow \mathbb{R}^V \ni u$ $\quad u(v) = \ln \gamma(v)$

$K: \{ \gamma \in \mathbb{R}^V_{>0} \mid \Sigma$

$K: \mathbb{R}^V \longrightarrow \mathbb{R}^V \qquad u \longmapsto$ Curvature of e^u at.

$K: \quad W = \{ u \in \mathbb{R}^V \mid \sum_{v \in V} u(v) = 0 \} \longrightarrow \mathbb{R}^V$ is 1-1.

Define $\Phi: \mathbb{R}^V \longrightarrow \mathbb{R}$

$$\Phi(u) = -\sum_{\sigma = \triangle v_i v_j v_k \in T} F(u_i, u_j, u_k) \qquad u_j = u(v_j) \text{ etc}$$

the "energy" of a CP metric u.

Φ is convex since it is the sum of convex functions

Key lemma. $\quad \nabla \Phi = K + 2\pi(1, 1, \dots, 1)$

$$\frac{\partial \Phi}{\partial u_i} = -\text{sum of angles at the } v_i\text{-th vertex!} = K(v_i) - 2\pi$$

This follows from the definitions.

Lemma 2 $\quad \Phi$ restricted to W is strictly convex.

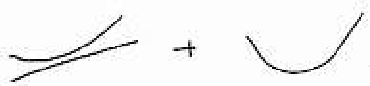
Indeed, if $\alpha \neq \beta$ in W,

$g(t) = \Phi(t\alpha + (1-t)\beta) : (0,1) \longrightarrow \mathbb{R}$ is strictly convex

$\alpha \neq \beta \Rightarrow \exists$ tri $\Delta = \Delta v_i v_j v_k$ s.t. $\underbrace{(\alpha(v_i), \alpha(v_j), \alpha(v_k))}_{\alpha_\Delta} \neq \underbrace{(\beta(v_i), \beta(v_j), \beta(v_k))}_{\beta_\Delta}$

$+(\lambda, \lambda, \lambda)$. $\Rightarrow F(t\alpha_\Delta + (1-t)\beta_\Delta)$ is strictly convex in t

so $\quad g(t)$ is the sum of convex functions and a strictly convex function $\Rightarrow$ done



$\Rightarrow \quad \nabla \Phi : W \longrightarrow \mathbb{R}^N$ is 1-1 done