Kuiper's Proof

We will show $\forall \varepsilon > 0, \quad vol(M) \geq (v_n - \varepsilon)\, \|M\|$

Lemma 1. It suffices to prove $\forall \varepsilon > 0, \ \exists \ c = \sum a_i \sigma_i \in \lambda \alpha_M$,

s.t $\quad a_i \geq 0$ and $vol(\sigma_i) \geq v_n - \varepsilon$.

pf. If so, due to $\partial c = 0 \Rightarrow \int_c dvol > 0 \Rightarrow \lambda > 0$

$$\Rightarrow c' = \frac{1}{\lambda} c = \sum \left(\frac{a_i}{\lambda}\right) \sigma_i \in \alpha_M$$

Now $\quad vol(M) = \int_{c'} dvol = \sum \frac{a_i}{\lambda} \int_{\sigma_i} dvol = \sum \left(\frac{a_i}{\lambda}\right) vol(\sigma_i)$

$$\geq (v_n - \varepsilon) \sum \left(\frac{a_i}{\lambda}\right) \geq (v_n - \varepsilon)\, \|M\|. \qquad \square$$

Let us work out $n = 3$ case, General case the same.

Lemma 2 $\forall \varepsilon > 0$, $\exists R_0 > 10$ s.t if $R > R_0$ and σ is a straight tetra where edge lengths are $\in [R-2, R+2]$, then $vol(\sigma) > v_3 - \varepsilon$.
This is due to the continuity of the volume on lengths.

Now produce a cell decomposition $X_1 \sqcup \cdots \sqcup X_k$ of M^3 s.t

(1) Each X_i is simply connected $\quad diam(X_i) \leq \min\{1, \frac{1}{10} diam(M)\}$

(2) $X_i \cap X_j = \phi$ if $i \neq j$

(3) Find $q_i \in \mathring{X}_i \quad \forall i$

Let $\pi: \mathbb{H}^3 \to M$ universal cover, $\pi^{-1}\{X_1, \cdots, X_k\} = \{\Omega_j\text{'s}\}$ $\quad \pi^{-1}\{q_i\} = P$
$\pi^{-1}\{X_i\}$ is a decomposition of $\mathbb{H}^3$ invariant under the action of $\pi_1(M)$.

Fix $R > R_0$ + a regular straight tetra $\tau_R = [u_1, \cdots, u_4]$ of edge lengths R $\quad \tau_R \subset \mathbb{H}^3$

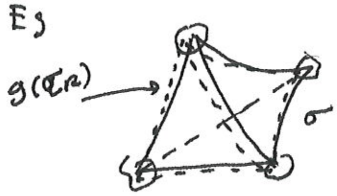
Now fix ε, R

<u>Def</u> A good tetra $\sigma: \Delta^3 \longrightarrow \mathbb{H}^3$ is a straight simplex (oriented)

s.t. (1) its vertices $p_1, \cdots, p_4 \in P$ say $p_i \in \Omega_i$ $i = 1,2,3,4$

(2) $\exists$ $g \in \mathrm{Iso}(\mathbb{H}^3)$ s.t. $g(u_i) \in \Omega_i$

Eg



<u>RM</u>. There are only finitely many good tetra up to $\pi_1(M)$ action

since there are only finitely many geodesic path in M of length $\leq R+2$ joining q_i to q_j's.

If σ is good, + m is the Haar measure on $\mathrm{Iso}(\mathbb{H}^3)$, Let

$$a(\sigma) = m\{\, g \in \mathrm{Iso}(\mathbb{H}^3) \mid g(u_i) \in \Omega_i \quad i = 1,2,3,4 \}$$

<u>Key lemma</u> The infinite chain $\tilde{\beta} = \sum_{\sigma \text{ good}} a(\sigma)\sigma$ is closed, i.e. $\partial\tilde{\beta} = 0$.

It is also invariant under $\pi_1(M)$ action.

<u>Pf</u> Define a good triangle $\delta: \Delta^2 \longrightarrow \mathbb{H}^3$ to be a straight triangle

s.t ~~its~~ (1) its vertices are $p_1, p_2, p_3 \in P$ $p_i \in \Omega_i$ $i = 1,2,3$

(2) $\exists$ $g \in \mathrm{Iso}(\mathbb{H}^3)$ s.t $g(u_i) \in \Omega_i$ $i = 1,2,3$

By definition $\partial\tilde{\beta} = \sum_{\tau \text{ good triangle}} \mathrm{Coff}(\tau)\,\tau$

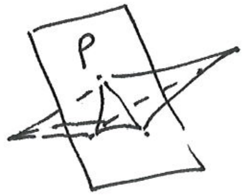
where

$\mathrm{Coff}(\tau) = m(A_R) - m(A_L)$ Let $G = \mathrm{Iso}(\mathbb{H}^3)$, P plane thru $p_1 p_2 p_3$

where $A_R = \{\, g \in G \mid g(u_i) \in \Omega_i,\ i = 1,2,3 \quad g(u_4) \in P_+ \}$

$A_L = \{\, g \in G \mid g(u_i) \in \Omega_i,\ i = 1,2,3, \quad g(u_4) \in P_- \}$

where P_+, P_- are the right + left half spaces bounded by P



We claim that $\exists\ \Phi \in Iso(\mathbb{H}^3)$, the hyperbolic reflection about the plane through u_1, u_2, u_3 $\Phi(u_i) = u_i$ + reflects it.

Define $F: A_R \longrightarrow A_L$ by $g \longmapsto g \circ \Phi$

It is a bijection $\Rightarrow$ $A_L = A_R \circ \Phi$ $\Rightarrow$ $m(A_L) = m(A_R \cdot \Phi) = M(A_R)$

by the right invariant property of the Haar measure.

$$\Rightarrow \partial \tilde{\beta} = 0.$$

To see $\forall\ \gamma \in \pi_1(M)$ $\gamma_\# \hat{\beta} = \hat{\beta}$. for a good tetra $\sigma = [p_1, \ldots, p_4]$

let $A_\sigma = \{ g \in G \mid g(u_i) \in \Omega_i \}$. Note that $a(\sigma) = m(A_\sigma)$

It is easy to see. $\forall\ \gamma \in \pi_1(M)$

$$\gamma \cdot A_\sigma = A(\gamma\sigma)$$

First σ good $\Rightarrow$ $\gamma\sigma$ is good since $\{\Omega_i\}$ is $\pi_1(M)$ invariant

<u>Next</u> $\gamma \cdot A_\sigma = \{ \gamma g \in G \mid \gamma g(u_i) \in \gamma \Omega_i \} \underset{h = \gamma g}{=} \{ h \in G \mid h(u_i) \in \gamma \Omega_i \}$

$= A_{\gamma\sigma}$ $\Longrightarrow$ $a(\gamma\sigma) = m(A_{\gamma\sigma}) = m(\gamma A_\sigma) \underset{\text{left inv}}{=} m(A_\sigma) = a(\sigma)$

Therefore

$$\gamma_\# \tilde{\beta} = \sum_{\sigma \text{ good}} a(\sigma)\, \gamma \cdot \sigma = \sum_{\sigma \text{ good}} a(\gamma\sigma)\, \gamma\sigma$$

$$= \sum_{\tau \text{ good}} a(\tau)\, \tau = \tilde{\beta}. \qquad \square$$

As a consequence, we have produce a finite chain

$$c = \sum_{[\sigma] \text{ good}} a(\sigma)\, [\sigma] \in \lambda\, \alpha_M$$

s.t $\operatorname{vol}[\sigma] > v_3 - \varepsilon$ + $a(\sigma) \geq 0$ + $\exists\ a(\sigma') > 0$

Where $[\sigma]$ is the projection of σ from $\mathbb{H}^3$ to M^3.

By the lemma 1, we are done

To prove Mostow's theorem, we need to organize Kuiper's argument further.

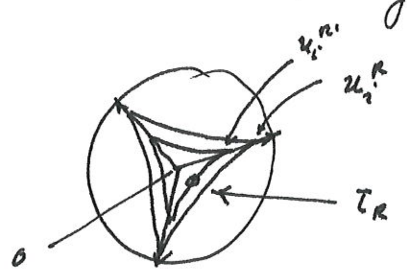
First, we organize the regular length R tetra $\tau_R = [u_1^R, u_2^R, u_3^R, u_4^R]$

s.t $0 \in \tau_1$ and is the symmetric center s.t

u_i^R is in the geodesic ray from 0 to u_i^1 $i=1,2,3,4$

($\Rightarrow$ 0 is also the center of τ_R).

$n=2$:



In particular, if $R' > R > 1$

$\tau_R \subset \tau_{R'}$

Next, the fundamental cycle produced by Kuiper is

$$C_R = \frac{\mathrm{vol}(M)}{\displaystyle\sum_{\substack{[\sigma] \\ \text{good } R\text{-tetra}}} a(\sigma)\,\mathrm{vol}(\sigma)} \sum_{[\sigma]} a(\sigma)[\sigma] \in \mathcal{C}_M$$

where σ — good R-tetra.

Lemma 3. Fix R,
$$\sum_{\substack{[\sigma] \\ \text{good } R\text{-tetra}}} a(\sigma) \leq \mathrm{vol}(G/\pi_1(M))$$

Pf We claim that if $A_\sigma \cap \gamma A_{\sigma'} \neq \phi \Rightarrow \sigma = \gamma\sigma'$, $\gamma \in \pi_1(M)$

Indeed, if $g \in A_\sigma \cap \gamma A_{\sigma'} \Rightarrow g(u_i^R) \in \Omega_i$ $i=1,2,3,4$

$g(u_i^R) \in \gamma\Omega_i$ $\sigma' = [p_1', p_2', p_3', p_4']$ $p_i' \in \Omega_i$

$\Rightarrow \gamma\sigma' = \sigma$

$\Rightarrow$ The map $G \to G/\pi_1(M)$ send for $[\sigma] \neq [\sigma']$ A_σ & $A_{\sigma'}$ disjointly

$$\Rightarrow \sum_{[\sigma]} a(\sigma) = \sum_{[\sigma]} \mathrm{vol}(A_\sigma) \leq \mathrm{vol}(G/\pi_1(M)) \quad \square$$

Lecture 4. Gromov's Proof of Mostow Rigidity

We now prove

Thm (Mostow) If $f: M^n \to N^n$ is a homeomorphism between two closed connected hyperbolic manifolds of dim $n \geq 3$, then $\exists$ an isometry $g: M^n \to N^n$ s.t $g \simeq f$.

Proof (Gromov) We may assume $\deg(f) = \deg(f^{-1}) = 1 \Rightarrow$

$$\|M\| \geq |\deg(f)| \, \|N\| \geq |\deg(f)| \, |\deg(f^{-1})| \, \|M\|$$

$$\Rightarrow \quad \|M\| = \|N\|$$

By Gromov + Thurston $\Rightarrow$ $\mathrm{vol}(M) = \mathrm{vol}(N)$.

Let $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ be a lift of f to the universal cover. st

$$\tilde{f}(\gamma(x)) = f_*(\gamma)\tilde{f}(x) \qquad \forall\, x \in \mathbb{H}^n, \ \gamma \in \pi_1(M) \qquad (1)$$

$f_*: \pi_1(M) \to \pi_1(N)$ is the isomorphism induced by f.

Since $\tilde{f}$ is a quasi-isometry $\Rightarrow$ $\tilde{f}$ extends continuously to

$$F: \overline{\mathbb{H}^n} \to \overline{\mathbb{H}^n} \quad \text{s.t} \quad (1) \text{ still holds}$$

Goal: $F|_{\partial \mathbb{H}^n} = \hat{g} \in \mathrm{Iso}(\mathbb{H}^n)$

Assuming this, $\Rightarrow$ $\hat{g}(\gamma(x)) = f_*(\gamma)\, g(x) \quad \forall\, x \in \partial\mathbb{H}^n, \ \forall \gamma \in \pi_1(M)$ (2)

$\Rightarrow$ (2) holds for all $x \in \mathbb{H}^n$

$\Rightarrow$ $\hat{g}$, induces an isometry $g: M^n \to N^n$

The fact that $f \simeq g$ follows from (2).

Let us focus on $n=3$ now

To see $\tilde{g} \in \mathrm{Iso}(\mathbb{H}^3) \Leftrightarrow \tilde{g}$ is a Möbius transformation.

Lemma (See Bennedetti-Petronio) $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a homeomorphism is a Möbius transformation iff $\forall$ regular set $\{v_1, v_2, v_3, v_4\} \in \hat{\mathbb{C}}$, $\{hv_1, hv_2, hv_3, hv_4\}$ is again regular.

Here $\{v_1, v_2, v_3, v_4\}$ is regular $\Leftrightarrow$ the ideal tetra $[v_1, v_2, v_3, v_4] \in \overline{\mathbb{H}}^3$ is regular.

We prove by contradiction. Suppose $\hat{g} = F|_{\partial \mathbb{H}^3}$ is not regular.

We will show that $\|M\| > \|N\|$ which contradicts Gromov-Thurston thm.

To this end, let us go back to Kuiper's proof & his construction.

First $\tilde{g} \notin$ Möbius $\Rightarrow \exists$ regular ideal tetra $[v_1, v_2, v_3, v_4] \in \mathbb{H}^3$ st

$$\mathrm{Vol}([g(v_1), \dots g(v_4)]) < V_3 - \delta \quad \text{for some } \delta > 0.$$

(Here we have used the fact that regular ideal tetra achieves the unique maximum point for volume)

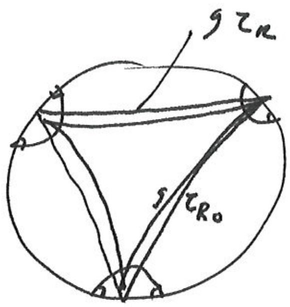
Find open disjoint half-space nbhd U_i of $v_i \in \overline{\mathbb{H}}^3$ s.t

(1) $\forall\ p_1, \dots, p_4 \in \overline{\mathbb{H}}^3$ $p_i \in U_i$ $i = 1,2,3,4 \Rightarrow \mathrm{Vol}([\hat{g}p_1, \dots, \hat{g}p_4]) < V_3 - \delta$

(2) $\exists\ R_0 > 0$ s.t if $R \geq R_0 \Rightarrow$

$$\emptyset \neq \{g \in G \mid g(U_i^{R_0}) \in U_i \ \ i = 1,2,3,4\} \subset \{g \in G \mid g(U_i^{R}) \in U_i \ \ i = 1,2,3,4\}$$

$n=3$



Now fix R_0 and $R > R_0$ as above, let

$$I_R = \{ \text{good } R\text{-tetra } \sigma = [p_1, \cdots p_4] \mid \exists\ \gamma \in \pi_1(M) \text{ st } \tilde{\eta}(\gamma(p_i)) \in U_i \ \ i=1,2,3,4\}$$

By the choice of R_0, $I_R \neq \phi$, furthermore

$\cancel{I_{R_0} \subseteq}$

By the Kuiper's construction, let $\eta = [\sigma]$ be R-good in M^{n3}

$$+ \quad b(\eta) = \frac{\mathrm{vol}(M)}{\sum\limits_{[\sigma]\ R\text{-good}} a(\sigma)\,\mathrm{vol}(\sigma))}\ a(\sigma).$$

then $$C_R = \sum_{\eta\ R\text{-good}} b(\eta)\,\eta \in \underline{\alpha_M}$$

Consider $$\widehat{\tilde{f}_\#}(C_R) = \sum_{\eta\ R\text{-good}} b(\eta)\,\eta' \qquad \eta' = \widehat{f \circ \eta} \quad \text{straightening}$$

It is in α_N by the assumption.

$$\Rightarrow \mathrm{vol}(N) = \sum_\eta b(\eta)\,\mathrm{vol}(\eta')$$

$$= \sum_{\mathrm{I}} b(\eta)\,\mathrm{vol}(\eta') + \sum_{\mathrm{II}} b(\eta)\,\mathrm{vol}(\eta') \qquad \boxed{\eta' = \pi(\sigma),\ \sigma \in I_R}$$

$$\leq V_3 \sum_{\mathrm{I}} b(\eta) + (V_3 - \delta) \sum_{\mathrm{II}} b(\eta)$$

$$= V_3 \sum_{\mathrm{I} \cup \mathrm{II}} b(\eta) - \delta \sum_{\mathrm{II}} b(\eta)$$

$$\leq V_3 \sum_\eta b(\eta) - \delta \sum_{\substack{\mathrm{II} \\ \cancel{R_0\text{-good}}}} b(\eta) \qquad \cancel{\eta = \pi(\sigma)\ \ \sigma \in I_{R_0}}$$

But $\quad \mathrm{vol}(N) = \mathrm{vol}(M)$

$$= \sum_{\eta} b(\eta)\,\mathrm{vol}(\eta) \geq (v_3 - \varepsilon) \sum b(\eta)$$

so we conclude that (for any $\varepsilon > 0$) R large

$$\delta \sum_{\substack{\mathrm{II}\ \eta \\ \cancel{R_0\text{-good}}}} b(\eta) \leq \varepsilon \sum_{\eta} b(\eta)$$

But

$$b(\eta) = \frac{\mathrm{vol}(M)}{\sum_{[\sigma]\ R\text{-good}} \sigma(\sigma)\,\mathrm{vol}(\sigma)}\ a(\eta)$$

$$\Rightarrow$$

$$\delta \left(\sum_{\mathrm{II}} a(\sigma) \right) \leq \varepsilon \sum_{[\sigma]\ R\text{-good}} a(\sigma) \to 0$$

by lemma 3 in lecture 3

This shows, as $R \to \infty$

$$\sum_{\mathrm{II}} a(\sigma) \to 0$$

But $\quad \sum_{\substack{\mathrm{II} \\ R\text{-good}}} a(\sigma) \geq \sum_{\substack{\mathrm{II} \\ R_0\text{-good}}} a(\sigma)$ by the choice of τ^{R}'s

This is impossible.

§4. ~~Gromov's Proof of Mostow Theorem~~

§4.1. A characterization of Möbius transformations

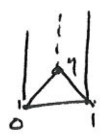
<u>Recall</u>: a Möbius transf of $\hat{\mathbb{C}}$ = compositions of inversions + reflection

It is $g \in PSL(2,\mathbb{C})$ $\quad \frac{az+b}{cz+d}$ or $\frac{a\bar{z}+b}{c\bar{z}+d}$. $\quad$ Möb$(\mathbb{C})$ = group of all Möb.

Let $\eta = e^{\pi i/3}$,

<u>Def</u> A set $\{v_1, v_2, v_3, v_4\} \subset \hat{\mathbb{C}}$ is called <u>regular</u> if $\exists\ g \in$ Möb$(\mathbb{C})$ st

$$\{v_1, v_2, v_3, v_4\} = \{g(0), g(1), g(\infty), g(\eta)\}$$



$\Leftrightarrow$ $\{v_1, v_2, v_3, v_4\}$ vertices of an ideal regular hyperbolic tetra

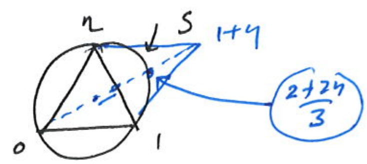
<u>Lemma</u> If $\{a, b, c, d\}$ and $\{a, b, c, d'\}$ are regular and $d' \neq d$, then $d' = \mathrm{Inv}_S(d)$ where S is the circle (or line) containing a, b, c

<u>Pf</u>. Indeed, if $h \in$ Möb st $h(0) = 0$ $h(1) = 1$ $h(\infty) = \infty$ $\Rightarrow$ $h = id$ or $h(z) = \bar{z}$. $\square$

<u>Eg</u> if v_1, v_2, v_3 vertis of a regular triagle $\Rightarrow$ $\{v_1, v_2, v_3, \infty\}$ and $\{v_1, v_2, v_3, \frac{v_1+v_2+v_3}{3}\}$ are regular

It is the image of $\{v_1, v_2, v_3, \infty\}$ under inversion about the circle through v_1, v_2, v_3 $\qquad |1+\eta| =$

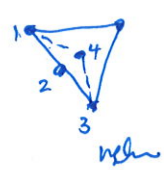
<u>Eg</u> $\{1, \eta, 1+\eta, \frac{1+\eta}{3}\}$ is <u>regular</u>



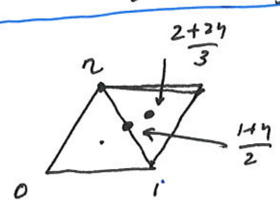
Let S be the circle through $0, 1, \eta$

Now $\{1, \eta, 1+\eta, \frac{1+\eta+(1+\eta)}{3}\}$ regular $\Rightarrow$ $\mathrm{Inv}_S \{1, \eta, 1+\eta, \frac{2+2\eta}{3}\}$ regular
(center)

<u>Which is</u>: $\{1, \eta, \frac{1+\eta}{2}, \frac{2+2\eta}{3}\}$ regular $\quad \left(\frac{2+2\eta}{3} \in S\right)$



$\Rightarrow$



<u>due to</u>: $\mathrm{Inv}_S(1+\eta) = \frac{1+\eta}{2}$

<u>Prop</u>. If $h: S^2 \to S^2$ homeomorphism preserving regular sets, then $h \in$ Möb.

Gromov's Proof

h preserves orientation $\quad h(\infty)=\infty \quad h(0)=0, \; h(1)=1, \; h(\eta)=\eta$

+ h preserves regular sets.

<u>Goal</u> $\quad h = id$.

<u>step 1</u> $\quad h\big|_{\mathbb{Z}+\eta\mathbb{Z}} = id$

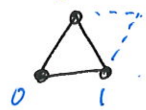
Indeed $\{0, 1, \eta, 1+\eta\}$ regular $\Rightarrow$ $h\{0, 1, \eta, 1+\eta\}$ regular

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{0, 1, \eta, h(1+\eta)\}$

<u>But</u> $\quad h(1+\eta) = 1+\eta$ or 0 due to lemma $\Rightarrow$

$$h(1+\eta) = 1+\eta$$



$\Rightarrow$

repeat $\Rightarrow$ $h(n+m\eta) = n+m\eta$

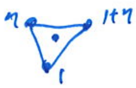
$\forall\, n, m \in \mathbb{Z}$

<u>step 2</u> $\quad h\big|_{\mathbb{Z}(\frac{1}{2}) + \mathbb{Z}(\frac{\eta}{2})} = id.$

$h(0)=0$

$\Rightarrow h\left(\frac{2+2\eta}{3}\right) =$

<u>Note</u> $\quad \left\{1, \eta, 1+\eta, \frac{1+\eta+(1+\eta)}{3}\right\}$ regular, apply h and step 1



$\Rightarrow \quad h\left(\frac{1+\eta+(1+\eta)}{3}\right) = \frac{1+\eta+(1+\eta)}{3} = \frac{2+2\eta}{3}$ ✓

<u>But</u>, $\quad \underline{\left\{1, \eta, \frac{1+\eta}{2}, \frac{2+2\eta}{3}\right\}}$ regular $\Rightarrow$ Apply h, SAME

$h\left(\frac{1+\eta}{2}\right) = \frac{1+\eta}{2}$ $\quad$ <u>since</u> $\quad h\left(\frac{1+\eta}{2}\right) = \begin{cases} \frac{1+\eta}{2} & \underline{\text{or}} \\ 1+\eta \end{cases}$

$\Rightarrow$

$\Rightarrow$ $\quad \underline{h(\tfrac{1}{2}) = \tfrac{1}{2}} \quad \underline{h(\tfrac{\eta}{2}) = \tfrac{\eta}{2}}$

$\Rightarrow$ the SAME as in step 1

Inductively $\Rightarrow$ $\quad h\big|_{\mathbb{Z}(\frac{1}{2^n}) + \mathbb{Z}(\frac{\eta}{2^n})} = id$ $\quad\Rightarrow$

But $\quad \bigcup_{n\in\mathbb{Z}_{\geq 1}} \left(\mathbb{Z}(\tfrac{1}{2^n}) + \mathbb{Z}(\tfrac{\eta}{2^n})\right) \subset \mathbb{C}$ is dense $\Rightarrow$ $h = id$

h continuous.

□

Key observation :