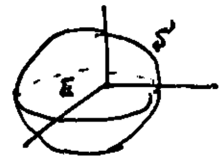
1) Use divergence theorem.

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_E \operatorname{div}(F)\, dV$$



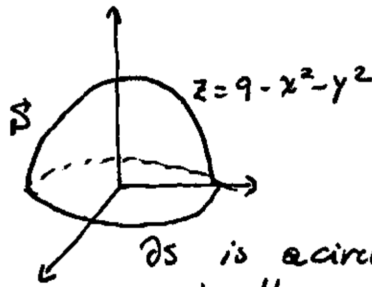
S is the sperical surface, E is the interior of the sphere.

$$\operatorname{div}(F) = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} x = 1$$

$$\iiint_E 1\, dV = \text{Volume of the sphere} = \frac{4}{3}\pi$$

2) Stokes' theorem:

$$\iint_S \nabla\times F\cdot dS = \oint_{\partial S} F\cdot dS$$



∂S is a circle of radius 3 in the xy-plane

Surface integral:

parametrization: use polar coords.

$$\phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 9-r^2)$$

$$\phi_r = (\cos\theta,\ \sin\theta,\ -2r)$$

$$\phi_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$$

$$N = \phi_r\times\phi_\theta = \langle 2r^2\cos\theta,\ 2r^2\sin\theta,\ r\rangle$$

← z coordinate is positive; upward orientation.

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 3y & 4z & -6x \end{vmatrix} = \langle -4, 6, -3\rangle$$

$$\iint_S \nabla\times F\, dS = \int_0^{2\pi}\int_0^3 (-8r^2\cos\theta + 12r^2\sin\theta - 3r)\, dr\, d\theta =$$

$$= \boxed{-27\pi}$$

Line integral: the boundary ∂S is a circle of radius 3 parametrized by

$r(t) = \langle 3\cos t,\ 3\sin t,\ 0\rangle$ oriented counterclockwise

$$\int_{\partial S} F\cdot ds = \int_0^{2\pi} \langle 9\sin t,\ 0,\ -18\cos t\rangle \cdot \langle -3\sin t,\ 3\cos t,\ 0\rangle\, dt =$$

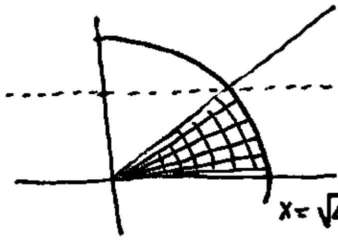
$$= \int_0^{2\pi} -27\sin^2 t\, dt = -27\int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos(2t)}{2}\right) dt =$$

$$= \boxed{-27\pi} \leftarrow \text{same as before } \checkmark$$

3) Solution online: math.rutgers.edu/~fiorda

4) $$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$$

the region of integration is $0 \le y \le \sqrt{2}$, $y \le x \le \sqrt{4-y^2}$



a sector of a circle. In polar coordinates, $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{4}$

the integral in polar coordinates becomes

$$\int_0^{\pi/4}\int_0^{2} \frac{1}{1+r^2}\, r\, dr\, d\theta =$$

$$= \frac{\pi}{4}\int_0^2 \frac{r}{1+r^2}\, dr = \frac{\pi}{4}\int_1^5 \frac{1}{2u}\, du = \frac{\pi}{8}\ln(5)$$

$u = r^2+1$
$du = 2r\,dr$

5) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$\nabla f = \langle 6xy - 6x, \; 3x^2 + 3y^2 - 6y \rangle$

$\nabla f = \langle 0, 0 \rangle$ if and only if $\begin{cases} 6xy - 6x = 0 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases}$

From 1st equation, $6x(y-1) = 0$

$\rightarrow \quad x = 0 \quad$ or $\quad y = 1$

$x = 0$: plug in 2nd equation

$3y^2 - 6y = 0 \longleftrightarrow 3y(y-2) = 0 \rightarrow \begin{matrix} y = 0 \\ y = 2 \end{matrix}$

critical points: $(0,0)$ $(0,2)$

$y = 1$: from 2nd eqn:

$3x^2 - 3 = 0 \longleftrightarrow x^2 = 1 \longleftrightarrow x = \pm 1$

crit points: $(1,1)$, $(-1,1)$

Four critical points: $(0,0)$ $(0,2)$ $(1,1)$ $(-1,1)$

$D = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix} = (6y-6)^2 - 36x^2$

at $(0,0)$, $D = 36 > 0$ $\quad f_{xx} = -6 \Rightarrow$ local maximum

$(0,2)$ $\quad D = 36 > 0$ $\quad f_{xx} = 6 \Rightarrow$ local minimum

$\left.\begin{matrix} (1,1) \quad D = -36 < 0 \\ (-1,1) \quad D = -36 < 0 \end{matrix}\right\}$ saddle points.

7) $z = 2x^2 + y^2$ let $f(x,y) = 2x^2 + y^2$

linearization of $f(x,y)$ at $(1,1)$:

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$f_x = 4x$ $\qquad f_y = 2y$

$$L(x,y) = 3 + 4(x-1) + 2(y-1)$$

$$= 4x + 2y - 3$$

8) $f(x,y) = x^2 y$ $\qquad g(x,y) = x^2 + 2y^2 - 6$

$\nabla f = \langle 2xy, x^2 \rangle$ $\qquad \nabla g = \langle 2x, 4y \rangle$

$$\begin{cases} 2xy = 2x\lambda & \leftrightarrow \quad 2x(y-\lambda) = 0 \\ x^2 = 4y\lambda \\ x^2 + 2y^2 = 6 \end{cases}$$

From the first equation, 2 cases: $x = 0$, $y = \lambda$

if $x=0$: from last equation, $2y^2 = 6$

$\Rightarrow$ $y = \pm\sqrt{3}$ two critical points, $(0,\sqrt{3})$, $(0,-\sqrt{3})$

if $y = \lambda$: from second equation, $x^2 = 4y^2$. Plug in last eqn:

$6y^2 = 6 \leftrightarrow y = \pm 1$

$x^2 = 4y^2 = 4 \leftrightarrow x = \pm 2$ four crit points, ~~$(\pm 2, \pm 1)$~~

$(\pm 2, \pm 1)$

$(\pm 2, \mp 1)$

$f(0,\sqrt{3}) =$ ~~[illegible]~~ $0 = f(0,-\sqrt{3})$ $\qquad f(\pm 2, 1) = 4$ maximum

$f(\pm 2, -1) = -4$ minimum

10)

a) $F(x,y) = (3+2xy)i + (x^2-3y^2)j$

$$\frac{\partial}{\partial x}(x^2-3y^2) = 2x = \frac{\partial}{\partial y}(3+2xy)$$

Cross partials are equal.

Since the domain of F is the whole $\mathbb{R}^2$, which is simply connected, F is conservative.

b) Look for potential $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = 3+2xy \quad \leftrightarrow \quad \phi(x,y) = 3x + x^2y + c(y)$$

$$\frac{\partial \phi}{\partial y} = x^2 + c'(y) = x^2 - 3y^2 \quad \leftrightarrow \quad c'(y) = -3y^2$$

$$\leftrightarrow \quad c(y) = -y^3$$

$$\phi(x,y) = 3x + x^2y - y^3$$

c)

$$\int_c F \cdot ds = \varphi(2,3) - \varphi(0,0) = 6 + 4\cdot 3 - 27$$

$$= -9$$

11) Heat flow:

$$-k \iint_S \nabla u \, dS$$

$\nabla u = \langle 0, 4y, 4z \rangle$

parametrization of cylinder: $\Phi(x,\theta) = (x, \sqrt{6}\cos\theta, \sqrt{6}\sin\theta)$

$\Phi_x = \langle 1, 0, 0 \rangle \qquad \Phi_\theta = \langle 0, -\sqrt{6}\sin\theta, \sqrt{6}\cos\theta \rangle$

$\Phi_x \times \Phi_\theta = \langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta \rangle$

take $N = \langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta \rangle$ to match orientation

$$-k \iint \nabla u \, dS = -6.5 \int_0^4 \int_0^{2\pi} \langle 0, 4\sqrt{6}\cos\theta, 4\sqrt{6}\sin\theta \rangle \cdot \langle 0, -\sqrt{6}\cos\theta, -\sqrt{6}\sin\theta \rangle \, d\theta \, dx$$

$$= 6.5 \int_0^4 \int_0^{2\pi} 24 \, d\theta \, dx =$$

$$= 6.5 \cdot 24 \cdot 4 \cdot 2\pi = 1248\pi$$

12) a)

$$\nabla \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

$$\operatorname{div}(\nabla \phi) = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} = \Delta \phi$$

b) follows from a)

c): $\phi(x,y) = 42 \qquad \phi(x,y) = x^2 - y^2$

$\phi(x,y) = \ln(x^2 + y^2)$