Problem 4. By changing to polar coordinates,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy = \int_0^{2\pi}\int_0^{+\infty} e^{-r^2} r dr d\theta$$

The inner integral can be solved by substituting $u = r^2$, $du = -2rdr$. Thus

$$\int_0^{+\infty} e^{-r^2} r dr = -\frac{1}{2}\int_0^{+\infty} e^{-u} du = -\frac{1}{2}\left[e^{-u}\right]_0^{+\infty} = \frac{1}{2}$$

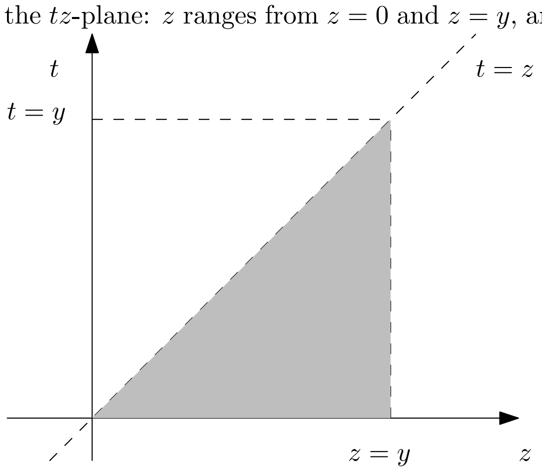
Integrating the result in the variable θ, we get

$$\int_0^{2\pi} \frac{1}{2} d\theta = \frac{1}{2}\cdot 2\pi = \pi$$

Problem 5. We need to interchange the order of integration twice. Let's start with the double integral

$$\int_0^y \int_0^z f(t)dtdz$$

note that here y is fixed. First draw a picture of the domain of integration D in the tz-plane: z ranges from $z = 0$ and $z = y$, and t ranges from $t = 0$ and $t = z$.



As a vertically simple region, $D = \{0 \le t \le y,\ t \le z \le y\}$ and the iterated integral can be rewritten as

$$\int_0^y \int_t^y f(t)dzdt = \int_0^y (y-t)f(t)dt$$

Hence,

$$\int_0^x \int_0^y \int_0^z f(t)dtdzdy = \int_0^x \int_0^y (y-t)f(t)dtdy$$

and changing again the order of integration,

$$\begin{aligned}\int_0^x \int_0^y (y-t)f(t)dtdy &= \int_0^x \int_t^x (y-t)f(t)dydt \\ &= \int_0^x \left[\frac{(y-t)^2}{2}\right]_{y=t}^{y=x} f(t)dt \\ &= \int_0^x \frac{(x-t)^2}{2} f(t)dt\end{aligned}$$

Problem 7. Change the coordinates to spherical. $x^2+y^2+z^2 \leq 1$ becomes $\rho^2 \leq 1$, so $0 \leq \rho \leq 1$.
The equation $z \leq 0$ becomes $\rho\cos(\phi) \leq 0$, and dividing by ρ, $\cos(\phi) \leq 0$. This implies $\frac{\pi}{2} \leq \phi \leq \pi$. The equations $x \leq 0$ and $y \leq 0$ together imply $\pi \leq \theta \leq \frac{3}{2}\pi$. Thus the integral becomes

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \rho \sin\theta \sin\phi \rho^2 \sin\phi \; drd\theta d\phi &= \int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \rho^3 \sin\theta \sin^2\phi \; drd\theta d\phi \\ = \int_{\frac{\pi}{2}}^{\pi} \sin^2\phi d\phi \cdot \int_{\pi}^{\frac{3\pi}{2}} sin\theta d\theta \cdot \int_0^1 \rho^3 d\rho &= \frac{\pi}{4} \cdot (-1) \cdot \frac{1}{4} = -\frac{\pi}{16}\end{aligned}$$