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1. Introduction

Let f be a regular mapping (see the definitions below) of the simple closed curve M^1 into the plane E^2. The resulting curve $C = f(M)$ may cut itself a number of times; if this number is finite, and the "positive" and "negative" self-intersections are distinguished, the algebraic number I_f of them is invariant under "regular deformations," which keep the mapping always regular.[1] We may determine I_f by considering the space $\mathfrak{T}$ of ordered pairs of points of M^1, mapping it into E^2 in a manner determined by f (see below), and counting the coverings of the origin. The object of this paper is to study the situation in n dimensions, for regular mappings of a manifold M^n into E^{2n}. The main theorem (which is trivial for $n = 1$ and well known for $n = 2$) is that M^n may be imbedded in E^{2n}.

An outline of the paper follows. Take $E^n \subset E^{2n}$. It is shown how E^n may be distorted near the origin, giving a mapping $f(E^n) \subset E^{2n}$ such that there is just one self-intersection; specific equations (2.2) for f are given. By introducing such self-intersections into the mapping f of a manifold M^n into E^{2n}, we may alter I_f as we please. As in the case $n = 1$, we express I_f in terms of a mapping F of $\mathfrak{T}$ into V^{2n} = all vectors in E^{2n}, provided that n is even and M is orientable. In any other case, I_f is only determined mod 2; we must identify (p, q) with (q, p) in $\mathfrak{T}$ $(p, q \in M)$, forming a space $\mathfrak{T}^*$, and let F^* map it into V^*, formed from V by identifying each vector with its negative. The determination of the "divided degree" needed is accomplished through a study of "locally derivable" mappings F^* of spaces into V^*; these are obtainable, locally, as the projection into V^* of mappings into V. We carry this theory somewhat further (in §§5, 6) than is needed in the present paper; the notions are capable of great generalization.

After the proof of the invariance of I_f under regular deformations comes the key theorem (Theorem 4) in the proof of the imbedding theorem; it says essentially that the number of self-intersections may be reduced by two, if the number present is greater than $| I_f |$ (or is $\geqq 2$, if I_f is taken mod 2). It is a curious fact that this theorem is proved only in the case $n \geqq 3$. The proof of the imbedding theorem is now immediate if M is closed, and is easily carried out if M is open. In an appendix, we prove that the complex projective plane (a manifold of di-

* Presented to the American Mathematical Society, Sept. 9, 1942. The numbers in brackets refer to the bibliography at the end of the following paper.

[1] See [2]. It is shown further there that if $I_{f_0} = I_{f_1}$, then there is a regular deformation of f_0 into f_1. It would be interesting to know if there is such a theorem in n dimensions.

mension 4) may be imbedded in E^7; since[2] its characteristic is odd, it cannot be imbedded in E^6.

We give a brief description of $\mathfrak{T}$ and F. The set of all pairs (p, u), $p \in M$, u a unit vector tangent to M at p, is the "tangent space" $\mathfrak{S}$ of M. The space $\mathfrak{T}$ consists of $\mathfrak{S}$, and all pairs (p, q), $p \neq q$ $(p, q \in M)$; if $q \to p$ in the direction of a unit vector u at p, we let $(p, q) \to (p, u)$. Thus $\mathfrak{T}$ is bounded by $\mathfrak{S}$. F maps (p, q) into a multiple of $f(q) - f(p)$, so chosen that $\mathfrak{S}$ is mapped into vectors $\neq 0$.

NOTATIONS. A mapping of a manifold M^n is *smooth* if it is of class C^1 (has continuous first partial derivatives). It is *regular* if the Jacobian matrix is everywhere of rank n, or, n independent vectors at any point are carried into n independent vectors, or, each non-zero vector is carried into a non-zero vector. It is an *imbedding* if it is proper (see §24 of the following paper; we may omit this if M is closed) and regular, and is one-one; it is an *immersion* if it is proper and regular. Let I denote the unit interval $0 \leqq x \leqq 1$. We use ∂ for boundary in both the point set and combinatorial sense. If f maps chains in the complex K into chains in K', its dual f' is defined by

$$f'(\sum a'_i \sigma'_i) = \sum_j a'_i \alpha_{ji} \sigma_j, \quad \text{where} \quad f(\sigma_i) = \sum_j \alpha_{ij} \sigma'_j.$$

Scalar products are defined by $(\sum \alpha_i \sigma^r_i) \cdot (\sum \beta_i \sigma^r_i) = \sum \alpha_i \beta_i$. A subscript 2 commonly denotes reducing mod 2.

I. COMBINATORIAL STUDY OF SELF-INTERSECTIONS

2. A particular self-intersection

We shall define a regular mapping f of class C^∞ of the interior Q^n of the unit sphere in E^n into E^{2n} such that there is just one self-intersection, and so that f equals the identity, with all derivatives, on the boundary of Q^n.

DEFINITIONS. Let f be a regular mapping of M^n into E^{2n}. Then for each $p \in M$, there is a plane $T_f(p)$ in E^{2n} tangent to $f(M)$ at $f(p)$, of dimension n. Suppose $f(p_1) = f(p_2)$. We say this point is a *regular self-intersection* if $T_f(p_1)$ and $T_f(p_2)$ have only $f(p_1)$ in common. Then if $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are sets of independent vectors in M at p_1 and at p_2 respectively, f carries these into a set of independent vectors in E^{2n} (for definitions, see (4.1) and (4.2)). Conversely, with two such sets of vectors, the self-intersection is regular. If f has only regular self-intersections, and there are no triple points $f(p) = f(q) = f(r)$, we say f is *completely regular*. See also the end of §4.

For $n = 1$, a curve with one loop, like the written letter e, gives the required self-intersection. Such a mapping of x-space into (y, z)-space may be expressed in the form

$$(2.1) \qquad y = x - \frac{2x}{1 + x^2}, \qquad z = \frac{1}{1 + x^2}.$$

[2] H. Seifert, *Algebraische Approximationen von Mannigfaltigkeiten*, Math. Zeitschrift, vol. 41 (1936), pp. 1-17, Satz 2; compare E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Comp. Math. Helvetici, vol. 8 (1936), Anhang II; also [3].

For $x = \pm 10$ say, these are approximately $y = x$, $z = 0$; hence we may flatten out $f(M)$ near $x = \pm 10$, to obtain exactly these equations there. For general n, the example will again be of a mapping near the identity for large values of the x's, and we shall not trouble to mention the final alterations necessary. (The detailed proof may be given with the help of Lemma 11 of the following paper.)

Consider E^4 as given by moving an E^3 parallel to itself in the y_2-direction. Take $E^2 \subset E^3 \subset E^4$, and take the above $f(M^1) \subset E^2$. As E^3 moves, say with increasing y_2, let the two parts of $f(M^1)$ near the self-intersection be pulled apart, into opposite sides of E^2 in E^3. As y_2 continues to increase, the curve may be flattened out into the line parallel to the y_1-axis. If we let y_2 decrease from 0, we pull the two parts of $f(M^1)$ away in the opposite directions, etc. This clearly defines a self-intersection of the required type. The case $n = 2$ in (2.2) is easily seen to define this mapping.

For general n, the equations are[3]

$$u = (1 + x_1^2) \cdots (1 + x_n^2);$$

$$(2.2) \qquad y_1 = x_1 - \frac{2x_1}{u}, \qquad y_i = x_i \qquad (i = 2, \cdots, n),$$

$$y_{n+1} = \frac{1}{u}, \qquad y_{n+i} = \frac{x_1 x_i}{u} \qquad (i = 2, \cdots, n).$$

The matrix of partial derivatives, for example for $n = 3$, is, transposed,

$$\begin{Vmatrix} 1 - \dfrac{2(1 - x_1^2)}{u(1 + x_1^2)} & 0 & 0 & \dfrac{-2x_1}{u(1 + x_1^2)} & \dfrac{x_2(1 - x_1^2)}{u(1 + x_1^2)} & \dfrac{x_3(1 - x_1^2)}{u(1 + x_1^2)} \\ \dfrac{4x_1 x_2}{u(1 + x_2^2)} & 1 & 0 & \dfrac{-2x_2}{u(1 + x_2^2)} & \dfrac{x_1(1 - x_2^2)}{u(1 + x_2^2)} & \dfrac{-2x_1 x_2 x_3}{u(1 + x_2^2)} \\ \dfrac{4x_1 x_3}{u(1 + x_3^2)} & 0 & 1 & \dfrac{-2x_3}{u(1 + x_3^2)} & \dfrac{-2x_1 x_2 x_3}{u(1 + x_3^2)} & \dfrac{x_1(1 - x_3^2)}{u(1 + x_3^2)} \end{Vmatrix}.$$

To show that f is regular, we must find, for each $p = (x_1, \cdots, x_n)$, a determinant of order n which is $\neq 0$. It is clearly sufficient to find an element $\neq 0$ in the first row. Suppose the element $\partial y_{n+1}/\partial x_1$ ($= \partial y_4/\partial x_1$) is 0. Then $x_1 = 0$, and hence $\partial y_1/\partial x_1 = 1 - 2/u$. If this is 0, then $u = 2$; hence not all the x_i are 0, and (since $x_1 = 0$) some $\partial y_{n+i}/\partial x_1$ is $\neq 0$.

Next we find the self-intersections. Suppose

$$f(x_1, \cdots, x_n) = f(x_1', \cdots, x_n'), \qquad (x_1, \cdots, x_n) \neq (x_1', \cdots, x_n').$$

Since $y_i' = y_i$ ($i = 2, \cdots, n$), $x_i' = x_i$ for $i > 1$. Since $y_{n+1}' = y_{n+1}$, $u' = u$; it follows that $x_1'^2 = x_1^2$; hence $x_1' = -x_1 \neq 0$. Since $y_{n+i}' = y_{n+i}$ ($i > 1$), $x_i =$

[3] The equations are seen to be closely related to those defining the singularity (3.3) of the following paper. It would be possible to define the required self-intersections and singularities in turn, mapping E^1 into E^2, E^2 into E^3, E^2 into E^4, E^3 into E^6, etc., using each time the preceding mapping.

$0 \ (i > 1)$. Since $y_1' = y_1$,

$$x_1 - \frac{2x_1}{u} = -x_1 + \frac{2x_1}{u}, \qquad u = 1 + x_1^2 = 2,$$

and $x_1 = \pm 1$. Thus the only self-intersection is

$$f(1, 0, \cdots, 0) = f(-1, 0, \cdots, 0).$$

To show that this is a regular intersection, we note that the matrix $\| \partial y_i / \partial x_j \|$ transposed, at $(\pm 1, 0, \cdots, 0)$, is (for instance for $n = 3$)

$$\begin{Vmatrix} 1 & 0 & 0 & \mp\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \pm\frac{1}{2} \end{Vmatrix}.$$

The elements of the i^{th} row are the components of the vector $\partial f / \partial x_i$ at $(\pm 1, 0, \cdots 0)$. The two matrices with the two sets of signs give a $2n$-rowed square matrix whose determinant we must prove $\neq 0$. If we subtract the i^{th} row from the $(n + i)^{\text{th}}$ row and expand in terms of the first n columns, we obtain a determinant with diagonal terms equal to 1 or -1, and zeros below the diagonal; hence it is $\neq 0$.

Finally, in the boundary of the region $\sum x_i^2 \leqq 100n$ for example, the equations are practically

$$y_i = x_i \ (i = 1, \cdots, n), \qquad y_i = 0 \ (i > n);$$

hence we may flatten out the mapping on the boundary.

3. The spaces $\mathfrak{T}$ and $\mathfrak{T}^*$

Consider M^n as lying in E^{2n+1} (see [1], Theorem 1). The points of $\mathfrak{S}$ are the pairs (p, v), $p \in M$, v a unit vector tangent to M at p. The points of $\mathfrak{T} - \mathfrak{S}$ are the pairs (p, q), $q \neq p$. We define neighborhoods in $\mathfrak{T} - \mathfrak{S}$ in the obvious fashion. To define neighborhoods in $\mathfrak{T}$ at points of $\mathfrak{S}$, we first define a subsidiary space $\mathfrak{T}'$ as follows. Choose a positive continuous function $\eta(p)$ (which may be made constant if M is compact) such that for each $p \in M$ and sphere $S(p)$ of radius $\eta(p)$ in the tangent plane $T(p)$ to M at p, $S(p)$ plus interior projects (along the planes perpendicular to $T(p)$) onto M in a one-one way. Let π_p be this projection. The points of $\mathfrak{T}'$ are the triples (p, v, λ), $p \in M$, v is a unit vector tangent to M at p, and $0 \leqq \lambda < \eta(p)$. The definition of neighborhoods in $\mathfrak{T}'$ is clear. We map $\mathfrak{T}'$ onto a portion of $\mathfrak{T}$ by setting

$$\Phi(p, v, 0) = (p, v) \in \mathfrak{S},$$

$$\Phi(p, v, \lambda) = (p, \pi_p(p + \lambda v)) \in \mathfrak{T} - \mathfrak{S} \qquad (\lambda > 0).$$

Neighborhoods in $\mathfrak{T}'$ map into neighborhoods in $\mathfrak{T}$.

If M is a manifold of class C^r, $\mathfrak{T}$ is a manifold with boundary, of class C^{r-1}. Clearly the choice of the imbedding of M^n in E^{2n+1} does not affect the definition of $\mathfrak{T}$, but only the particular neighborhoods chosen.

The space $\mathfrak{S}^*$ is formed from $\mathfrak{S}$ by identifying "opposite points" (p, v) and $(p, -v)$. The space $\mathfrak{T}^*$ is formed from $\mathfrak{T}$ by the above identification, and also the identification $(p, q) = (q, p)$.

LEMMA 1. *$\mathfrak{T}$ is orientable if and only if M is.*

Let C be a closed curve in $\mathfrak{T} - \mathfrak{S}$ running through (p, q). Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be independent sets of vectors in M at p and at q respectively. Then keeping q fixed and letting p move in the direction of u_i defines a vector $\bar{u}_i$ in $\mathfrak{T}$; similarly $\bar{v}_i$ is defined, with p fixed. Now the ordered set

$$(\bar{u}_1, \cdots, \bar{u}_n, \bar{v}_1, \cdots, \bar{v}_n) \tag{3.1}$$

defines an orientation of $\mathfrak{T}$ near (p, q). As we follow around C, the points p and q move along closed paths C_1 and C_2 in M.

Suppose first that M is orientable. Then we may carry the vectors of each set around so that they come back to the same vectors, showing that $\mathfrak{T} - \mathfrak{S}$ is orientable.[4] Now take any closed path in $\mathfrak{T}$. Clearly any points on $\mathfrak{S}$ may be pushed away from $\mathfrak{S}$ (replace $\lambda = 0$ by $\lambda > 0$ in $\mathfrak{T}'$). Applying the above proof shows that $\mathfrak{T}$ is orientable.

If M is non-orientable, let C_1 be a closed curve in M reversing the orientation. Then $n > 1$, so that we may choose a point q not on C_1. As p runs around C_1, (p, q) runs arouhd a curve C in $\mathfrak{T}$, which clearly reverses the orientation in $\mathfrak{T}$.

LEMMA 2. *If n is even and M is orientable, $\mathfrak{T}^*$ is orientable; otherwise, $\mathfrak{T}^*$ is non-orientable.*

As before, we need only consider $\mathfrak{T}^* - \mathfrak{S}^*$.

If M is not orientable, the proof above shows that $\mathfrak{T}^*$ is not. Now take M orientable.

Suppose $n > 1$ is odd. Let σ^n be a cell in M, represented with coordinates $x_1, \cdots, x_n$. Let e_i be the unit vector parallel to the x_i-axis in E^n; this maps into a vector $u_i(p)$ at p for each $p \in \sigma$. Let C be a simple closed curve in σ, and let p and q be distinct points of C. Then $(u_1(p), \cdots, u_1(q), \cdots)$ defines an orientation of $\mathfrak{T}^*$ at (p, q). Now move p and q along the two arcs of C into q and p respectively; these vectors are carried into $u_1(q), \cdots, u_1(p), \cdots$ respectively. Thus a permutation of the vectors is defined, formed by n^2 transpositions; n^2 is odd. Since $(q, p) = (p, q)$, we have defined a closed curve in $\mathfrak{T}^*$ reversing the orientation. Note that if n is even, then C preserves the orientation. If $n = 1$, essentially the same proof applies, with $C = M$.

Suppose finally that M is orientable and n is even. Let C^* be any closed curve in $\mathfrak{T}^* - \mathfrak{S}^*$. Take $(p, q) = (q, p)$ in C^*. As we run around C^*, the points p, q separately may be followed in M, and thus (p, q) may be followed in $\mathfrak{T}$. If (p, q) comes back into (p, q) in $\mathfrak{T}$, the proof in the last lemma shows that the orientation is unchanged. If (p, q) comes back into (q, p) in $\mathfrak{T}$, let C be a simple closed curve in M containing p and q, and as in the proof just above, carry p and q into q and p respectively in σ^n, forming a curve D^* in $\mathfrak{T}^*$. We just saw

[4] This follows also at once from the fact that $\mathfrak{T} - \mathfrak{S}$ is part of the cartesian product $M \times M$. We shall need later the vectors defined here.

that C^* followed by D^* does not reverse the orientation, and we noted above that D^* does not either; hence C^* does not.

DEFINITION. If $\mathfrak{T}$ or $\mathfrak{T}^*$ is orientable and M is oriented, the *corresponding orientation* of $\mathfrak{T}$ or $\mathfrak{T}^*$ is defined by the vectors (3.1).

4. The mappings of $\mathfrak{T}$ and $\mathfrak{T}^*$ corresponding to a mapping f of M into an E^m

Take $M \subset E^{2n+1}$ again. Then a vector u tangent to M at p may be defined as follows by a parametrized curve $P(\lambda)$ starting from p:

$$u = \lim_{\lambda \to 0} \frac{P(\lambda) - p}{\lambda}. \tag{4.1}$$

If ∇f is the mapping of vectors induced by f, than

$$\nabla f(u, p) = \lim_{\lambda \to 0} \frac{f(P(\lambda)) - f(p)}{\lambda}. \tag{4.2}$$

Note that ∇f is a generalization of the gradient of a real-valued function.

Let $|u|$ denote the length of u; $|q - p|$ is the distance from p to q in E^{2n+1}. Define

$$F(p, q) = \frac{f(q) - f(p)}{|q - p|}, \qquad p \neq q, \tag{4.3}$$

$$F(p, u) = \nabla f(u, p) \qquad \text{if } |u| = 1; \tag{4.4}$$

then, taking $P(\lambda)$ as before, if $|u| = 1$ in (4.1),

$$\lim_{\lambda \to 0} F(p, P(\lambda)) = \lim_{\lambda \to 0} \frac{f(P(\lambda)) - f(p)}{|P(\lambda) - p|} = \lim_{\lambda \to 0} \frac{f(P(\lambda) - f(p)}{\lambda}$$
$$= \nabla f(u, p) = F(p, u).$$

Thus it is clear that F is a continuous mapping of $\mathfrak{T}$ into the space V^m of vectors in E^m.

Clearly F defines a mapping F^* of $\mathfrak{T}^*$ into V^{m*}.

LEMMA 3. *If f is regular, then F maps no point of $\mathfrak{S}$ into O, and maps a point (p, q) of $\mathfrak{T} - \mathfrak{S}$ into O if and only if $f(p) = f(q)$.*

This is obvious.

Suppose $f(p) = f(q)$, $p \neq q$, and u is a vector in M at p. Then if $\bar{u}$ in $\mathfrak{T}$ is defined as in §3,

$$\nabla F(\bar{u}, (p, q)) = \lim_{\lambda \to 0} \frac{F(P(\lambda), q) - F(p, q)}{\lambda}$$
$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\frac{f(q) - f(P(\lambda))}{|q - P(\lambda)|} - \frac{f(q) - f(p)}{|q - p|} \right];$$

clearly

$$\nabla F(\bar{u}_i, (p, q)) = -\frac{1}{|q - p|} \nabla f(u_i, p). \tag{4.5}$$

Similarly

$$\nabla F(\bar{v}_i, (p, q)) = \frac{1}{|q - p|} \nabla f(v_i, q). \tag{4.6}$$

LEMMA 4. *Let f be a completely regular mapping of M^n into E^{2n}. Then if $f(p) = f(q)$ $(p \neq q)$, F is regular in a neighborhood of (p, q), and maps it over O in V^{2n} with the degree ± 1.*

Since f is completely regular, the vectors

$$\nabla f(u_1, p), \cdots, \nabla f(u_n, p), \qquad \nabla f(v_1, q), \cdots, \nabla f(v_n, q) \tag{4.7}$$

are independent; hence so are the $\nabla F(\bar{u}_i, (p, q))$ and $\nabla F(\bar{v}_i, (p, q))$. The last part of the lemma is a consequence of this.

We consider briefly manifolds with boundary. If f is regular in M and on the boundary ∂M of M (with the obvious definitions, assuming for example that ∂M is a manifold), and $f(p) = f(q)$, $p \neq q$, implies that neither p nor q is in ∂M, we call f *completely regular*. If this is so, F maps no part of $\mathfrak{T}$ coming from points on or near ∂M into O; we need not be explicit as to the definitions of $\mathfrak{T}$ and $\mathfrak{S}$ for such points. Similarly for F^*.

5. Mappings into a vector space with opposite vectors identified

Let $V = V^m$ be the space of vectors in E^m. Let ω be the mapping $\omega(v) = -v$ of V into itself. Form $V^* = V^{m*}$ from V by identifying v with $-v$ for all v; let $\pi(v) = \pi(-v)$ be the corresponding point of V^*. Let O be the zero vector of V, and set $O^* = \pi(O)$.

An arc in $V - O$ from $v \neq O$ to $-v$ maps under π into a closed curve, defining the single element γ not the identity of the fundamental group of $V^* - O^*$.

Let T be an $(m - 1)$-plane in V through O; set $\pi(T) = T^*$. For any subdivision K^* of $V^* - O^*$ with no vertex in T^*, let W^{*1} be the cocycle mod 2 containing those 1-cells cutting T^*; let $\mathbf{W}^{*1}$ be its cohomology class (which is invariantly determined).

DEFINITION. Let f^* map the space R into V^*. If there is a mapping f of R into V such that $f^* = \pi f$, we say f^* is *derivable from a mapping into* V, or is *derivable* simply.

The following lemmas are easily proved.

LEMMA 5. *f^* is derivable if and only if it is derivable over each component of $f^{*-1}(V^* - O^*)$.*

LEMMA 6. *A mapping f^* of a connected complex K into $V^* - O^*$ is derivable if and only if any of the following equivalent conditions are satisfied:*

(α) *Each closed curve in K maps into a curve in $V^* - O^*$ defining the identity element of the fundamental group.*

(β) *Each closed curve in K maps into a curve cutting T^* an even number of times (if this number is finite).*

(γ) *Each 1-cycle of K maps into a bounding 1-cycle of $V^* - O^*$.*

(δ) $f^{*\prime}\mathbf{W}^{*1} = 0$.

DEFINITION. A mapping f^* of a space R into V^* is *locally derivable* (in terms of the f_i below) if there exist open sets $U_1, U_2, \cdots$, covering R and mappings f_i of U_i into V such that:

(a) For each i, $f^* = \pi f_i$ in U_i.

(b) For each i and j, either $f_i = f_j$ or $f_i = \omega f_j$ in $U_i \cap U_j$.

REMARKS. If $\{U_i\}$ and $\{f_i\}$ are replaced by $\{U'_i\}$ and $\{f'_i\}$, the definition is considered as equivalent if (b) holds for the old and new sets together. (See Example 3 below.) Say a set $A \subset R$ is *r-thin* if any singular r-cell in R may be moved away from A by an arbitrarily slight deformation. If $f^{*-1}(O^*)$ is 1-thin, and each non-void $U_i \cap U_j$ is connected, then clearly (b) is a consequence of (a), and any sets of f_i are equivalent. Any mapping of a space into $V^* - O^*$ is clearly locally derivable.

EXAMPLES. The first four illustrate the meaning of local derivability; the fifth illustrates its use in this paper. S and S^* are defined in §6.

(1) Let f_1^* be the identity mapping of $Q^* = S^*$ plus interior into itself. Let f_t map each radius from O^* into itself, shrinking it by the factor t; then $f_0(p) = O^*$ (all p). Then f_0 is locally derivable, while (as proved in Example 9) f_t is not if $t > 0$.

(2) Take f_1 as before, and let f_t map all points of any radius within a distance $1 - t$ of O^* into O^*, extending f_t in an obvious manner over Q^*; again $f_0(p) = O^*$. Then f_1^* is not locally derivable, while f_t is if $t < 1$. However, if a slight change is made for $t > 0$, forming g_t^*, such that each $g_t^{*-1}(O^*)$ is 1-thin in Q^*, then no g_t^* $(t > 0)$ is locally derivable; for the corollary to Lemma 7 below would be contradicted. Hence, by a Remark above, we cannot make (a) hold.

(3) Let $R_1, R_2, \cdots$ be a sequence of disjoint spheres plus interiors in some E^r, approaching a point p_0. Let f map all of E^r but these interiors into O, and map the interiors into $V - O$; set $f^* = \pi f$. Then if $p_0 \in U_i$, there are an infinite number of choices for f_i, making f^* locally derivable in distinct ways.

(4) Let $C_1, C_2, \cdots$ be a sequence of simple closed curves in E^r, with disjoint neighborhoods $U'_1, U'_2, \cdots$. Map U'_i into V^* so that $f^*(C_i)$ defines γ and so that if $f^*(p) = O^*$ $(p \in E^r - \sum U'_i)$, then f^* is continuous. If the C_i converge smoothly to a simple closed curve C, then f^* is locally derivable (but see a remark in Example 2); if the C_i converge to a line segment, it is easily seen that f^* is not locally derivable, even though we can make (a) hold.

(5) Let Q^2 be the unit circle S^1 plus interior in E^2, and let g be the identity mapping of Q^2 into E^2. (We could equally well map Q^r into E^m, $2 \leqq r \leqq m$.) If we identify opposite points of S^1, Q^2 becomes a projective plane P^2; πg becomes a mapping f^* of P^2 into V^*. f^* is locally derivable but not derivable. A diameter of Q^2 becomes a "projective line" L of P^2; set $f^*(L) = L^*$. Subdivide P^2 so that L lies on no vertex a_i, and set $U_i = St(a_i)$ (St = star). If f_i in U_1 is chosen so that $f^* = \pi f_1$ there, then there is a unique choice for each f_i such that if $a_i a_j$ is a simplex in P^2 not cutting L, then $f_i = f_j$ in $St(a_i a_j)$. Now the W^1 of (5.1) below contains just those $a_i a_j$ cutting L; $\mathbf{W}^1 \neq 0$. If we omit a small concentric disc from Q^2, and hence from P^2, a Möbius strip M^2 illustrating Lemma 6 is formed, with $f'\mathbf{W}^{*1}$ being the cohomology class of the part of the above W^1 lying in M^2.

(6) Take Q^2, S^1, as in (5). Map S^1 into a curve in $V^* - O^*$ so as to define γ; map radii of Q^2 into segments from O^*. Then f^* is not locally derivable; in the notations of Lemma 8 below,

$$\delta f^{*\prime} W^1 \cdot Q^2 = f^{*\prime} W^1 \cdot \partial Q^2 = W^1 \cdot f^* S^1 = 1_2 .$$

If f^* is any mapping of Q_2 into $V^* - O^*$ with an inner point p going into O^*, then a slight alteration of f^* may be made which will give a mapping g^* for which a neighborhood of p is mapped as above; then g^* is not locally derivable.

DEFINITION. Let f^* be a locally derivable mapping of the complex K into V^* such that f^{*-1} $(O^*)$is 0-thin, i.e. nowhere dense in K. Then the cohomology class W^1 of f^* is defined as follows. Take the U_i and f_i as before. Let K_1 be a subdivision of K so fine[5] that each star $St_1(a_i)$ of a vertex a_i of K_1 is in some U_{λ_i}. Set

$$(5.1) \qquad W^1(a_i a_j) = \begin{cases} 0_2 & \text{if } f_{\lambda_i} = f_{\lambda_j} \text{ in } St_1(a_i a_j), \\ 1_2 & \text{if } f_{\lambda_i} = w f_{\lambda_j} \text{ in } St(a_i a_j). \end{cases}$$

Then $\mathbf{W}^1$ is the cohomology class of $W^1 = \sum W^1(a_i a_j) a_i a_j$.

LEMMA 7. *W^1 is well defined, and is a cocycle (coefficients* mod 2). *Altering the f_i alters it by a coboundary. Its class $\mathbf{W}^1$ is independent of the choice of K, the U_i and the f_i, provided that the new f_i are equivalent to the old. If f^{*-1} (O^*) is 1-thin, then f^* is derivable if and only if $\mathbf{W}^1 = 0$.*

REMARKS. Any such W^1 is called a *characteristic cocycle* of f^*. By standard theory, that $\mathbf{W}^1$ is independent of K is a consequence of the following statement: If K_1 and K_2 are such that each $St_2(a_{2i})$ lies in some $St_1(a_{1\nu_i})$, so that a simplicial mapping $\phi(a_{2i}) = a_{1\nu_i}$ of K_2 into K_1 is defined, and W_1^1 is a characteristic cocyle· of K_1, then $W_2^1 = \phi' \, W_1^1$ (ϕ' = dual of ϕ) is a characteristic cocycle of K_2; by definition, $W_2^1(a_{2i}a_{2j}) = W_1^1(\phi(a_{2i}a_{2j})) = W_1^1(a_{1\nu_i}a_{1\nu_j})$.

First, since $f^{*-1}(O^*)$ is nowhere dense in K_1, the definition of W^1 is obviously unique. Next, suppose a definite f_k is changed to ωf_k. Then if $i_1, \cdots, i_r$ are the i such that $\lambda_{i_s} = k$, $W^1(a_i a_j)$ is changed if and only if just one of i, j lies among the i_s; thus, if δ denotes coboundary,

$$\text{new } W^1 = \text{old } W^1 + \delta \sum (a_{i_s})_2 .$$

Now for any sets $\{U_i\}$, $\{U'_i\}$ and equivalent $\{f_i\}$, $\{f'_i\}$, choose K_1 so that $St_1(a_i) \subset U_{\lambda_i} \cap U'_{\mu_i}$. Let f_{ij} and f'_{ij} denote f_{λ_i} and f'_{μ_i} respectively, considered in $U_{\lambda_i} \cap U'_{\mu_j}$ (if this is not void); all the f are equivalent. Clearly the f_{λ_i} and the f_{ij}, also the f'_{μ_i} and f'_{ij}, define the same W^1. The proof above shows that changing the f_{ij} to the f'_{ij} alters W^1 by a coboundary.

Now suppose K_1 and K_2 are as described above. In using (5.1) to define W_2^1, since $St_2(a_{2i}) \subset St_1(a_{1\nu_i})$, we may take for the f_k defined about a_{2i} the same as the f_k defined about $a_{1\nu_i}$; then clearly $W_2^1(a_{2i}a_{2j}) = W_1^1(a_{1\nu_i}a_{1\nu_j})$.

[5] From the corollary below it is clear that the f_i may be defined in the stars in K, so that no subdivision is necessary in reality.

Suppose $f^* = \pi f$. Since $f^{*-1}(O^*)$ is 1-thin, the given set $\{f_i\}$ is equivalent to the single mapping f; hence W^1 may be defined using f. Since $W^1 = 0$, $\mathbf{W}^1 = 0$.

Suppose finally that $\mathbf{W}^1 = 0$. For a given subdivision K_1, and sets $\{U_i\}$, $\{f_i\}$, we have $W^1 = \delta X^0$ for some $X^0 = \sum \alpha_i a_i$ ($\alpha_i = 0_2$ or 1_2). If $St_1(a_i) \subset U_{\lambda_i}$, set $f'_i = f_{\lambda_i}$ in $St_1(a_i)$. For each i with $\alpha_i = 1_2$, alter f'_i to $\omega f'_i$; then (see a proof above) the new $W^1 = 0$. Now for any p in K, say $p \in St(a_i)$; set $f(p) = f'_i(p)$. This makes f continuous in K, and $f^* = \pi f$.

EXAMPLE (7). Let f map a circle C into V so that just one point p_0 goes into O, and set $f^* = \pi f$. Let U_1 and U_2 cover C, p not being in U_2. Then f_1 and f_2 may be chosen so that W^1 is not $\backsim 0$, though f^* is derivable. Note that $f^{*-1}(O^*) = p_0$, which is not 1-thin in C.

COROLLARY. *Any locally derivable mapping f^* of a cell (in fact, of a complex with vanishing* 1*-cohomology group* mod 2) *is derivable, if $f^{*-1}(O^*)$ is* 1*-thin in K.*

A direct proof for a cell may be given simply if we build it up as in (b) of Lemma 12.

We shall give a lemma which shows immediately whether or not a mapping is locally derivable or derivable, provided that $f^{*-1}(O^*)$ is 1-thin. Since it involves notions[6] not yet generally known, and the lemma is not needed here, we shall be rather brief.

A geometric *r-G*-cochain $X = X^r$ in a point set R is a pair $\bar{X} \supset \bar{X}'$ (nucleus and nuclear boundary) of closed sets, $(r - 1)$-thin and r-thin respectively, and a group of homomorphisms $X \cdot A^r$ of those integral singular chains A^r for which $A \cap \bar{X}' = \partial A \cap \bar{X} = 0$ into G, with the properties $X \cdot A = 0$ if $A \cap \bar{X} = 0$, $X \cdot \partial B^{r+1} = 0$ if $B \cap \bar{X}' = 0$. If G and H are paired to J, set $X \cdot \sum h_i \sigma_i^r = \sum (X \cdot \sigma_i^r) \cdot h_i$. The coboundary δX of X is the pair of sets $\bar{X}'$, 0, with the definition $(\delta X) \cdot A^{r+1} = X \cdot \partial A^{r+1}$. To any cochain X^r in a complex corresponds in a fairly obvious fashion a geometric cochain with nucleus and nuclear boundary formed from duals of r- and $(r + 1)$-cells.

Since T^* and O^* are 0-thin and 1-thin in V^* respectively, they give a geometric cochain W^1 of V^*, such that $W^1 \cdot \sigma^1$ counts the number of times mod 2 that σ^1 crosses T^*. This is the geometric analogue of the W^1 first defined. If f^* is as in the lemma below, then

$$W^1_{f^*} \cdot A^1 = W^1 \cdot f^* A^1$$

defines a geometric cocycle $W^1_{f^*} = f^{*\prime} W^1$ in K. Note that $\delta f^{*\prime} W^1 = f^{*\prime} \delta W^1$.

LEMMA 8. *Let f^* be any mapping of a complex K into V^* such that $f^{*-1}(T^*)$ and $f^{*-1}(O^*)$ are* 0*-thin and* 1*-thin respectively. Then f^* is locally derivable if and only if $\delta f^{*\prime} W^1 = 0$, and is derivable if and only if $f^{*\prime} W^1 \backsim 0$.*

REMARK. Using ordinary cocyles, we may replace the condition $\delta f^{*\prime} W^1 = 0$ by the following: For any arbitrarily fine subdivision K_1 of K there is an arbitrarily small deformation of f^* into f_1^* such that $f_1^{*\prime} W^1$ is defined and is a cocycle.

Suppose f^* is locally derivable, using $\{U_i\}$ and $\{f_i\}$. Take any singular chain A^2, with $\partial A^2 \cap f^{*-1}(O^*) = 0$, so that $\delta W^1_{f^*} \cdot A$ is defined. Using a fine

[6] See H. Whitney, *Geometric methods in combinatorial topology,* not yet published.

subdivision of A, we may write $A = \sum \alpha_i \sigma_i^r$, each σ_i^r in some U_{λ_i}; since $f^{*^{-1}}(O^*)$ is 1-thin, we may deform A slightly so that $\sum \alpha_i(\delta W_{f^*}^1 \cdot \sigma_i^r)$ is defined; it equals $\delta W_{f^*}^1 \cdot A$. Using f_{λ_i}, it is clear that $\delta W_{f^*}^1 \cdot \sigma_i^r = W^1 \cdot \partial f^* \sigma_i^r = O_2$; hence $\delta W_{f^*}^1 = 0$.

Suppose $\delta W_{f^*}^1 = 0$. Each point $p \in K$ is in a connected neighborhood U such that any 1-cycle in U bounds. Choose $p_0 \in U$ so that $f^*(p_0) \neq O^*$; choose $f(p_0)$ so that $\pi f(p_0) = f^*(p_0)$. For any $p' \in U$ with $f^*(p') \neq O^*$, choose an arc C in U from p_0 to p' with $f^*(C) \subset V^* - O^*$ (which is possible, since $f^{*^{-1}}(O^*)$ is 1-thin); $f(p')$ is uniquely definable so that $f(p'')$ exists and is continuous in C, and $f^* = \pi f$ there. We must show that going around a closed curve A^1 in U and through p_0 ($f^*(A^1) \subset V^* - O^*$) brings $f(p_0)$ back to its original value. Say $A^1 = \partial B^2$. Then $W_{f^*}^1 \cdot A^1 = \delta W_{f^*}^1 \cdot B^2 = 0$, from which the statement clearly follows. Thus we prove (a); since $f^{*^{-1}}(O^*)$ is 1-thin, (b) follows also for suitable $\{U_i\}$.

Suppose f^* is derivable; say $f^* = \pi f$. Then $\mathbf{W}_{f^*}^1 = f'\pi'\mathbf{W}^1 = f'0 = 0$, since the cohomology groups of V vanish.

Suppose finally $W_{f^*}^1 = \delta W^0$. Then for any closed curve A^1 in K with $f^*(A^1) \subset V^* - O^*$, $W_{f^*}^1 \cdot A^1 = W^0 \cdot \partial A^1 = 0$; it follows by Lemma 5 and Lemma 6 (β) that f is definable throughout K.

LEMMA 9. *Let f_0^* be a locally derivable mapping of the complex K into V^*, with $f^{*^{-1}}(O^*)$ nowhere dense in K. Then there is an arbitrarily slight deformation f_t^* of f_0^* $(0 \leqq t \leqq 1)$ such that $F^*(t, p) = f_t(p)$ is locally derivable, with mappings F_i equivalent in $0 \times K$ to the given $f_{0,i}$ there, and such that $f_1^*(K^{2n-1}) \subset V^* - O^*$. If A is a closed subset of K with $f^*(A) \subset V^* - O^*$, we may make $f_t^* = f_0^*$ in A.*

REMARK. As a consequence, it is easily seen (using the various derived of K) that $f_1^{*^{-1}}(O^*)$ may be made $(2n - 1)$-thin in K.

We may suppose $f_{0,i}$ is defined in $St(a_i)$; we shall define F_i in $I \times St(a_i)$. Let $St'(a_i)$ be the barycentric star of a_i, containing all points on cells of the first derived of K which have a_i as a vertex. Then $\overline{St}'(a_i) \subset St(a_i)$ and $\sum \overline{St}'(a_i) = K$. Take a fixed i. We may define $f_{t,i}$ in $\overline{St}(a_i)$ for $0 < t \leqq t_1$ so that

$$f_{t_1,i}(K^{2n-1} \cap \overline{St}'(a_i)) \subset V^* - O^*,$$

$$f_{t,i}(p) = f_{0,i}(p) \qquad (p \in \partial St(a_i) \cup (St(a_i) \cap A)),$$

with $f_{t,i}(p)$ arbitrarily close to $f_{0,i}(p)$. For each j such that $a_i a_j$ is a 1-cell of K, set

$$f_{t,j}(p) = f_{t,i}(p) \text{ or } \omega f_{t,i}(p) \qquad (p \in St(a_i a_j)),$$

according as which holds for $t = 0$, and set $f_{t,j}(p) = f_{0,j}(p)$ elsewhere. This clearly defines $F^*(t, p)$ for $p \in K$ and $0 < t \leqq t_1$. Do this for each a_i in turn, using successive subintervals of (0, 1). (K may be infinite, if it is "locally finite.") If the successive alterations are small enough, we will have automatically $F^*(t, p) \subset V^* - O^*$ $(p \in K^{2n-1} \cap \overline{St}'(a_i))$ for previous i. Thus all of F^* is defined.

6. Locally derivable mappings into V^{2n*}

DEFINITIONS. Let $S = S^{2n-1}$ be the unit sphere in V^{2n}, and set $S^* = S^{2n-1*} = \pi S$. Then S^* is a projective space, which is orientable, since $2n - 1$ is odd. We may suppose the spaces oriented so that, combinatorially, $\pi V = 2V^*$ and $\pi S = 2S^*$; thus π has the degree 2. Let P and P^* be the projections of $V - O$ and $V^* - O^*$ along rays from O and O^* onto S and S^* respectively.

Let L^* be a ray from O^* in V^*. Given any subdivision K^* of $V^* - O^*$ with $(K^*)^{2n-2} \subset V^* - L^*$, we may define a cocycle $W = W^{2n-1}$ in K^* as follows. For any $\sigma = \sigma^{2n-2}$, $W^{2n-1}(\sigma) = KI(\sigma, L^*)$. This is the algebraic number of times that $P^*\sigma$ covers $L^* \cap S^*$ in S^* or, that L^* cuts through σ.

Since $L^* - O^*$ is $(2n - 2)$-thin in $V^* - O^*$, we can consider W as a geometric cocycle in $V^* - O^*$. Since O^* is not $(2n - 1)$-thin in V^* for $n \geqq 2$, (in fact, not 2-thin; compare Example (6) and the corollary to Lemma 7), we cannot properly consider it as a cochain in V^*, whose coboundary has the nucleus O^*. In fact, δW^* may be considered in a general or in a restricted way:

(A) If a singular chain $A^{2n} \subset V^*$ has $\partial A^{2n} \subset V^* - O^*$, we may set $\delta W \cdot A^{2n} = W \cdot \partial A^{2n}$. For example, if $A^{2n} = S^*$ plus interior, then $\partial A = S^*$, and $\delta W \cdot A = 1$.

(B) If we define $\delta W \cdot A^{2n}$ only for singular chains A which may be expressed as $A = \sum \alpha_i \sigma_i^{2n}$, with $\partial \sigma_i^{2n} \subset V^* - O^*$, then we have always $\delta W \cdot A = W \cdot \partial A \equiv 0 \pmod 2$; see the next lemma.

DEFINITIONS. For any singular cycle $A^{2n-1} \subset V^* - O^*$, the *degree* $d(A)$ of A^{2n-1} is $W^{2n-1} \cdot A^{2n-1}$. If f^* maps a cycle A^{2n-1} into $V^* - O^*$, the *degree* $d_{f^*}(A)$ *of* f^* *in* A *about* O^* is $d(f^*A)$. This is clearly invariant under deformations; furthermore, combinatorially,

$$P^* f^* A = \alpha S^*, \qquad \alpha = d_{f^*}(A). \tag{6.1}$$

For any singular chain A^{2n} with $\partial A^{2n} \subset V^* - O^*$, the *degree* $d(A^{2n})$ *of* A^{2n} *over* O^* is $W^{2n-1} \cdot \partial A^{2n}$ $(= \delta W \cdot A)$; define $d_{f^*}(A^{2n})$ similarly. These are obvious generalizations of the degree about O or over O in V.

LEMMA 10. *Suppose* $n \geqq 2$. *If* f^* *maps a complex* K *into* V^* *so that* $f^*K^{2n-1} \subset V^* - O^*$, *then* $\delta f^{*\prime} W^{2n-1} \equiv 0 \pmod 2$, *so that*

$$\tilde{W}_{f^*}^{2n} = \tfrac{1}{2}\delta f^{*\prime} W^{2n-1} \quad \textit{exists.} \tag{6.2}$$

The strict meaning of $\tilde{W}_{f^*}^{2n}$ is given by its action on any $2n$-G-chain $A^{2n} = \sum g_i \sigma_i^{2n}$, which is as follows:

$$\tilde{W}_{f^*}^{2n} \cdot \sum g_i \sigma_i^{2n} = \sum g_i (\tilde{W}_{f^*}^{2n} \cdot \sigma_i^{2n}), \tag{6.3}$$

$$\tilde{W}_{f^*}^{2n} \cdot \sigma = \tfrac{1}{2}\delta f^{*\prime} W^{2n-1} \cdot \sigma = \tfrac{1}{2}(W^{2n-1} \cdot f^* \partial \sigma). \tag{6.4}$$

Since $2n - 1 > 1$, $\partial\sigma$ is simply connected; hence there is a mapping f of $\partial\sigma$ into $V - O$ such that $f^* = \pi f$. Say $P f \partial\sigma = \alpha S$. Then

$$P^* f^* \partial\sigma = P^* \pi f \partial\sigma = \pi P f \partial\sigma = 2\alpha S^*;$$

hence

$$\delta f^{*\prime} W^{2n-1} \cdot \sigma = W^{2n-1} \cdot f^* \partial \sigma = W^{2n-1} \cdot P^* f^* \partial \sigma = 2\alpha,$$

from which the lemma follows.

REMARKS. By Lemma 9 and the remark following it, a slight deformation of a locally derivable f^* will give an f_1^* such that $f_1^{*\prime} W^{2n-1}$ and $\tilde{W}_{f_1^*}^{2n}$ are geometric cochains.

It is not true that $\partial A^{2n} = 0$ implies $\tilde{W}_{f^*}^{2n} \cdot A = \frac{1}{2}(\delta f^{*\prime} W^{2n-1} \cdot A) = \frac{1}{2}(W^{2n-1} \cdot f^* \partial A) = 0$; see Example (8) below. The second expression has in general no meaning, say if the coefficients are integers mod 2. But it holds for integral A, as is apparent from (6.4).

DEFINITION. Given a locally derivable mapping f^* of K into V^*, and a $2n$-G-chain A in K such that $f^* \partial A \subset V^* - O^*$, make a slight deformation of f^* into f_1^* such that $f_1^*(K^{2n-1}) \subset V^* - O^*$, as in Lemma 9, keeping ∂A^* in $V^* - O^*$; let the *divided degree* of A under f be

$$\tilde{d}_{f^*}(A) = W_{f_1^*}^{2n} \cdot A \qquad \text{(which is an element of } G). \tag{6.5}$$

Note that this defines $\tilde{d}_{f^*}(A)$ for locally derivable singular chains A with $\partial A \subset V^* - O^*$. It need not vanish for cycles (see Example (8) below).

LEMMA 11. *The above degree is independent of the choice of f_1^*, vanishes for chains mapped into $V^* - O^*$, and is invariant under deformations f_t^* such that $F^*(t, p) = f_t^*(p)$ is locally derivable in $I \times K$ and $f_t^*(\partial A) \subset V^* - O^*$. It vanishes for all boundaries ∂B^{2n+1}, in fact, for all "cycles of the first kind," expressible as $\sum g_i A_i^{2n}$ ($g_i \subset G$, A_i^{2n} integral cycles).*

We prove the last part first. Supposing $f_1^*(K^{2n-1}) \subset V^* - O^*$, we have

$$\tilde{W}_{f_1^*}^{2n} \cdot \sum g_i A_i^{2n} = \sum g_i (\tilde{W}_{f_1^*}^{2n} \cdot A_i^{2n}) = \sum g_i [\tfrac{1}{2}(W^{2n-1} \cdot f^* \partial A_i^{2n})] = 0;$$

also, if $A^{2n} = \partial B^{2n+1}$, say $B^{2n+1} = \sum g_i \sigma_i^{2n+1}$, then $A^{2n} = \sum g_i \partial \sigma_i^{2n+1}$ is a cycle of the first kind, so that $\tilde{W}_{f_1^*}^{2n} \cdot A^{2n} = 0$.

To prove the rest, it is sufficient to show that if

$$f_t^* \partial A \cup f_0^*(K^{2n-1}) \cup f_1^*(K^{2n-1}) \subset V^* - O^*,$$

and $F^*(t, p) = f_t^*(p)$ is locally derivable, then

$$\tilde{W}_{f_1^*}^{2n} \cdot A = \tilde{W}_{f_0^*}^{2n} \cdot A.$$

By a slight deformation of F^*, keeping $F^*(0, p)$ and $F^*(1, p)$ fixed, we may suppose that F^* maps $(I \times K)^{2n-1}$ into $V^* - O^*$, and is locally derivable (see Lemma 9); suppose this is done. The chain A determines chains $0 \times A$, $1 \times A$, $I \times A$ in $I \times K$ (if we use the usual subdivision of $I \times K$). Using the fact just proved, we have $\tilde{W}_{F^*}^{2n} \cdot [\partial(I \times A)] = 0$; hence, since $\partial(I \times A) = 1 \times A - 0 \times A - I \times \partial A$ and $F^*(I \times \partial A) \subset V^* - O^*$, $\tilde{W}_{F^*}^{2n} \cdot (1 \times A) = W_{F^*}^{2n} \cdot (0 \times A)$, which give the result.

MORE EXAMPLES. (8) (Compare Examples (1) and (5).) Let Q^{2n} be S^{2n-1} plus interior R^{2n}, and form P^{2n} by identifying opposite points of S^{2n-1}. Thus

$P^{2n} = S^*$ plus R^{2n}. Set $\bar{f}$ = identity in Q^{2n}, and $\bar{f}^* = \pi\bar{f}$; this gives a mapping f^* of P^{2n} into V^*. Now P^{2n} is a projective space, of even dimension $2n$, and hence is non-orientable; it has a fundamental $2n$-cycle A^{2n} mod 2. If we subdivide it so that O is interior to a $2n$-cell σ_0^{2n} of Q^{2n}, it is clear that

$$\tilde{W}_{f^*}^{2n} \cdot A^{2n} = [\tfrac{1}{2}(W^{2n-1} \cdot \pi\partial\sigma_0^{2n})]_2 = 1_2 .$$

Of course A^{2n} is not a cycle of the first kind. In the case $n = 1$, this type of mapping illustrates Theorem 1; see the discussion there for $n = 1$.

We may deform f^* by contracting $f^*(K)$ to O^*, and then pulling along a single ray away from O^*. This shows that the assumption in Lemma 11 that F^* is locally derivable cannot be omitted.

(9) Let f^* be the identity mapping of S^* into itself. We may consider S^* as an integral cycle; $W^{2n-1} \cdot S^* = 1$. Of course S^* bounds its interior (combinatorially); but by Lemma 11, there exists no complex K containing S^* in which S^* bounds, such that there is a locally derivable mapping of K into V^* which equals the identity in S^*.

7. The intersection number of a mapping

Definitions. Let f be a completely regular mapping of M^n (with or without boundary) into E^{2n}; we suppose E^{2n} oriented. Suppose first that M is orientable and n is even. Choose an orientation of M. Suppose $f(p) = f(q)$; let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be sets of vectors in M as in §3, each set determining the positive orientation of M. Then we say this self-intersection is of *positive* or *negative* type according as the vectors (4.7) determine the positive or negative orientation of E^{2n}. Also, by (4.5) and (4.6) (see the end of §3), the type is positive or negative according as the degree in Lemma 4 is 1 or -1. Note that this is independent of the orientation chosen for M. The *intersection number* I_f of f is the algebraic number of self-intersections in this case, and the number of self-intersections mod 2 if M is non-orientable or n is odd. In the case $n = 1$, it is possible to determine I_f in a certain sense as an integer; see the discussion following Theorem 1.

Define F and F^* as in §4. Clearly F^* is locally derivable. Considering $\mathfrak{T}^*$ as a singular chain, with integers or integers mod 2 as coefficients according as it is orientable or not, define the degree $d_F(\mathfrak{T})$ and divided degree $\tilde{d}_{F^*}(\mathfrak{T}^*)$ as in §6. Let D denote the distant intersection of M (see [3], §18).

Theorem 1. *Let f be a completely regular mapping of M^n into E^{2n} (M may have a boundary). Let the number of self-intersections be finite. Then*

(a) *If M is orientable and n is even,*

(7.1) $$I_f = \tilde{d}_{F^*}(\mathfrak{T}^*) = \tfrac{1}{2}d_F(\mathfrak{T}) = \tfrac{1}{2}(D \cdot M^n) \qquad (\textit{an integer}).$$

(b) *If M is non-orientable or n is odd,*

(7.2) $$I_f = \tilde{d}_{F^*}(\mathfrak{T}^*) \qquad (\textit{an integer} \text{ mod } 2).$$

(c) *If M is orientable and n is odd,*

$$d_F(\mathfrak{T}) = D\cdot M^n = 0. \tag{7.3}$$

The theorem shows that F^* has more interest than F in this connection. We discuss the case $n = 1$ further later.

Suppose first that M is orientable and n is even. Take a self-intersection $f(p) = f(q)$, $p \neq q$. The points of $\mathfrak{T}$ near (p, q) are mapped by F over O with the degree 1 or -1, according as the intersection is of positive or negative type. The same is true for the points of $\mathfrak{T}$ near (q, p). Thus each positive [negative] self-intersection contributes 2 [-2] to d_F, and $I_f = \frac{1}{2}d_F$, which is part of (7.1). That $I_f = \frac{1}{2}(D\cdot M)$ follows easily from the definition of D.

Next we prove (c). Take (p, q) again, and define the vectors u_i and v_i as before. Neighborhoods of (p, q) and (q, p) are oriented by $\bar{u}_1, \cdots, \bar{v}_n$ and $\bar{v}_1, \cdots, \bar{u}_n$ respectively; since n is odd, they are mapped over O in V^{2n} with opposite degrees; hence they contribute 0 to d_F. Thus $d_F = 0$. Similarly $D\cdot M = 0$.

Now consider F^* in $\mathfrak{T}^*$, $\mathfrak{T}^*$ orientable; clearly F^* is locally derivable. The only points mapped into O^* by F^* are points $(p, q) = (q, p), f(p) = f(q), p \neq q$. Let U be a neighborhood of (p, q) in $\mathfrak{T}$; it corresponds to a neighborhood U^* in $\mathfrak{T}^*$, say under the mapping ϕ. Then $F^* = \pi F\phi^{-1}$ in U^*. Let $\sigma^* = \sigma^{*2n}$ be a cell in U^* about (p, q), oriented like $\mathfrak{T}^*$. If $\sigma = \phi^{-1}\sigma^*$, then as noted above, F maps σ over O with the degree 1 or -1, according as the self-intersection is positive or negative. Since π is of degree 2, πF maps σ over O^* with the degree 2 or -2, i.e.

$$W^{2n-1}\cdot\pi F\partial\sigma = W^{2n-1}\cdot F^*\partial\sigma^* = 2 \text{ or } -2.$$

Hence, by (6.4),

$$\tilde{W}^{2n}_{F^*}\cdot\sigma^* = \tfrac{1}{2}(W^{2n-1}\cdot F^*\partial\sigma^*) = 1 \text{ or } -1;$$

using (6.5), we see that σ^* contributes 1 or -1 to $\tilde{d}_{F^*}$. Thus $I_f = \tilde{d}_{F^*}$ (M orientable and n even), proving the rest of (7.1).

Finally we prove (b). In this case, $\mathfrak{T}^*$ is non-orientable, so that it is considered as a chain mod 2 in forming $\tilde{d}_{F^*}(\mathfrak{T}^*)$. As before, $\tilde{W}^{2n}_{F^*}\cdot\sigma^* = 1$ or -1, so that (using (6.3)) $\tilde{W}^{2n}_{F^*}\cdot(1_2\sigma^*) = 1_2$; clearly (7.2) follows.

THE CASE $n = 1$. Though $\mathfrak{T}^*$ is non-orientable, we shall use integer coefficients. For M we may take the unit circle in the plane E^2, described by an angle θ ($0 \leqq \theta < 2\pi$). $\mathfrak{T}^*$ is a Möbius strip with boundary curve $\mathfrak{S}^*$. Any point of $\mathfrak{T}^* - \mathfrak{S}^*$ is a pair (θ, θ'); we may take $\theta < \theta'$. We may let (θ, θ) denote the points of $\mathfrak{S}^*$. If we express $\mathfrak{T}^*$ as the image of the triangle (x, y), $0 \leqq x \leqq y \leqq 1$, the vertical and horizontal sides will each map into the set $\mathfrak{S}'$ of points $(0, \theta)$. Let $\mathfrak{T}^*_1$ be an integral chain covering $\mathfrak{T}^*$; then, combinatorially, $\partial\mathfrak{T}^*_1 = \mathfrak{S}^* + 2\mathfrak{S}'$, so that by (6.3) and (6.4), with $W^1 = W^{2n-1}$ as in §6, using (7.1) to define I_f as an integer,

$$I_f = \tilde{W}^2_{F^*}\cdot\mathfrak{T}^*_1 = \tfrac{1}{2}(W^1\cdot F^*\partial\mathfrak{T}^*_1) = \tfrac{1}{2}(W^1\cdot F^*\mathfrak{S}^*) + W^1\cdot F^*\mathfrak{S}'.$$

$W^1 \cdot A^1$ is the degree of A^1 about O^* in V^{2*}; clearly $\frac{1}{2}(W^1 \cdot F^* \mathfrak{S}^*)$ is the number of times $F\mathfrak{S}^*$ (or more properly, ∇F^v, $v \in \mathfrak{S}$) winds about O in V^2, i.e. is the "rotation number" of f in M. Now if the point $\theta = 0$ is chosen so that $f(0)$ is on a line L of support to $f(M)$ in E^2 (i.e. $f(M)$ lies on one side of L), it is easy to see that $F\mathfrak{S}'$ makes half a turn about O, so that $F^*\mathfrak{S}'$ makes one turn about O^*, and $W^1 \cdot F^*\mathfrak{S}' = \pm 1$. Hence I_f always differs by unity from the rotation number of $f(M)$, again illustrating the warning in §6.

THEOREM 2. *Let M^n be compact (closed, or compact and with boundary). Then the intersection number I_f is invariant under regular deformations; if M has a boundary, we assume that during the deformation, the boundary is not carried across M:*

$$F_t(\partial M) \cap f_t(M - \partial M) = 0 \qquad \text{(all } t). \tag{7.4}$$

This means $I_{f_1} = I_{f_0}$, f_0 and f_1 being completely regular; other f_t need only be regular.

REMARKS. If we omit (7.4), it would be simple to give a formula expressing the change in I_f in terms of the crossings of ∂M through M. Thus if M^1 is a closed arc, mapped into the written letter e, $I_f = \pm 1$; if we pull it out into a line segment, $I_f = 0$, and ∂M^1 has crossed M^1 once.

If M is not compact (and no further conditions are assumed), the theorem fails. For example, let M^1 be the open segment $0 < \lambda < 4$. For $0 \leqq t \leqq 1$, let f_t be the mapping given by $(x, y) = f_t(\lambda)$:

$$x = \lambda, \qquad y = t/\lambda \qquad (0 < \lambda < 2),$$

and let the rest of M^1 be carried counter-clockwise around onto the line $y = 1$, and back across the y-axis along $y = 1$. Then $I_{f_t} = -1$ if $t > 0$, but $I_{f_0} = 0$.

Of course I_f is not invariant under deformations not assumed regular.

Suppose first that M is a manifold (without boundary). Since each F_t^* maps $\mathfrak{S}^* = \partial\mathfrak{T}^*$ into $V^* - O^*$, and is continuous, $\tilde{d}_{F_1^*} = \tilde{d}_{F_0^*}$, by Lemma 11. By the last theorem, $I_{f_0} = I_{f_1}$.

Now suppose M has a boundary. Then $\partial\mathfrak{T}^*$ consists of points of $\mathfrak{S}^*$ and points (p, q), $p \in \partial M$ or $q \in \partial M$. The hypothesis shows again that F_t leaves $\partial\mathfrak{T}^*$ in $V^* - O^*$, and again $I_{f_0} = I_{f_1}$.

II. THE IMBEDDING THEOREM

8. Discussion of the theorem

We first give theorems on the possibility of altering I_f, and of deforming f (keeping I_f fixed) so as to reduce the number of self-intersections. The imbedding theorem follows from these.

THEOREM 3. *Given any compact M (with or without boundary), there is a proper completely regular mapping f of M^n into E^{2n} with any desired I_f.*

Let f_0 be a completely regular mapping of M^n into E^{2n} (see [1], Theorem 3; it is not hard to extend this theorem to the case that M has a boundary, for example if ∂M is a manifold). Now take a small piece of M, flatten it, cut it out, and replace it by a piece with a single self-intersection, defined in §2, or by this piece altered by a reflection in E^{2n}. Then I_{f_0} is changed by 1 or -1 at will. Thus I_f may be increased or decreased by 1, and hence may be brought to any desired value.

THEOREM 4. *Let n be $\geqq 3$, and let f_0 be a proper regular mapping of a closed M^n into E^{2n}, with at most a finite number of self-intersections. Then there is a regular deformation f_t of f into a proper completely regular mapping f_1, such that the number of self-intersections under f_1 is increased by 2. If the number is $> |I_f|$, or is >0 if n is odd or M is non-orientable, we may decrease the number by 2. If M has a boundary, we may keep ∂M fixed, and keep M from cutting through ∂M.*

PROBLEM. Does the theorem hold for $n = 2$?

For the moment we shall prove merely that the number of self-intersections may be increased by 2. The rest of the proof will occupy §§9 through 12.

First, f may be made completely regular, as in [1]. Suppose this is done. Take a small nearly flat portion σ^n of $f(M)$. Take $p \in \sigma^n$, and pull p out away from σ^n, around and towards another point q of σ^n, pulling a small portion of σ^n along with it so we always have a regular mapping. We may make the moving part of M avoid any other points of M or ∂M. Say σ^n was approximately in the $(x_1, \cdots, x_n)$-plane in E^{2n}, and say q is the origin O, and we are pulling p around in the (x_1, x_{n+1})-plane, and down along the x_{n+1}-axis. We may tip the moving portion of σ^n near p to be parallel to the $(x_1, x_{n+2}, \cdots, x_{2n})$-axis, as we move p. Now pull p through O. Then if the mapping f_1 is obtained, the new self-intersections under f_1 are clearly on the x_1-axis T. The part of $f_1(\sigma^n)$ near p intersects T at points on each side of O. These latter points are new self-intersections, as required.

The rest of the proof of Theorem 4 runs as follows. We need merely show that two self-intersections of opposite types (or any two, if M is non-orientable or n is odd) can be gotten rid of by a regular deformation. Say

$$f(p_1) = f(p_2) = q, \qquad f(p_1') = f(p_2') = q',$$

these being of opposite types. Let C_1 and C_2 be non-intersecting curves in M, C_i joining p_i and p_i', neither passing through any other point where f has a self-intersection. Then $B_i = f(C_i)$ joins q to q', and $B = B_1 \cup B_2$ is a simple closed curve in $f(M)$. We let B bound a smooth 2-cell σ, which touches M only at B, and show that the part of M near C_2 may be deformed along near σ so that it passes beyond B_1, thus removing the two self-intersections. Until §12 we assume that M is orientable and n is even.

THEOREM 5. *Any smooth n-manifold may be imbedded in E^{2n}.*

REMARK. This theorem may be generalized to the case of manifolds with boundary, but we shall not discuss in what generality. For example, if ∂M is a manifold, the extension is not difficult. For the proof, see the proof of Case IV of Theorem 6 in the following paper.

For $n = 1$, the theorem is trivial. For $n = 2$, we imbed the sphere, projective plane, or Klein bottle, in E^4, and add the necessary number of handles to obtain the given manifold. (An imbedding of the projective plane is easily found from [3] p. 108.)

Now we suppose that $n \geqq 3$. First, by [1], Theorem 3, there is a proper completely regular mapping f_0 of M^n into E^{2n}. Suppose M is closed. Then, by Theorem 3, we alter f_0 to f_1 so that $I_{f_1} = 0$, and by Theorem 4, we may get rid of all of the intersections, giving the required imbedding.

Suppose now that M is open. We may suppose that M is connected. Let $(p_i, q_i)(i = 1, 2, \cdots)$ be the points with $f_0(p_i) = f_0(q_i)$. For each i, let C_i be a smooth curve in M from p_i to infinity, and let U_i be a neighborhood of C_i, so that U_i touches no other U_j or q_j. Let

$$(p, r, s)(p \in S_0^{n-2}, 0 \leqq r \leqq 1, 0 \leqq s < 1)$$

represent the points of U_i (supposing $n \geqq 2$), with $r = 0$ and $2\delta \leqq s < 1$ representing C_i; the points with $r = 1$ or $s = 0$ give the boundary of U_i. Deform U_i into itself by setting

$$\phi_t(p, r, s) = [p, r, \{1 - (1 - \delta)(1 - r)t\}s];$$

this leaves the boundary of U_i fixed. It may easily be replaced by a smooth deformation. Doing this for each i gives a mapping f with $f(M) \subset f_0(M) - \sum C_i$. Clearly f has no self-intersections, and is still proper.

PROBLEM. Does there exist an imbedding, for M open, with no limit set?

9. A lemma on bundles of vector spaces

A bundle $\mathfrak{B}$ *of vector spaces* is defined as follows. Let V_0^m be a fixed vector space of dimension n. Let A be a topological space, the *base space*, and let $\{F_i\}$ be a covering of A by closed sets (we could use open sets). To each $p \in A$ there corresponds a set of points $V^m(p)$; $V^m(p) \cap V^m(q) = 0$ if $p \neq q$. For each F_i, and each $p \in F_i$, $\xi_{F_i}(p, q)$ $(q \in V_0^m)$ is a one-one mapping of V_0^m onto $V^m(p)$. Let $q = \xi_{F_i}^{-1}(p, r)$ mean $r = \xi_{F_i}(p, q)$. For $p \in F_i \cap F_j$, we assume $\xi_{F_i}^{-1}(p, \xi_{F_j}(p, q))$ is a linear mapping of V_0^m onto itself, which varies continuously with p. Because of this, each $V^m(p)$ may be considered as a vector space. The ξ_{F_i} are the *coordinate systems*; the set of all points on all $V^m(p)$ is the *total space*. We write $\mathfrak{B} = (\mathfrak{B}; V_0^m, A)$. If there exists a coordinate system $\zeta(p, q)$, defined for p in all of A, then $\mathfrak{B}$ is *simple*; ζ expresses the total space as the topological product of A and V_0^m.

If we replace V_0^m by a sphere S_0^{m-1}, and linear mappings by orthogonal mappings, we have a *sphere bundle* (see [3]).

LEMMA 12. *Let* $(\mathfrak{B}; V_0^m, K)$ *be a bundle with a complex* K *for base space. Let* K' *be a (possibly void) subcomplex of* K. *Let* $e_1, \cdots, e_m$ *be the unit vectors in* V_0^m. *Let* $\zeta(p, e_1), \cdots, \zeta(p, e_{i-1})$ *be defined for* $p \in K$, *and let* $\zeta(p, e_i)$ *be defined for* p *in* K', *so that these are continuous and independent where defined. Then*

$\zeta(p, e_i)$ may be extended continuously over K so as to be independent of the other $\zeta(p, e_i)$ if either of the following two conditions hold:

(a) $\dim(K) \leqq m - i$.

(b) *K can be built up from K' by adding cells, each having a cell, or nothing, in common with the subcomplex of K' plus the cells already chosen.*

REMARKS. In (b), it follows that $\mathfrak{B}$ is simple. If $\dim(K) = m - i + 1$, the existence of $\zeta(p, e_i)$ is expressible simply in terms of one of the "characteristic classes" of $\mathfrak{B}$ (see [3]).

The proof in Case (b) is due essentially to Wazewski.[7] In Case (a), we extend $\zeta(p, e_i)$ over cells of K of higher and higher dimensions. That this can be done at each step is a simple consequence of the dimensions under consideration (see a similar proof in §10), and is fundamental in the existence of the characteristic classes (which exist equally well for bundles of spheres or vector spaces).

10. The 2-cell σ

Choose the C_i as described in §8; this is possible, since $n \geqq 3$. Let M_1 and M_2 be neighborhoods of C_1 and C_2 in M. Let p_{it} $(0 \leqq t \leqq 1)$ move from p_i along C_i to p_i' as t goes from 0 to 1. We can suppose p_{it} is an imbedding of the interval (0, 1) in M. Then $q_{it} = f(p_{it})$ is an imbedding.

Let E^2 have coordinates x, y. Let A_1 be the interval $0 \leqq x \leqq 1$, $y = 0$. Let A_2 be the arc of a circle of radius 1 and with center at $(1/2, -\sqrt{3}/2)$, which joins the ends r and r' of A_1 and lies in the half plane $y \geqq 0$. Let $A = A_1 \cup A_2$, let τ' be a small neighborhood of A in E^2, and let τ be τ' plus the interior region τ''. We shall find an imbedding ψ of τ in E^{2n} such that

$$\psi(r) = q, \qquad \psi(r') = q', \qquad \psi(A_i) = B_i \qquad (i = 1, 2)$$

and such that $\psi(\tau) \cap f(M) = B$, and no tangent plane to $\psi(\tau)$ at a point q^* of B lies in a tangent plane to $f(M)$ at q^*. We do this first with τ replaced by τ'.

Let T_1^n and T_2^n be the planes tangent to $f(M_1)$ and $f(M_2)$ at q. Let T^2 be the plane tangent to B_1 and B_2 at q; it cuts T_1^n and T_2^n in lines T_1^1 and T_2^1. By a slight deformation of f, we may suppose that f maps a neighborhood of p_i in M_i onto a neighborhood of q in T_i^n, and maps a neighborhood of C_i (extended) in M_i onto a neighborhood of T_i^1 $(i = 1, 2)$. We may map a closed neighborhood $\bar{V}$ of r in τ linearly onto a closed neighborhood of q in T^2 so that a closed neighborhood of r in A_i goes into a closed neighborhood of q in T_i^1 (and hence in B_i). To this end, we suppose the two ends of A_2 have been straightened. Call this mapping ϕ, and define ϕ similarly near r'.

Let r_{it} run from r to r' along A_i, choosing it so that $\phi(r_{it}) = q_{it}$ where ϕ is defined. Let u_{it} be a smooth vector function defined in A_i and with values in τ such that u_{i0} and u_{i1} point in the forward and backward directions along A_j at r and r' $(j \neq i)$, and letting u_{1t} and u_{2t} turn in the positive and negative sense respectively as t runs from 0 to 1. Then u_{it} is not tangent to A_i at r_{it}. Le

[7] W. Wazewski, Compositio Math. vol. 2 (1935), pp. 63–68.

R_{it} be a segment of length ρ and center r_{it}, in the direction of u_{it}. We may suppose $\bar{V}$ and $\bar{V}'$ chosen so that part of their boundaries are formed by segments R_{it} extended, and may choose ρ so that no R_{1t_1} touches an R_{2t_2} if r_{1t_1} and r_{2t_2} are outside $\bar{V} \cup \bar{V}'$.

Set $v_{it} = \nabla\phi(u_{it}, r_{it})$ at points r_{it} of $\bar{V} \cup \bar{V}'$. Extend v_{it} over the rest of B_i so that it is smooth and not tangent to $f(M_i)$ at q_{it}. This may be done as follows (or we could use Lemma 12, (a)). Extend v_{it} so as to be smooth over all of B_i except a small piece D_i. With a curvilinear coordinate system about D_i, we may suppose D_i lies on the y_1-axis, and $f(M_i)$ lies in the $(y_1, \cdots, y_n)$-plane E^n. Now $v_{it} = v'_{it} + v''_{it}$, v'_{it} and v''_{it} being tangent and orthogonal to E^n respectively; v''_{it} is $\neq 0$ at the ends of D_i. Extend v'_{it} to be smooth over D_i. Since $2n - n > 1 = \dim (D_i)$, we may extend v''_{it} over D_i so as to be smooth and $\neq 0$ there. Now set $v_{it} = v'_{it} + v''_{it}$ in D_i.

Since ϕ is linear so far,

$$\phi(r_{it} + \alpha u_{it}) = q_{it} + \alpha v_{it} \quad (r_{it} \in \bar{V} \cup \bar{V}', \; |\alpha| \leqq \rho).$$

We may use this formula to extend ϕ over a closed neighborhood $\bar{\tau}'$ of A.

Let ψ' be a continuous extension of ϕ over τ. Since $2n \geqq 5$, we may[8] replace ψ' by an imbedding ψ such that $\psi(r)$ is arbitrarily close to $\psi'(r)$, and as we approach points of A, ψ approaches ψ' closer and closer, together with first derivatives. It follows that ψ is an imbedding, and no tangent plane to $\psi(\tau)$ at a point $\psi(r^*)$ $(r^* \in A)$ lies in a tangent plane to $f(M)$ at $\psi(r^*)$. Further, since $n + 2 < 2n$, we may suppose that $\psi(\tau) \cap f(M) = B$. Now $\sigma = \psi(\tau)$ is the required 2-cell.

11. The neighborhood of σ

It is here we shall use the fact that the two self-intersections are of opposite types. Our present purpose is to define smooth vector functions $w_1(q^*), \cdots, w_{2n}(q^*)$, for $q^* \in \sigma$, so that these are independent, and if $q^* = \psi(r^*)$,

(a) $w_1(q^*) = \nabla\psi(e_1, r^*)$, $w_2(q^*) = \nabla\psi(e_2, r^*)$,

(b) $w_3(q^*), \cdots, w_{n+1}(q^*)$ are tangent to $f(M_1)$ at q^* for $q^* \in B_1$,

(c) $w_{n+2}(q^*), \cdots, w_{2n}(q^*)$ are tangent to $f(M_2)$ at q^* for $q^* \in B_2$.

These show how σ, $f(M_1)$ and $f(M_2)$ lie together in E^{2n}.

Since ψ is regular, $w_1(q^*)$ and $w_2(q^*)$ are independent. Let V_0^{2n} be the space of vectors in E^{2n}. Let $e_3, \cdots, e_{n+1}$ and $e_{n+2}, \cdots, e_{2n}$ determine the subspaces V_1^{n-1} and V_2^{n-1} respectively. Consider, at each point $q^* \in B_1$, the vector space $V_1^{n-1}(q^*)$ of vectors tangent to $f(M_1)$ but orthogonal to B_1. Considering $V_1^{n-1}(q^{**})$ as disjoint from $V_1^{n-1}(q^*)$ if $q^{**} \neq q^*$, we have a bundle of vector spaces $\mathfrak{B}_1$ with the cell B_1 as base space; use $\mathfrak{B}_1 = (\mathfrak{B}_1; V_1^{n-1}, B_1)$. By Lemma 12, (b), we may find a coordinate system ζ_1 in it; choose it so that $w_1(q^*), w_2(q^*), \cdots, w_{n+1}(q^*)$ determine the positive orientation of $f(M)$ at $q^* \in B_1$, if we set

$$w_i(q^*) = \zeta_1(q^*, e_i) \quad (q^* \in B_1; i = 3, \cdots, n + 1). \tag{11.1}$$

[8] H. Whitney, Trans. Am. Math. Soc. vol. 36 (1934), pp. 63–89, Lemma 6; compare [1], Theorem 5, or Theorem 2, with the remarks following it.

Since the only vectors at $q^* \in B_1$ which are tangent to both τ and $f(M)$ are tangent to B_1, the w_i are independent so far, and all properties hold.

Similarly, taking $V_2^{n-1}(q^*)$ tangent to $f(M)$ but orthogonal to B_2 at $q^* \in B_2$ gives a bundle $(\mathfrak{B}_2 ; V_2^{n-1}, B_2)$ with a coordinate system ζ_2. Define $w_i(q^*)$ by (11.1), with ζ_2, for $q^* \in B_2$, $i \geqq n + 2$. If $w_2'(q^*)$ points positively along B_2 at q^*, we may suppose $w_2'(q^*), w_{n+2}(q^*), \cdots, w_{2n}(q^*)$ determine the positive orientation of $f(M_2)$ at q^*. The properties still hold. (At q and q', clearly all the vectors are independent.)

Let $(\mathfrak{B}; V_0^{2n}, \sigma)$ be the bundle with each $V(q^*) = V_0^{2n}$. (The total space is $V_0^{2n} \times \sigma$.) In the subbundle $(\mathfrak{B}; V_0^{2n}, B_2)$ over B_2, the vectors $w_1(q^*), w_2(q^*), w_{n+2}(q^*), \cdots, w_{2n}(q^*)$ are independent; $w_1(q^*), \cdots, w_{2n}(q^*)$ are independent for $q^* = q$ and for $q^* = q'$. By Lemma 12, using (a), with $K = B_2$, $K' = q \cup q'$, we may extend $w_3(q^*), \cdots, w_n(q^*)$ over B_2 so that the $2n - 1$ vectors are independent. Now recall that the two self-intersections are of opposite types (we are supposing that M is orientable and n is even). By the choice of ζ_1 and ζ_2, this means that

$$w_1(q^*), \qquad w_3(q^*), \cdots, w_{n+1}(q^*), \qquad w_2'(q^*), \qquad w_{n+2}(q^*), \cdots, w_{2n}(q^*),$$

define opposite orientations of E^{2n} for $q^* = q$ and for $q^* = q'$. Now $w_2(q)$ and $w_2(q')$ may be deformed into $w_2'(q)$ and $-w_2'(q')$ respectively, keeping them tangent to σ and independent of B_1; they therefore remain independent of the other vectors listed above. Hence

$$w_1(q^*), w_2(q^*), \cdots, w_{2n}(q^*)$$

define the same orientation of E^{2n} at $q^* = q$ as at $q^* = q'$. It follows that $w_{n+1}(q^*)$ may be defined over B_2 so as to remain independent of the other vectors. (If we deform $w_{n+1}(q^*)$ for q^* near q and q' to become a unit vector orthogonal to the other vectors, then $w_{n+1}(q^*)$ is uniquely determined so as to be continuous and have this property in the rest of B_2.)

By Lemma 12, using (a) again, with $K = \sigma$ and $K' = B$, since dim $(\sigma) = 2 \leqq 2n - (n + 1)$, we may extend $w_3(q^*), \cdots, w_{n+1}(q^*)$ over σ so that $w_1(q^*), \cdots, w_{n+1}(q^*)$ are independent there. Finally, by Lemma 12, using (b), since σ and B_2 clearly have the property of K and K', we may extend $w_{n+2}(q^*), \cdots, w_{2n}(q^*)$ over σ. The $w_i(q^*)$ determine a neighborhood of σ in E^{2n}, as will be apparent from (12.1).

12. Completion of the proof of Theorem 4

Consider $\tau \subset E^2$ as lying in E^{2n}. For each point $(a_1, a_2, a_3, \cdots, a_{2n})$, set $r^* = (a_1, a_2, 0, \cdots, 0)$, and

$$\psi\left(r^* + \sum_{i=3}^{n} a_i e_i\right) = \psi(r^*) + \sum_{i=3}^{n} a_i w_i(\psi(r^*)). \tag{12.1}$$

Since $w_1(q^*), \cdots, w_{2n}(q^*)$ are independent and smooth, this is a mapping with non-vanishing Jacobian; hence, in a neighborhood of the interior of σ, we may invert, and define ψ^{-1}. (The $w_i(q^*)$ are taken of class C^2 for this to hold.) If we set

$$N_1 = \psi^{-1}(f(M_1)), \qquad N_2 = \psi^{-1}(f(M_2)),$$

then N_1 and N_2 will lie in a neighborhood U of τ. If we define a deformation of N_2 in U which brings it into a position not intersecting N_1, applying ψ defines a corresponding deformation of f; there will then be two self-intersections fewer, as required.

Set $\pi(x_1, \cdots, x_{2n}) = (x_1, 0, x_3, \cdots, x_{2n})$. By (a), (b) and (c) of §11, and (12.1), for any $r^* \in A_1$, the tangent plane to N_1 and hence to $\pi(N_1)$ at r^* lies in the $(x_1, x_3, \cdots, x_{n+1})$-plane, and the tangent plane to $\pi(N_2)$ at r^* lies in the $(x_1, x_{n+2}, \cdots, x_{2n})$-plane, it follows that

(12.2) $$\pi(N_1) \cap \pi(N_2) \quad \text{is on the } x_1\text{-axis.}$$

Let $\mu(x_1)$ be a smooth real function whose graph $x_2 = \mu(x_1)$ lies just above A_2, and lies on the x_1-axis just outside A_2. Take $\epsilon > 0$ so that the points of N_2 within ϵ of the (x_1, x_2)-plane E^2 form a set interior to N_2. Let $\nu(\lambda)$ be a smooth real function with

$$| \nu(\lambda) | \leqq 1, \qquad \nu(0) = 1, \qquad \nu(\lambda) = 0 \quad \text{for} \quad | \lambda | \geqq \epsilon^2.$$

For $r^* = (x_1, \cdots, x_{2n})$, set

(12.3) $$\theta_t(r^*) = r^* - t\nu(x_3^2 + \cdots + x_{2n}^2)\mu(x_1)e_2,$$

and consider this as applying only to points of N_2. This is clearly a regular deformation. For $t = 1$, it moves the part of N_2 lying on A_2 into a position with $x_1 < 0$. Since $\pi(\theta_t(N_2)) = \pi(N_2)$, (12.2) gives

$$N_1 \cap \theta_1(N_2) \quad \text{is on the } x_1\text{-axis.}$$

But $\theta_1(N_2)$ does not touch the x_1-axis; hence this set is void. Furthermore, since $\psi(\tau) \cap f(M) = B$, taking ϵ small enough will insure that no new self-intersections are introduced. This completes the proof in the case that M is orientable and n is even.

In the other cases, we can get rid of any pair of self-intersections by a regular deformation. To show this, it is only necessary to show that C_1 and C_2 may be chosen in such a fashion that M_1 and M_2, when each is oriented, intersect in points of opposite types.

Suppose M is not orientable. Try a pair C_1, C_2. If the points are of the same type, choose a curve C_2' such that $C_2 \cup C_2'$ reverses the orientation in M; then C_1 and C_2' will do.

Suppose n is odd, and M is orientable. If C_1 and C_2 will not do, choose C_1' and C_2' so that (with the notation of §9) C_1' joins p_1 and p_2', and C_2' joins p_2 and p_1'. We may suppose C_i' coincides with C_i near one end and with C_j $(j \neq i)$ at the other, and $M_i = M_i'$ near p_i, and $M_i = M_j'$ $(j \neq i)$ near p_i'. Orient the M_i, and the M_i' accordingly (near p_i). Then the intersection of the pair (M_1', M_2') is like that of (M_2, M_1) at q'; hence (compare the proof in Lemma 2) the intersection is of opposite type from that of (M_1, M_2) at q', as required. The theorem is now proved.

Appendix

Imbedding the complex projective plane M^4 in E^7

Let M^6 be the space of all triples (z_1, z_2, z_3) of complex numbers, not all 0. If we identify any two triples (z_1, z_2, z_3), $(\lambda z_1, \lambda z_2, \lambda z_3)$ ($\lambda \neq 0$ complex), M^4 is obtained. Let $[z_1, z_2, z_3]$ denote points of M^4.

We first give briefly a topological description of M^4, then discuss how an imbedding may be found from the topological point of view, and finally give equations which are proved to be an imbedding. In terms of the real variables, the imbedding will be analytic; it of course cannot be so in terms of the complex variables, since 7 is odd.

Set $z_k = x_k + iy_k$, and

$$N(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + |z_3|^2 = \sum (x_k^2 + y_k^2). \tag{1}$$

We remark that, since $[z_1, z_2, z_3]$ may be normalized to make $N = 1$, uniquely except for a factor λ with $|\lambda| = 1$, the 5-sphere $N = 1$ is expressed as the total space of a bundle of 1-spheres with M^4 as base space. A similar fact holds for the $(2n + 1)$-sphere.

First we divide M^4 into three subsets; the points with $z_3 \neq 0$, those with $z_3 = 0$ but $z_2 \neq 0$, and those with $z_3 = z_2 = 0$. These sets, with normal forms for their points, are:

$$\begin{aligned} &Q^4\text{: all } [z_1, z_2, 1],\\ &\left.\begin{aligned} &Q^2\text{: all } [z_1, 1, 0],\\ &Q^0\text{: all } [1, 0, 0]; \end{aligned}\right\} S^2. \end{aligned}$$

Since these are cells, the Euler-Poincaré characteristic of M^4 is 3. S^2 is complex projective 1-space, equivalent to the extended complex plane, and is topologically a 2-sphere. For any $[z_1, z_2, 0]$ in S^2, set

$$P^*[z_1, z_2, 0] = \text{all } [z_1, z_2, z_3].$$

Set

$$\rho[z_1, z_2, z_3] = |z_3|^2/(|z_1|^2 + |z_2|^2), \tag{2}$$

(ρ may be ∞), and define the subsets of M^4

$$A_-\text{: } \rho < 1, \qquad A\text{: } \rho = 1, \qquad A_+\text{: } \rho > 1.$$

Set $P_-[z_1, z_2, 0] = P^*[z_1, z_2, 0] \cap A_-$, and define P, P_+ similarly.

Clearly A is a 3-sphere (see below), and A_+ is a 4-cell bounded by it; each is in Q^4. Since the sets $P_-(p) \cup P(p)$ are disjoint, these form a bundle of closed 2-cells, with S^2 as base space. The $P(p)$ form a bundle of 1-spheres with S^2

as base space; the total space is the 3-sphere A. We remark that this proves the 2-dimensional invariant of the bundle to have the value ± 1; also, that mapping each point q of A into $p \in S^2$ such that $q \in P(p)$ is exactly Hopf's essential mapping of a 3-sphere into a 2-sphere.[9]

We now discuss the imbedding problem. If we set $F^*[z_1, z_2, 1] = (z_1, z_2)$, this imbeds Q^4 in 4-space. Of course F^* cannot be extended over S^2. To make this possible, let us pull back towards the origin those points with small ρ; we may set

$$F'[z_1, z_2, 1] = \frac{2}{N}(z_1, z_2) \tag{3}$$

or equivalently (the last expression allowable even for $z_3 = 0$)

$$F'[z_1, z_2, z_3] = \frac{2\,|z_3|^2}{\cdot N}\left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right) = \frac{2}{N}(z_1\bar{z}_3, z_2\bar{z}_3). \tag{4}$$

This mapping fails to be regular at points of $\dot{A}$, where the vectors pointing away from [0, 0, 1] (using (3)) are mapped into the origin.

Let us introduce three new coordinates into the image space, map S^2 into the unit sphere in this space, and extend the mapping over M^4. Using stereographic projection, and one complex and one real coordinate, we find

$$F''[z_1, z_2, 0] = \left(2\frac{z_1\bar{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}\right),$$

or more generally,

$$F''[z_1, z_2, z_3] = \frac{1}{N}(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

Combining this with (4), with a slight change, we define our mapping by

$$F[z_1, z_2, z_3] = \frac{1}{N}(z_2\bar{z}_3, z_3\bar{z}_1, z_1\bar{z}_2, |z_1|^2 - |z_2|^2), \tag{5}$$

using three complex and one real coordinate, N being defined by (1). In terms of the seven real variables,

$$\begin{aligned} F = \frac{1}{N}(&x_2x_3 + y_2y_3, y_2x_3 - x_2y_3, x_1x_3 + y_1y_3, x_1y_3 - y_1x_3, \\ &x_1x_2 + y_1y_2, y_1x_2 - x_1y_2, x_1^2 + y_1^2 - x_2^2 - y_2^2). \end{aligned} \tag{6}$$

From (5) it is clear that $F[z_1, z_2, z_3] = F[\lambda z_1, \lambda z_2, \lambda z_3]$, λ complex, i.e. it is independent of the manner of writing a point in M^4.

We now prove that F is regular. Consider first the F' of (3). Since the set

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[9] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Annalen, vol. 104 (1931), pp. 637–665. The mapping also is that determined in the Remark following (1).

of points (z_1, z_2) with $|z_1|^2 + |z_2|^2 = r^2$, which forms a 3-sphere of radius r if $r > 0$, is mapped into a concentric sphere, F' is regular wherever the vector $v_{z_1 z_2}$ at $[z_1, z_2, 1]$ pointing away from the origin is mapped into a non-zero multiple of itself. Set

$$\alpha = x_1^2 + y_1^2 + x_2^2 + y_2^2, \qquad N_\lambda = 1 + \lambda^2\alpha^2 \quad (\lambda \text{ real}).$$

Then the image of $\frac{1}{2}v_{z_1 z_2}$ is the value when $\lambda = 1$ of

$$\frac{\partial}{\partial\lambda}\left[\frac{1}{N_\lambda}(\lambda x_1, \lambda y_1, \cdots)\right] = -\frac{2\lambda\alpha^2}{N_\lambda^2}(\lambda x_1, \cdots) + \frac{1}{N_\lambda}(x_1, \cdots)$$
$$= -\frac{1}{N_\lambda^2}(2\lambda^2\alpha^2 - N_\lambda)(x_1, y_1, x_2, y_2).$$

Taking $\lambda = 1$, we see that F' is regular except possibly on a set B_3, where $2\alpha^2 - N = 0$, i.e. $2\alpha^2 = 1 + \alpha^2$, $\alpha = 1$, or else $z_3 = 0$. With $[z_1, z_2, z_3]$, we see that B_3 is given by the condition:

(7) Either $z_3 = 0$ or $|z_3|^2 = |z_1|^2 + |z_2|^2$,

i.e. $B_3 = S^2 \cup A$.

Define B_1, B_2 similarly, and set $B = B_1 \cap B_2 \cap B_3$. Then replacing F' by F (where we need not consider the last variable $|z_1|^2 - |z_2|^2$), we see that F is regular except possibly at points of B. Now B contains only the points

(8) $p_{1z} = [0, 1, z], \quad p_{2z} = [z, 0, 1], \quad p_{3z} = [1, z, 0], \quad |z| = 1.$

For $[0, 0, 1]$ is not in B_3; similarly for $[0, 1, 0]$ and $[1, 0, 0]$. In case at most one $z_i = 0$, say $|z_1| \leqq |z_2| \leqq |z_3|$; then $z_2 \neq 0$. If $z_1 \neq 0$, or if $z_1 = 0$ and $|z_3| > |z_2|$, then $|z_2|^2 < |z_1|^2 + |z_3|^2$, and $[z_1, z_2, z_3]$ is not in B_2, while if $z_1 = 0$ and $|z_3| = |z_2|$, then $[z_1, z_2, z_3] = [0, 1, z_3/z_2] = p_{1z}$, $z = z_3/z_2$.

We must prove still that F is regular at the points p_{iz}. To do this, it is sufficient to show that the $7 \times 6 = 42$ partial derivatives of the components of F, considered as mapping M^6 into E^7, form a matrix of rank 4 at each such point. For this proves that the tangent space to M^6 at any such point is mapped into a space of dimension 4 in E^7 by F; but the tangent space to M^4 is clearly mapped into the same space in E^7. It is of course sufficient to find a submatrix of rank 4.

For points of M^6 representing points p_{iz}, we choose points with $N = 1$, obtained from (8) by multiplying by a real positive factor. Now

$$p_{1z} = (0, 0, x_2, 0, x_3, y_3), \qquad x_2^2 = x_3^2 + y_3^2 = \tfrac{1}{2},$$
$$p_{2z} = (x_1, y_1, 0, 0, x_3, 0), \qquad x_3^2 = x_1^2 + y_1^2 = \tfrac{1}{2},$$
$$p_{3z} = (x_1, 0, x_2, y_2, 0, 0), \qquad x_1^2 = x_2^2 + y_2^2 = \tfrac{1}{2}.$$

The corresponding values of F are:

$$\begin{array}{llllllll}
\text{at } p_{1z}: & (x_2x_3, & -x_2y_3, & 0, & 0, & 0, & 0, & -x_2^2 \quad), \\
\text{at } p_{2z}: & (\ 0, & 0, & x_1x_3, & -y_1x_3, & 0, & 0, & x_1^2 + y_1^2 \quad), \\
\text{at } p_{3z}: & (\ 0, & 0, & 0, & 0, & x_1x_2, & -x_1y_2, & x_1^2 - x_2^2 - y_2^2).
\end{array}$$

Consider first a point p_{1z}. Form F_1 from F by omitting the third and fourth of the seven variables. Then since $N = 1$ and

$$-2x_2^2x_3 + x_3 = 2x_2^2y_3 - y_3 = 0$$

at the point under consideration, differentiating (6) gives the following matrix for $\partial F_1/\partial x_1, \cdots, \partial F_1/\partial y_2$:

$$\begin{Vmatrix} 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 2x_2(x_2^2 - 1) \\ y_3 & x_3 & 0 & 0 & -2y_2 \end{Vmatrix}.$$

This is obviously of rank 4. Similarly, at p_{2z}, let F_2 omit the fifth and sixth variables from F. Then the matrix of $\partial F_2/\partial x_2, \cdots, \partial F_2/\partial y_3$ is

$$\begin{Vmatrix} x_3 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3(x_1^2 + x_1^2) \\ 0 & 0 & y_1 & x_1 & 0 \end{Vmatrix},$$

again of rank 4. At p_{3z}, letting F_3 omit the first and second variables, the matrix of $\partial F_3/\partial x_1, \partial F_3/\partial y_1, \partial F_3/\partial x_3, \partial F_3/\partial y_3$ is

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 2x_1 \\ 0 & 0 & y_2 & x_2 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \end{Vmatrix},$$

again of rank 4. This completes the proof that F is regular.

To show that F is one-one, take any point

$$q = (w_1, w_2, w_3, \gamma) = (u_1, v_1, \cdots, v_3, \gamma) = F(p)$$

for some $p \in M^4$; we shall show that $p = [z_1, z_2, z_3]$ is uniquely determined, i.e. the z_i are determined up to a complex factor.

Suppose first that $w_3 = 0$, $\gamma = 0$. Then one of z_1, z_2 is 0, and since $|z_1|^2 - |z_2|^2 = 0$, both are 0; hence p is unique.

Suppose next that $w_3 = 0$, $\gamma \neq 0$. If $\gamma > 0$, then $z_2 = 0$, $|z_1| > 0$ (since $N > 0$). Since $z_1 \neq 0$, we can normalize p to $[1, 0, z_3]$. Now

$$\gamma = \frac{|z_1|^2}{N} = \frac{1}{N}, \qquad w_2 = \frac{z_3\bar{z}_1}{N} = \gamma z_3;$$

hence $z_3 = w_2/\gamma$ is determined. If $\gamma < 0$, then $z_1 = 0$, $z_2 \neq 0$, etc.

Suppose now that $w_3 \neq 0$; then $z_1 \neq 0$, $z_2 \neq 0$. In this case we normalize so that

$$N = 1, \qquad z_2 = x_2 > 0.$$

If first $w_2 \neq 0$, then

$$w_1 w_3 = x_2^2 z_1 \bar{z}_3 = x_2^2 \bar{w}_2, \qquad x_2^2 = \frac{w_1 w_3}{\bar{w}_2},$$

determining x_2. Now

$$z_3 = \frac{\bar{w}_1}{x_2}, \qquad z_1 = \frac{w_3}{x_2},$$

so that p is determined.

Suppose finally that $w_3 \neq 0$, $w_2 = 0$. Normalize as before. Now $z_3 = 0$; we find, from $w_3 = x_2 z_1$, $\gamma = |z_1|^2 - x_2^2$,

$$|z_1|^2 = \frac{|w_3|^2}{x_2^2} = \gamma + x_2^2,$$

$$x_2^2 = \tfrac{1}{2}[-\gamma \pm (\gamma^2 + 4|w_3|^2)^{1/2}].$$

Since the square root is $> |\gamma|$, the minus sign is impossible, and x_2^2 and hence $z_2 = x_2$ is uniquely determined, as is $z_1 = w_3/x_2$. This completes the proof.

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