Generalized Poincare's Conjecture in Dimensions Greater Than Four

Stephen Smale

The Annals of Mathematics, Second Series, Volume 74, Issue 2 (Sep., 1961), 391-406.

Your use of the JSTOR database indicates your acceptance of JSTOR's Terms and Conditions of Use. A copy of JSTOR's Terms and Conditions of Use is available at http://www.jstor.org/about/terms.html, by contacting JSTOR at jstor-info@umich.edu, or by calling JSTOR at (888)388-3574, (734)998-9101 or (FAX) (734)998-9113. No part of a JSTOR transmission may be copied, downloaded, stored, further transmitted, transferred, distributed, altered, or otherwise used, in any form or by any means, except: (1) one stored electronic and one paper copy of any article solely for your personal, non-commercial use, or (2) with prior written permission of JSTOR and the publisher of the article or other text.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The Annals of Mathematics is published by The Annals of Mathematics. Please contact the publisher for further permissions regarding the use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/annals.html.

The Annals of Mathematics
©1961 The Annals of Mathematics

JSTOR and the JSTOR logo are trademarks of JSTOR, and are Registered in the U.S. Patent and Trademark Office. For more information on JSTOR contact jstor-info@umich.edu.

©1999 JSTOR

http://www.jstor.org/
Sun Oct 3 18:30:13 1999

ANNALS OF MATHEMATICS
Vol. 74, No. 2, September, 1961
Printed in Japan

GENERALIZED POINCARÉ'S CONJECTURE IN DIMENSIONS GREATER THAN FOUR

BY STEPHEN SMALE*

(Received October 11, 1960)
(Revised March 27, 1961)

Poincaré has posed the problem as to whether every simply connected closed 3-manifold (triangulated) is homeomorphic to the 3-sphere, see [18] for example. This problem, still open, is usually called Poincaré's conjecture. The generalized Poincaré conjecture (see [11] or [28] for example) says that every closed n-manifold which has the homotopy type of the n-sphere S^n is homeomorphic to the n-sphere. One object of this paper is to prove that this is indeed the case if $n \geqq 5$ (for differentiable manifolds in the following theorem and combinatorial manifolds in Theorem B).

THEOREM A. *Let M^n be a closed C^∞ manifold which has the homotopy type of S^n, $n \geqq 5$. Then M^n is homeomorphic to S^n.*

Theorem A and many of the other theorems of this paper were announced in [20]. This work is written from the point of view of differential topology, but we are also able to obtain the combinatorial version of Theorem A.

THEOREM B. *Let M^n be a combinatorial manifold which has the homotopy of S^n, $n \geqq 5$. Then M^n is homeomorphic to S^n.*

J. Stallings has obtained a proof of Theorem B (and hence Theorem A) for $n \geqq 7$ using different methods (*Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc., 66 (1960), 485–488).

The basic theorems of this paper, Theorems C and I below, are much stronger than Theorem A.

A *nice* function f on a closed C^∞ manifold is a C^∞ function with non-degenerate critical points and, at each critical point β, $f(\beta)$ equals the index of β. These functions were studied in [21].

THEOREM C. *Let M^n be a closed C^∞ manifold which is $(m-1)$-connected, and $n \geqq 2m$, $(n, m) \neq (4, 2)$. Then there is a nice function f on M with type numbers satisfying $M_0 = M_n = 1$ and $M_i = 0$ for $0 < i < m$, $n - m < i < n$.*

Theorem C can be interpreted as stating that a cellular structure can be imposed on M^n with one 0-cell, one n-cell and no cells in the range $0 < i < m$, $n - m < i < n$. We will give some implications of Theorem C.

*The author is an Alfred P. Sloan Fellow.

First, by letting $m = 1$ in Theorem C, we obtain a recent theorem of M. Morse [13].

THEOREM D. *Let M^n be a closed connected C^∞ manifold. There exists a (nice) non-degenerate function on M with just one local maximum and one local minimum.*

In § 1, the handlebodies, elements of $\mathcal{H}(n, k, s)$ are defined. Roughly speaking if $H \in \mathcal{H}(n, k, s)$, then H is defined by attaching s-disks, k in number, to the n-disk and "thickening" them. By taking $n = 2m + 1$ in Theorem C, we will prove the following theorem, which in the case of 3-dimensional manifolds gives the well known Heegard decomposition.

THEOREM F. *Let M be a closed C^∞ $(2m + 1)$-manifold which is $(m-1)$-connected. Then $M = H \cup H'$, $H \cap H' = \partial H = \partial H'$ where $H, H' \in \mathcal{H}(2m + 1, k, m)$ are handlebodies (∂V means the boundary of the manifold V).*

By taking $n = 2m$ in Theorem C, we will get the following.

THEOREM G. *Let M^{2m} be a closed $(m-1)$-connected C^∞ manifold, $m \neq 2$. Then there is a nice function on M whose type numbers equal the corresponding Betti numbers of M. Furthermore M, with the interior of a $2m$-disk deleted, is a handlebody, an element of $\mathcal{H}(2m, k, m)$ where k is the m^{th} Betti number of M.*

Note that the first part of Theorem G is an immediate consequence of the Morse relation that the Euler characteristic is the alternating sum of the type numbers [12], and Theorem C.

The following is a special case of Theorem G.

THEOREM H. *Let M^{2m} be a closed C^∞ manifold $m \neq 2$ of the homotopy type of S^{2m}. Then there exists on M a non-degenerate function with one maximum, one minimum, and no other critical point. Thus M is the union of two $2m$-disks whose intersection is a submanifold of M, diffeomorphic to S^{2m-1}.*

Theorem H implies the part of Theorem A for even dimensional homotopy spheres.

Two closed C^∞ oriented n-dimensional manifolds M_1 and M_2 are *J-equivalent* (according to Thom, see [25] or [10]) if there exists an oriented manifold V with ∂V diffeomorphic to the disjoint union of M_1 and $-M_2$, and each M_i is a deformation retract of V.

THEOREM I. *Let M_1 and M_2 be $(m-1)$-connected oriented closed C^∞ $(2m+1)$-dimensional manifolds which are J-equivalent, $m \neq 1$. Then M_1 and M_2 are diffeomorphic.*

We obtain an orientation preserving diffeomorphism. If one takes M_1 and M_2 J-equivalent disregarding orientation, one finds that M_1 and M_2 are diffeomorphic.

In studying manifolds under the relation of J-equivalence, one can use the methods of cobordism and homotopy theory, both of which are fairly well developed. The importance of Theorem I is that it reduces diffeomorphism problems to J-equivalence problems for a certain class of manifolds. It is an open question as to whether arbitrary J-equivalent manifolds are diffeomorphic (see [10, Problem 5])(Since this was written, Milnor has found a counter-example).

A short argument of Milnor [10, p. 33] using Mazur's theorem [7] applied to Theorem I yields the odd dimensional part of Theorem A. In fact it implies that, if M^{2m+1} is a homotopy sphere, $m \neq 1$, then M^{2m+1} minus a point is diffeomorphic to euclidean $(2m+1)$-space (see also [9, p. 440]).

Milnor [10] has defined a group $\mathcal{H}^n$ of C^∞ homotopy n-spheres under the relation of J-equivalence. From Theorems A and I, and the work of Milnor [10] and Kervaire [5], the following is an immediate consequence.

THEOREM J. *If n is odd, $n \neq 3$, $\mathcal{H}^n$ is the group of classes of all differentiable structures on S^n under the equivalence of diffeomorphism. For n odd there are a finite number of differentiable structures on S^n. For example*:

n	3	5	7	9	11	13	15
Number of Differentiable Structures on S^n	0	0	28	8	992	3	16256

Previously it was known that there are a countable number of differentiable structures on S^n for all n (Thom), see also [9, p. 442]; and unique structures on S^n for $n \leqq 3$ (e.g., Munkres [14]). Milnor [8] has also established lower bounds for the number of differentiable structures on S^n for several values of n.

A group Γ^n has been defined by Thom [24] (see also Munkres [14] and Milnor [9]). This is the group of all diffeomorphisms of S^{n-1} modulo those which can be extended to the n-disk. A group A^n has been studied by Milnor as those structures on the n-sphere which, minus a point, are diffeomorphic to euclidean space [9]. The group Γ^n can be interpreted (by Thom [22] or Munkres [14]) as the group of differentiable structures on S^n which admit a C^∞ function with the non-degenerate critical points, and hence one has the inclusion map $i\colon \Gamma^n \to A^n$ defined. Also, by taking J-equivalence classes, one gets a map $p\colon A^n \to \mathcal{H}^n$.

THEOREM K. *With notation as in the preceding paragraph, the following sequences are exact*:

(a) $A^n \xrightarrow{p} \mathcal{H}^n \longrightarrow 0$, $\quad n \neq 3, 4$

(b) $\Gamma^n \xrightarrow{i} A^n \longrightarrow 0$, $\quad n$ even $\neq 4$

(c) $0 \longrightarrow A^n \xrightarrow{p} \mathcal{H}^n$, $\quad n$ odd $\neq 3$.

Hence, if n is even, $n \neq 4$, $\Gamma^n = A^n$ and, if n is odd $\neq 3$, $A^n = \mathcal{H}^n$.

Here (a) follows from Theorem A, (b) from Theorem H, and (c) from Theorem I.

Kervaire [4] has also obtained the following result.

THEOREM L. *There exists a manifold with no differentiable structure at all.*

Take the manifold W_0 of Theorem 4.1 of Milnor [10] for $k = 3$. Milnor shows ∂W_0 is a homotopy sphere. By Theorem A, ∂W_0 is homeomorphic to S^{11}. We can attach a 12-disk to W_0 by a homeomorphism of the boundary onto ∂W_0 to obtain a closed 12 dimensional manifold M. Starting with a triangulation of W_0, one can easily obtain a triangulation of M. If M possessed a differentiable structure it would be almost parallelizable, since the obstruction to almost parallelizability lies in $H^6(M, \pi_5(\mathrm{SO}(12))) = 0$. But the index of M is 8 and hence by Lemma 3.7 of [10] M cannot possess any differentiable structure. Using Bott's results on the homotopy groups of Lie groups [1], one can similarly obtain manifolds of arbitrarily high dimension without a differentiable structure.

THEOREM M. *Let C^{2m} be a contractible manifold, $m \neq 2$, whose boundary is simply connected. Then C^{2m} is diffeomorphic to the $2m$-disk. This implies that differentiable structures on disks of dimension $2m$, $m \neq 2$, are unique. Also the closure of the bounded component C of a C^∞ imbedded $(2m-1)$-sphere in euclidean $2m$-space, $m \neq 2$, is diffeomorphic to a disk.*

For these dimensions, the last statement of Theorem M is a strong version of the Schoenflies problem for the differentiable case. Mazur's theorem [7] had already implied C was homeomorphic to the $2m$-disk.

Theorem M is proved as follows from Theorems C and I. By Poincaré duality and the homology sequence of the pair $(C, \partial C)$, it follows that ∂C is a homotopy sphere and J-equivalent to zero since it bounds C. By Theorem I, then, ∂C is diffeomorphic to S^n. Now attach to C^{2m} a $2m$-disk by a diffeomorphism of the boundary to obtain a differentiable manifold V. One shows easily that V is a homotopy sphere and, hence by Theorem H, V is the union of two $2m$-disks. Since any two $2m$ sub-disks of V are

equivalent under a diffeomorphism of V (for example see Palais [17]), the original $C^{2m} \subset V$ must already have been diffeomorphic to the standard $2m$-disk.

To prove Theorem B, note that $V = (M$ with the interior of a simplex deleted) is a contractible manifold, and hence possesses a differentiable structure [Munkres 15]. The double W of V is a differentiable manifold which has the homotopy type of a sphere. Hence by Theorem A, W is a topological sphere. Then according to Mazur [7], ∂V, being a differentiable submanifold and a topological sphere, divides W into two topological cells. Thus V is topologically a cell and M a topological sphere.

THEOREM N. *Let C^{2m}, $m \neq 2$, be a contractible combinatorial manifold whose boundary is simply connected. Then C^{2m} is combinatorially equivalent to a simplex. Hence the Hauptvermutung* (see [11]) *holds for combinatorial manifolds which are closed cells in these dimensions.*

To prove Theorem N, one first applies a recent result of M. W. Hirsch [3] to obtain a compatible differentiable structure on C^{2m}. By Theorem M, this differentiable structure is diffeomorphic to the $2m$-disk D^{2m}. Since the standard $2m$-simplex σ^{2m} is a C^1 triangulation of D^{2m}, Whitehead's theorem [27] applies to yield that C^{2m} must be combinatorially equivalent to σ^{2m}.

Milnor first pointed out that the following theorem was a consequence of this theory.

THEOREM O. *Let M^{2m}, $m \neq 2$, be a combinatorial manifold which has the same homotopy type as S^{2m}. Then M^{2m} is combinatorially equivalent to S^{2m}. Hence, in these dimensions, the Hauptvermutung holds for spheres.*

For even dimensions greater than four, Theorems N and O improve recent results of Gluck [2].

Theorem O is proved by applying Theorem N to the complement of the interior of a simplex of M^{2m}.

Our program is the following. We introduce handlebodies, and then prove "the handlebody theorem" and a variant. These are used together with a theorem on the existence of "nice functions" from [21] to prove Theorems C and I, the basic theorems of the paper. After that, it remains only to finish the proof of Theorems F and G of the Introduction.

The proofs of Theorems C and I are similar. Although they use a fair amount of the technique of differential topology, they are, in a certain sense, elementary. It is in their application that we use many recent results.

A slightly different version of this work was mimeographed in May 1960. In this paper J. Stallings pointed out a gap in the proof of the handlebody theorem (for the case $s=1$). This gap happened not to affect our main theorems.

Everything will be considered from the C^∞ point of view. All imbeddings will be C^∞. A differentiable isotopy is a homotopy of imbeddings with continuous differential.

$$E^n = \{x = (x_1, \cdots, x_n)\},\ \| x \| = \left(\sum_{i=1}^n x_i^2\right)^{1/2},$$
$$D^n = \{x \in E^n \mid \| x \| \leqq 1\},\ \partial D^n = S^{n-1} = \{x \in E^n \mid \| x \| = 1\};$$
$$D_i^n \text{ etc. are copies of } D^n.$$

A. Wallace's recent article [26] is related to some of this paper.

1. Let M^n be a compact manifold, Q a component of ∂M and

$$f_i\colon \partial D_i^s \times D_i^{n-s} \to Q,\ i = 1, \cdots, k$$

imbeddings with disjoint images, $s \geqq 0$, $n \geqq s$. We define a new compact C^∞ manifold $V = \chi(M, Q; f_1, \cdots, f_k; s)$ as follows. The underlying topological space of V is obtained from M, and the $D_i^s \times D_i^{k-s}$ by identifying points which correspond under some f_i. The manifold thus defined has a natural differentiable structure except along corners $\partial D_i^s \times \partial D_i^{n-s}$ for each i. The differentiable structure we put on V is obtained by the process of "straightening the angle" along these corners. This is carried out in Milnor [10] for the case of the product of manifolds W_1 and W_2 with a corner along $\partial W_1 \times \partial W_2$. Since the local situation for the two cases is essentially the same, his construction applies to give a differentiable structure on V. He shows that this structure is well-defined up to diffeomorphism.

If $Q = \partial M$ we omit it from the notation $\chi(M, Q; f_1, \cdots, f_k; s)$, and we sometimes also omit the s. We can consider the "handle" $D_i^s \times D_i^{n-s} \subset V$ as differentiably imbedded.

The next lemma is a consequence of the definition.

(1.1) LEMMA. *Let $f_i\colon \partial D_i^s \times D_i^{n-s} \to Q$ and $f_i'\colon \partial D_i^s \times D_i^{n-s} \to Q$, $i=1, \cdots, k$ be two sets of imbeddings each with disjoint images, Q, M as above. Then $\chi(M, Q; f_1, \cdots, f_k; s)$ and $\chi(M, Q; f_1', \cdots, f_k'; s)$ are diffeomorphic if*

(a) *there is a diffeomorphism $h\colon M \to M$ such that $f_i' = hf_i$, $i = 1, \cdots, k$; or*

(b) *there exist diffeomorphisms $h_i\colon D^s \times D^{n-s} \to D^s \times D^{n-s}$ such that $f_i' = f_i h_i$, $i = 1, \cdots, k$; or*

(c) *the f_i' are permutations of the f_i.*

If V is the manifold $\chi(M, Q; f_1, \cdots, f_k; s)$, we say $\sigma = (M, Q; f_1, \cdots, f_k; s)$

is a presentation of V.

A *handlebody* is a manifold which has a presentation of the form $(D^n; f_1, \cdots, f_k; s)$. Fixing n, k, s the set of all handlebodies is denoted by $\mathcal{H}(n, k, s)$. For example, $\mathcal{H}(n, k, 0)$ consists of one element, the disjoint union of $(k+1)$ n-disks; and one can show $\mathcal{H}(2, 1, 1)$ consists of $S^1 \times I$ and the Möbius strip, and $\mathcal{H}(3, k, 1)$ consists of the classical handlebodies [19; Henkelkörper], orientable and non-orientable, or at least differentiable analogues of them. The following is one of the main theorems used in the proof of Theorem C. An analogue in § 5 is used for Theorem I.

(1.2) HANDLEBODY THEOREM. *Let* $n \geqq 2s + 2$ *and, if* $s = 1$, $n \geqq 5$; *let* $H \in \mathcal{H}(n, k, s)$, $V = \chi(H; f_1, \cdots, f_r; s + 1)$, *and* $\pi_s(V) = 0$. *Also, if* $s = 1$, *assume* $\pi_1(\chi(H; f_1, \cdots, f_{r-k}; 2)) = 1$. *Then* $V \in \mathcal{H}(n, r - k, s + 1)$. (We do not know if the special assumption for $s = 1$ is necessary.)

The next three sections are devoted to a proof of (1.2).

2. Let $G_r = G_r(s)$ be the free group on r generators $D_1, \cdots, D_r$ if $s=1$, and the free abelian group on r generators $D_1, \cdots, D_r$ if $s > 1$. If $\sigma = (M, Q; f_1, \cdots, f_r; s + 1)$ is a presentation of a manifold V, define a homomorphism $f_\sigma: G_r \to \pi_s(Q)$ by $f_\sigma(D_i) = \varphi_i$, where $\varphi_i \in \pi_s(Q)$ is the homotopy class of $\bar{f}_i: \partial D^{s+1} \times 0 \to Q$, the restriction of f_i. To take care of base points in case $\pi_1(Q) \neq 1$, we will fix $x_0 \in \partial D^{s+1} \times 0$, $y_0 \in Q$, Let U be some cell neighborhood of y_0 in Q, and assume $\bar{f}_i(x_0) \in U$. We say that the homomorphism f_σ is *induced* by the presentation σ.

Suppose now that $F: G_r \to \pi_s(Q)$ is a homomorphism where Q is a component of the boundary of a compact n-manifold M. Then we say that a manifold V realizes F if some presentation of V induces F. Manifolds realizing a given homomorphism are not necessarily unique.

The following theorem is the goal of this section.

(2.1) THEOREM. *Let* $n \geqq 2s + 2$, *and if* $s = 1$, $n \geqq 5$; *let* $\sigma = (M, Q; f_1, \cdots, f_r; s + 1)$ *be a presentation of a manifold* V, *and assume* $\pi_1(Q) = 1$ *if* $n = 2s + 2$. *Then for any automorphism* $\alpha: G_r \to G_r$, V *realizes* $f_\sigma\alpha$.

Our proof of (2.1) is valid for $s = 1$, but we have application for the theorem only for $s > 1$. For the proof we will need some lemmas.

(2.2) LEMMA. *Let* Q *be a component of the boundary of a compact manifold* M^n *and* $f_1: \partial D^s \times D^{n-s} \to Q$ *an imbedding. Let* $\bar{f}_2: \partial D^s \times 0 \to Q$ *be an imbedding, differentiably isotopic in* Q *to the restriction* $\bar{f}_1$ *of* f_1 *to* $\partial D^s \times 0$. *Then there exists an imbedding* $f_2: \partial D^s \times D^{n-s} \to Q$ *extending* $\bar{f}_2$ *and a diffeomorphism* $h: M \to M$ *such that* $hf_2 = f_1$.

PROOF. Let $\bar{f}_t: \partial D^s \times 0 \to Q$, $1 \leqq t \leqq 2$, be a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Then by the covering homotopy property for spaces of differentiable imbeddings (see Thom [23] and R. Palais, Comment. Math. Helv. 34 (1960)), there is a differentiable isotopy $F_t: \partial D^s \times D^{n-s} \to Q$, $1 \leqq t \leqq 2$, with $F_1 = f_1$ and F_t restricted to $\partial D^s \times 0 = \bar{f}_t$. Now by applying this theorem again, we obtain a differentiable isotopy $G_t: M \to M$, $1 \leqq t \leqq 2$, with G_1 equal the identity, and G_t restricted to image F_1 equal $F_t F_1^{-1}$. Then taking $h = G_2^{-1}$, F_2 satisfies the requirements of f_2 of (2.2); i.e., $hf_2 = G_2^{-1}F_2 = F_1F_2^{-1}F_2 = f_1$.

(2.3) THEOREM (H. Whitney, W.T. Wu). *Let $n \geqq \max(2k+1, 4)$ and $f, g: M^k \to X^n$ be two imbeddings, M closed, M connected and X simply connected if $n = 2k+1$. Then, if f and g are homotopic, they are differentiably isotopic.*

Whitney [29] proved (2.3) for the case $n \geqq 2k+2$. W.T. Wu [30] (using methods of Whitney) proved it where X^n was euclidean space, $n = 2k+1$. His proof also yields (2.3) as stated.

(2.4) LEMMA. *Let Q be a component of the boundary of a compact manifold M^n, $n \geqq 2s+2$ and if $s = 1$, $n \geqq 5$, and $\pi_1(Q) = 1$ if $n = 2s+2$. Let $f_1: \partial D^{s+1} \times D^{n-s-1} \to Q$ be an imbedding, and $\bar{f}_2: \partial D^{s+1} \times 0 \to Q$ an imbedding homotopic in Q to $\bar{f}_1$, the restriction of f_1 to $\partial D^{s+1} \times 0$. Then there exists an imbedding $f_2: \partial D^{s+1} \times D^{n-s-1} \to Q$ extending $\bar{f}_2$ such that $\chi(M, Q; f_2)$ is diffeomorphic to $\chi(M, Q; f_1)$.*

PROOF. By (2.3), there exists a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Apply (2.2) to get $f_2: \partial D^{s+1} \times D^{n-s-1} \to Q$ extending $\bar{f}_2$, and a diffeomorphism $h: M \to M$ with $hf_2 = f_1$. Application of (1.1) yields the desired conclusion.

See [16] for the following.

(2.5) LEMMA (Nielson). *Let G be a free group on r-generators $\{D_1, \cdots, D_r\}$, and $\mathcal{A}$ the group of automorphisms of G. Then $\mathcal{A}$ is generated by the following automorphisms:*

$$R: D_1 \to D_1^{-1}, \quad D_i \to D_i \qquad i > 1$$

$$T_i: D_1 \to D_i, \quad D_i \to D_1, \quad D_j \to D_j \qquad j \neq 1, j \neq i, i = 2, \cdots, r$$

$$S: D_1 \to D_1 D_2, \quad D_i \to D_i, \qquad i > 1.$$

The same is true for the free abelian case (well-known).

It is sufficient to prove (2.1) with α replaced by the generators of $\mathcal{A}$ of (2.5).

First take $\alpha = R$. Let $h: D^{s+1} \times D^{n-s-1} \to D^{s+1} \times D^{n-s-1}$ be defined by

$h(x, y) = (r, x, y)$ where $r: D^{s+1} \to D^{s+1}$ is a reflection through an equatorial s-plane. Then let $f_i' = f_i h$. If $\sigma' = (M, Q; f_1', f_2, \cdots, f_r; s+1)$, $\chi(\sigma')$ is diffeomorphic to V by (1.1). On the other hand $\chi(\sigma')$ realizes $f_{\sigma'} = f_\sigma \alpha$.

The case $\alpha = T_i$ follows immediately from (1.1). So now we proceed with the proof of (2.1) with $\alpha = S$.

Define V_1 to be the manifold $\chi(M, Q; f_2, \cdots, f_r; s+1)$ and let $Q_1 \subset \partial V_1$ be $Q_1 = \partial V_1 - (\partial M - Q)$. Let $\varphi_i \in \pi_s(Q)$, $i = 1, \cdots, r$ denote the homotopy class of $f_i: \partial D_i^{s+1} \times 0 \to Q$, the restriction of f_i. Let $\gamma: \pi_s(Q \cap Q_1) \to \pi_s(Q)$ and $\beta: \pi_s(Q \cap Q_1) \to \pi_s(Q_1)$ be the homomorphisms induced by the respective inclusions.

(2.6) LEMMA. *With notations and conditions as above,* $\varphi_2 \in \gamma \operatorname{Ker} \beta$.

PROOF. Let $q \in \partial D_2^{n-s-1}$ and $\psi: \partial D_2^{s+1} \times q \to Q \cap Q_1$ be the restriction of f_2. Denote by $\bar{\psi} \in \pi_s(Q \cap Q_1)$ the homotopy class of ψ. Since ψ and $\bar{f}_2$ are homotopic in Q, $\gamma\bar{\psi} = \varphi_2$. On the other hand $\beta\bar{\psi} = 0$, thus proving (2.6).

By (2.6), let $\bar{\psi} \in \pi_s(Q \cap Q_1)$ with $\gamma\bar{\psi} = \varphi_2$ and $\beta\bar{\psi} = 0$. Let $g = y + \bar{\psi}$ (or $y\bar{\psi}$ in case $s = 1$; our terminology assumes $s > 1$) where $y \in \pi_s(Q \cap Q_1)$ is the homotopy class of $\bar{f}_1: \partial D_1^{s+1} \times 0 \to Q \cap Q_1$. Let $\bar{g}: \partial D^{s+1} \times 0 \to Q \cap Q_1$ be an imbedding realizing g (see [29]).

If $n = 2s + 2$, then from the fact that $\pi_1(Q) = 1$, it follows that also $\pi_1(Q_1) = 1$. Then since $\bar{g}$ and $\bar{f}_1$ are homotopic in Q_1, i.e., $\beta g = \beta y$, (2.4) applies to yield an imbedding $e: \partial D^{s+1} \times D^{n-s-1} \to Q_1$ extending $\bar{g}$ such that $\chi(V_1, Q_1; e)$ and $\chi(V_1, Q_1; f_1)$ are diffeomorphic.

On one hand $V = \chi(V, Q; f_1, \cdots, f_r) = \chi(V_1, Q_1; f_1)$ and, on the other hand, $\chi(V, Q; e, f_2, \cdots, f_r) = \chi(V_1, Q_1; e)$, so by the preceding statement, V and $\chi(V, Q; e, f_2, \cdots, f_r)$ are diffeomorphic. Since $\gamma g = g_1 + g_2, f_\sigma\alpha(D_1) = f_\sigma(D_1 + D_2) = g_1 + g_2, f_\sigma'(D_1) = gD_1 = g_1 + g_2, f_\sigma\alpha = f_{\sigma'}$, where $\sigma' = (V, Q; e, f_2, \cdots, f_r)$. This proves (2.1).

3. The goal of this section is to prove the following theorem.

(3.1) THEOREM. *Let* $n \geqq 2s + 2$ *and, if* $s = 1$, $n \geqq 5$. *Suppose* $H \in \mathcal{H}(n, k, s)$. *Then given* $r \geqq k$, *there exists an epimorphism* $g: G_r \to \pi_s(H)$ *such that every realization of* g *is in* $\mathcal{H}(n, r - k, s + 1)$.

For the proof of 3.1, we need some lemmas.

(3.2) LEMMA. *If* $\mathcal{H}(n, k, s)$ *then* $\pi_s(H)$ *is*

(a) *a set of* $k + 1$ *elements if* $s = 0$,

(b) *a free group on* k *generators if* $s = 1$,

(c) *a free abelian group on* k *generators if* $s > 1$.

Furthermore if $n \geqq 2s + 2$, *then* $\pi_i(\partial H) \to \pi_i(H)$ *is an isomorphism for* $i \leqq s$.

PROOF. We can assume $s > 0$ since, if $s = 0$, H is a set of n-disks $k+1$

in number. Then H has as a deformation retract in an obvious way the wedge of k s-spheres. Thus (b) and (c) are true. For the last statement of (3.2), from the exact homotopy sequence of the pair $(H, \partial H)$, it is sufficient to show that $\pi_i(H, \partial H) = 0$, $i \leqq s + 1$.

Thus let $f: (D^i, \partial D^i) \to (H, \partial H)$ be a given continuous map with $i \leqq s + 1$. We want to construct a homotopy $f_r: (D^i, \partial D^i) \to (H, \partial H)$ with $f_0 = f$ and $f_1(D^i) \subset \partial H$.

Let $f_1: (D^i, \partial D^i) \to (H, \partial H)$ be a differentiable approximation to f. Then by a radial projection from a point in D^n not in the image of f_1, f_1 is homotopic to a differentiable map $f_2: (D^i, \partial D^i) \to (H, \partial H)$ with the image of f_2 not intersecting the interior of $D^n \subset H$. Now for dimensional reasons f_2 can be approximated by a differentiable map $f_3: (D^i, \partial D^i) \to (H, \partial H)$ with the image of f_3 not intersecting any $D^s_i \times 0 \subset H$. Then by other projections, one for each i, f_3 is homotopic to a map $f_4: (D^i, \partial D^i) \to (H, \partial H)$ which sends all of D^i into ∂H. This shows $\pi_i(H, \partial H) = 0$, $i \leqq s + 1$, and proves (3.2).

If $\beta \in \pi_{s-1}(\mathrm{O}(n - s))$, let H_β be the $(n - s)$-cell bundle over S^s determined by β.

(3.3) LEMMA. *Suppose* $V = \chi(H_\beta; f; s + 1)$ *where* $\beta \in \pi_{s-1}(\mathrm{O}(n - s))$, $n \geqq 2s + 2$, *or if* $s = 1$, $n \geqq 5$. *Let also* $\pi_s(V) = 0$. *Then* V *is diffeomorphic to* D^n.

PROOF. The zero-cross-section $\sigma: S^s \to H_\beta$ is homotopic to zero, since $\pi_s(V) = 0$, and so is regularly homotopic in V to a standard s-sphere S^s_0 contained in a cell neighborhood by dimensional reasons [29]. Since a regular homotopy preserves the normal bundle structure, $\sigma(S^s)$ has a trivial normal bundle and thus $\beta = 0$. Hence H_β is diffeomorphic to the product of S^s and D^{n-s}.

Let $\sigma_1: S^s \to \partial H_\beta$ be a differentiable cross section and $\bar{f}: \partial D^{s+1} \times 0 \to \partial H_\beta$ the restriction of $f: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$. Then σ_1 and $\bar{f}$ are homotopic in ∂H_β (perhaps after changing f by a diffeomorphism of $D^{s+1} \times D^{n-s-1}$ which reverses orientation of $\partial D^{s+1} \times 0$) since $\pi_s(V) = 0$, and hence differentiably isotopic. Thus we can assume $\bar{f}$ and s_1 are the same.

Let f_ε be the restriction of f to $\partial D^{s+1} \times D^{n-s-1}_\varepsilon$ where D^{n-s-1}_ε denotes the disk $\{x \in D^{n-s-1} \mid \| x \| \leqq \varepsilon\}$, and $\varepsilon > 0$. Then the imbedding $g_\varepsilon: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ is differentiably isotopic to f where $g_\varepsilon(x, y) = f_\varepsilon r_\varepsilon(x, y)$ and $r_\varepsilon(x, y) = (x, \varepsilon y)$. Define $k_\varepsilon: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ by $p_x g_\varepsilon(x, y)$ where p_x: $g_\varepsilon(x \times D^{n-s-1}) \to F_x$ is projection into the fibre F_x of ∂H_β over $\sigma^{-1} g_\varepsilon(x, 0)$. If ε is small enough, k_ε is well-defined and an imbedding. In fact if ε is small enough, we can even suppose that for each x, k_ε maps $x \times D^{n-s-1}$ linearly onto image $k_\varepsilon \cap F_x$ where image $k_\varepsilon \cap F_x$ has a linear structure

induced from F_x.

It can be proved k_ε and g_ε are differentiably isotopic. (The referee has remarked that there is a theorem, Milnor's "tubular neighborhood theorem", which is useful in this connection and can indeed be used to make this proof clearer in general.)

We finish the proof of (3.3) as follows. Suppose V is as in (3.3) and $V' = \chi(H_\beta; f'; s+1)$, $\pi_\varepsilon(V') = 0$. It is sufficient to prove V and V' are diffeomorphic since it is clear that one can obtain D^n by choosing f' properly and using the fact that H_β is a product of S^s and D^{n-s}. From the previous paragraph, we can replace f and f' by k_ε and k'_ε with those properties listed. We can also suppose without loss of generality that the images of k_ε and k'_ε coincide. It is now sufficient to find a diffeomorphism h of H_β with $hf = f'$. For each x, define h on image $f \cap F_x$ to be the linear map which has this property. One can now easily extend h to all of H_β and thus we have finished the proof of (3.3).

Suppose now M_1^n and M_2^n are compact manifolds and $f_i\colon D^{n-1} \times i \to \partial M_i$ are imbeddings for $i = 1$ and 2. Then $\chi(M_1 \cup M_2; f_1 \cup f_2; 1)$ is a well defined manifold, where $f_1 \cup f_2\colon \partial D^1 \times D^{n-1} \to \partial M_1 \cup \partial M_2$ is defined by f_1 and f_2, the set of which, as the f_i vary, we denote by $M_1 + M_2$. (If we pay attention to orientation, we can restrict $M_1 + M_2$ to have but one element.) The following lemma is easily proved.

(3.4) LEMMA. *The set $M^n + D^n$ consists of one element, namely M^n.*

(3.5) LEMMA. *Suppose an imbedding $f\colon \partial D^s \times D^{n-s} \to \partial M^n$ is null-homotopic where M is a compact manifold, $n \geqq 2s + 2$ and, if $s = 1$, $n \geqq 5$. Then $\chi(M; f) \in M + H_\beta$ for some $\beta \in \pi_{s-1}(\mathrm{O}(n-s))$.*

PROOF OF (3.5). Let $\bar{f}\colon \partial D^s \times q \to \partial M$ be the restriction of f where q is a fixed point in ∂D^{n-s}. Then by dimensional reasons [29], $\bar{f}$ can be extended to an imbedding $\varphi\colon D^s \to \partial M$ where the image of φ intersects the image of f only on $\bar{f}$. Next let T be a tubular neighborhood of $\varphi(D^s)$ in M. This can be done so that T is a cell, $T \cup (D^s \times D^{n-s})$ is of the form H_β and $V \in M + H_\beta$. We leave the details to the reader.

To prove (3.1), let $H = \chi(D^n; f_1, \cdots, f_k; s)$. Then f_i defines a class $\bar{\gamma}_i \in \pi_s(H, D^n)$. Let $\gamma_i \in \pi_s(\partial H)$ be the image of γ_i under the inverse of the composition of the isomorphisms $\pi_s(\partial H) \to \pi_s(H) \to \pi_s(H, D^n)$ (using (3.2)). Define g of (3.1) by $gD_i = \gamma_i$, $i \leqq k$, and $gD_i = 0$, $i > k$. That g satisfies (3.1) follows by induction from the following lemma.

(3.6) LEMMA. $\chi(H; g_1; s+1) \in \mathcal{H}(n, k-1, s)$ *if the restriction of g_1 to $\partial D^{s+1} \times 0$ has homotopy class $\gamma_1 \in \pi_s(\partial H)$.*

Now (3.6) follows from (3.3), (3.4) and (3.5), and the fact that g_1 is dif-

ferentiably isotopic to g_1' whose image is in $\partial H_\beta \cap \partial H$, where H_β is defined by (3.5) and f_1.

4. We prove here (1.2). First suppose $s = 0$. Then $H \in \mathcal{H}(n, k, 0)$ is the disjoint union of n-disks, $k+1$ in number, and $V=\chi(H; f_1, \cdots, f_r; 1)$. Since $\pi_0(V) = 1$, there exists a permutation of $1, \cdots, r$, $i_1, \cdots, i_r$ such that $Y = \chi(H; f_{i_1}, \cdots, f_{i_k}; 1)$ is connected. By (3.4), Y is diffeomorphic to D^n. Hence $V = \chi(Y; f_{i_{k+1}}, \cdots, f_{i_r}; 1)$ is in $\mathcal{H}(n, r-k, 1)$.

Now consider the case $s = 1$. Choose, by (3.1), $g: G_k \to \pi_1(\partial H)$ such that every manifold derived from g is diffeomorphic to D^n. Let $Y = \chi(H; f_1, \cdots, f_{r-k})$. Then $\pi_1(Y)=1$ and by the argument of (3.2), $\pi_1(\partial Y) = 1$. Let $\bar{g}_i: \partial D^2 \times 0 \to \partial H$ be disjoint imbeddings realizing the classes $g(D_i) \in \pi_1(\partial H)$ which are disjoint from the images of all f_i, $i = 1, \cdots, k$. Then by (2.4) there exist imbeddings $g_1, \cdots, g_k: \partial D^2 \times D^{n-2} \to \partial H$ extending the $\bar{g}_i$ such that $V = \chi(Y; f_{r-k+1}, \cdots, f_r)$ and $\chi(Y; g_1, \cdots, g_k)$ are diffeomorphic. But

$$\begin{aligned}\chi(Y, g_1, \cdots, g_k) &= \chi(H; g_1, \cdots, g_k, f_1, \cdots, f_{r-k}) \\ &= \chi(D^n, f_1, \cdots, f_{r-k}) \in \mathcal{H}(n, r-k, 2) .\end{aligned}$$

Hence so does V.

For the case $s > 1$, we use an algebraic lemma.

(4.1) LEMMA. *If $f, g: G \to G'$ are epimorphisms where G and G' are finitely generated free abelian groups, then there exists an automorphism $\alpha: G \to G$ such that $f\alpha = g$.*

PROOF. Let G'' be a free abelian group of rank equal to $\operatorname{rank} G - \operatorname{rank} G'$, and let $p: G' + G'' \to G'$ be the projection. Then, identifying elements of G and $G' + G''$ under some isomorphism, it is sufficient to prove the existence of α for $g = p$. Since the groups are free, the following exact sequence splits

$$0 \longrightarrow f^{-1}(0) \longrightarrow G \xrightarrow{f} G' \longrightarrow 0 .$$

Let $h: G \to f^{-1}(0)$ be the corresponding projection and let $k: f^{-1}(0) \to G''$ be some isomorphism. Then $\alpha: G \to G' + G''$ defined by $f + kh$ satisfies the requirements of (4.1).

REMARK. Using Grusko's Theorem [6], one can also prove (4.1) when G and G' are free groups.

Now take $\sigma = (H; f_1, \cdots, f_r; s + 1)$ of (1.2) and $g: G_r \to \pi_s(\partial H)$ of (3.1). Since $\pi_s(V) = 0$, and $s > 1$, $f_\sigma: G_r \to \pi_s(\partial H)$ is an epimorphism. By (3.2) and (4.1) there is an automorphism $\alpha: G_r \to G_r$ such that $f_\sigma \alpha = g$. Then (2.1) implies that V is in $\mathcal{H}(n, r-k, s+1)$ using the main property of g.

5. The goal of this section is to prove the following analogue of (1.2).

(5.1) THEOREM. *Let $n \geqq 2s+2$, or if $s=1$, $n \geqq 5$, M^{n-1} be a simply connected, $(s-1)$-connected closed manifold and $\mathcal{H}_M(n, k, s)$ the set of all manifolds having presentations of the form $(M\times[0,1], M\times 1; f_1 \cdots, f_k; s)$. Now let $H \in \mathcal{H}_M(n, k, s)$, $Q = \partial H - M\times 0$, $V = \chi(H, Q; g_1, \cdots, g_r; s+1)$ and suppose $\pi_s(M\times 0)\to\pi_s(V)$ is an isomorphism. Also suppose if $s=1$, that $\pi_1(\chi(H, Q; g_1, \cdots, g_{r-k}; 2)) = 1$. Then $V \in \mathcal{H}_M(n, r-k, s+1)$.*

One can easily obtain (1.2) from (5.1) by taking for M, the $(n-1)$-sphere. The following lemma is easy, following (3.2).

(5.2) LEMMA. *With definitions and conditions as in* (5.1), $\pi_s(Q) = G_k$ *if* $s=1$, *and if* $s>1$, $\pi_s(Q) = \pi_s(M) + G_k$.

Let $p_1\colon \pi_s(Q) \to \pi_s(M)$, $p_2\colon \pi_s(Q) \to G_k$ be the respective projections.

(5.3) LEMMA. *With definitions and conditions as in* (5.1), *there exists a homomorphism* $g\colon G_r \to \pi_s(Q)$ *such that* $p_1 g$ *is trivial,* $p_2 g$ *is an epimorphism, and every realization of* g *is in* $\mathcal{H}_M(n, r-k, s+1)$, *each* $r \geqq k$.

The proof follows (3.1) closely.

We now prove (5.1). The cases $s=0$ and $s=1$ are proved similarly to these cases in the proof of (1.2). Suppose $s>1$. From the fact that $\pi_s(M\times 0)\to\pi_s(V)$ is an isomorphism, it follows that $p_1 f_\sigma$ is trivial and $p_2 f_\sigma$ is an epimorphism where $\sigma = (H, Q; g_1, \cdots, g_r, s+1)$. Then apply (4.1) to obtain an automorphism $\alpha\colon G_r \to G_r$ such that $p_2 f_\sigma \alpha = p_2 g$ where g is as in (5.3). Then $f_\sigma \alpha = g$, hence using (2.1), we obtain (5.1).

6. The goal of this section is to prove the following two theorems.

(6.1) THEOREM. *Suppose f is a C^∞ function on a compact manifold W with no critical points on $f^{-1}[-\varepsilon, \varepsilon] = N$ except k non-degenerate ones on $f^{-1}(0)$, all of index λ,* and $N \cap \partial W = \varnothing$. Then $f^{-1}[-\infty, \varepsilon]$ has a presentation of the form $(f^{-1}[-\infty, -\varepsilon], f^{-1}(-\varepsilon); f_1, \cdots, f_k; \lambda)$.

(6.2) THEOREM. *Let $(M, Q; f_1, \cdots, f_k; s)$ be a presentation of a manifold V, and g be a C^∞ function on M, regular, in a neighborhood of Q, and constant with its maximum value on Q. Then there exists a C^∞ function G on V which agrees with g outside a neighborhood of Q, is constant and regular on $\partial V - (\partial M - Q)$, and has exactly k new critical points, all non-degenerate, with the same value and with index s.*

SKETCH OF PROOF OF (6.1). Let β_i denote the critical points of f at level zero, $i = 1, \cdots, k$ with disjoint neighborhoods V_i. By a theorem of Morse [13] we can assume V_i has a coordinate system $x=(x_1, \cdots, x_n)$ such that for $\| x \| \leqq \delta$, some $\delta > 0$, $f(x) = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$. Let E_1 be the $(x_1, \cdots, x_\lambda)$ plane of V_i and E_2 the $(x_{\lambda+1}, \cdots, x_n)$ plane. Then for $\varepsilon_1 > 0$ sufficiently small $E_1 \cap f^{-1}[-\varepsilon_1, \varepsilon_1]$ is diffeomorphic to D^λ. A sufficiently

small tubular neighborhood T of E_1 will have the property that $T' = T \cap f^{-1}[-\varepsilon_1, \varepsilon_1]$ is diffeomorphic to $D^\lambda \times D^{n-\lambda}$ with $T \cap f^{-1}(-\varepsilon_1)$ corresponding to $\partial D^\lambda \times D^{n-\lambda}$.

As we pass from $f^{-1}[-\infty, -\varepsilon_1]$ to $f^{-1}[-\infty, \varepsilon_1]$, it happens that one such T' is added for each i, together with a tubular neighborhood of $f^{-1}(-\varepsilon_1)$ so that $f^{-1}[-\infty, \varepsilon_1]$ is diffeomorphic to a manifold of the form $\chi(f^{-1}[-\infty, -\varepsilon_1], f^{-1}(-\varepsilon_1); f_1, \cdots, f_k; \lambda)$. Since there are no critical points between $-\varepsilon$ and $-\varepsilon_1$, ε_1 and ε, ε_1 can be replaced by ε in the preceding statement thus proving (6.1).

Theorem (6.2) is roughly a converse of (6.1) and a sketch of the proof can be constructed similarly.

7. In this section we prove Theorems C and I of the Introduction.

The following theorem was proved in [21].

(7.1) THEOREM. *Let V^n be a C^∞ compact manifold with ∂V the disjoint union of V_1 and V_2, each V_i closed in ∂V. Then there exists a C^∞ function f on V with non-degenerate critical points, regular on ∂V, $f(V_1) = -(1/2), f(V_2) = n + (1/2)$ and at a critical point β of f, $f(\beta) =$* index β.

Functions described in (7.1) are called *nice* functions.

Suppose now M^n is a closed C^∞ manifold and f is the function of (7.1). Let $X_s = f^{-1}[0, s + (1/2)]$, $s = 0, \cdots, n$.

(7.2) LEMMA. *For each s, the manifold X_s has a presentation of the form $(X_{s-1}; f_1, \cdots, f_k; s)$.*

This follows from (6.1).

(7.3). LEMMA. *If $H \in \mathcal{H}(n, k, s)$, then there exists—a C^∞ non-degenerate function f on H, $f(\partial H) = s+(1/2)$, f has one critical point of index 0, value 0, k critical points of index s, value s and no other critical points.*

This follows from (6.2).

The proof of Theorem C then goes as follows. Take a nice function f on M by (7.1), with X_s defined as above. Note that $X_0 \in \mathcal{H}(n, q, 0)$ and $\pi_0(X_1) = 0$, hence by (7.2) and (1.2), $X_1 \in \mathcal{H}(n, k, 1)$. Suppose now that $\pi_1(M) = 1$ and $n \geqq 6$. The following argument suggested by H. Samelson simplifies and replaces a complicated one of the author. Let X_2' be the sum of X_2 and k copies $H_1, \cdots, H_k$ of $D^{n-2} \times S^2$. Then since $\pi_1(X_2) = 0$, (1.2) implies that $X_2' \in H(n, r, 2)$. Now let $f_i: \partial D^3 \times D^{n-3} \to \partial H_i \cap \partial X_2'$ for $i = 1, \cdots, k$ be differentiable imbeddings such that the composition

$$\pi_2(\partial D^3 \times D^{n-3}) \to \pi_2(\partial H_i \times \partial X_2') \to \pi_2(\partial H_i)$$

is an isomorphism. Then by (3.3) and (3.4), $\chi(X_2', f_1, \cdots, f_k; 3)$ is diffeomorphic to X_2. Since $X_3 = \chi(X_2; g_1, \cdots, g_l; 3)$ we have

$$X_3 = \chi(X_2', f_1, \cdots, f_k, g_1, \cdots, g_l; 3) ,$$

and another application of (1.2) yields that $X_3 \in H(n, k + l - r, 3)$.

Iteration of the argument yields that $X_m' \in \mathcal{H}(n, r, m)$. By applying (7.3), we can replace g by a new nice function h with type numbers satisfying $M_0 = 1$, $M_i = 0$, $0 < i < m$. Now apply the preceding arguments to $-h$ to yield that $h^{-1}[n - m - (1/2), n] = X_m^* \in \mathcal{H}(n, k_1, m)$. Now we modify h by (7.3) on X_m^* to get a new nice function on M agreeing with h on $M - X_m^*$ and satisfying the conditions of Theorem C.

The proof of Theorem I goes as follows. Let V^n be a manifold with $\partial V = V_1 - V_2$, $n = 2m + 2$. Take a nice function f on V by (7.1) with $f(V_1) = -(1/2)$ and $f(V_2) = n + (1/2)$.

Following the proof of Theorem C, replacing the use of (1.2) with (5.1), we obtain a new nice function g on V with $g(V_1) = -(1/2)$, $g(V_2) = n + (1/2)$ and no critical points except possibly of index $m + 1$. The following lemma can be proved by the standard methods of Morse theory [12].

(7.4) LEMMA. *Let V be as in* (7.1) *and f be a C^∞ non-degenerate function on V with the same boundary conditions as in* (7.1). *Then*

$$\chi_V = \sum(-1)^q M_q + \chi_{V_1} ,$$

where χ_V, χ_{V_1} are the respective Euler characteristics, and M_q denote the q^{th} type number of f.

This lemma implies that our function g has no critical points, and hence V_1 and V_2 are diffeomorphic.

8. We have yet to prove Theorems F and G. For Theorem F, observe by Theorem C, there is a nice function f on M with vanishing type numbers except in dimensions M_0, M_m, M_{m+1}, M_n, and $M_0 = M_n = 1$. Also, by the Morse relation, observe that the Euler characteristic is the alternating sum of the type numbers, $M_m = M_{m+1}$. Then by (7.2), $f^{-1}[0, m + (1/2)]$, $f^{-1}[m + (1/2), 2m + 1] \in \mathcal{H}(2m + 1, M_m, m)$ proving Theorem F.

All but the last statement of Theorem G has been proved. For this just note that $M - D^{2m}$ is diffeomorphic to $f^{-1}[0, m + (1/2)]$ which by (7.2) is in $\mathcal{H}(2m, k, m)$.

UNIVERSISTY OF CALIFORNIA, BERKELEY

REFERENCES

1. R. BOTT, *The stable homotopy of the classical groups*, Ann. of Math., 70 (1959), 313–337.

2. H. GLUCK, *The weak Hauptvermutung for cells and spheres*, Bull. Amer. Math. Soc., 66 (1960), 282-284.
3. MORRIS W. HIRSCH, *On combinatorial submanifolds of differentiable manifolds*, to appear.
4. M. KERVAIRE, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv., 34 (1960), 304-312.
5. ——— and J. MILNOR, *Groups of homotopy spheres*, to appear.
6. K. A. KUROSH, Theory of Groups, v. 2, New York, 1956.
7. B. MAZUR, *On embeddings of spheres*, Bull. Amer. Math. Soc., 65 (1959), 59-65.
8. J. MILNOR, *Differentiable structures on spheres*, Amer. J. Math., 81 (1959), 962-972.
9. ———, *Sommes de variétés différentiables et structures différentiables des sphères*, Bull. Soc. Math. France, 87 (1959), 439-444.
10. ———, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton University, 1959.
11. E. E. MOISE, Certain classical problems of euclidean topology, Lectures of Summer Institute on Set Theoretic Topology, Madison, 1955.
12. M. MORSE, The calculus of variations in the large, Amer. Math. Soc. Colloq. Publications, v. 18, New York, 1934.
13. ———, *The existence of polar non-degenerate functions on differentiable manifolds*, Ann. of Math., 71 (1960), 352-383.
14. J. MUNKRES, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*. Bull. Amer. Math. Soc., 65 (1959), 332-334.
15. ———, *Obstructions to imposing differentiable structures*, Abstract, Notices of the Amer. Math. Soc., 7 (1960), 204.
16. J. NIELSON, *Über die Isomorphismen unendlicher Gruppen ohne Relationen*, Math. Ann., 9 (1919), 269-272.
17. R. PALAIS, *Extending diffeomorphisms*, Proc. Amer. Math. Soc., 11 (1960), 274-277.
18. C. D. PAPAKYRIAKOPOULOS, *Some problems on 3-dimensional manifolds*, Bull. Amer. Math. Soc., 64 (1958), 317-335.
19. H. SEIFERT and W. THRELLFALL, Lehrbuch der Topologie, Teubner, Leipzig, 1934.
20. S. SMALE, *The generalized Poincaré conjecture in higher dimensions*, to appear.
21. ———, *On gradient dynamical systems*, Ann. of Math., 74 (1961), 199-206.
22. R. THOM, *Les structures différentiables des boules et des sphères*, to appear.
23. ———, La classification des immersions, Séminaire Bourbaki, Paris, December, 1957.
24. ———, Des variétés triangulées aux variétés différentiables, Proceedings of the International Congress of Mathematicians, 1958, Cambridge University Press, 1960.
25. ———, Les classes caracteristiques de Pontrjagin des variétés triangulées, Topologia Algebraica, Mexico, 1958, pp. 54-67.
26. A. H. WALLACE, *Modifications and cobounding manifolds*, Canad. J. Math., 12 (1960), 503-528.
27. J. H. C. WHITEHEAD, *On C^1-complexes*, Ann. of Math., 41 (1940), 809-824.
28. ———, *On the homotopy type of manifolds*, Ann. of Math., 41 (1940), 825-832.
29. H. WHITNEY, *Differentiable manifolds*, Ann. of Math., 37 (1936), 645-680.
30. W. T. WU, *On the isotopy of C^r manifolds of dimension n in euclidean $(2n + 1)$-space*, Science Record, Now Ser. v. II (1958), 271-275.