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GENERALIZED POINCARÉ'S CONJECTURE IN DIMENSIONS GREATER THAN FOUR

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Poincaré has posed the problem as to whether every simply connected closed 3-manifold (triangulated) is homeomorphic to the 3-sphere, see [18] for example. This problem, still open, is usually called Poincaré's conjecture. The generalized Poincaré conjecture (see [11] or [28] for example) says that every closed n-manifold which has the homotopy type of the n-sphere S^n is homeomorphic to the n-sphere. One object of this paper is to prove that this is indeed the case if $n \geqq 5$ (for differentiable manifolds in the following theorem and combinatorial manifolds in Theorem B).

THEOREM A. *Let M^n be a closed C^∞ manifold which has the homotopy type of S^n, $n \geqq 5$. Then M^n is homeomorphic to S^n.*

Theorem A and many of the other theorems of this paper were announced in [20]. This work is written from the point of view of differential topology, but we are also able to obtain the combinatorial version of Theorem A.

THEOREM B. *Let M^n be a combinatorial manifold which has the homotopy of S^n, $n \geqq 5$. Then M^n is homeomorphic to S^n.*

J. Stallings has obtained a proof of Theorem B (and hence Theorem A) for $n \geqq 7$ using different methods (*Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc., 66 (1960), 485–488).

The basic theorems of this paper, Theorems C and I below, are much stronger than Theorem A.

A *nice* function f on a closed C^∞ manifold is a C^∞ function with non-degenerate critical points and, at each critical point β, $f(\beta)$ equals the index of β. These functions were studied in [21].

THEOREM C. *Let M^n be a closed C^∞ manifold which is $(m-1)$-connected, and $n \geqq 2m$, $(n, m) \neq (4, 2)$. Then there is a nice function f on M with type numbers satisfying $M_0 = M_n = 1$ and $M_i = 0$ for $0 < i < m$, $n - m < i < n$.*

Theorem C can be interpreted as stating that a cellular structure can be imposed on M^n with one 0-cell, one n-cell and no cells in the range $0 < i < m$, $n - m < i < n$. We will give some implications of Theorem C.

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First, by letting $m = 1$ in Theorem C, we obtain a recent theorem of M. Morse [13].

THEOREM D. *Let M^n be a closed connected C^∞ manifold. There exists a (nice) non-degenerate function on M with just one local maximum and one local minimum.*

In § 1, the handlebodies, elements of $\mathcal{H}(n, k, s)$ are defined. Roughly speaking if $H \in \mathcal{H}(n, k, s)$, then H is defined by attaching s-disks, k in number, to the n-disk and "thickening" them. By taking $n = 2m + 1$ in Theorem C, we will prove the following theorem, which in the case of 3-dimensional manifolds gives the well known Heegard decomposition.

THEOREM F. *Let M be a closed C^∞ $(2m + 1)$-manifold which is $(m-1)$-connected. Then $M = H \cup H'$, $H \cap H' = \partial H = \partial H'$ where $H, H' \in \mathcal{H}(2m + 1, k, m)$ are handlebodies (∂V means the boundary of the manifold V).*

By taking $n = 2m$ in Theorem C, we will get the following.

THEOREM G. *Let M^{2m} be a closed $(m-1)$-connected C^∞ manifold, $m \neq 2$. Then there is a nice function on M whose type numbers equal the corresponding Betti numbers of M. Furthermore M, with the interior of a 2m-disk deleted, is a handlebody, an element of $\mathcal{H}(2m, k, m)$ where k is the m^{th} Betti number of M.*

Note that the first part of Theorem G is an immediate consequence of the Morse relation that the Euler characteristic is the alternating sum of the type numbers [12], and Theorem C.

The following is a special case of Theorem G.

THEOREM H. *Let M^{2m} be a closed C^∞ manifold $m \neq 2$ of the homotopy type of S^{2m}. Then there exists on M a non-degenerate function with one maximum, one minimum, and no other critical point. Thus M is the union of two 2m-disks whose intersection is a submanifold of M, diffeomorphic to S^{2m-1}.*

Theorem H implies the part of Theorem A for even dimensional homotopy spheres.

Two closed C^∞ oriented n-dimensional manifolds M_1 and M_2 are *J-equivalent* (according to Thom, see [25] or [10]) if there exists an oriented manifold V with ∂V diffeomorphic to the disjoint union of M_1 and $-M_2$, and each M_i is a deformation retract of V.

THEOREM I. *Let M_1 and M_2 be $(m - 1)$-connected oriented closed C^∞ $(2m+1)$-dimensional manifolds which are J-equivalent, $m \neq 1$. Then M_1 and M_2 are diffeomorphic.*

We obtain an orientation preserving diffeomorphism. If one takes M_1 and M_2 J-equivalent disregarding orientation, one finds that M_1 and M_2 are diffeomorphic.

In studying manifolds under the relation of J-equivalence, one can use the methods of cobordism and homotopy theory, both of which are fairly well developed. The importance of Theorem I is that it reduces diffeomorphism problems to J-equivalence problems for a certain class of manifolds. It is an open question as to whether arbitrary J-equivalent manifolds are diffeomorphic (see [10, Problem 5])(Since this was written, Milnor has found a counter-example).

A short argument of Milnor [10, p. 33] using Mazur's theorem [7] applied to Theorem I yields the odd dimensional part of Theorem A. In fact it implies that, if M^{2m+1} is a homotopy sphere, $m \neq 1$, then M^{2m+1} minus a point is diffeomorphic to euclidean $(2m+1)$-space (see also [9, p. 440]).

Milnor [10] has defined a group $\mathcal{H}^n$ of C^∞ homotopy n-spheres under the relation of J-equivalence. From Theorems A and I, and the work of Milnor [10] and Kervaire [5], the following is an immediate consequence.

THEOREM J. *If n is odd, $n \neq 3$, $\mathcal{H}^n$ is the group of classes of all differentiable structures on S^n under the equivalence of diffeomorphism. For n odd there are a finite number of differentiable structures on S^n. For example*:

n	3	5	7	9	11	13	15
Number of Differentiable Structures on S^n	0	0	28	8	992	3	16256

Previously it was known that there are a countable number of differentiable structures on S^n for all n (Thom), see also [9, p. 442]; and unique structures on S^n for $n \leqq 3$ (e.g., Munkres [14]). Milnor [8] has also established lower bounds for the number of differentiable structures on S^n for several values of n.

A group Γ^n has been defined by Thom [24] (see also Munkres [14] and Milnor [9]). This is the group of all diffeomorphisms of S^{n-1} modulo those which can be extended to the n-disk. A group A^n has been studied by Milnor as those structures on the n-sphere which, minus a point, are diffeomorphic to euclidean space [9]. The group Γ^n can be interpreted (by Thom [22] or Munkres [14]) as the group of differentiable structures on S^n which admit a C^∞ function with the non-degenerate critical points, and hence one has the inclusion map $i\colon \Gamma^n \to A^n$ defined. Also, by taking J-equivalence classes, one gets a map $p\colon A^n \to \mathcal{H}^n$.

THEOREM K. *With notation as in the preceding paragraph, the following sequences are exact*:

(a) $A^n \xrightarrow{p} \mathcal{H}^n \longrightarrow 0$, $n \neq 3, 4$

(b) $\Gamma^n \xrightarrow{i} A^n \longrightarrow 0$, n even $\neq 4$

(c) $0 \longrightarrow A^n \xrightarrow{p} \mathcal{H}^n$, n odd $\neq 3$.

Hence, if n is even, $n \neq 4$, $\Gamma^n = A^n$ and, if n is odd $\neq 3$, $A^n = \mathcal{H}^n$.

Here (a) follows from Theorem A, (b) from Theorem H, and (c) from Theorem I.

Kervaire [4] has also obtained the following result.

THEOREM L. *There exists a manifold with no differentiable structure at all.*

Take the manifold W_0 of Theorem 4.1 of Milnor [10] for $k = 3$. Milnor shows ∂W_0 is a homotopy sphere. By Theorem A, ∂W_0 is homeomorphic to S^{11}. We can attach a 12-disk to W_0 by a homeomorphism of the boundary onto ∂W_0 to obtain a closed 12 dimensional manifold M. Starting with a triangulation of W_0, one can easily obtain a triangulation of M. If M possessed a differentiable structure it would be almost parallelizable, since the obstruction to almost parallelizability lies in $H^6(M, \pi_5(\mathrm{SO}(12))) = 0$. But the index of M is 8 and hence by Lemma 3.7 of [10] M cannot possess any differentiable structure. Using Bott's results on the homotopy groups of Lie groups [1], one can similarly obtain manifolds of arbitrarily high dimension without a differentiable structure.

THEOREM M. *Let C^{2m} be a contractible manifold, $m \neq 2$, whose boundary is simply connected. Then C^{2m} is diffeomorphic to the $2m$-disk. This implies that differentiable structures on disks of dimension $2m$, $m \neq 2$, are unique. Also the closure of the bounded component C of a C^∞ imbedded $(2m - 1)$-sphere in euclidean $2m$-space, $m \neq 2$, is diffeomorphic to a disk.*

For these dimensions, the last statement of Theorem M is a strong version of the Schoenflies problem for the differentiable case. Mazur's theorem [7] had already implied C was homeomorphic to the $2m$-disk.

Theorem M is proved as follows from Theorems C and I. By Poincaré duality and the homology sequence of the pair $(C, \partial C)$, it follows that ∂C is a homotopy sphere and J-equivalent to zero since it bounds C. By Theorem I, then, ∂C is diffeomorphic to S^n. Now attach to C^{2m} a $2m$-disk by a diffeomorphism of the boundary to obtain a differentiable manifold V. One shows easily that V is a homotopy sphere and, hence by Theorem H, V is the union of two $2m$-disks. Since any two $2m$ sub-disks of V are equivalent under a diffeomorphism of V (for example see Palais [17]), the original $C^{2m} \subset V$ must already have been diffeomorphic to the standard $2m$-disk.

To prove Theorem B, note that $V = (M$ with the interior of a simplex deleted) is a contractible manifold, and hence possesses a differentiable structure [Munkres 15]. The double W of V is a differentiable manifold which has the homotopy type of a sphere. Hence by Theorem A, W is a topological sphere. Then according to Mazur [7], ∂V, being a differentiable submanifold and a topological sphere, divides W into two topological cells. Thus V is topologically a cell and M a topological sphere.

THEOREM N. *Let C^{2m}, $m \neq 2$, be a contractible combinatorial manifold whose boundary is simply connected. Then C^{2m} is combinatorially equivalent to a simplex. Hence the Hauptvermutung* (see [11]) *holds for combinatorial manifolds which are closed cells in these dimensions.*

To prove Theorem N, one first applies a recent result of M. W. Hirsch [3] to obtain a compatible differentiable structure on C^{2m}. By Theorem M, this differentiable structure is diffeomorphic to the $2m$-disk D^{2m}. Since the standard $2m$-simplex σ^{2m} is a C^1 triangulation of D^{2m}, Whitehead's theorem [27] applies to yield that C^{2m} must be combinatorially equivalent to σ^{2m}.

Milnor first pointed out that the following theorem was a consequence of this theory.

THEOREM O. *Let M^{2m}, $m \neq 2$, be a combinatorial manifold which has the same homotopy type as S^{2m}. Then M^{2m} is combinatorially equivalent to S^{2m}. Hence, in these dimensions, the Hauptvermutung holds for spheres.*

For even dimensions greater than four, Theorems N and O improve recent results of Gluck [2].

Theorem O is proved by applying Theorem N to the complement of the interior of a simplex of M^{2m}.

Our program is the following. We introduce handlebodies, and then prove "the handlebody theorem" and a variant. These are used together with a theorem on the existence of "nice functions" from [21] to prove Theorems C and I, the basic theorems of the paper. After that, it remains only to finish the proof of Theorems F and G of the Introduction.

The proofs of Theorems C and I are similar. Although they use a fair amount of the technique of differential topology, they are, in a certain sense, elementary. It is in their application that we use many recent results.

A slightly different version of this work was mimeographed in May 1960. In this paper J. Stallings pointed out a gap in the proof of the handlebody theorem (for the case $s=1$). This gap happened not to affect our main theorems.

Everything will be considered from the C^∞ point of view. All imbeddings will be C^∞. A differentiable isotopy is a homotopy of imbeddings with continuous differential.

$$E^n = \{x = (x_1, \cdots, x_n)\},\ \| x \| = (\textstyle\sum_{i=1}^n x_i^2)^{1/2} ,$$
$$D^n = \{x \in E^n \mid \| x \| \leqq 1\},\ \partial D^n = S^{n-1} = \{x \in E^n \mid \| x \| = 1\} ;$$
$$D_i^n \text{ etc. are copies of } D^n.$$

A. Wallace's recent article [26] is related to some of this paper.

1. Let M^n be a compact manifold, Q a component of ∂M and

$$f_i\colon \partial D_i^s \times D_i^{n-s} \to Q,\ i = 1, \cdots, k$$

imbeddings with disjoint images, $s \geqq 0$, $n \geqq s$. We define a new compact C^∞ manifold $V = \chi(M, Q; f_1, \cdots, f_k; s)$ as follows. The underlying topological space of V is obtained from M, and the $D_i^s \times D_i^{k-s}$ by identifying points which correspond under some f_i. The manifold thus defined has a natural differentiable structure except along corners $\partial D_i^s \times \partial D_i^{n-s}$ for each i. The differentiable structure we put on V is obtained by the process of "straightening the angle" along these corners. This is carried out in Milnor [10] for the case of the product of manifolds W_1 and W_2 with a corner along $\partial W_1 \times \partial W_2$. Since the local situation for the two cases is essentially the same, his construction applies to give a differentiable structure on V. He shows that this structure is well-defined up to diffeomorphism.

If $Q = \partial M$ we omit it from the notation $\chi(M, Q; f_1, \cdots, f_k; s)$, and we sometimes also omit the s. We can consider the "handle" $D_i^s \times D_i^{n-s} \subset V$ as differentiably imbedded.

The next lemma is a consequence of the definition.

(1.1) LEMMA. *Let $f_i\colon \partial D_i^s \times D_i^{n-s} \to Q$ and $f_i'\colon \partial D_i^s \times D_i^{n-s} \to Q$, $i=1,\cdots,k$ be two sets of imbeddings each with disjoint images, Q, M as above. Then $\chi(M, Q; f_1,\cdots, f_k; s)$ and $\chi(M, Q; f_1',\cdots, f_k'; s)$ are diffeomorphic if*

(a) *there is a diffeomorphism $h\colon M \to M$ such that $f_i' = hf_i$, $i = 1,\cdots, k$; or*

(b) *there exist diffeomorphisms $h_i\colon D^s \times D^{n-s} \to D^s \times D^{n-s}$ such that $f_i' = f_i h_i$, $i = 1, \cdots, k$; or*

(c) *the f_i' are permutations of the f_i.*

If V is the manifold $\chi(M, Q; f_1,\cdots, f_k; s)$, we say $\sigma = (M, Q; f_1,\cdots, f_k; s)$

is a presentation of V.

A *handlebody* is a manifold which has a presentation of the form $(D^n; f_1, \cdots, f_k; s)$. Fixing n, k, s the set of all handlebodies is denoted by $\mathcal{H}(n, k, s)$. For example, $\mathcal{H}(n, k, 0)$ consists of one element, the disjoint union of $(k+1)$ n-disks; and one can show $\mathcal{H}(2, 1, 1)$ consists of $S^1 \times I$ and the Möbius strip, and $\mathcal{H}(3, k, 1)$ consists of the classical handlebodies [19; Henkelkörper], orientable and non-orientable, or at least differentiable analogues of them. The following is one of the main theorems used in the proof of Theorem C. An analogue in § 5 is used for Theorem I.

(1.2) HANDLEBODY THEOREM. *Let* $n \geqq 2s + 2$ *and, if* $s = 1$, $n \geqq 5$; *let* $H \in \mathcal{H}(n, k, s)$, $V = \chi(H; f_1, \cdots, f_r; s + 1)$, *and* $\pi_s(V) = 0$. *Also, if* $s = 1$, *assume* $\pi_1(\chi(H; f_1, \cdots, f_{r-k}; 2)) = 1$. *Then* $V \in \mathcal{H}(n, r - k, s + 1)$. (We do not know if the special assumption for $s = 1$ is necessary.)

The next three sections are devoted to a proof of (1.2).

2. Let $G_r = G_r(s)$ be the free group on r generators $D_1, \cdots, D_r$ if $s=1$, and the free abelian group on r generators $D_1, \cdots, D_r$ if $s > 1$. If $\sigma = (M, Q; f_1, \cdots, f_r; s + 1)$ is a presentation of a manifold V, define a homomorphism f_σ: $G_r \to \pi_s(Q)$ by $f_\sigma(D_i) = \varphi_i$, where $\varphi_i \in \pi_s(Q)$ is the homotopy class of $\bar{f}_i$: $\partial D^{s+1} \times 0 \to Q$, the restriction of f_i. To take care of base points in case $\pi_1(Q) \neq 1$, we will fix $x_0 \in \partial D^{s+1} \times 0$, $y_0 \in Q$, Let U be some cell neighborhood of y_0 in Q, and assume $\bar{f}_i(x_0) \in U$. We say that the homomorphism f_σ is *induced* by the presentation σ.

Suppose now that F: $G_r \to \pi_s(Q)$ is a homomorphism where Q is a component of the boundary of a compact n-manifold M. Then we say that a manifold V realizes F if some presentation of V induces F. Manifolds realizing a given homomorphism are not necessarily unique.

The following theorem is the goal of this section.

(2.1) THEOREM. *Let* $n \geqq 2s + 2$, *and if* $s = 1$, $n \geqq 5$; *let* $\sigma = (M, Q; f_1, \cdots, f_r; s + 1)$ *be a presentation of a manifold* V, *and assume* $\pi_1(Q) = 1$ *if* $n = 2s + 2$. *Then for any automorphism* α: $G_r \to G_r$, V *realizes* $f_\sigma \alpha$.

Our proof of (2.1) is valid for $s = 1$, but we have application for the theorem only for $s > 1$. For the proof we will need some lemmas.

(2.2) LEMMA. *Let* Q *be a component of the boundary of a compact manifold* M^n *and* f_1: $\partial D^s \times D^{n-s} \to Q$ *an imbedding. Let* $\bar{f}_2$: $\partial D^s \times 0 \to Q$ *be an imbedding, differentiably isotopic in* Q *to the restriction* $\bar{f}_1$ *of* f_1 *to* $\partial D^s \times 0$. *Then there exists an imbedding* f_2: $\partial D^s \times D^{n-s} \to Q$ *extending* $\bar{f}_2$ *and a diffeomorphism* h: $M \to M$ *such that* $hf_2 = f_1$.

PROOF. Let $\bar{f}_t\colon \partial D^s \times 0 \to Q$, $1 \leqq t \leqq 2$, be a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Then by the covering homotopy property for spaces of differentiable imbeddings (see Thom [23] and R. Palais, Comment. Math. Helv. 34 (1960)), there is a differentiable isotopy $F_t\colon \partial D^s \times D^{n-s} \to Q$, $1 \leqq t \leqq 2$, with $F_1 = f_1$ and F_t restricted to $\partial D^s \times 0 = \bar{f}_t$. Now by applying this theorem again, we obtain a differentiable isotopy $G_t\colon M \to M$, $1 \leqq t \leqq 2$, with G_1 equal the identity, and G_t restricted to image F_1 equal $F_t F_1^{-1}$. Then taking $h = G_2^{-1}$, F_2 satisfies the requirements of f_2 of (2.2); i.e., $hf_2 = G_2^{-1}F_2 = F_1F_2^{-1}F_2 = f_1$.

(2.3) THEOREM (H. Whitney, W.T. Wu). *Let $n \geqq \max(2k+1, 4)$ and $f, g\colon M^k \to X^n$ be two imbeddings, M closed, M connected and X simply connected if $n = 2k+1$. Then, if f and g are homotopic, they are differentiably isotopic.*

Whitney [29] proved (2.3) for the case $n \geqq 2k + 2$. W.T. Wu [30] (using methods of Whitney) proved it where X^n was euclidean space, $n = 2k + 1$. His proof also yields (2.3) as stated.

(2.4) LEMMA. *Let Q be a component of the boundary of a compact manifold M^n, $n \geqq 2s + 2$ and if $s = 1, n \geqq 5$, and $\pi_1(Q) = 1$ if $n = 2s + 2$. Let $f_1\colon \partial D^{s+1} \times D^{n-s-1} \to Q$ be an imbedding, and $\bar{f}_2\colon \partial D^{s+1} \times 0 \to Q$ an imbedding homotopic in Q to $\bar{f}_1$, the restriction of f_1 to $\partial D^{s+1} \times 0$. Then there exists an imbedding $f_2\colon \partial D^{s+1} \times D^{n-s-1} \to Q$ extending $\bar{f}_2$ such that $\chi(M, Q\colon f_2)$ is diffeomorphic to $\chi(M, Q; f_1)$.*

PROOF. By (2.3), there exists a differentiable isotopy between $\bar{f}_1$ and $\bar{f}_2$. Apply (2.2) to get $f_2\colon \partial D^{s+1} \times D^{n-s-1} \to Q$ extending $\bar{f}_2$, and a diffeomorphism $h\colon M \to M$ with $h f_2 = f_1$. Application of (1.1) yields the desired conclusion.

See [16] for the following.

(2.5) LEMMA (Nielson). *Let G be a free group on r-generators $\{D_1, \cdots, D_r\}$, and $\mathcal{A}$ the group of automorphisms of G. Then $\mathcal{A}$ is generated by the following automorphisms:*

$$R\colon D_1 \to D_1^{-1}, \quad D_i \to D_i \qquad i > 1$$

$$T_i\colon D_1 \to D_i, \quad D_i \to D_1, \quad D_j \to D_j \qquad j \neq 1, j \neq i, i = 2, \cdots, r$$

$$S\colon D_1 \to D_1D_2, \quad D_i \to D_i, \qquad i > 1.$$

The same is true for the free abelian case (well-known).

It is sufficient to prove (2.1) with α replaced by the generators of $\mathcal{A}$ of (2.5).

First take $\alpha = R$. Let $h\colon D^{s+1} \times D^{n-s-1} \to D^{s+1} \times D^{n-s-1}$ be defined by

$h(x, y) = (r, x, y)$ where $r\colon D^{s+1} \to D^{s+1}$ is a reflection through an equatorial s-plane. Then let $f_i' = f_1 h$. If $\sigma' = (M, Q; f_1', f_2, \cdots, f_r; s+1)$, $\chi(\sigma')$ is diffeomorphic to V by (1.1). On the other hand $\chi(\sigma')$ realizes $f_{\sigma'} = f_\sigma \alpha$.

The case $\alpha = T_i$ follows immediately from (1.1). So now we proceed with the proof of (2.1) with $\alpha = S$.

Define V_1 to be the manifold $\chi(M, Q; f_2, \cdots, f_r; s+1)$ and let $Q_1 \subset \partial V_1$ be $Q_1 = \partial V_1 - (\partial M - Q)$. Let $\varphi_i \in \pi_s(Q)$, $i = 1, \cdots, r$ denote the homotopy class of $f_i\colon \partial D_i^{s+1} \times 0 \to Q$, the restriction of f_i. Let $\gamma\colon \pi_s(Q \cap Q_1) \to \pi_s(Q)$ and $\beta\colon \pi_s(Q \cap Q_1) \to \pi_s(Q_1)$ be the homomorphisms induced by the respective inclusions.

(2.6) Lemma. *With notations and conditions as above,* $\varphi_2 \in \gamma \operatorname{Ker} \beta$.

Proof. Let $q \in \partial D_2^{n-s-1}$ and $\psi\colon \partial D_2^{s+1} \times q \to Q \cap Q_1$ be the restriction of f_2. Denote by $\bar\psi \in \pi_s(Q \cap Q_1)$ the homotopy class of ψ. Since ψ and $\bar f_2$ are homotopic in Q, $\gamma\bar\psi = \varphi_2$. On the other hand $\beta\bar\psi = 0$, thus proving (2.6).

By (2.6), let $\bar\psi \in \pi_s(Q \cap Q_1)$ with $\gamma\bar\psi = \varphi_2$ and $\beta\bar\psi = 0$. Let $g = y + \bar\psi$ (or $y\bar\psi$ in case $s = 1$; our terminology assumes $s > 1$) where $y \in \pi_s(Q \cap Q_1)$ is the homotopy class of $\bar f_1\colon \partial D_1^{s+1} \times 0 \to Q \cap Q_1$. Let $\bar g\colon \partial D^{s+1} \times 0 \to Q \cap Q_1$ be an imbedding realizing g (see [29]).

If $n = 2s + 2$, then from the fact that $\pi_1(Q) = 1$, it follows that also $\pi_1(Q_1) = 1$. Then since $\bar g$ and $\bar f_1$ are homotopic in Q_1, i.e., $\beta g = \beta y$, (2.4) applies to yield an imbedding $e\colon \partial D^{s+1} \times D^{n-s-1} \to Q_1$ extending $\bar g$ such that $\chi(V_1, Q_1; e)$ and $\chi(V_1, Q_1; f_1)$ are diffeomorphic.

On one hand $V = \chi(V, Q; f_1, \cdots, f_r) = \chi(V_1, Q_1; f_1)$ and, on the other hand, $\chi(V, Q; e, f_2, \cdots, f_r) = \chi(V_1, Q_1; e)$, so by the preceding statement, V and $\chi(V, Q; e, f_2, \cdots, f_r)$ are diffeomorphic. Since $\gamma g = g_1 + g_2$, $f_\sigma \alpha(D_1) = f_\sigma(D_1 + D_2) = g_1 + g_2$, $f'_\sigma(D_1) = gD_1 = g_1 + g_2$, $f_\sigma \alpha = f_{\sigma'}$, where $\sigma' = (V, Q; e, f_2, \cdots, f_r)$. This proves (2.1).

3. The goal of this section is to prove the following theorem.

(3.1) Theorem. *Let* $n \geqq 2s + 2$ *and, if* $s = 1$, $n \geqq 5$. *Suppose* $H \in \mathcal{H}(n, k, s)$. *Then given* $r \geqq k$, *there exists an epimorphism* $g\colon G_r \to \pi_s(H)$ *such that every realization of* g *is in* $\mathcal{H}(n, r-k, s+1)$.

For the proof of 3.1, we need some lemmas.

(3.2) Lemma. *If* $\mathcal{H}(n, k, s)$ *then* $\pi_s(H)$ *is*

(a) *a set of* $k + 1$ *elements if* $s = 0$,

(b) *a free group on* k *generators if* $s = 1$,

(c) *a free abelian group on* k *generators if* $s > 1$.

Furthermore if $n \geqq 2s + 2$, *then* $\pi_i(\partial H) \to \pi_i(H)$ *is an isomorphism for* $i \leqq s$.

Proof. We can assume $s > 0$ since, if $s = 0$, H is a set of n-disks $k+1$

in number. Then H has as a deformation retract in an obvious way the wedge of k s-spheres. Thus (b) and (c) are true. For the last statement of (3.2), from the exact homotopy sequence of the pair $(H, \partial H)$, it is sufficient to show that $\pi_i(H, \partial H) = 0$, $i \leqq s + 1$.

Thus let $f\colon (D^i, \partial D^i) \to (H, \partial H)$ be a given continuous map with $i \leqq s + 1$. We want to construct a homotopy $f_r\colon (D^i, \partial D^i) \to (H, \partial H)$ with $f_0 = f$ and $f_1(D^i) \subset \partial H$.

Let $f_1\colon (D^i, \partial D^i) \to (H, \partial H)$ be a differentiable approximation to f. Then by a radial projection from a point in D^n not in the image of f_1, f_1 is homotopic to a differentiable map $f_2\colon (D^i, \partial D^i) \to (H, \partial H)$ with the image of f_2 not intersecting the interior of $D^n \subset H$. Now for dimensional reasons f_2 can be approximated by a differentiable map $f_3\colon (D^i, \partial D^i) \to (H, \partial H)$ with the image of f_3 not intersecting any $D_i^s \times 0 \subset H$. Then by other projections, one for each i, f_3 is homotopic to a map $f_4\colon (D^i, \partial D^i) \to (H, \partial H)$ which sends all of D^i into ∂H. This shows $\pi_i(H, \partial H) = 0$, $i \leqq s + 1$, and proves (3.2).

If $\beta \in \pi_{s-1}(\mathrm{O}(n - s))$, let H_β be the $(n - s)$-cell bundle over S^s determined by β.

(3.3) Lemma. *Suppose* $V = \chi(H_\beta; f; s + 1)$ *where* $\beta \in \pi_{s-1}(\mathrm{O}(n - s))$, $n \geqq 2s + 2$, *or if* $s = 1$, $n \geqq 5$. *Let also* $\pi_s(V) = 0$. *Then* V *is diffeomorphic to* D^n.

Proof. The zero-cross-section $\sigma\colon S^s \to H_\beta$ is homotopic to zero, since $\pi_s(V) = 0$, and so is regularly homotopic in V to a standard s-sphere S_0^s contained in a cell neighborhood by dimensional reasons [29]. Since a regular homotopy preserves the normal bundle structure, $\sigma(S^s)$ has a trivial normal bundle and thus $\beta = 0$. Hence H_β is diffeomorphic to the product of S^s and D^{n-s}.

Let $\sigma_1\colon S^s \to \partial H_\beta$ be a differentiable cross section and $\bar{f}\colon \partial D^{s+1} \times 0 \to \partial H_\beta$ the restriction of $f\colon \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$. Then σ_1 and $\bar{f}$ are homotopic in ∂H_β (perhaps after changing f by a diffeomorphism of $D^{s+1} \times D^{n-s-1}$ which reverses orientation of $\partial D^{s+1} \times 0$) since $\pi_s(V) = 0$, and hence differentiably isotopic. Thus we can assume $\bar{f}$ and s_1 are the same.

Let f_ε be the restriction of f to $\partial D^{s+1} \times D_\varepsilon^{n-s-1}$ where D_ε^{n-s-1} denotes the disk $\{x \in D^{n-s-1} \mid \| x \| \leqq \varepsilon\}$, and $\varepsilon > 0$. Then the imbedding $g_\varepsilon\colon \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ is differentiably isotopic to f where $g_\varepsilon(x, y) = f_\varepsilon r_\varepsilon(x, y)$ and $r_\varepsilon(x, y) = (x, \varepsilon y)$. Define $k_\varepsilon\colon \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ by $p_x g_\varepsilon(x, y)$ where $p_x\colon g_\varepsilon(x \times D^{n-s-1}) \to F_x$ is projection into the fibre F_x of ∂H_β over $\sigma^{-1} g_\varepsilon(x, 0)$. If ε is small enough, k_ε is well-defined and an imbedding. In fact if ε is small enough, we can even suppose that for each x, k_ε maps $x \times D^{n-s-1}$ linearly onto image $k_\varepsilon \cap F_x$ where image $k_\varepsilon \cap F_x$ has a linear structure induced from F_x.

It can be proved k_ε and g_ε are differentiably isotopic. (The referee has remarked that there is a theorem, Milnor's "tubular neighborhood theorem", which is useful in this connection and can indeed be used to make this proof clearer in general.)

We finish the proof of (3.3) as follows. Suppose V is as in (3.3) and $V' = \chi(H_\beta; f'; s+1)$, $\pi_s(V') = 0$. It is sufficient to prove V and V' are diffeomorphic since it is clear that one can obtain D^n by choosing f' properly and using the fact that H_β is a product of S^s and D^{n-s}. From the previous paragraph, we can replace f and f' by k_ε and k'_ε with those properties listed. We can also suppose without loss of generality that the images of k_ε and k'_ε coincide. It is now sufficient to find a diffeomorphism h of H_β with $hf=f'$. For each x, define h on image $f \cap F_x$ to be the linear map which has this property. One can now easily extend h to all of H_β and thus we have finished the proof of (3.3).

Suppose now M_1^n and M_2^n are compact manifolds and $f_i\colon D^{n-1} \times i \to \partial M_i$ are imbeddings for $i = 1$ and 2. Then $\chi(M_1 \cup M_2; f_1 \cup f_2; 1)$ is a well defined manifold, where $f_1 \cup f_2\colon \partial D^1 \times D^{n-1} \to \partial M_1 \cup \partial M_2$ is defined by f_1 and f_2, the set of which, as the f_i vary, we denote by $M_1 + M_2$. (If we pay attention to orientation, we can restrict $M_1 + M_2$ to have but one element.) The following lemma is easily proved.

(3.4) LEMMA. *The set $M^n + D^n$ consists of one element, namely M^n.*

(3.5) LEMMA. *Suppose an imbedding $f\colon \partial D^s \times D^{n-s} \to \partial M^n$ is null-homotopic where M is a compact manifold, $n \geqq 2s + 2$ and, if $s = 1$, $n \geqq 5$. Then $\chi(M; f) \in M + H_\beta$ for some $\beta \in \pi_{s-1}(\mathrm{O}(n-s))$.*

PROOF OF (3.5). Let $\bar{f}\colon \partial D^s \times q \to \partial M$ be the restriction of f where q is a fixed point in ∂D^{n-s}. Then by dimensional reasons [29], $\bar{f}$ can be extended to an imbedding $\varphi\colon D^s \to \partial M$ where the image of φ intersects the image of f only on $\bar{f}$. Next let T be a tubular neighborhood of $\varphi(D^s)$ in M. This can be done so that T is a cell, $T \cup (D^s \times D^{n-s})$ is of the form H_β and $V \in M + H_\beta$. We leave the details to the reader.

To prove (3.1), let $H = \chi(D^n; f_1, \cdots, f_k; s)$. Then f_i defines a class $\bar{\gamma}_i \in \pi_s(H, D^n)$. Let $\gamma_i \in \pi_s(\partial H)$ be the image of γ_i under the inverse of the composition of the isomorphisms $\pi_s(\partial H) \to \pi_s(H) \to \pi_s(H, D^n)$ (using (3.2)). Define g of (3.1) by $gD_i = \gamma_i$, $i \leqq k$, and $gD_i = 0$, $i > k$. That g satisfies (3.1) follows by induction from the following lemma.

(3.6) LEMMA. $\chi(H; g_1; s+1) \in \mathcal{H}(n, k-1, s)$ *if the restriction of g_1 to $\partial D^{s+1} \times 0$ has homotopy class $\gamma_1 \in \pi_s(\partial H)$.*

Now (3.6) follows from (3.3), (3.4) and (3.5), and the fact that g_1 is differentiably isotopic to g_1' whose image is in $\partial H_\beta \cap \partial H$, where H_β is defined by (3.5) and f_1.

4. We prove here (1.2). First suppose $s = 0$. Then $H \in \mathcal{H}(n, k, 0)$ is the disjoint union of n-disks, $k+1$ in number, and $V = \chi(H; f_1, \cdots, f_r; 1)$. Since $\pi_0(V) = 1$, there exists a permutation of $1, \cdots, r$, $i_1, \cdots, i_r$ such that $Y = \chi(H; f_{i_1}, \cdots, f_{i_k}; 1)$ is connected. By (3.4), Y is diffeomorphic to D^n. Hence $V = \chi(Y; f_{i_{k+1}}, \cdots, f_{i_r}; 1)$ is in $\mathcal{H}(n, r-k, 1)$.

Now consider the case $s = 1$. Choose, by (3.1), $g\colon G_k \to \pi_1(\partial H)$ such that every manifold derived from g is diffeomorphic to D^n. Let $Y = \chi(H; f_1, \cdots, f_{r-k})$. Then $\pi_1(Y) = 1$ and by the argument of (3.2), $\pi_1(\partial Y) = 1$. Let $\bar{g}_i\colon \partial D^2 \times 0 \to \partial H$ be disjoint imbeddings realizing the classes $g(D_i) \in \pi_1(\partial H)$ which are disjoint from the images of all f_i, $i = 1, \cdots, k$. Then by (2.4) there exist imbeddings $g_1, \cdots, g_k\colon \partial D^2 \times D^{n-2} \to \partial H$ extending the $\bar{g}_i$ such that $V = \chi(Y; f_{r-k+1}, \cdots, f_r)$ and $\chi(Y; g_1, \cdots, g_k)$ are diffeomorphic. But

$$\begin{aligned}\chi(Y, g_1, \cdots, g_k) &= \chi(H; g_1, \cdots, g_k, f_1, \cdots, f_{r-k}) \\ &= \chi(D^n, f_1, \cdots, f_{r-k}) \in \mathcal{H}(n, r-k, 2) .\end{aligned}$$

Hence so does V.

For the case $s > 1$, we use an algebraic lemma.

(4.1) LEMMA. *If $f, g\colon G \to G'$ are epimorphisms where G and G' are finitely generated free abelian groups, then there exists an automorphism $\alpha\colon G \to G$ such that $f\alpha = g$.*

PROOF. Let G'' be a free abelian group of rank equal to $\operatorname{rank} G - \operatorname{rank} G'$, and let $p\colon G' + G'' \to G'$ be the projection. Then, identifying elements of G and $G' + G''$ under some isomorphism, it is sufficient to prove the existence of α for $g = p$. Since the groups are free, the following exact sequence splits

$$0 \longrightarrow f^{-1}(0) \longrightarrow G \xrightarrow{\ f\ } G' \longrightarrow 0 .$$

Let $h\colon G \to f^{-1}(0)$ be the corresponding projection and let $k\colon f^{-1}(0) \to G''$ be some isomorphism. Then $\alpha\colon G \to G' + G''$ defined by $f + kh$ satisfies the requirements of (4.1).

REMARK. Using Grusko's Theorem [6], one can also prove (4.1) when G and G' are free groups.

Now take $\sigma = (H; f_1, \cdots, f_r; s+1)$ of (1.2) and $g\colon G_r \to \pi_s(\partial H)$ of (3.1). Since $\pi_s(V) = 0$, and $s > 1$, $f_\sigma\colon G_r \to \pi_s(\partial H)$ is an epimorphism. By (3.2) and (4.1) there is an automorphism $\alpha\colon G_r \to G_r$ such that $f_\sigma \alpha = g$. Then (2.1) implies that V is in $\mathcal{H}(n, r-k, s+1)$ using the main property of g.

5. The goal of this section is to prove the following analogue of (1.2).

(5.1) THEOREM. *Let* $n \geqq 2s + 2$, *or if* $s = 1$, $n \geqq 5$, M^{n-1} *be a simply connected,* $(s-1)$*-connected closed manifold and* $\mathcal{H}_M(n, k, s)$ *the set of all manifolds having presentations of the form* $(M\times[0, 1], M\times 1; f_1 \cdots, f_k; s)$. *Now let* $H \in \mathcal{H}_M(n, k, s)$, $Q = \partial H - M\times 0$, $V = \chi(H, Q; g_1, \cdots, g_r; s+1)$ *and suppose* $\pi_s(M\times 0)\to\pi_s(V)$ *is an isomorphism. Also suppose if* $s = 1$, *that* $\pi_1(\chi(H, Q; g_1, \cdots, g_{r-k}; 2)) = 1$. *Then* $V \in \mathcal{H}_M(n, r-k, s+1)$.

One can easily obtain (1.2) from (5.1) by taking for M, the $(n-1)$-sphere. The following lemma is easy, following (3.2).

(5.2) LEMMA. *With definitions and conditions as in* (5.1), $\pi_s(Q) = G_k$ *if* $s = 1$, *and if* $s > 1$, $\pi_s(Q) = \pi_s(M) + G_k$.

Let $p_1: \pi_s(Q) \to \pi_s(M)$, $p_2: \pi_s(Q) \to G_k$ be the respective projections.

(5.3) LEMMA. *With definitions and conditions as in* (5.1), *there exists a homomorphism* $g: G_r\to\pi_s(Q)$ *such that* $p_1 g$ *is trivial,* $p_2 g$ *is an epimorphism, and every realization of* g *is in* $\mathcal{H}_M(n, r-k, s+1)$, *each* $r \geqq k$.

The proof follows (3.1) closely.

We now prove (5.1). The cases $s = 0$ and $s = 1$ are proved similarly to these cases in the proof of (1.2). Suppose $s > 1$. From the fact that $\pi_s(M\times 0)\to\pi_s(V)$ is an isomorphism, it follows that $p_1 f_\sigma$ is trivial and $p_2 f_\sigma$ is an epimorphism where $\sigma = (H, Q; g_1, \cdots, g_r, s+1)$. Then apply (4.1) to obtain an automorphism $\alpha: G_r \to G_r$ such that $p_2 f_\sigma \alpha = p_2 g$ where g is as in (5.3). Then $f_\sigma \alpha = g$, hence using (2.1), we obtain (5.1).

6. The goal of this section is to prove the following two theorems.

(6.1) THEOREM. *Suppose* f *is a* C^∞ *function on a compact manifold* W *with no critical points on* $f^{-1}[-\varepsilon, \varepsilon] = N$ *except* k *non-degenerate ones on* $f^{-1}(0)$, *all of index* λ, and $N \cap \partial W = \varnothing$. Then $f^{-1}[-\infty, \varepsilon]$ has a presentation of the form $(f^{-1}[-\infty, -\varepsilon], f^{-1}(-\varepsilon); f_1, \cdots, f_k; \lambda)$.

(6.2) THEOREM. *Let* $(M, Q; f_1, \cdots, f_k; s)$ *be a presentation of a manifold* V, *and* g *be a* C^∞ *function on* M, *regular, in a neighborhood of* Q, *and constant with its maximum value on* Q. *Then there exists a* C^∞ *function* G *on* V *which agrees with* g *outside a neighborhood of* Q, *is constant and regular on* $\partial V - (\partial M - Q)$, *and has exactly* k *new critical points, all non-degenerate, with the same value and with index* s.

SKETCH OF PROOF OF (6.1). Let β_i denote the critical points of f at level zero, $i = 1, \cdots, k$ with disjoint neighborhoods V_i. By a theorem of Morse [13] we can assume V_i has a coordinate system $x=(x_1, \cdots, x_n)$ such that for $\| x \| \leqq \delta$, some $\delta > 0$, $f(x) = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$. Let E_1 be the $(x_1, \cdots, x_\lambda)$ plane of V_i and E_2 the $(x_{\lambda+1}, \cdots, x_n)$ plane. Then for $\varepsilon_1 > 0$ sufficiently small $E_1 \cap f^{-1}[-\varepsilon_1, \varepsilon_1]$ is diffeomorphic to D^λ. A sufficiently small tubular neighborhood T of E_1 will have the property that $T' = T \cap f^{-1}[-\varepsilon_1, \varepsilon_1]$ is diffeomorphic to $D^\lambda \times D^{n-\lambda}$ with $T \cap f^{-1}(-\varepsilon_1)$ corresponding to $\partial D^\lambda \times D^{n-\lambda}$.

As we pass from $f^{-1}[-\infty, -\varepsilon_1]$ to $f^{-1}[-\infty, \varepsilon_1]$, it happens that one such T' is added for each i, together with a tubular neighborhood of $f^{-1}(-\varepsilon_1)$ so that $f^{-1}[-\infty, \varepsilon_1]$ is diffeomorphic to a manifold of the form $\chi(f^{-1}[-\infty, -\varepsilon_1], f^{-1}(-\varepsilon_1); f_1, \cdots, f_k; \lambda)$. Since there are no critical points between $-\varepsilon$ and $-\varepsilon_1$, ε_1 and ε, ε_1 can be replaced by ε in the preceding statement thus proving (6.1).

Theorem (6.2) is roughly a converse of (6.1) and a sketch of the proof can be constructed similarly.

7. In this section we prove Theorems C and I of the Introduction.

The following theorem was proved in [21].

(7.1) THEOREM. *Let V^n be a C^∞ compact manifold with ∂V the disjoint union of V_1 and V_2, each V_i closed in ∂V. Then there exists a C^∞ function f on V with non-degenerate critical points, regular on ∂V, $f(V_1) = -(1/2), f(V_2) = n + (1/2)$ and at a critical point β of $f, f(\beta) =$* index β.

Functions described in (7.1) are called *nice* functions.

Suppose now M^n is a closed C^∞ manifold and f is the function of (7.1). Let $X_s = f^{-1}[0, s + (1/2)]$, $s = 0, \cdots, n$.

(7.2) LEMMA. *For each s, the manifold X_s has a presentation of the form $(X_{s-1}; f_1, \cdots, f_k; s)$.*

This follows from (6.1).

(7.3). LEMMA. *If $H \in \mathcal{H}(n, k, s)$, then there exists—a C^∞ non-degenerate function f on $H, f(\partial H) = s+(1/2)$, f has one critical point of index 0, value 0, k critical points of index s, value s and no other critical points.*

This follows from (6.2).

The proof of Theorem C then goes as follows. Take a nice function f on M by (7.1), with X_s defined as above. Note that $X_0 \in \mathcal{H}(n, q, 0)$ and $\pi_0(X_1) = 0$, hence by (7.2) and (1.2), $X_1 \in \mathcal{H}(n, k, 1)$. Suppose now that $\pi_1(M) = 1$ and $n \geqq 6$. The following argument suggested by H. Samelson simplifies and replaces a complicated one of the author. Let X'_2 be the sum of X_2 and k copies $H_1, \cdots, H_k$ of $D^{n-2} \times S^2$. Then since $\pi_1(X_2) = 0$, (1.2) implies that $X'_2 \in H(n, r, 2)$. Now let $f_i\colon \partial D^3 \times D^{n-3} \to \partial H_i \cap \partial X'_2$ for $i = 1, \cdots, k$ be differentiable imbeddings such that the composition

$$\pi_2(\partial D^3 \times D^{n-3}) \to \pi_2(\partial H_i \times \partial X'_2) \to \pi_2(\partial H_i)$$

is an isomorphism. Then by (3.3) and (3.4), $\chi(X_2', f_1, \cdots, f_k; 3)$ is diffeomorphic to X_2. Since $X_3 = \chi(X_2; g_1, \cdots, g_l; 3)$ we have

$$X_3 = \chi(X_2', f_1, \cdots, f_k, g_1, \cdots, g_l; 3) ,$$

and another application of (1.2) yields that $X_3 \in H(n, k + l - r, 3)$.

Iteration of the argument yields that $X_m' \in \mathcal{H}(n, r, m)$. By applying (7.3), we can replace g by a new nice function h with type numbers satisfying $M_0 = 1, M_i = 0, 0 < i < m$. Now apply the preceding arguments to $-h$ to yield that $h^{-1}[n - m - (1/2), n] = X_m^* \in \mathcal{H}(n, k_1, m)$. Now we modify h by (7.3) on X_m^* to get a new nice function on M agreeing with h on $M - X_m^*$ and satisfying the conditions of Theorem C.

The proof of Theorem I goes as follows. Let V^n be a manifold with $\partial V = V_1 - V_2, n = 2m + 2$. Take a nice function f on V by (7.1) with $f(V_1) = -(1/2)$ and $f(V_2) = n + (1/2)$.

Following the proof of Theorem C, replacing the use of (1.2) with (5.1), we obtain a new nice function g on V with $g(V_1) = -(1/2)$, $g(V_2) = n + (1/2)$ and no critical points except possibly of index $m + 1$. The following lemma can be proved by the standard methods of Morse theory [12].

(7.4) LEMMA. *Let V be as in* (7.1) *and f be a C^∞ non-degenerate function on V with the same boundary conditions as in* (7.1). *Then*

$$\chi_V = \sum(-1)^q M_q + \chi_{V_1} ,$$

where χ_V, χ_{V_1} are the respective Euler characteristics, and M_q denote the q^{th} type number of f.

This lemma implies that our function g has no critical points, and hence V_1 and V_2 are diffeomorphic.

8. We have yet to prove Theorems F and G. For Theorem F, observe by Theorem C, there is a nice function f on M with vanishing type numbers except in dimensions M_0, M_m, M_{m+1}, M_n, and $M_0 = M_n = 1$. Also, by the Morse relation, observe that the Euler characteristic is the alternating sum of the type numbers, $M_m = M_{m+1}$. Then by (7.2), $f^{-1}[0, m + (1/2)]$, $f^{-1}[m + (1/2), 2m + 1] \in \mathcal{H}(2m + 1, M_m, m)$ proving Theorem F.

All but the last statement of Theorem G has been proved. For this just note that $M - D^{2m}$ is diffeomorphic to $f^{-1}[0, m + (1/2)]$ which by (7.2) is in $\mathcal{H}(2m, k, m)$.

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