Math 151 Workshop Problems, Fall 2001, Week 14

1. Let the function $f(x)$ be defined by $f(x) = 2 - x$ for $0 \le x \le 1$, $f(x) = 1$ for $1 < x \le 2$, and $f(x) = x - 1$ for $2 < x \le 3$.

a) Draw the graph of $y = f(x)$ for $0 \le x \le 3$.

b) Let $0 \le b \le 3$. Define $F(b)$ to be the area under the graph of $y = f(x)$ between $x = 0$ and $x = b$. Use elementary geometry to find a formula for $F(b)$. You will have to consider the cases $0 \le b \le 1$, $1 < b \le 2$ and $2 < b \le 3$ separately. Draw the graph of $y = F(b)$.

c) Where is F (not) continuous? Where is it (not) differentiable?

d) Calculate $F'(x)$ for $0 \le x \le 3$ (except where F isn't differentiable!) and draw the graph of $y = F'(x)$. Have you seen this graph recently? (This is not a coincidence; it's a consequence of the Fundamental Theorem of Calculus.)

2. Interpret each definite integral below as signed area (draw a picture in each case). Then using only formulas from geometry, scissors and paste, compute the value of each expression.

a) $\displaystyle\int_0^4 \sqrt{16 - x^2}\, dx$ b) $\displaystyle\int_0^{\pi/2} \sin x\, dx + \int_0^1 \arcsin x\, dx$ c) $\displaystyle\int_{-1}^1 \left(\sqrt{1 - x^2} - \frac{1}{2}\right) dx$

3. Let $f(x) = \dfrac{8}{x^2 + 4}$.

a) What is the largest value of $f(x)$ on the interval $0 \le x \le 2$? The smallest value?

b) Use your answers to a) and the geometric meaning of the definite integral as an area (draw a picture!) to show that

$$2 \le \int_0^2 \frac{8}{x^2 + 4}\, dx \le 4.$$

c) By cutting the interval $0 \le x \le 2$ into two pieces and repeating a) and b) for each piece, show that

$$2.6 \le \int_0^2 \frac{8}{x^2 + 4}\, dx \le 3.6$$

d) Now cut the interval $0 \le x \le 2$ into four pieces and repeat a) and b) for each piece to find closer upper and lower estimates for the integral. Show the graphical intepretation.

e) Use the `fnInt(` program in your calculator to calculate the numerical value of the integral. (This program is in the `MATH` menu; see your calculator's instruction manual for examples.) Note: the exact value of the integral, computed by the fundamental theorem, is 4 arctan (2/2), which is better known by another name.

4. The symbol $\int f(x)\,dx$ means a function $F(x)$ whose derivative is $f(x)$; we call $F(x)$ an *antiderivative* or an *indefinite integral* of $f(x)$.

a) Evaluate five of the following six indefinite integrals.

$$\int 3xe^{x^2}\,dx \qquad \int (2x^8+1)^4 x^9\,dx \qquad \int 5(16-x)^{-3}\,dx$$

$$\int x^{-9}\sqrt{2x^{-8}+1}\,dx \qquad \int x\cos(\pi x^2)\,dx \qquad \int \frac{1}{\sqrt{1-4x^2}}\,dx$$

b) Modify the integral that you omitted to change it into one that you can evaluate (there are many ways to do this–find at least two such modifications).

5. In each case, make an educated guess of the number of functions F satisfying the given conditions. Base your guess on the number and type of conditions (think about how many constants of integration there are), and explain how you arrived at your guess. Then use integration actually to find all the functions F satisfying the given conditions. Observe if your guess was correct, and explain what went wrong if it was incorrect.

a) $F'(x) = \dfrac{4x^3+3x^2+7}{x^4+x^3+7x+12}$ for all $x > 0$; $F(1) = 2$.

b) $F'(x) = \dfrac{1}{2x+0.25}$ for all $x > 0$; $F(0.375) = 0$ and $F(0.875) = 1$.

c) $F''(x) = \sin(2x - \frac{\pi}{4})$; $F(0) = 0$ and $F(\pi) = 0$.

d) $F''(x) = \cos 2x + e^{3x}$ for all $x > 0$; $F(\pi) = 0$.

6. a) Compute the area of the bounded region enclosed by the curve $y = e^x$, the line $y = 12$ and the y-axis.

b) How does this area compare with the value of the integral $\displaystyle\int_1^{12} \ln x\,dx$? Explain your answer.

7. Below is the graph of a function f whose domain is $[-3, 5]$. The part of the curve from $x = -2$ to $x = 0$ is a parabola under which the area is $4/3$ square units.
Let F be the function defined by

$$F(x) = \int_0^x f(t)\,dt.$$

As best you can, sketch the graph of F. Where are the x-intercepts of F? Where is F continuous? Where is F differentiable? Where is F increasing? decreasing? concave up? concave down? Relate all these answers to the graph of f.

8. Calculate $\int \sin x \cos x \, dx$ in several different ways: a) by substituting $u = \sin x$; b) by substituting $u = \cos x$; c) by using the identity $\sin 2x = \sin x \cos x$. Then reconcile your three different-looking answers.

9. There is a finite region bounded by the three curves $y = x^2 + x$, $y = -\dfrac{x^2}{2} + x$, and $x + y = 1$. Sketch the region and find its area.