Math 151 Workshop Problems, Fall 2001, Week 13

1. You are $6'$ tall. A $5'$-high painting is mounted on a wall so that its lower edge is $1'$ above your eye when you stand next to the wall.

a) Suppose that you stand x feet away from the wall to look at the painting. Find a formula for your viewing angle θ between the top and bottom of the painting, as a function of x. (*Hint:* Draw a horizontal line at eye level.)

b) Graph θ as a function of x and estimate from the graph where it has a maximum. Now use calculus to find the value of x that maximizes θ.

2. A charged particle moves along the x-axis under the influence of an electric field. The field strength varies with time, and as a result the velocity of the particle seems complicated. Let the position of the particle at time t be denoted by $x = x(t)$, and the velocity of the particle at time t be denoted $v = v(t)$.
Suppose that $x(0) = 0$, and

$$v(t) = \begin{cases} 8t - 2, & \text{if } 0 \le t \le 1 \\ 9t - 3t^2, & \text{if } 1 \le t \le 2 \\ 8 - t, & \text{if } 2 \le t \le 3 \end{cases}$$

Sketch the graph of $v = v(t)$. What is $x(1)$? What is $x(2)$? What is $x(3)$? Sketch the graph of $x = x(t)$.

3. Find a function $f(x)$ such that $f''(x) = e^{-x}$ and the graph of $y = f(x)$ passes through the point $(0, 3)$ and has a horizontal tangent there. Do these conditions determine f completely?

4. a) A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 40 ft/s^2. What is the distance covered before the car comes to a stop?

b) A car braked with a constant deceleration of 40 ft/s^2 and produced skid marks measuring 160 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

5. (a) Draw the trapezoid with vertices $(2, 0)$, $(4, 0)$, $(2, 5)$ and $(4, 11)$, compute its area A, and find an equation for the top edge.

(b) Draw a figure and explain geometrically why the number

$$\frac{1}{100} \sum_{n=1}^{200} \left[3\left(2 + \frac{n}{100}\right) + 1 \right]$$

represents an area which is just slightly larger than A.

(c) Move the "error" pieces in your figure around to show that the difference between the two areas is no more than 0.06 square units.

6. Let the function $f(x)$ be defined by $f(x) = 2 - x$ for $0 \le x \le 1$, $f(x) = 1$ for $1 < x \le 2$, and $f(x) = x - 1$ for $2 < x \le 3$.

a) Draw the graph of $y = f(x)$ for $0 \le x \le 3$.

b) Let $0 \le b \le 3$. Define $F(b)$ to be the area under the graph of $y = f(x)$ between $x = 0$ and $x = b$. Use elementary geometry to find a formula for $F(b)$. You will have to consider the cases $0 \le b \le 1$, $1 < b \le 2$ and $2 < b \le 3$ separately. Draw the graph of $y = F(b)$.

c) Where is F (not) continuous? Where is it (not) differentiable?

d) Calculate $F'(x)$ for $0 \le x \le 3$ (except where F isn't differentiable!) and draw the graph of $y = F'(x)$. Have you seen this graph recently? (This is not a coincidence; it's a consequence of the Fundamental Theorem of Calculus.)

7. Interpret each definite integral below as signed area (draw a picture in each case). Then using only formulas from geometry, scissors and paste, compute the value of each expression.

a) $\displaystyle\int_0^4 \sqrt{16 - x^2}\, dx$ b) $\displaystyle\int_0^{\pi/2} \sin x\, dx + \int_0^1 \arcsin x\, dx$ c) $\displaystyle\int_{-1}^1 \left(\sqrt{1 - x^2} - \frac{1}{2}\right) dx$

8. Let $f(x) = \dfrac{8}{x^2 + 4}$.

a) What is the largest value of $f(x)$ on the interval $0 \le x \le 2$? The smallest value?

b) Use your answers to a) and the geometric meaning of the definite integral as an area (draw a picture!) to show that

$$2 \le \int_0^2 \frac{8}{x^2 + 4}\, dx \le 4.$$

c) By cutting the interval $0 \le x \le 2$ into two pieces and repeating a) and b) for each piece, show that

$$2.6 \le \int_0^2 \frac{8}{x^2 + 4}\, dx \le 3.6$$

d) Now cut the interval $0 \le x \le 2$ into four pieces and repeat a) and b) for each piece to find closer upper and lower estimates for the integral. Show the graphical intepretation.

e) Use the `fnInt(` program in your calculator to calculate the numerical value of the integral. (This program is in the `MATH` menu; see your calculator's instruction manual for examples.) Note: the exact value of the integral, computed by the fundamental theorem, is 4 arctan (2/2), which is better known by another name.

9. The symbol $\int f(x)\,dx$ means a function $F(x)$ whose derivative is $f(x)$; we call $F(x)$ an *antiderivative* or an *indefinite integral* of $f(x)$.

a) Evaluate five of the following six indefinite integrals.

$$\int 3xe^{x^2}\,dx \qquad \int (2x^8+1)^4 x^9\,dx \qquad \int 5(16-x)^{-3}\,dx$$

$$\int x^{-9}\sqrt{2x^{-8}+1}\,dx \qquad \int x\cos(\pi x^2)\,dx \qquad \int \frac{1}{\sqrt{1-4x^2}}\,dx$$

b) Modify the integral that you omitted to change it into one that you can evaluate (there are many ways to do this–find at least two such modifications).

10. In each case, make an educated guess of the number of functions F satisfying the given conditions. Base your guess on the number and type of conditions (think about how many constants of integration there are), and explain how you arrived at your guess. Then use integration actually to find all the functions F satisfying the given conditions. Observe if your guess was correct, and explain what went wrong if it was incorrect.

a) $F'(x) = \dfrac{4x^3+3x^2+7}{x^4+x^3+7x+12}$ for all $x>0$; $\quad F(1)=2$.

b) $F'(x) = \dfrac{1}{2x+0.25}$ for all $x>0$; $\quad F(0.375)=0$ and $F(0.875)=1$.

c) $F''(x) = \sin(2x-\frac{\pi}{4})$; $F(0)=0$ and $F(\pi)=0$.

d) $F''(x) = \cos 2x + e^{3x}$ for all $x>0$; $\quad F(\pi)=0$.

11. a) Compute the area of the bounded region enclosed by the curve $y=e^x$, the line $y=12$ and the y-axis.

b) How does this area compare with the value of the integral $\displaystyle\int_1^{12} \ln x\,dx$? Explain your answer.