Math 151 Workshop Problems, Fall 2001, Week 10

1. For functions f and g, we use the symbolism

$f(x) >> g(x)$ as $x \to a$ if $\displaystyle\lim_{x\to a}\left|\frac{f(x)}{g(x)}\right| = +\infty$

$f(x) \sim g(x)$ as $x \to a$ if $\displaystyle\lim_{x\to a}\left|\frac{f(x)}{g(x)}\right|$ is a positive real number

$f(x) << g(x)$ as $x \to a$ if $\displaystyle\lim_{x\to a}\left|\frac{f(x)}{g(x)}\right| = 0$

For each of the following pairs of functions f and g, determine whether $f(x) >> g(x)$, $f(x) \sim g(x)$ or $f(x) << g(x)$ as x approaches the indicated value.

a) e^x and 3^x as $x \to +\infty$

b) e^x and x^{30} as $x \to +\infty$

c) $e^x + e^{-3x}$ and $e^{3x} + e^{-x}$ as $x \to +\infty$; as $x \to -\infty$

d) $e^{1/x}$ and $e^{1/(3x)}$ as $x \to +\infty$

e) $x^{3.0001}$ and $x^3 \ln x$ as $x \to +\infty$

f) xe^{x^3} and x^3e^x as $x \to +\infty$

g) $\exp(1/x^2)$ and $\exp(1/x^3)$ as $x \to 0$

h) x^{-2} and $(x^{-2} + 2x^{-1})$ as $x \to 0^+$

2. Suppose that you know that $h'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$. What are the critical numbers of h? Which of them are local extrema, and what kind of local extrema are they? You don't need to know a formula for $h(x)$ to answer these questions, so don't waste your time trying to find one.

3. For any constant c, let $f_c(x) = x^3 + 2x^2 + cx$. Thus $f_1(x) = x^3 + 2x^2 + x$, $f_5(x) = x^3 + 2x^2 + 5x$, etc. *

(a) Graph $y = f_c(x)$ for the following values of the parameter c: $c = -1, 0, 1, 2, 3, 4$. What are the similarities and differences among the graphs, and how do the graphs change as the parameter increases?

(b) For what values of the parameter c will f_c have one local maximum and one local minimum? Use calculus. As c increases, what happens to the distance between the local maximum and the local minimum?

(c) For what values of the parameter c will f_c have no local maximum or local minimum? Use calculus.

(d) Are there any values of the parameter c for which f_c will have exactly one horizontal tangent line?

4. a) Suppose that you know the following about a differentiable function f defined for all x:

$$7 \le f'(x) \le 11 \text{ for all } x \text{ in } [3,5], \text{ and } f(3) = 17,$$

* In this situation, there are many functions, one for each possible constant c; although c is a constant, we may imagine different values for it. Such "unspecified constants which may be assigned different values" are sometimes called "parameters".

and you know *nothing else* about f. Use the Mean Value Theorem to find numbers L and U such that $f(5)$ must lie between L and U. Illustrate by sketching a wedge formed by two lines through $(3,17)$ within which the graph of f must lie (at least for $3 \le x \le 5$).

(b) Suppose that the numbers 3, 5, 7, and 11 in (a) are replaced by numbers a, b, α, β. Determine numbers L and U such that it must be true that $L \le f(b) \le U$. Your L and U should be expressed in terms of a, b, α, β and $f(a)$, and you should use the Mean Value Theorem to show that $L \le f(b) \le U$.

c) Now suppose you know that $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$, and that $f(0) = 3$, but that you have no more information about the function f. Use your answer to (b), with $a = 0$ and $b = 2$, to name as narrow an interval as you can in which the number $f(2)$ must lie. Don't try to find a formula for $f(x)$. You might be able to, with a lot of work, but it would sidestep the point of this problem.

d) Get a better (narrower) answer to the question in (c), by first narrowing down $f(1)$ and then narrowing down $f(2)$. (Note: you must adjust the a and b and α and β you use here carefully; you will be doing the process twice.)

5. Consider the functions given by the equation $f_c(x) = (x^2 + c)e^x$, where c is a parameter.

(a) Using the calculator, observe the curves for $-4 \le x \le 1$, for the values $c = 0, 1, 2$. Do all three curves have the same number of horizontal tangents? the same number of inflection points? You may have to ZOOM in to investigate this.

(b) Use calculus to determine the location of all the inflection points of the graph of $y = f_c(x)$. Your answer will of course depend on c.

(c) At what values of c does the number of inflection points change? What are the values of c (if any) for which there is exactly one inflection point?

6. Imagine using the linearization of $\arccos x$ at $x_0 = 1/2$ to calculate an approximate value for arccos(.49). But do not do the linearization until part (c).

(a) What do the first and second derivatives tell you about the graph of $\arccos x$ near $x = 1/2$?

(b) Use your answer to (a) to decide if the approximate value calculated by linearization is an overestimate or an underestimate of the true value. Explain your decision.

(c) Now do the linearization and compare the approximate value with the true value to check your answer to (b).

7. A piece of wire $18'$ long is to be cut into two pieces, one to be bent into a square and the other to be bent into an equilateral triangle. Let A be the sum of the areas of the square and the triangle.

(a) How should the wire be cut to maximize A?

(b) How should the wire be cut to minimize A?

(c) How would your answers differ if it were permitted not to cut the wire, but to use the whole wire for a square or the whole wire for a triangle?

8. A castle is surrounded by a moat which is uniformly $27'$-wide. At the outer edge of the moat is an $8'$-foot fence. The castle abuts the inner edge of the moat. What is the length of the shortest ladder that will reach across the fence and moat to the castle? (Hints: Think of the ladder in two pieces. One choice for the independent variable is the angle between the ladder and the ground.)

9. If f is a function, then a real number x_0 is called a *fixed point* of f if and only if $f(x_0) = x_0$.

(a) Find all the fixed points of the following functions to two-place accuracy.

$$f(x) = x^2 \qquad g(x) = x^5 \qquad h(x) = \frac{2}{3}\arctan x.$$

(b) Illustrate your answers graphically, using the graphs of $y = x$ and $y = f(x)$.

(c) Suppose that f is a differentiable function and $f'(x) < 1$ for all x. Use the MVT to explain why f can have no more than one fixed point. To which of the functions in (a) does this general statement apply?