Math 151 Workshop Problems, Fall 2001, Week 9

1. Suppose that we measure the density ρ of perfectly cylindrical log by measuring the diameter D, height h and weight W of the log and using the formula

$$\rho = \frac{4W}{\pi D^2 h}.$$

In terms of *relative* error, is the result most sensitive to errors in the measurement of D, of h, or of W? That is, which is the worst: a 1% error in D, a 1% error in h, or a 1% error in W? Arrive at your answer as follows:

(a) Use linear approximation to estimate the relationship between the relative errors $\Delta D/D$ and $\Delta\rho/\rho$, treating h and W as constants for the purposes of this calculation.

(b) Do the same for the other two variables, instead of D, and compare the three results.

2. The following statements are true:

$$(10,000,000,000)^{\left(\frac{1}{10,000,000,000}\right)} = 1.\ 00000\ 000\underline{23}\ \underline{02585}\ \underline{09}56\cdots$$

and

$$\ln 10 = \underline{2}.\underline{30258}\ \underline{509}2\cdots$$

Explain the amazing coincidence of the digits.

Hint Use the linearization of e^x around $x = 0$.

3. Consider the equation $x^4 - 18x^2 - 15 = 0$. Show that solving this equation by Newton's method leads to the recursion $x_{n+1} = \dfrac{3[x_n^4 - 6x_n^2 + 5]}{4x_n(x_n^2 - 9)}$.

(a) Carry out two recursions starting with $x_0 = 4$. Draw the graph of $y = x^4 - 18x^2 - 15$ and interpret your calculations on the graph, drawing a tangent line at $x = 4$, etc., etc. What do you think happens to x_n as $n \to \infty$?

(b) Same question if $x_0 = 1/2$.

(c) Same question if $x_0 = \sqrt{5}$.

4. (a) Graph the curves $y_1 = \sin x$ and $y_2 = 3\cos x$. Use the cursor and the ZOOM BOX to find the the point of intersection of the curves that has the smallest positive x-value. Read off the approximate coordinates (x_0, y_0) of this point.

(b) Write out the Newton's method iteration to find the solutions to the equation $\sin x - 3\cos x = 0$. Give a formula for x_{n+1} in terms of x_n for any n.

(c) Starting with the value x_0, calculate x_1 and x_2 using your formula in (b). This is easy if you use the SEQUENCE mode in your calculator (see the manual). Compare your results with the approximate values that you can get for the solution by ZOOMing in on the point of intersection using the calculator.

5. Find the limits for the following, which are in the indeterminate form "$\infty - \infty$".

a) $\lim_{x\to 0} \frac{1}{\tan x} - \frac{1}{x}$

b) $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} - \frac{1}{x}$

c) $\lim_{x\to 0} \frac{e^x}{x} - \frac{e^{-x}}{x}$

Note that three completely different types of behavior occur.

6. For functions f and g, we use the symbolism

$f(x) >> g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = +\infty$

$f(x) \sim g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right|$ is a positive real number

$f(x) << g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = 0$

For each of the following pairs of functions f and g, determine whether $f(x) >> g(x)$, $f(x) \sim g(x)$ or $f(x) << g(x)$ as x approaches the indicated value.

a) e^x and 3^x as $x \to +\infty$

b) e^x and x^{30} as $x \to +\infty$

c) $e^x + e^{-3x}$ and $e^{3x} + e^{-x}$ as $x \to +\infty$; as $x \to -\infty$

d) $e^{1/x}$ and $e^{1/(3x)}$ as $x \to +\infty$

e) $x^{3.0001}$ and $x^3 \ln x$ as $x \to +\infty$

f) xe^{x^3} and $x^3 e^x$ as $x \to +\infty$

g) $\exp(1/x^2)$ and $\exp(1/x^3)$ as $x \to 0$

h) x^{-2} and $(x^{-2} + 2x^{-1})$ as $x \to 0^+$

7. Suppose that you know that $h'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$. What are the critical numbers of h? Which of them are local extrema, and what kind of local extrema are they? You don't need to know a formula for $h(x)$ to answer these questions, so don't waste your time trying to find one.

8. For any constant c, let $f_c(x) = x^3 + 2x^2 + cx$. Thus $f_1(x) = x^3 + 2x^2 + x$, $f_5(x) = x^3 + 2x^2 + 5x$, etc. *

(a) Graph $y = f_c(x)$ for the following values of the parameter c: $c = -1, 0, 1, 2, 3, 4$. What are the similarities and differences among the graphs, and how do the graphs change as the parameter increases?

(b) For what values of the parameter c will f_c have one local maximum and one local minimum? Use calculus. As c increases, what happens to the distance between the local maximum and the local minimum?

* In this situation, there are many functions, one for each possible constant c; although c is a constant, we may imagine different values for it. Such "unspecified constants which may be assigned different values" are sometimes called "parameters".

(c) For what values of the parameter c will f_c have no local maximum or local minimum? Use calculus.

(d) Are there any values of the parameter c for which f_c will have exactly one horizontal tangent line?

9. a) Suppose that you know the following about a differentiable function f defined for all x:

$$7 \leq f'(x) \leq 11 \text{ for all } x \text{ in } [3,5], \text{ and } f(3) = 17,$$

and you know *nothing else* about f. Use the Mean Value Theorem to find numbers L and U such that $f(5)$ must lie between L and U. Illustrate by sketching a wedge formed by two lines through $(3, 17)$ within which the graph of f must lie (at least for $3 \leq x \leq 5$).

(b) Suppose that the numbers 3, 5, 7, and 11 in (a) are replaced by numbers a, b, α, β. Determine numbers L and U such that it must be true that $L \leq f(b) \leq U$. Your L and U should be expressed in terms of a, b, α, β and $f(a)$, and you should use the Mean Value Theorem to show that $L \leq f(b) \leq U$.

c) Now suppose you know that $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$, and that $f(0) = 3$, but that you have no more information about the function f. Use your answer to (b), with $a = 0$ and $b = 2$, to name as narrow an interval as you can in which the number $f(2)$ must lie. Don't try to find a formula for $f(x)$. You might be able to, with a lot of work, but it would sidestep the point of this problem.

d) Get a better (narrower) answer to the question in (c), by first narrowing down $f(1)$ and then narrowing down $f(2)$. (Note: you must adjust the a and b and α and β you use here carefully; you will be doing the process twice.)