Math 151 Workshop Problems, Fall 2000, Week 7

1. Moving along the s-axis for all time, a particle has the position $s = t^3 + 9t^2 + 27t + 81$. There is a certain velocity which the particle attains just once in its trip.

(a) What is this velocity, and when is it attained?

(b) Graph s versus t and explain what your answer to (a) tells you about the graph.

2. From the point $(8, 0)$ you can draw tangent lines to the parabola $y = x^2 - x$. How many such tangent lines are there, and where are the points of tangency? Graph the parabola and the tangent lines.

3. Two particles P_1 and P_2 move along the s-axis starting at time $t = 0$. Their respective position functions are $s_1(t) = 8e^{2t} - e^{4t}$ and $s_2(t) = \sin^2 2t - \sin^2 t$.

(a) Find on your calculator a numerical approximation for the time and place at which they (first) collide. Is it a head-on collision?

(b) Find the time and place at which P_1 changes direction for the first time. Then do the same for P_2. Give exact answers (in terms of π, e, square roots, logs, etc.) and also give 2-place decimal approximations.

4. a) In this problem you will consider two families of curves. The first family consists of the curve $xy^2 = 1$, the curve $xy^2 = 2$, etc., that is, the curves $xy^2 = C$, one for each constant C. Pick a few values of C and sketch the curves. The second family consists of the curves $2x^2 - y^2 = C$, again one for each constant C. Sketch a few of these too, in the same window.

b) Show that any curve in the first family and any curve in the second family are perpendicular wherever they happen to meet. For this reason the two families are said to be "orthogonal". Orthogonal families play a very important role in the study of electromagnetism and heat flow. Besides, they make elegant pictures.

c) Now consider a different family of curves, the one defined by the equations $xy^3 = C$. Can you find a family orthogonal to it? (Hint: if you choose the right constant A, the family $Ax^2 - y^2 = C$ will work.) Check your answer with a few graphs.

5. Let a be a positive constant and consider the functions

$$f(x) = \arcsin\left(\frac{x}{a}\right) \quad \text{and} \quad g(x) = a \arctan\left(\frac{x}{a}\right).$$

Find the derivatives of f and g and express them in as simple a form as possible. There is a certain value of a for which the lines tangent to the graphs of these two functions at $x = 1$ are parallel lines. Find that value of a to 3-place accuracy. (Find an exact equation satisfied by a, and then get an accurate enough solution from your calculator.)

6. Find the following derivatives:

(a) $\dfrac{d}{dx}\left(\left[\ln(1+x^2)\right]\left[\ln(1-x^2)\right]\right)$ and $\dfrac{d}{dx}\left(\ln\left[(1+x^2)(1-x^2)\right]\right)$

(b) $\dfrac{d}{dx}\left(\dfrac{x^x\sqrt{1+e^2}}{e^{2x}\sqrt{1+x^2}}\right)$. (For (b), use the logarithmic differentiation trick: $y' = y(\ln y)'$)

7. An unidentified object moves along the s-axis, with displacement $s = s(t)$ (meters), velocity $v = v(t)$ (m/sec) and acceleration $a = a(t)$ (m/sec^2). It so happens that the velocity and displacement are related by the equation $v = \sqrt{8s + 16}$. Moreover, at the instant $t = 0$, the object is observed at $s = 6$.

a) Show that a is constant, and find its value.

b) Graph v as a function of s.

c) Graph v as a function of t.

8. An object is moving along the parabola $y = 3x^2$.

(a) When it passes through the point $(2, 12)$, its "horizontal" velocity is $\dfrac{dx}{dt} = 3$. What is its "vertical" velocity at that instant?

(b) If it travels in such a way that $\dfrac{dx}{dt} = 3$ for all t, then what happens to $\dfrac{dy}{dt}$ as $t \to +\infty$?

(c) If, however, it travels in such a way that $\dfrac{dy}{dt}$ remains constant, then what happens to $\dfrac{dx}{dt}$ as $t \to +\infty$?

9. A soccer player is dribbling while running parallel to the sideline and directly toward the point on the goal line which is $30'$ from the near edge of the goal. The goal mouth is $20'$ wide. Does the goal mouth seem to be getting bigger or smaller to her (that is, is the angle θ increasing or decreasing) when she is a) $40'$ from the goal line; b) $35'$ from the goal line; c) $30'$ from the goal line? Make sure that your answers fit with common sense. (The question is about $\dfrac{d\theta}{dt}$.)