**Math 151: Exam 1** NAME: SOLUTIONS

**Tuesday 9:50-11:10am, October 9, 2001** in **ARC 108**

**1.** Find an equation for the tangent line to the curve $x^3+y^3=6xy$ at the point $(3,3)$.
*Solution* . Using implicit differentiation with respect to $x$, we get

$$3x^2+3y^2y'=6(xy'+y).$$

Solving for $y'$ gives

$$y'(3y^2-6x)=6y-3x^2,$$

and so

$$y'=\frac{2y-x^2}{y^2-2x}.$$

Hence, at $(x,y)=(3,3)$, we have

$$y'(3)=\frac{6-9}{9-6}=-1.$$

The equation for the tangent line through $(3,3)$ with slope $y'(3)=-1$ is then $y-3=-(x-3)$, or

$$y=-x+6.$$

**2.** Compute the derivatives of the following functions:

(a) $y=e^{x\cos x}$. *Solution:* $y'=e^{x\cos x}(-x\sin x+\cos x)$.
(b) $y=(x^3+4x)^7$. *Solution:* $y'=7(x^3+4x)^6(3x^2+4)$.
(c) $y=\dfrac{e^x}{x+\ln x}$. *Solution:* $y'=((x+\ln x)e^x-e^x(1+1/x))/(x+\ln x)^2$.

**3.** Compute the following limits:

(a) $\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x}$. *Solution:*

$$\begin{aligned}
&\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x}\\
&=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\\
&=\lim_{x\to\infty}\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\\
&=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\\
&=\lim_{x\to\infty}\frac{2+x^{-1}}{\sqrt{1+x^{-1}+x^{-2}}+\sqrt{1-x^{-1}}} \qquad \text{(Note: We can assume } x>0\text{, since } x\to+\infty)\\
&=2.
\end{aligned}$$

(b) $\lim_{x\to 0}x^2\cos(1/x^2)$. *Solution:* Since $-1\le\cos(1/x^2)\le 1$ for all $x\neq 0$, and $\lim_{x\to0}\pm x^2=0$, the Squeeze Theorem gives $\lim_{x\to0}x^2\cos(1/x^2)=0$.

(c) $\lim_{x\to\infty} \dfrac{5x^3 - x^2 + 2}{2x^3 + x - 3}$. *Solution:*

$$\lim_{x\to\infty} \frac{5x^3 - x^2 + 2}{2x^3 + x - 3} = \lim_{x\to\infty} \frac{5 - x^{-1} + 2x^{-3}}{2 + x^{-2} - 3x^{-3}} = \frac{5}{2}.$$

(d) $\lim_{x\to 0} \dfrac{\sin(6x)}{\sin(8x)}$. *Solution:*

$$\lim_{x\to 0} \frac{\sin(6x)}{\sin(8x)} = \lim_{x\to 0} \frac{\sin(6x)}{6x} \frac{8x}{\sin(8x)} \frac{6x}{8x} = \frac{6}{8}.$$

**4.** Either compute the following limits if they exist or explain why they do not exist:

(a) $\lim_{x\to 0} \dfrac{\sin x}{|x|}$. *Solution:* Limit does not exist because

$$\lim_{x\to 0^+} \frac{\sin x}{|x|} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \quad \text{while} \quad \lim_{x\to 0^-} \frac{\sin x}{|x|} = \lim_{x\to 0^-} \frac{\sin x}{-x} = -1.$$

(b) $\lim_{x\to -\infty} \dfrac{\sqrt{4x^2+1}}{x+1}$. *Solution:* We can suppose $x < 0$, so $|x| = -x$ and

$$\begin{aligned} \lim_{x\to -\infty} \frac{\sqrt{4x^2+1}}{x+1} &= \lim_{x\to -\infty} \frac{\sqrt{x^2(4+x^{-2})}}{x+1} \\ &= \lim_{x\to -\infty} \frac{|x|\sqrt{4+x^{-2}}}{x+1} \\ &= \lim_{x\to -\infty} \frac{2|x|}{x} \\ &= -2. \end{aligned}$$

(c) $\lim_{x\to 0} \cos(1/x)$. *Solution:* Limit does not exist, since for $x = 1/\pi, 1/3\pi, 1/5\pi, \ldots$, we have $\cos(1/x) = -1$, while for $x = 1/2\pi, 1/4\pi, 1/6\pi, \ldots$, we have $\cos(1/x) = 1$.

**5.** Please answer the following:

(a) Explain carefully what it meant by saying that a function $y = f(x)$ is continuous at a point $x = a$. (It is not enough to give a graphical explanation.) *Solution:* A function $y = f(x)$ is continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

(b) Is it possible to find an $x$ in the interval $[0, 4]$ such that $\sin x = x^2 - 0.3$? Though not necessary to actually solve for such an $x$, if it exists, justify your answer. *Solution:* Let $f(x) = x^2 - 0.3 - \sin x$ and observe that $f(0) = -0.3 < 0 < f(4) = 16 - 0.3 - \sin 4 \approx 16$. Since $f$ is continuous on $[0, 4]$, the Intermediate Value Theorem implies that there exists a number $c$ so that $0 < c < 4$ and $f(c) = 0$.

**6.** The functions $f$ and $g$ are differentiable, with values and derivatives

$$f(3) = 2, \quad f'(3) = 7, \quad g(3) = 3, \quad g'(3) = 5.$$

Compute the following quantities, explaining your work (an answer alone will *not* receive full credit):

(a) $(f+g)'(3)$. *Solution:* $f'(3)+g'(3)=7+5=12$.
(b) $(f\cdot g)'(3)$. *Solution:* $f(3)\cdot g'(3)+f'(3)\cdot g(3)=2\cdot 5+7\cdot 3=31$.
(c) $\left(\dfrac{f}{g}\right)'(3)$. *Solution:* $(g(3)\cdot f'(3)-f(3)\cdot g'(3))/g(3)^2=(3\cdot 7-2\cdot 5)/3^2=11/9$.
(d) $(f\circ g)'(3)$. *Solution:* $f'(g(3))\cdot g'(3)=f'(3)\cdot g'(3)=7\cdot 5=35$.

**7.** Write the definition of the derivative as a limit and use the definition to compute the derivative of $f(x)=\sqrt{x}-3x$. *Solution:* Recall that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

*Solution* . We have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-3(x+h)-(\sqrt{x}-3x)}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}-\lim_{h\to 0}\frac{3(x+h)-3x}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)-3\\ &= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}-3\\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}-3.\\ &= \frac{2}{\sqrt{x}}-3.\end{aligned}$$

**8.** Suppose a function $f$ has inverse $f^{-1}=g$, with $f(2)=5$ and $f'(2)=17$. Find a formula for $g'(x)$ in terms of $f'$ and $g$ and evaluate $g'(5)$. [Hint: Use implicit differentiation.] *Solution:* Since $f(g(x))=x$, the chain rule gives

$$f'(g(x))\cdot g'(x)=1,$$

so that $g'(x)=1/f'(g(x))$. Hence, noting that $g(5)=g(f(2))=2$, we have

$$g'(5)=1/f'(g(5))=1/f'(2)=1/17.$$