Take-Home Problem Solutions. Mathematical Finance; Fall, 2005; Due November 3.

1. Consider a one-period market with one bond and one stock. The bond has interest rate $\bar{r}$ per period: thus, if $R(t)$ denotes the value at t of \$1 invested at time 0, $R(0) = 1$ and $R(1) = 1 + \bar{r}$. The stock price is denoted by $S(t)$. The market has three states, ω_1, ω_2, and ω_3. The stock prices at time 1 are $S(1)(\omega_1) = dS(0)$, $S(1)(\omega_2) = S(0)$, $S(1)(\omega_3) = uS(0)$. Assume $d < 1 < u$.

(a). Find a condition on $\bar{r}$, d, and u that ensures no arbitrage and prove the validity of your condition.

We claim there is no arbitrage if and only if $d < 1 + \bar{r} < u$. Many students solved this assuming $\bar{r} > 0$ so that they needed only to check $\bar{r}$. This was perfectly acceptable. We show here that if negative $\bar{r}$ is allowed (think deflation), but at least $1 + \bar{r} > 0$, then $d < 1 + \bar{r} < u$ is necessary and sufficient.

We give two methods. First, we show that if $1 + \bar{r} \leq d$ or $u \leq 1 + \bar{r}$, there is arbitrage. Indeed, suppose that $1 + \bar{r} \leq d$. Then borrow a dollar and invest it in $(1/S(0))$ units of stock. After the next period the stock will be worth $(1/S(0))S(1)(\omega_i)$ and this is greater than $1+\bar{r}$ in state ω_1 and strictly greater in states ω_2 and ω_3. This is then an arbitrage. If $u \leq 1 + \bar{r}$, selling a stock short and investing the money at the risk free rate achieves an arbitrage.

Now assume that there is an arbitrage portfolio (π_1, π_2). Then

$$\begin{array}{rcll}
\pi_1 + \pi_2 S(0) & \leq & 0 & (i) \\
\pi_1(1 + \bar{r}) + \pi_2 dS(0) & \geq & 0 & (ii) \\
\pi_1(1 + \bar{r}) + \pi_2 S(0) & \geq & 0 & (iii) \\
\pi_1(1 + \bar{r}) + \pi_2 uS(0) & \geq & 0 & (iv)
\end{array}$$

and not all the quantities listed are non-negative. Obviously this situation can only exist if $\pi_2 \not0$ since we are assuming $1 + \bar{r} > 0$. Inequalities (i) and (ii) and imply that

$$\pi_2 S(0)(d - (1 + \bar{r})) \geq 0, \qquad \pi_2 S(0)(u - (1 + \bar{r})) \geq 0 \tag{1}$$

(subtract $1+\bar{r}$ times the first equation from the second and fourth.) Now suppose that $d < 1 + \bar{r}$. It then follows that from the first inequality in (1) and from $\pi_2 \neq 0$ that $\pi_2 < 0$. But this then implies from the second inequaity in (1) that $u \leq 1 + \bar{r}$. Likewise, if $1 + \bar{r} < u$, (1) implies $1 + \bar{r} \leq d$. This shows that if there is no arbitrage then either $1 + \bar{r} \leq d$ or $1 + \bar{r} \geq u$.

We could also establish the condition, using the condition that characterizes no arbitrage by the existence of state-price vector. This is a 3-vector ψ of all positive components, such that

$$\begin{pmatrix} 1 \\ S(0) \end{pmatrix} = \begin{pmatrix} 1+\bar{r} & 1+\bar{r} & 1+\bar{r} \\ dS(0) & S(0) & uS(0) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{2}$$

Then general solution can be written as follows; choose ψ_2 arbitrarily and then set

$$\begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix} = \frac{1}{(1+\bar{r})(u-d)} \begin{pmatrix} u-(1+\bar{r})+\psi_2(1+\bar{r})(1-u) \\ (1+\bar{r})-d+\psi_1(1+\bar{r})(d-1) \end{pmatrix}$$

We need necessary and sufficient conditions that this system has a solution in which all terms are positive. If $\psi_2 > 0$, then the terms $\psi_2(1+\bar{r})(1-u)$ and $\psi_1(1+\bar{r})(d-1)$ are negative. So clearly there is no solution all of whose terms is positive if $u \leq (1+r)$ or $(1+r) \leq d$. On the other hand if $d < (1+r) < u$. It is clear that if $\psi_2 > 0$ but is small enough, so also will ψ_1 and ψ_3 be positive.

For all remaining parts of the problem, assume that the no-arbitrage condition is satisfied.

(b). Show that the market is not complete. If

$$V = \begin{pmatrix} V(\omega_1) \\ V(\omega_2) \\ V(\omega_3) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

show that V can be replicated if and only if V is orthogonal to the vector

$$Z = \begin{pmatrix} 1-u \\ u-d \\ d-1 \end{pmatrix}$$

A portfolio (π_1, π_2) replicates the payoff V if

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 1+\bar{r} & dS(0) \\ 1+\bar{r} & S(0) \\ 1+\bar{r} & uS(0) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

The range of the matrix multiplication as π ranges over all 2-vectors is 2 dimensional proper subspace of the space of 3-vectors and so the model is not complete.

In general a matrix equation $x = Ay$ has a solution if and only if x is perpendicular to the null space of A^*, the transpose of A. However the null space of A^* is spanned by the vector $Z = (1-u, u-d, d-1)^*$, as one can easily check. So a contingent claim V is replicable if and only if it is perpendicular to Z.

(c). Compute the no arbitrage price of a replicable contingent claim.

One can give a variety of formulas. Let V be a replicable contingent claim. Then we know

$$\begin{pmatrix} V_1 \\ V_3 \end{pmatrix} = \begin{pmatrix} 1+\bar{r} & dS(0) \\ 1+\bar{r} & uS(0) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix},$$

and by assumption of replicability we don't have to worry about V_2. This is exactly the equation for replicability of a claim in the one-period binomial model and we can deduce from the work there that the price of the option must be

$$\frac{1}{1+\bar{r}}\left[\tilde{q}V_1 + \tilde{p}V_3\right]$$

where $\tilde{p} = ((1+\bar{r}) - d)/(u-d)$, $\tilde{q} = 1 - \tilde{p}$, as in the binomial model.

(d). What condition must hold in order that a European call with strike price K be replicable?

The European call will be replicable only in two cases, the trivial case, in which $K \geq uS(0)$ and so the payoff of the option is always zero and hence its value is 0, or the case $K \leq dS(0)$. In the latter case, the payoff vector is $(dS(0) - K, S(0) - K, uS(0) - K)$ and a straightforward computation shows this is perpendicular to the vector Z identified in part (b). On the other hand, if $dS(0) < K < uS(0)$, then $Z \cdot V = (u-d)\max\{S(0) - K, 0\} + (d-1)(uS(0) - K) \neq 0$.

2. (From Chapter 1, Volume 1 of Shreve). The background for this problem is in Lectures 4, 5, 6 available on the course web page. We consider a three period binomial tree model with $S(0) = 4$, $u = 2$, $d = 1/2$, and interest rate $r = 1/4$. With these numbers, $\tilde{p} = 1/2 = \tilde{q}$. We will study the **Asian option.** This is an option written on the average price of a stock over its history. Define

$$Y(0) = 0, \quad Y(t) = \sum_{i=1}^{t} S_(i), \quad 1 \leq t \leq 3.$$

The Asian option with strike price K pays

$$V(3) = \max\{Y(3)/3 - K, 0\}$$

at the terminal time $T = 3$.

Note: I mistated this problem because I should have written

$$Y(0) = S(0), \quad Y(t) = \sum_{i=0}^{t} S_(i), \quad 1 \leq t \leq 3.$$

and $V(3) = \max\{Y(3)/4 - K, 0\}$.

I will present the solution with the old definitions though so as not to confuse the issue even further!)

(a). Let $V(t)(\omega|_t)$ denote the price of the Asian option at time t in state ω. Write down a recursive backward equation for $V(t)(\omega|_t)$.

This is just the equation we derived in class, which applies to all contingent claims. Note that for our specific problem $\tilde{p} = \tilde{q} = 1/2$, and $1 + \bar{r} = 5/4$.

$$\begin{aligned} V(t)(\omega|_t) &= \frac{1}{1+\bar{r}} \left[\tilde{q}V(t+1)(\omega|_t, -1) + \tilde{p}V(t+1)(\omega|_t, 1)\right] \\ &= \frac{2}{5} \left[V(t+1)(\omega|_t, -1) + V(t+1)(\omega|_t, 1)\right] \end{aligned} \tag{3}$$

It turns out that one can express $V(\omega|_t)$ as a function $v(t)(S(t), Y(t))$ of the current price $S(t)$ and current accumulated sum of prices $Y(t)$. This is true by definition for $T = s$, with $v(3)(s, y) = \max\{y/3 - K, 0\}$. Derive a recursive, backward algorithm expressing $v(t)(s, y)$ in terms of $v(t+1)(s, y)$.

If the stock price at time t is $S(t) = s$ and if $Y(t) = y$ and if $\xi_{t+1} = -1$, then $S(t+1) = ds$ and $Y(t+1) = y + ds$. On the other hand, if $\xi_{t+1} = -1$, then $S(t+1) = us$ and $Y(t+1) = y + us$. Then it is clear from (3) that if we can represent the option price at time $t+1$ as a function $v(t+1)(S(t+1), Y(t+1))$, then

$$v(t)(s, y) = \frac{2}{5} \left[v(t+1)(ds, y + ds) + v(t+1)(us, y + us)\right]. \tag{4}$$

The terminal condition is, of course,

$$v(3)(s, y) = \max\{(y/3) - K, 0\}. \tag{5}$$

(b). Use the result of (a) to find the price of the option at time $t = 0$. (We will do the calculation with strike price $K = 5$.) It is necessary to write down the possible pairs of values $(S(t), Y(t))$ for $t = 0, 1, 2$. At time $t = 0$, $S(0) = 4$, $Y(0) = 0$. At time $t = 1$, there are two possibilities for $(S(1), Y(1)$: either $(8, 8)$ or $(2, 2)$. At time $t = 2$, there are three possibilies: $(16, 24)$ (corresponding to $(\xi_1, \xi_2) = (1, 1)$), $(4, 12)$ (corresponding to $(\xi_1, \xi_2) = (1, -1)$), $(4, 6)$ (corresponding to $(\xi_1, \xi_2) = (-1, 1)$), and $(1, 3)$ (corresponding to $(\xi_1, \xi_2) = (-1, -1)$). Apply the terminal condition (5) and the recursion (4),

$$\begin{aligned} v(2)(16, 24) &= \frac{2}{5} \left[v(3)(8, 32) + v(3)(32, 56)\right] = \frac{116}{15}, \\ v(2)(4, 12) &= \frac{2}{5} \left[v(3)(2, 14) + v(3)(8, 20)\right] = \frac{2}{3}, \end{aligned}$$

$$
\begin{aligned}
v(2)(4,6) &= \frac{2}{5}\,[v(3)(2,8)+v(3)(8,14)] = 0, \\
v(2)(1,3) &= \frac{2}{5}\,[v(3)(2,5)+v(3)(.5,3.5)] = 0.
\end{aligned}
$$

From these values, $v(1)$ can be computed by the recursion:

$$
\begin{aligned}
v(1)(8,8) &= \frac{2}{5}\,[v(2)(4,12)+v(2)(16,24)] = \frac{84}{25}, \\
v(1)(2,2) &= \frac{2}{5}\,[v(2)(1,3)+v(2)(4,6)] = 0.
\end{aligned}
$$

Finally, $v(0)(4,0) = (2/5)v(1)(8,8) = 168/125$.

(c). Let $\pi(t)(s,y)$ denote the number of shares of stock that should be held during period $[t,t+1]$ in the replicating portfolio if the stock price is s and if $Y(t)=y$.

According the delta hedging formula,

$$
\pi(t)(s,y) = \frac{v(t+1)(us,y+us) - v(t+1)(ds,y+ds)}{s(u-d)}.
$$

3. *Invariance of Brownian motion under certain time and space scalings.* Let W be a Brownian motion. By checking the conditions defining a Brownian motion, show that each of the following is a Brownian motion:

(a) $Y(t) = W(1-t) - W(1)$, $0 \le t \le 1$; (time-reversed Brownian motion.) (This process is restricted to the time interval $0 \le t \le 1$; show it is a Brownian motion on this interval.

It is necessary to check that the process Y is continuous, has independent increments, and that the distribution of $Y(t+s)-Y(t)$ is normal with mean 0 and variance s. The process clearly has continuous paths because B does. If $0 \le t_1 < t_2 < \cdots < t_n \le 1$,

$$
\begin{aligned}
&(Y(t_1), Y(t_2)-Y(t_1), \ldots, Y(t_n)-Y(t_{n-1})) \\
&\quad = \; (B(1)-B(1-t_1), B(1-t_1)-B(1-t_2), \ldots, B(1-t_n)-B(1-t_{n-1}))
\end{aligned}
$$

The components of the vector on the right are increments of B over disjoint time intervals and hence are independent. Furthermore, $Y(t+s)-Y(t) = B(1-t)-B(1-t-s)$; the last expression is a Brownian motion increment over an interval of length s and hence is normal with mean 0 and variance s.

(b) $Y(t) = cW(t/c^2)$.

Sample path continuity of Y follows directly from the continuity for W. As the time parameter of Y is just a linear scaling, the increments of Y are increments of Brownian motion, and hence Y has the independent increments property and the increments are normal with zero mean. It remains only to check the increments have the correct variance. But

$$\mathrm{Var}\,(Y(t+s) - Y(s)) = c^2 \mathrm{Var}\left(W((t+s)/c^2) - W(s/c^2)\right) = c^2(s/c^2) = s.$$

(c). $Y(t) = tW(1/t)$. (Set $Y(0) = 0$. For complete rigor, one needs to show that $\lim_{t\downarrow 0} Y(t) = Y(0) = 0$. You may assume this.)

It is helpful to remember that jointly normal random variables are independent if their covariance is zero.

The process Y is again continuous by continuity of W. Consider any increment $Y(t_2) - Y(t_1)$, $t_2 > t_1$. As a linear combination of zero mean, jointly normal random variables–namely $W(1/t_1)$ and $W(1/t_2)$, it is normal and has mean zero. To calculate its variance, we use the fundamental property, $E[W(t,s)] = t \wedge s$, where $t \wedge s = \min\{t, s\}$.

$$\begin{aligned} E\left[(Y(t_2) - Y(t_1))^2\right] &= E\left[t_2^2 W^2(1/t_2) - 2t_2 t_1 W(1/t_2) W(1/t_1) + t_1^2 W^2(1/t_1)\right] \\ &= t_2 - 2t_2 t_1 (t_2^{-1} \wedge t_1^{-1}) + t_1 = t_2 - t_1. \end{aligned}$$

Now let $r \le s < t$. We will show that the covariance of $Y(t) - Y(s)$ and $Y(r)$ is zero. It then will follow that Y has independent increments, because then the covariance of $Y(t) - Y(S)$ with any linear combination of $Y(r_1), \ldots, Y(r_m)$ for times $r_i \le s$ is zero, which by normality implies independence. But

$$E\left[Y(r)(Y(t) - Y(s))\right] = rt(r^{-1} \wedge t^{-1}) - rs(r^{-1} \wedge s^{-1}) = r - r = 0.$$

4. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and assume that Y is $\mathcal{G}$ measurable; equivalently, the σ-algebra $\sigma(Y)$ generated by Y is contained in $\mathcal{G}$. Assume $E|X| < \infty$.

Suppose that there is a function ϕ such that

$$E\left[X \mid \mathcal{G}\right] = \phi(Y).$$

Using the properties defining conditional expectation, show that

$$E\left[X \,\middle|\, Y\right] = E\left[X \mid \mathcal{G}\right].$$

This is an application of the tower propery. Let $\sigma(Y)$ be the σ-algebra generated by Y. Since Y is $\mathcal{G}$-measurable, $\sigma(Y) \subset \mathcal{G}$. Thus

$$\begin{aligned} E\left[X \,\middle|\, Y\right] &= E\left[E\left[X \mid \mathcal{G}\right] \mid Y\right] \\ &= E\left[\phi(Y) \mid Y\right] = \phi(Y) \\ &= E\left[X \mid \mathcal{G}\right]. \end{aligned}$$

The second-to-last equality follows because obviously Y is $\sigma(T)$-measurable, and the last equality follows by hypothesis.