Problem Solutions. Mathematical Finance; Fall, 2005

Exercise 4.10, Shreve, volume 2.

(i). Apply Itô's rule formally to $X(t) = S(t)\triangle(t) + \Gamma(t)M(t)$. This means approximating the differential $d(YZ)$ without dropping any terms: $d(YZ) = (Y + dY)(Z + dZ) - YZ = YdZ + ZdY + dYdZ$. To simplify notation, the argument t will be dropped.

$$dX = Sd\triangle + \triangle dS + d\triangle\, dS + Md\Gamma + \Gamma dM + dM\, d\Gamma. \tag{1}$$

Note that $dM = rMdt$ since $M = e^r t$ and hence $\Gamma dM = rM\Gamma\, dt = r(X - \triangle S)\, dt$. On the other hand, we have

$$dX = \triangle dS + r(X - \triangle S)\, dt = \triangle dS + \Gamma dM \tag{2}$$

Subtract equation (2) from (1). The result is:

$$0 = Sd\triangle + d\triangle\, dS + Md\Gamma + dM\, d\Gamma. \tag{3}$$

(ii). According to the problem statement, $N(t) = c(t, S(t)) - \triangle(t)S(t)$ satisfies the equation

$$\begin{aligned} dN(t) &= c_t(t, S(t))\, dt + c_x(t, S(t))\, dS(t) + \frac{1}{2}c_{xx}(t, S(t))\, dS(t)\, dS(t) \\ &\quad -\triangle(t)\, dS(t) - S(t)\, d\triangle(t) - d\triangle(t)\, dS(t). \end{aligned} \tag{4}$$

It is assumed that $S(t)$ follows the Black-Scholes price model $dS = S(\alpha\, dt + \sigma\, dW)$ and hence by the formal rules for operating with differentials $dS(t)\, dS(t) = \sigma^2 s^2(t)$. Because $N(t) = M(t)\Gamma(t)$, another expression for $dN(t)$ is

$$dN(t) = M(t)\, d\Gamma(t) + \Gamma(t)\, dM(t) + dM(t)\, d\Gamma(t) \tag{5}$$

Now equate the right-hand sides of equations (4) and (5) and use the self-financing condition of (3). The result is

$$\Gamma(t)\, dM(t) = \left[c_t(t, S(t)) + \frac{1}{2}\sigma^2 s^2(t) c_{xx}(t, S(t))\right] dt + \left[c_x(t, S(t)) - \triangle(t)\right] dS(t). \tag{6}$$

To eliminate risk, impose the delta hedging portfolio, $\triangle(t) = c_x(t, S(t))$. Also, use $\Gamma(t)\, dM(t) = r\Gamma(t)M(t)\, dt = rN(t)\, dt$, and $N(t) = c(t, S(t)) - \triangle(t)S(t) = c(t, S(t)) - S(t)c_x(t, S(t))$. Equation (6) then becomes

$$\left[rc(t, S(t)) - rS(t)c_x(t, S(t)\right]\, dt = \left[c_t(t, S(t)) + \frac{1}{2}\sigma^2 s^2(t) c_{xx}(t, S(t))\right] dt.$$

Equating coefficents of dt for all possible values of $S(t)$ gives the Black-Scholes differential equation

$$c_t(t,s) + rsc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) = rc(t,s). \tag{7}$$

Exercise 4.11, Shreve. Let $c(t,s)$ denote the price of a European call computed according to the Black-Scholes formula assuming that the volatility of the underlying is σ_1^2 Then c solves the PDE given as equation (7) in the previous problem solution, but with σ_1 in place of σ.

Suppose that the price however actually follows

$$dS(t) = S(t)\left[\alpha\, dt + \sigma_2\, dW(t)\right], \tag{8}$$

where $\sigma_2 > \sigma_1$. Set up a portfolio long one European call and short $c_x(t, S(t))$ share of stock–this portfolio follows the delta hedging rule determined by c so the investor using this portfolio thinks σ_1 is the volatility. Start with initial wealth $X(0) = 0$. At each time, any remaining cash is invested at the risk free rate r, (or if cash is needed to maintain the position, borrowed at rate r.) and at the same time withdraw money from the portfolio at rate $(1/2)(\sigma_2^2 - \sigma_1^2)s^2(t)c_{xx}(t, S(t))$. Let $X(t)$ denote the wealth at time t generated by this investment/consumption strategy. The value of the portfolio at any time t is $c(t, S(t)) - c_x(t, S(t))S(t)$, so the cash on hand at time t is $X(t) - c(t, S(t)) + c_x(t, S(t))S(t)$. Thus, the change in wealth in infinitesimal time dt is,

$$\begin{aligned} dX(t) &= \left[r\left(X(t) - c(t,S(t)) + c_x(t,S(t))S(t)\right) - (1/2)(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))\right] dt \\ &\quad + dc(t,S(t)) - c_x(t,S(t))\, dS(t). \end{aligned}$$

By Itô's rule and equation (8) for $S(t)$, this reduces to

$$\begin{aligned} dX(t) &= \left[r\left(X(t) - c(t,S(t)) + c_x(t,S(t))S(t)\right) - (1/2)(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t,S(t))\right] dt \\ &\quad \left[c_t(t,S(t)) + (1/2)\sigma_2^2 S^2(t)c_{xx}(t,S(t))\right] dt \\ &= \left[c_t(t,s) + rsc_x(t,s) + (1/2)\sigma^2 s^2 c_{xx}(t,s) - rc(t,s)\right]\Big|_{s=S(t)}\, dt + rX(t)\, dt \qquad (9) \end{aligned}$$

But c satisfies the Black-Scholes PDE:

$$c_t(t,s) + rsc_x(t,s) + \frac{1}{2}\sigma^2 s^2 c_{xx}(t,s) = rc(t,s).$$

Thus, equation (9) for $X(t)$ reduces to

$$dX(t) = rX(t)\, dt.$$

whose solution is $X(t) = X(0)e^{rt}$. But $X(0) = 0$ and so $X(t) \equiv 0$. Thus, while maintaining 0 total wealth of the portfolio and cash position, over the course of the time interval $[0, T]$ we have withdrawn $\int_0^T (1/2)(\sigma_2^2 - \sigma_1^2)c_{xx}(t, S(t))\, dt$ dollars. As $sigma_2^2 > \sigma_1^2$ and $c_{xx}(t, s) > 0$ for al (t, s), we have achieved an arbitrage.