610:621: Mathematical Finance: Lectures 4 and 5, September 13 and 15

Multi-period binomial tree model.

1. Preliminary framework. In this lecture we formulate and study a market model that has T periods, so that the set of trading times is $\mathcal{T} = \{0, 1, 2, \ldots, T\}$. The market will consist of one risk-free bond, yielding a return of $1 + \bar{r}$ per period, and one stock. We assume that the stock does not pay dividends, so that the return or loss on the stock is due totally to its price changes. As in the one-period model, it will be assumed that in each period, the return on the stock is either u or d, $d < u$. But before specifying this more fully, we define the portfolio strategies allowed. For convenience of discussion, assume throughout that monetary value is measured in US dollars.

2. Self-financing portfolio processes and the wealth equation. The discussion of this section does not depend on the model for the set of states Ω, nor the movements of the stock prices. Here, $S(t)(\omega)$ denotes the price of the stock at time t, if state ω occurs, consistent with previous notation. Explicit dependence of prices, portfolios, etc. on ω will often be suppressed. For example, we often write just $S(t)$ if the dependence on ω does not need to be discussed.

Since transactions are allowed at each time t, $t = 0, 1, 2, \ldots, T$, the portfolio can be rebalanced for each time period. We shall let $\pi(t)(\omega)$ denote the portfolio chosen at time t, $t = 0, 1, \ldots, T-1$, for investment in period $t+1$. Taken in its totality, as a function of t and ω, π is called a portfolio process. Since there are only two investment possibilities, $\pi(t)$ is a vector with two components; for consistency with previous notation, let $\pi_1(t)$ be the dollar amount invested at the risk-free interest rate, and $\pi_2(t)$ the number of shares invested in the stock.

The total value of the portfolio invested at time t will be denoted $X(t)(\omega)$, and called the **wealth**. We have

$$X(t) = \pi_1(t) + \pi_2(t)S(t) \tag{1}$$

Because the amount invested in the bond is $\pi_1(t) = X(t) - \pi_2(t)S(t)$, we may and shall characterize the portfolio process by specifying $\pi_2(t)$ and $X(t)$, rather than $\pi_1(t)$ and $\pi_2(t)$.

It is natural, for an investment model, to demand that a portfolio process be **self-financing.** This means that at each time t, the wealth $X(t)$ must be only the initial wealth $X(0)$ plus what the investor has earned (or lost) from the investments, either from the risk-free interest, or from price movements of the stock. Mathematically, this means the following. Suppose that the wealth at time t is $X(t)$, and we have chosen a portfolio $\pi(t)$ with total value $X(t)$, to invest in period $t+1$. Then at time $t+1$ one period later, the value of this portfolio is $(1+\bar{r})(X(t)-\pi_2(t)S(t))+\pi_2(t)S(t{+}1)$, since $\pi_2(t)$ units of stock are owned and the return on the remainder of $X(t)$ invested risk-free is $1+\bar{r}$. For a self-financing strategy, we insist that this value is what we have to invest for period $t{+}2$. That is,

$$X(t{+}1) = (1+\bar{r})(X(t)-\pi_2(t)S(t))+\pi_2(t)S(t{+}1). \tag{2}$$

This equation is called the wealth equation. Once an initial $X(0)$ and a sequence of investments $\pi_2(0), \pi_2(1), \ldots, \pi_2(T{-}1)$ are given, (2) determines recursively a unique wealth process $X(0), X(1), \ldots, X(T)$, and hence completes the specification of the portfolio process. To summarize, a self-financing portfolio process is specified by an initial endowment $X(0)$, the amounts $\{\pi_2(t); 0 \le t \le T{-}1\}$, and the wealth equation.

There is one more general constraint on the portfolio process—a causality constraint. In any multi-period market model there is a natural flow of information. At time t, the investor has observed the market only up to time t. His or her choice of $\pi_2(t)(\omega)$ can only depend on this information. In general a state ω encodes the entire history of the market, for all times before and after t. The structure of $\pi_2(t)$ as a function of ω must be somehow constrained so that it only depends on past information. We do not yet have the mathematical technology to do this in general. We will specify below what the causality constraint is for the special case of binomial trees.

3. Contingent claims. In the T period model, suppose we have a contingent claim that promises to pay $V(T)(\omega)$ at time T if the market is in state ω.

One example is the European call option with expiration date T and strike price K. This gives the holder of the option the right, but not the obligation, to buy the stock at price K at time T. The holder will exercise if the price $S(T)(\omega) > K$, for a profit of $S(T)(\omega) - K$, but otherwise will not. The pay-off for the European call is thus $V(T)(\omega) = \max\{S(T)(\omega) - K, 0\}$.

Another example is the European put option, with expiration date T and strike price K. This gives the holder the right, but not the obligation, to sell the stock to the option writer for price K. This time the holder will exercise if $K > S(T)$ for a profit of $K - S(T)(\omega)$. Thus, the payoff is $V(T)(\omega) = \max\{K - S(T)(\omega), 0\}$.

There are many other option types. The American option, which we will discuss later, allows the holder to exercise the option at any time up to the expiration date. In this option the pay-off is not paid out at T, but we can define an equivalent option with pay-off at T by including an appropriate interest rate factor. So assume there is a risk-free return of $1+\bar{r}$ per period. If the state is ω, the pay-off of an American call option at strike price K is $V(T)(\omega) = \max\{0, (S(0)-K)(1+\bar{r})^T, (S(1)(\omega)-K)(1+\bar{r})^{T-1}, \ldots, S(T)(\omega) - K\}$. The American put has pay-off $V(T)(\omega) = \max\{0, K - S(0)(1+\bar{r})^T, (K-S(1)(\omega))(1+\bar{r})^{T-1}, \ldots, K - S(T)(\omega)\}$.

Again, in the future, we shall often write $V(T)$ without ω.

4. Binomial tree model: definition and the no arbitrage pricing principle. In this model, at each time t, if the price of the stock is $S(t)$, at time $t{+}1$ it can either be $uS(t)$ or $dS(t)$, where $0 < d < u$. To encode the stock movements for all time, we shall, use a sequence $\omega = (\xi_1, \ldots, \xi_T)$. Each ξ_t can equal either -1 or 1; $\xi_t = 1$ means $S(t{+}1) = uS(t)$, while $\xi_i = -1$ means $S(t{+}1) = dS(0)$. Thus, the model for the set Ω of all possible states of the market will be the set $\{-1, 1\}^T$ of all such sequences ω.

With this notation, the value of $S(t)(\omega)$ can be read off of ω;

$$S(t)(\xi_1, \ldots, \xi_T) = u^j d^{t-j} S(0), \qquad \text{where } j \text{ is the number of occurences of 1 in } (\xi_1, \ldots, \xi_t),$$

because j is the number of periods, up to period t, in which the return was u, while in the $t-j$ remaining, the return was d. For each t, $(1+\xi_t)/2$ equals 1 if $\xi_t = 1$, and it equals 0 if $\xi_t = -1$. Thus the number j in the formula for $S(t)$ is $\sum_{s=1}^{t}(1+\xi_s)/2$, and we can write

$$S(t)(\xi_1, \ldots, \xi_T) = u^{\sum_1^t (1+\xi_s)/2} d^{\sum_1^t (1-\xi_s)/2} S(0). \tag{3}$$

Or we can write a difference equation for the price:

$$S(t{+}1)(\omega) = u^{(1+\xi_{t+1})/2} d^{(1-\xi_{t+1})/2} S(t)(\omega), \quad 0 \le t \le T{-}1. \tag{4}$$

These formulae for the price are not too important technically for the pricing of derivatives, but there are two conceptual points in them.

(i) At each time t, $S(t)(\xi_1, \ldots, \xi_T)$ depends only on the values $\xi_1, \ldots, \xi_t$, marking the market movements up to time t. While this is obvious, it is an example of how to express the property that at time t, a function of ω depend only on the history of the market up to time t. If F is a function of $\omega = (\xi_1, \ldots, \xi_T)$ determined just by the values $\xi_1, \ldots, \xi_t$, we shall abuse notation slightly and write $F(\omega) = F(\xi_1, \ldots, \xi_t)$. To save writing, it is also convenenient to define $\omega|_t = (\xi_1, \ldots, \xi_t)$.

(ii) Equation (4) expresses the price as the solution of a difference equation driven by a "random" input sequence $\xi_1, \ldots, \xi_T$. This will be generalized in the continuous-time theory to a stochastic differential equation model for a stock price.

Before going on, we return to the theme of remark (i) and make several definitions. A function $\{u(t)(\omega) : t \in \{0, 1, \ldots, T\}, \omega \in \Omega\}$, abbreviated $\{u(t)\}$, is called a *process.* If, for every t, $u(t)(\omega)$ depends only on the values in $\omega|_t = (\xi_1, \ldots, \xi_t)$, then $\{u(t)\}$ is said to be **adapted**. This terminology allows us to complete the definition of a self-financing portfolio process, begun in section 2, by demanding that it be adapted.

We know from the definition of a self-financing portfolio process, that it is determined by an initial wealth $X(0)$ and an adapted process $\{\pi_2(t)\}$ specifying the number of shares of stock to invest in each period. To simplify notation and make it consistent with Shreve's, from now on we will use $\triangle(t)$ to denote $\pi_2(t)$. Thus, to repeat, a self-financing portfolio process is specified by an initial wealth, or initial endowment, $X(0)$, and adapted process $\triangle(0), \triangle(1), \ldots, \triangle(T-1)$, and the solution of the wealth equation (2) (with π_2 replaced by $\triangle$).

The binomial tree model should be viewed as a sequence of successive one period models. Indeed, suppose that we are at time t and we have observed the sequence of prices $S(0), S(1)(\omega), \ldots, S(t)(\omega)$. This is equivalent to observing $\omega|_t = (\xi_1, \ldots, \xi_t)$, encoding the price movements "up" (from S to uS) or "down" (from S to dS), up to time t. Given only this information, what happens from t to $t+1$ is described by the one-period model starting from price $S(t)(\omega|_t)$; either $\xi_{t+1} = 1$, so that the next price is

$$S(t+1)(\omega|_t, 1) = uS(t)(\omega|_t),$$

or $\xi_{t+1} = -1$, so that the next price is

$$S(t+1)(\omega|_t, -1) = dS(t)(\omega|_t).$$

An arbitrage at time t, given history $(\xi_1, \ldots, \xi_t)$ is a portfolio, with investment choices perhaps dependent on $(\xi_1, \ldots, \xi_t)$, that achieves an arbitrage for this one-period market going from t to $t+1$. We know that if $d < 1+\bar{r} < u$, no such arbitrage can exist, independent of t or $(\xi_1, \ldots, \xi_t)$.

With the model now defined, we can develop the no-arbitrage pricing theory for contingent claims. Given a function $G(\omega)$, defined on Ω, we write $G \geq 0$ to say $G(\omega) \geq 0$ for every ω in Ω, and we write $G > 0$ to say $G(\omega) \geq 0$ for every ω in Ω but G is not the function which is identically zero.

Definition. A self-financing portfolio process is an arbitrage if either $X(0) < 0$ and $X(T) \geq 0$, or $X(0) = 0$ and $X(T) > 0$, where X is the wealth process associated to the portfolio.

Theorem 1 *If $d < 1 + \bar{r} < u$, then the multi-period, binomial tree model does not admit arbitrage.*

The proof of this theorem is to observe that if there is a self-financing portfolio process that achieves an arbitrage, then there must be a $(\xi_1, \ldots, \xi_t)$ for which the portfolio process is an arbitrage for the one-period model starting at $(\xi_1, \ldots, \xi_t)$. But this is not allowed by the condition $d < 1 + \bar{r} < u$.

Henceforth, we assume that this no arbitrage condition, $d < 1+\bar{r} < u$, is in force.

Definition. Let $V(T)$ be a contingent claim. A replicating portfolio process for $V(T)$ is a self-financing portfolio process such that $X(T) = V(T)$ (shorthand for $X(T)(\omega) = V(T)(\omega)$ for all ω in Ω.)

We will prove in a bit that *every* contingent claim for the binomial tree model has a replicating portfolio. Thus the binomial tree model defines a **complete market**.

Given a contigent claim, we are of course interested in how to price it at time 0. We are also interested in pricing it at intermediate times t, when we know that $(\xi_1, \ldots, \xi_t)$ has happened. And of course, we want prices so that if the contingent claim is allowed as an investment opportunity additional to the stock and bond, no arbitrage possibilities arise. Let $V(T)$ be a contingent claim. Then the no-aribitrage price of $V(T)$ at time T is just $V(T)$itself. We shall use the notation $V(0), V(1), \ldots, V(T-1), V(T)$ to denote a sequence of arbitrage-free prices of the claim. In this sequence, $V(t)$ is really a function $V(t)(\xi_1, \ldots, \xi_t)$ depending on the sequence of stock price movements up to time t. Hence $\{V(t)\}$ is really a *price process.* To say that $V(t)$ is an arbitrage-free price means that for every $(\xi_1, \ldots, \xi_t)$, the $T-t$ period market starting from time t, ending at time T, with $(\xi_1, \ldots, \xi_t)$ fixed, and allowing investments in bond, stock, or contingent claim, does not admit arbitrage.

No arbitrage pricing principle. Let $V(T)$ be a contingent claim with a replicating portfolio process. Then $V(t) = X(t)$, $0 \leq t \leq T$, defines the unique, no-arbitrage price process for the claim, where $\{X(t)\}$ is the wealth process of the replicating portfolio.

This statement of principle is really a theorem. Its proof, as in the one period case, is as follows. If, say, $X(0)$ were not the price of the claim at time 0, then by either shorting the claim and going long in the replicating portfolio, or shorting the portfolio and going long in the claim, one can achieve an arbitrage. The price process must be unique, because if there were another price process that differed from it one could arbitrage the price differences.

4. Binomial tree model: solution. In this section, we present a general algorithm and formula for computing the no-arbitrage price process of a contingent claim for the binomial tree model. Of course, it is assumed always that $d < 1 + \bar{r} < u$.

Let us recall the definitions

$$\tilde{q} = \frac{u - (1+\bar{r})}{u-d}, \qquad \tilde{p} = \frac{1+\bar{r}-d}{u-d},$$

and the formulae

$$V(0) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(\omega_1) + \tilde{p}V(\omega_2)\right], \tag{5}$$

for the no arbitrage price of claim V in the one-period model, and

$$\delta = \frac{V(\omega_2) - V(\omega_1)}{S(\omega_2) - S(\omega_1)}, \tag{6}$$

the delta hedging formula, for the number of shares to invest in stock in the replicating portfolio. These formula were derived and discussed in previous lectures.

The key idea for the multi-period model is backward induction. Consider time $T-1$ and suppose $(\xi_1, \ldots, \xi_{T-1})$ has occured. Then in the next time step the state is either $(\xi_1, \ldots, \xi_{T-1}, -1)$, or it is $(\xi_1, \ldots, \xi_{T-1}, 1)$; the first case corresponds to state ω_1 in the one-period model, that is, a return on the stock of d, and the second corresponds to state ω_2 and a return of u. The pay-off of the claim in the first case is $V(T)(\xi_1, \ldots, \xi_{T-1}, -1)$ and in the second is $V(T)(\xi_1, \ldots, \xi_{T-1}, 1)$. What total wealth do we need at time $T-1$ and history $(\xi_1, \ldots, \xi_{T-1})$ to replicate this claim? Equation (5) says that the answer is

$$V(T-1)(\xi_1, \ldots, \xi_{T-1}) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(T)(\xi_1, \ldots, \xi_{T-1}, -1) + \tilde{p}V(T)(\xi_1, \ldots, \xi_{T-1}, 1)\right],$$

and if this is the wealth supplied, we can replicate the claim by investing in

$$\triangle(T-1)(\xi_1, \ldots, \xi_{T-1}) = \frac{V(T)(\xi_1, \ldots, \xi_{T-1}, 1) - V(T)(\xi_1, \ldots, \xi_{T-1}, -1)}{uS(T-1)(\xi_1, \ldots, \xi_{T-1}) - S(T-1)(\xi_1, \ldots, \xi_{T-1})}$$

shares of stock, and investing the remaining

$$V(T-1)(\xi_1, \ldots, \xi_{T-1}) - \triangle(T-1)(\xi_1, \ldots, \xi_{T-1})S(T-1)(\xi_1, \ldots, \xi_{T-1})$$

dollars in the bond. In this way, using the results of the one-period model we can derive the no-arbitrage price of the claim at time $T-1$ for each possible history of price movements, $(\xi_1, \ldots, \xi_{T-1})$, up to time $T-1$, and also the replicating portfolio we need to use.

Now consider time $T-2$ with a history $(\xi_1, \ldots, \xi_{T-2})$. How much money do we need to replicate $V(T)$ from this position? At time $T-1$, the state will be either $(\xi_1, \ldots, \xi_{T-2}, -1)$, and in that case we will need $V(T-1)(\xi_1, \ldots, \xi_{T-2}, -1)$ to replicate $V(T)$, or the state will be $(\xi_1, \ldots, \xi_{T-2}, 1)$ and we will need $V(T-1)(\xi_1, \ldots, \xi_{T-2}, 1)$ to replicate $V(T)$. Thus the amount of money we need to replicate $V(T)$ starting from $T-2$ is the same that we would need to replicate a one-period claim that pays off $V(T-1)(\xi_1,\ldots,\xi_{T-2},-1)$ in state ω_1 and $V(T-1)(\xi_1,\ldots,\xi_{T-2},1)$ in state ω_2. Applying formula (5) again:

$$V(T-2)(\xi_1,\ldots,\xi_{T-2}) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(T-1)(\xi_1,\ldots,\xi_{T-2},-1) + \tilde{p}V(T-1)(\xi_1,\ldots,\xi_{T-2},1)\right].$$

Likewise, the delta hedging formula specifies how many shares of the portfolio should be dedicated to stock in period $T-1$ to achieve this.

Continuing this procedure leads to a recursive equation solved backwards in time for the no-arbitrage prices and the replicating portfolio. To simplify notation, go back to writing $\omega|_t$ for $(\xi_1,\ldots,\xi_t)$. Then, for all t, $0 < t \leq T$, and for each ω_t, the equation for the price process is

$$V(t-1)(\omega|_{t-1}) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(t)(\omega|_{t-1},-1) + \tilde{p}V(t)(\omega|_{t-1},1)\right], \tag{7}$$

and the formula for the portfolio process is

$$\triangle(t-1)(\omega|_{t-1}) = \frac{V(t)(\omega|_{t-1},1) - V(t)(\omega|_{t-1},-1)}{uS(t-1)(\omega|_{t-1}) - dS(t-1)(\omega|_{t-1})}. \tag{8}$$

We summarize this solution in the following theorem, paraphrasing the theorem stated in Chapter 1 of Shreve, Volume 1.

Theorem 2 *Consider the binomial tree model with $d < 1+\bar{r} < u$. Let $V(T)$ be a contingent claim. Let $V(0), V(1), \ldots, V(T-1)$ be the solution to (7) and let $\triangle(0),\ldots,\triangle(T-1)$ be defined by (8). Consider the self-financing portfolio process defined by $\{\triangle(t)\}$ and initial wealth $X(0) = V(0)$. Then this portfolio process replicates $V(T)$ and in fact $X(t) = V(t)$ for all t. Thus $\{V(t)\}$ is the no-arbitrage price process.*

Example. Consider a two-period, European put at strike price 52. Assume the price of the stock at time 0 is \$50, that $\bar{r} = 0.05$ and that in each period the stock can increase by 20% or decrease by 20% per period. Recall that the pay-off to the holder of the put will be $V(2) = \max\{52 - S(T), 0\}$. Determine the price of the put at time 0.

Increase or decrease by 20% per period means $u = 1.2$ and $d = .8$. Thus

$$\tilde{q} = \frac{1.2 - 1.05}{1.2 - 0.8} = \frac{3}{8} \quad \text{and} \quad \tilde{p} = \frac{1.05 - 0.8}{1.2 - 0.8} = \frac{5}{8}.$$

The price process is $S(0) = 50$, $S(1)(1) = (1.2)50 = 60$, $S(1)(-1) = (0.8)50 = 40$¡ $S(2)(1,1) = 1.2S(1) = 72$, $S(2)(1,-1) = S(2)(-1,1) = (1.2)S(1)(-1) = 48$, and $S(2)(-1,-1) = (0.8)S(1)(-1) = 32$. (It helps to write all these number on the graph of a tree, with a node for each price, but I am too lazy to figure out how to typeset this!). The pay-offs at time $T = 2$ are

$$\begin{aligned} V(2)(1,1) &= \max\{52 - S(2)(1,1), 0\} = 0 \\ V(2)(1,-1) &= V(2)(-1,1) = \max\{52 - S(2)(1,-1), 0\} = 4 \\ V(2)(-1,-1) &= \max\{52 - S(2)(-1,-1), 0\} = 20. \end{aligned}$$

Apply (7). Then

$$\begin{aligned} V(1)(1) &= \frac{1}{1.05}\left[(3/8)V(2)(1,-1) + (5/8)V(2)(1,1)\right] = 1.43 \\ V(1)(-1) &= \frac{1}{1.05}\left[(3/8)V(2)(-1,-1) + (5/8)V(2)(-1,1)\right] = \frac{10}{1.05} = 9.52 \end{aligned}$$

Repeating,

$$V(0) = \frac{1}{1.05}\left[(3/8)V(1)(-1) + (5/8)V(1)(1)\right] = 4.76.$$

A person speculating on a price drop in a stock might purchase a call option. Suppose you buy this put at \$4.76 and the price drops from \$50 to \$32 in two periods, a total drop of %36. Then then return pay-off is \$20. The return in this situation is $20/4.76 = 4.2$. This example demonstrate that the percent return on an option can, if the circumstances are favorable, be much greater percent-wise, than the fluctation of the price of the underlying.