610:621: Mathematical Finance: Lecture 6, September 21.

Multi-period binomial tree model, continued.

The equation for pricing a contingent claim.

The context of this lecture is the binomial tree model established in the lecture notes for lectures 4 and 5. In this lecture we study equation (7) of the previous lecture for the price process $V(t)(\xi_1, \ldots, \xi_t)$ for a contingent claim paying $V(T)$ at time T. Writing $\omega|_t$ as shorthand for $(\xi_1, \ldots, \xi_t)$, this equation is

$$V(t)(\omega|_t) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(t{+}1)(\omega|_t, -1) + \tilde{p}V(t{+}1)(\omega|_t, 1)\right], \qquad t \le T-1. \tag{1}$$

(I have rewritten the equation here by replacing $t-1$ of equation (7) in the previous lecture by t.) Since $V(T)$ is given, this equation allows one to solve for the price process by a backward (in time) recursion. Equation (1) applies to a general contingent claim. For each time t, there are 2^t values of $V(t)$ to compute since there are 2^t sequences $(\xi_1, \ldots, \xi_t)$ of 1's and -1's. Thus to fully specify the price process, we need to apply equation (1) a total of $1+2+\ldots+2^{T-1} = 2^T - 1$ times, which, for large T can be computationally intensive.

A simpler and less computationally intensive equation for pricing can be obtained if the contingent claim is a function of the price of the stock: that is $V(T) = C(S(T))$. This happens, for example, for the European call, where $C(s) = \max\{s-K, 0\}$, or the European put, where $C(s) = \max\{K-s, 0\}$. Then we have the following result. For each time t, $V(t)(\omega|_t) = v(t, S(t)(\omega|_t))$ (more simply, $V(t) = v(t, S(t))$) where $v(t, \cdot)$ is the solution of the backward equation,

$$v(T, s) = C(s) \tag{2}$$

$$v(t, s) = \frac{1}{1+\bar{r}}\left[\tilde{q}v(t{+}1, ds) + \tilde{p}v(t{+}1, us)\right], \qquad t \le T{-}1. \tag{3}$$

The solution of this equation can be carried out on the binomial tree for the prices. At time t there are only $t{+}1$ possible different prices, namely $u^j d^{t-j} S(0)$ for $0 \le j \le t$. Thus the number of prices on the tree out to time T is $1 + 2 + \ldots + (T{+}1) = (T{+}1)(T{+}2)/2$, and so only this many applications of (3) are required to characterize the price process.

The proof that $V(t) = v(t, S(t))$ is by backward induction on t. The representation $V(T) = v(T, S(T))$ is immediate by the fact that $V(T) = C(S(T))$ and the definition of $v(T, \cdot)$ in (2). Now let us do the induction step. Assume $V(t{+}1) = v(t{+}1, S(t{+}1))$.

We will show then that $V(t) = v(t, S(t))$. The key to this is the simple observation, from the dynamics for the price process defined in the previous lecture, that

$$S(t+1)(\omega|_t, -1) = dS(t)(\omega|_t) \quad \text{and} \quad S(t+1)(\omega|_t, 1) = uS(t)(\omega|_t).$$

Applying (1) and the induction assumption

$$\begin{aligned} V(t)(\omega|_t) &= \frac{1}{1+\bar{r}}\left[\tilde{q}v\left(t+1, S(t+1)(\omega|_t, -1)\right) + \tilde{p}v\left(t+1, S(t+1)(\omega|_t, 1)\right)\right] \\ &= \frac{1}{1+\bar{r}}\left[\tilde{q}v\left(t+1, dS(t)(\omega|_t)\right) + \tilde{p}v\left(t+1, uS(t)(\omega|_t)\right)\right]. \end{aligned}$$

But this last expression is just $v(t, S(t))$, which completes the induction step.

Remark. Equation (3) is the binomial tree version of a partial differential equation for an option price that we will later derive in the continuous-time model.

Next we will write down a closed formula that expresses the general solution to (1). To define this, first let us define the function

$$J(\xi_1, \ldots, \xi_s) = \sum_1^s (1+\xi_j)/2 = \text{number of 1's in } \xi_1, \ldots, \xi_s,$$

which acts on any sequence $(\xi_1, \ldots, \xi_s) \in \{-1, 1\}^s$, for any positive integer s. Thus, if s is a time and if $(\xi_1, \ldots, \xi_s)$ represents a sequence of up and down movements of a stock up to time s, $J(\xi_1, \ldots, \xi_s)$ is the number of times it has gone up.

Then, if $V(t)$ solves (1), we have, for $t < T$, the solution

$$V(t)(\omega|_t) = \frac{1}{(1+\bar{r})^{T-t}} \sum_{\eta=(\eta_{t+1},\ldots,\eta_T)\in\{-1,1\}^{T-t}} \tilde{p}^{J(\eta)}\tilde{q}^{T-J(\eta)} V(T)(\omega|_t, \eta_{t+1}, \ldots, \eta_T). \tag{4}$$

The proof of this formula is again by backward induction. For $t = T-1$, formula (4) translates to

$$\begin{aligned} V(T-1)(\omega|_{T-1}) &= \frac{1}{(1+\bar{r})^1} \sum_{\eta\in\{-1,1\}} \tilde{p}^{J(\eta)}\tilde{q}^{T-J(\eta)} V(T)(\omega|_{T-1}, \eta) \\ &= \frac{1}{1+\bar{r}}\left[\tilde{q}V(T)(\omega|_{T-1}, -1) + \tilde{p}V(T)(\omega|_{T-1}, 1)\right]. \end{aligned}$$

But this is exactly what is given by the backward recursion (1) and so it is correct. Now assume that the representation (4) is valid for $t+1$. Then use this in (1), rearrange terms slightly and you'll get (4). I'll leave it as an exercise—I haven't the fortitude to try to typeset the calculation!

It is interesting to look at formula (4) for $t\!=\!0$. It is:

$$V(0) = \frac{1}{(1+\bar{r})^T} \sum_{\eta \in \{-1,1\}^T} \tilde{p}^{J(\eta)} \tilde{q}^{T-J(\eta)} V(T)(\eta). \tag{5}$$

This can be intepreted as an expectation. For each $\omega = (\xi_1, \dots, \xi_T)$ in the state space Ω for the binomial tree model, assign ω the probability

$$\tilde{\mathbb{P}}\left(\{\omega\}\right) = p^{J(\omega)} q^{T-J(\omega)}.$$

One can check that this is a valid probability assignment in that

$$\sum_{\omega \in \Omega} p^{J(\omega)} q^{T-J(\omega)} = (\tilde{p} + \tilde{q})^T = 1,$$

since $\tilde{p} + \tilde{q} = 1$. The probability assignment $\tilde{\mathbb{P}}$ is called the *risk neutral measure* on Ω. It generalizes the risk neutral probability measure we defined previously for the one-period model. Once this probability measure is defined, we can think of $V(T)$ as a random variable. Then formula (5) states that $V(0)$ is the expectation of $V(T)$ under this risk-free measure, discounted by $(1+\bar{r})^T$, the discount due to the risk-free interest rate over T periods:

$$V(0) = \frac{1}{(1+\bar{r})^T} \tilde{E}\left[V(T)\right].$$

Part of the reason $\tilde{\mathbb{P}}$ is called the risk-free measure is that,. in fact,

$$S(0) = \frac{1}{(1+\bar{r})^T} \tilde{E}\left[S(T)\right]. \tag{6}$$

In effect, the probabilities assigned by the risk-free measure provide the analogue of state-price vector, as they express the price at time 0 in terms of its price at time T. To prove (6), use the formula we stated last lecture (here expressed using J), $S(T)(\omega) = u^{J(\omega)} d^{T-J(\omega)} S(0)$. Also note as a preliminary that

$$d\tilde{q} + u\tilde{p} = \frac{d(u - (1+\bar{r})) + u(1+\bar{r}-d)}{u-d} = 1+\bar{r}.$$

Now,

$$\begin{aligned}
\frac{1}{(1+\bar{r})^T} \tilde{E}\left[S(T)\right] &= \frac{1}{(1+\bar{r})^T} \sum_{\omega \in \Omega} \tilde{p}^{J(\omega)} \tilde{q}^{T-J(\omega)} u^{J(\omega)} d^{T-J(\omega)} S(0) \\
&= S(0) \frac{1}{(1+\bar{r})^T} \left(u\tilde{p} + d\tilde{q}\right)^T \\
&= S(0).
\end{aligned}$$

Later, we will interpret the formula (4) for intermediate times t as a conditional expectation involving the risk-free measure. And we will show that $(1+\bar{r})^{-t}S(t)$, the discounted price process, is a martingale with respect to the risk-free measure. If you don't know what a martingale is, we will get to it, so this remark is for those that do know! In fact, the fundamental theorem of asset pricing for multi-period (discrete-time) markets says that no-arbitrage is equivalent to the existence of a probability measure under which the discounted price process is a martingale. This is the appropriate generalization of the result we stated for the one-period model that no-arbitrage is equivalent to the existence of a risk-neutral probability measure.

Some final remarks on this risk-neutral measure $\tilde{\mathbb{P}}$ on Ω. When we assign probabilities, we can then think of $J(\omega)$, which counts the number of 1's in ω, as a random variable. What is its distribution? We calculate the probability $\tilde{\mathbb{P}}\,(J=j)$ that there are exactly j 1's. Well, there are $\binom{T}{j}$ sequences ω for which $J(\omega)=j$ and, according to the definition of $\tilde{\mathbb{P}}$, each has probability $\tilde{p}^{J(\omega)}\tilde{q}^{T-J(\omega)}$. So,

$$\tilde{\mathbb{P}}\,(J=j) = \binom{T}{j}\tilde{p}^{J(\omega)}\tilde{q}^{T-J(\omega)}.$$

Thus J is a binomial random variable with parameters $\tilde{p}$ and T. It has the same distribution as the number of heads in T independent tosses of a coin, the probability of heads on each toss being $\tilde{p}$. The risk-neutral measure is assigning probabilities as if the price movements $\xi_1, \ldots, \xi_T$ were determined by independent Bernoulli random variables (essentially, coin tosses) with probability that $\xi_i = 1$ being $\tilde{p}$, and the probability that $\xi_i = -1$ being $\tilde{q} = 1 - \tilde{p}$.

Finally, suppose that $V(T) = C(S(T))$, that is, the contingent claim is a function of the price of the stock at time T. Under the risk-neutral measure the probability that $S(T) = d^j u^{T-j} S(0)$ is the probability that $J = j$, which we have just calculated. Thus

$$V(0) = \frac{1}{(1+\bar{r})^T}\tilde{E}\left[V(T)\right] = \frac{1}{(1+\bar{r})^T}\sum_{j=0}^{T}\binom{T}{j}\tilde{p}^{J(\omega)}\tilde{q}^{T-J(\omega)}C\left(d^j u^{T-j}S(0)\right).$$

This is a nice, simple formula for pricing. It can be used as the basis for a derivation of the famous Black-Scholes formul—but this is for a later lecture.