642:621: Mathematical Finance: Summary of Lecture 3, September 8

The one-period/finite state model The beginning of this lecture covers what is in the handout. We repeat in summary to make our notation clear. We consider a model with p basic assets and m states, gathered in the space $\Omega = \{\omega_1, \ldots, \omega_m\}$. The prices of the assets at time $t = 0$ are collected in the vector

$$A(0) = \begin{pmatrix} S_1(0) \\ \vdots \\ S_p(0) \end{pmatrix}.$$

The prices $S_k(1)(\omega_i)$, of the assets at time 1, if the state is ω_i, are collected in the vector

$$A(1)(\omega_i) = \begin{pmatrix} S_1(1)(\omega_i) \\ \vdots \\ S_p(1)(\omega_i) \end{pmatrix}.$$

These vectors are in turn put into a matrix whose entries describe the price of each asset at time 1 for each state:

$$\mathcal{A} \triangleq \begin{pmatrix} A(1)(\omega_1) & \cdots & A(1)(\omega_m) \end{pmatrix} = \begin{pmatrix} S_1(1)(\omega_1) & \cdots & S_1(1)(\omega_m) \\ S_2(1)(\omega_1) & \cdots & S_2(1)(\omega_m) \\ \vdots & \vdots & \vdots \\ S_p(1)(\omega_1) & \cdots & S_p(1)(\omega_m) \end{pmatrix}$$

A *state-price* vector is a vector $\psi = (\psi_1, \ldots, \psi_m)^*$ in $\mathbb{R}^m$ such that

(i) Each component $\psi_i > 0$;

(ii) For each asset k, $1 \le k \le p$,

$$S_k(0) = \sum_{i=1}^{m} \psi_i S_k(1)(\omega_i).$$

A succinct way to state the second condition using our notation is

$$A(0) = \mathcal{A}\,\psi. \tag{1}$$

We will explain the significance of the state-price vector in a bit.

A portfolio is represented by a vector $\triangle$ in $\mathbb{R}^p$; component $\triangle_k$ of this vector is the number of shares held by the portfolio in asset k. The value of the portfolio at time 0 is

$$A^*(0)\,\triangle = \begin{pmatrix} S_1(0) & \cdots & S_p(0) \end{pmatrix} \begin{pmatrix} \triangle_1 \\ \vdots \\ \triangle_p \end{pmatrix} = \sum_{k=1}^{p} \triangle_k S_k(0).$$

The vector in $\mathbb{R}^m$

$$\mathcal{A}^* \triangle = \begin{pmatrix} A^*(1)(\omega_1)\,\triangle \\ \vdots \\ A^*(1)(\omega_1)\,\triangle \end{pmatrix}$$

lists the values of the portfolio at time 1 for each different state.

If z is a vector, the notation $z \geq 0$ means that each component of the vector is non-negative; the notation $z > 0$ means that each component of the vector is non-negative and $z \neq \underline{0}$. With this notation, an arbitrage is a portfolio such that

$$\text{either } A^*(0)\triangle < 0 \text{ and } \mathcal{A}^*\triangle \geq 0, \text{ or} \tag{2}$$

$$A^*(0) = 0 \text{ and } \mathcal{A}^*\triangle > 0. \tag{3}$$

This may be state succinctly as

$$\begin{pmatrix} -A^*(0)\pi \\ \mathcal{A}^*\pi \end{pmatrix} = \begin{pmatrix} -A^*(0)\pi \\ A^*(1)(\omega_1)\pi \\ \vdots \\ A^*(1)(1)(\omega_m)\pi \end{pmatrix} > 0. \tag{4}$$

Let M be the set of all vectors of the form in (4), as $\triangle$ ranges over all vectors in $\mathbb{R}^p$. Then (4) says there is an arbitrage if and only if M intersects the strictly positive orthant $K \stackrel{\triangle}{=} \{z \in \mathbb{R}^{m+1}; z_i \geq 0 \;\text{ for each component } z_i \text{ of } z\}$. Equivalently, there is no arbitrage if and only if K and M intersect only in the zero vector $\underline{0}$. Since K is a closed cone containing no lines and M is a subspace, the separation theorem for convex cones stated in the handout shows that K and M can be separated by a hyperplane in $\mathbb{R}^{m+1}$ through the origin. It turns out that the vector perpendicular to this hyperplane can be used to define a state-price vector. This is the main mathematical point of the of the following theorem, whose proof is given in detail in the handout notes on separation of convex sets.

Theorem 1 *The market model defined by vector $A(0)$ and matrix $\mathcal{A}$ admits no arbitrage if and only if there is a state-price vector.*

This theorem is very important conceptually as well as for deriving pricing formulas. It is the version for the one-period model of the fundamental theorem of asset pricing. To understand its significance we must explore the meaning of and consequences of the existence of a state-price vector. Before doing this let us apply the example to the one-period, binomial model. This will be an important calculation, as it will give us the state-price vector, which we will use often.

2. The state price vector for the one-period, two-state model. We shall re-derive the no-arbitrage condition for this model using Theorem 1. We recall the price vectors and matrices of this model:

$$A(0) = \begin{pmatrix} 1 \\ S(0) \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 1+\bar{r} & 1+\bar{r} \\ dS(0) & uS(o) \end{pmatrix},$$

To determine if there is a state-price vector, we look for solutions to the matrix equation:

$$\begin{pmatrix} 1 \\ S(0) \end{pmatrix} = \begin{pmatrix} 1+\bar{r} & 1+\bar{r} \\ dS(0) & uS(o) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix}. \tag{5}$$

Some linear algebra shows that the unique solution is

$$\begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} = \frac{1}{1+\bar{r}} \begin{pmatrix} \frac{u-(1+\bar{r})}{u-d} \\ \frac{1+\bar{r}-d}{u-d} \end{pmatrix}. \tag{6}$$

By inspection, both entries are positive if and only if $d < 1+\bar{r} < u$, so there is no arbitrage if and only if this condition, which we derived before by direct analysis, holds.

Let us note the following identity for the state-price vector:

$$\psi_1 + \psi_2 = \frac{1}{1+\bar{r}}. \tag{7}$$

We can verify this either by using the solution given in (6) or, more simply, noting that the first equation in the linear system (5) for ψ is

$$1 = (1+\bar{r})\,(\psi_1 + \psi_2)\,.$$

The fraction $1/(1+\bar{r})$ is called the discount rate (per period); because of the risk-free interest rate, \$1 paid out one period in the future, is worth only \$ $1/(1+\bar{r})$ today. Thus we can summarize (7) by saying that the sum of the entries of the state-price vector is the discount rate. We will see that this fact extends to the general, one-period model. ◇

3. Interpretation of the state-price vector. In this section we are back in the general model of section 1. It will be assumed that *there is no arbitrage, and hence there is a state-price vector* ψ.

The first fact will generalize what we just derived about the state-price vector of the two-state model. First, suppose we can find a portfolio θ such that

$$\mathcal{A}^*\,\theta = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{8}$$

Remember that the left side is the vector listing the value of portfolio at time $T = 1$ θ for each possible state $\omega_1, \ldots, \omega_m$ of the market. Thus portfolio θ has the same value, namely 1, for each different state. Therefore, it is risk-free. Since the value of the portfolio θ at time 0 is $A^*(0)\,\theta = \sum_1^p \theta_i S_i(0)$, the portfolio has the risk-free return

$$\frac{1}{A^*(0)\theta}$$

Therefore, $A^*(0)\theta$ is the appropriate discount rate for the market.

Such a risk-free portfolio will exist if the market is complete (for, by definition, completeness means that there is a replicating portfolio for every contingent claim, including the claim that pays out 1 in every state), even if there is no asset in the market which by itself is risk free. Of course, if there is a risk-free asset with return $1 + \bar{r}$, as in our two-state model, there is a risk-free portfolio with this return: simply invest everything in the risk-free asset.

With these preliminaries we can state:

Assume that there is a risk-free portfolio θ as in (8). Let ψ be the state-price vector. Then

$$\sum_{1=1}^{p} \psi = A^*(0)\theta = \text{market discount rate.} \tag{9}$$

The proof of this is simple. By definition of the state price vector, we know that $A(0) = \mathcal{A}\psi$. Thus, using (8),

$$A^*(0)\,\theta = \psi^*\mathcal{A}^*\theta = \begin{pmatrix} \psi_1 & \cdots & \psi_m \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{p} \psi.$$

The second important thing to observe about the state-price vector is that it tells us how to compute the no-arbitrage price of any replicable contingent claim.

Let $V(1)$ define a contingent claim with pay-off at time $T = 1$. Thus $V(1)$ is a function that assigns to each possible state ω a pay-off $V(1)(\omega)$. We shall think of $V(1)$ as an m- vector

$$V(1) = \begin{pmatrix} V(1)(\omega_1) \\ \vdots \\ V(1)(\omega_m) \end{pmatrix}.$$

Assume that there is a replicating portfolio π for $V(1)$. Recall that this means that π is a portfolio with the same pay-off as $V(1)$ *for every state*:

$$\sum_1^p \pi_k S_k(\omega) = V(1)(\omega) \quad \text{for every } \omega,$$

or, using vectors and matrices,

$$V(1) = \mathcal{A}\,\pi. \tag{10}$$

Then the no-arbitrage price of the claim is

$$V(0) = \sum_1^m \psi_i V(1)(\omega_i). \tag{11}$$

Notice this formula contains no mention of the replicating portfolio. Compare this to the condition (ii) defining a state-price vector:

$$S_k(0) = \sum_{i=1}^m \psi_i S_k(1)(\omega_i), \quad 1 \le k \le p.$$

The same system ψ of weights that gives the prices of the underlying assets at time 0 in terms of their possible prices at time 1 also prices any contingent claim.

To show why (11) is true, recall that the no-arbitrage price of $V(1)$ is given by the value of its replicating portfolio at time 0: $V(0) = A^*(0)\pi$. But $A(0) = \mathcal{A}\psi$. Hence,

$$V(0) = \psi^*\mathcal{A}^*\pi = \psi^* V(1) = \begin{pmatrix} \psi_1 & \cdots & \psi_m \end{pmatrix} \begin{pmatrix} V(1)(\omega_1) \\ \vdots \\ V(1)(\omega_m) \end{pmatrix} = \sum_1^m \psi_i V(1)(\omega_i).$$

The two-state model again. We return to our model with one bond(risk free asset), one stock, and two states. We derived the state-price vector above. If $V(1) = (V(1)(\omega_1), V(1)(\omega_2))^*$ is a congingent claim then we get the price

$$V(0) = \frac{1}{1+\bar{r}}\left[\frac{u-(1+\bar{r})}{u-d}V(\omega_1) + \frac{(1+\bar{r})-d}{u-d}V(\omega_2)\right]. \tag{12}$$

We have already derived this in lecture 2 (see equation (3) on page 3) from the replicating portfolio. ◇

4. The risk-neutral measure. Assume no arbitrage and let ψ be the state-price vector. Let us define

$$\tilde{p}_j = \frac{\psi_j}{\sum_1^m \psi_i}, \quad 1 \le j \le m.$$

These are all positive numbers and $\sum_1^m \tilde{p}_j = 1$. Now let us consider the set of possible states $\Omega = \{\omega_1, \ldots, \omega_m\}$ as the outcome space of a random experiment and assign probability p_j to ω_j for each j, $1 \leq j \leq m$—since these p_j sm to 1 this makes mathematical sense. We are not saying that this probability assignment reflects the "real" probabilities that the different states occur. We merely want to analyze what would happen if the $\tilde{p}_j$'s were the probabilities. The use of $\tilde{}$ in the notation is to remind us of the special origin of these probabilities. This assignment of probabilities based on the state-price vector is called the **risk-neutral probability measure,** for reasons to be explained presently.

Now imagine an experiment in which the ω_j's are chosen randomly according to the probabilities $\tilde{p}_j$. Any function of the states—such as the price $S_k(1)(\omega)$ of an underlying asset at time 1, or the payout $V(1)(\omega)$ of a contingent claim, may then be viewed a random variable; ω is chosen at random and the value $S_k(1)(\omega)$ or $V(1)(\omega)$ is thus random. Considering, for example $S_k(1)$, what would be its expectation? Since it takes on value $S_k(1)(\omega_j)$ with probability p_j, $1 \leq k \leq m$, its expected value would be

$$\tilde{E}\left[S_k(1)\right] = \sum_1^m S_k(1)(\omega_j)\tilde{p}_j. \tag{13}$$

(The $\tilde{}$ in $\tilde{E}$ reminds that the risk-neutral measure is being used.) Of course, this extends to any function X on ω. Considered as a random variable,

$$\tilde{E}\left[X\right] = \sum_1^m X(\omega_j)\tilde{p}_j.$$

If we write out what the right-hand side of (13) is, using the definition of p_j and remember that ψ is a state-price vector.

$$E\left[S_k(1\right] = \frac{1}{\sum_1^m \psi_i} \sum_1^m S_k(1)(\omega_j)\psi_j = \frac{1}{\sum_1^m \psi_j} S_k(0).$$

Rearranging,

$$S_k(0) = \left(\sum_1^m \psi_i\right) \tilde{E}\left[S_k(1)\right]. \tag{14}$$

This is true for every asset, and, because of the pricing formula (11), the no-arbitrage price of claim $V(1)$ is also given by

$$V(0) = \left(\sum_1^m \psi_i\right) \tilde{E}\left[S_k(1)\right]. \tag{15}$$

Remembering the interpretation of $\sum_1^m \psi_i$ from the previous section, one can describe formula (14) in words as saying that that price of each asset at time 0 is the discounted expected value of its price at time 0. The same is true for the no-option price of a contingent claim. The *expected return* of each asset is thus

$$\frac{\tilde{E}\left[S_k(1)\right]}{S_k(0)} = \frac{1}{\sum_1^m \psi_i},$$

and we know this expected return to be just the risk-free rate (per period) of return. Thus, under the risk neutral measure, all assets have the same expected return; an investor who was interested only in expected return, but was neutral (that is, indifferent) toward risk, would not prefer any one asset over another.

Let us in general say that a probability assignment p_j, $1 \leq j \leq m$, on $\{\omega_1, \ldots, \omega_m\}$ is risk-neutral, if $p_j > 0$ for every j, and the expected return

$$\frac{E\left[S_k(1)\right]}{S_k(0)} \triangleq \frac{\sum_1^m p_j S_k(1)(\omega_j)}{S_k(0)}$$

is the same for every asset k , $1 \leq k \leq p$. We have shown that if there is a state-price vector, there is a risk-neutral measure. The converse is also obviously true: if there is a risk-neutral measure, then there is a state-price vector. Therefore, Theorem 1, the fundamental theorem of asset pricing for the one-period/finite-state model can be reformulated as:

Theorem 2 *There is no arbitrage in the one-period/finite-state model if and only if there is a risk-neutral probability measure.*

Once we get our hands on the risk-neutral measure, formula (15) tells us how to price contingent claims.

Interpretative remarks: A. There is no claim here that the risk-neutral probabilities reflect the "real" probabilities of the different states to occur. The idea that there are real probabilities out there is consistent with the frequentist view of the meaning of a probability. Going with this view, let us imagine that we can observe the market over many periods. The frequentist probability of ω_i is the long run (limiting) frequency of occurences of ω_i. Let us suppose the probablities are $p_1, p_2, \ldots, p_m$. Let us suppose also there is a risk-free return of $1+\bar{r}$ per period. The expected return of asset k in one period is then

$$\frac{\sum_1^m p_j S_k(1)(\omega_j)}{S_k(0)}.$$

If k is a risky asset, we should expect that this return is strictly greater than $(1 + \bar{r})$. If it were not, we should not expect there to be much demand for the asset, because it is risky, but, on the average, returns no more than a risk-free investment. Ideally, higher expected returns of risky assets are premiums to reward the investor for undertaking risk. In fact, this appears to be true empirically; average returns on stock are historically higher than those available from investing money at the going interest rate.

B. The pricing formula (15) does not require knowledge of the actual probabilities that the different states occur. All it requires is agreement on which states have positive probability of occuring. At first this seems counterintuitive. Imagine a European call option on a stock with price $S(0)$ in the one-period/two-state model. Let the strike price be K. Suppose that the price can increase to $uS(0) > K$ or decrease to $dS(0) < K$. The holder of this option (call her A) can exercise the right to buy the stock at time $T = 1$ for price K and will do so if the price goes up to $uS(0)$, because then she can make an immediate profit of $uS(0) - K$. However if the stock price goes down to $dS(0)$ she will not exercise the option. It would seem natural that the higher the probability that the stock price increases to $uS(0)$, the higher should be the price of the option. But this is not true. The reason is that there is a replicating portfolio. Think of the point of view of the seller of the option, call him B. He receives $V(0)$ for the option. The delta hedging formula (see lecture 2 notes) tells him how much to invest part of this in the stock and the remainder at the risk free interest rate. He will earn the payoff (to the holder of the option). Thus, if the stock goes up, he will collect the profit $uS(0) - K$. Now A will choose to exercise the option and she will pay him K. These two payments together give him $uS(0)$ with which he can buy a unit of stock and give it to K, as he is obligated to do. He loses no money. If the stock goes down, B's replicating portfolio returns 0, but A does not exercise the option and B gets to keep the $V(0)$ paid him at the outset. Whether the option is more or less likely to go up, the same $V(0)$, if invested according to the replicating portfolio, will always give B what he needs to cover the contract, and so the price does not depend on the relative likelihoods of up and down. The price depends only on the set of possible stock movements allowed by the model.

One can also look at probabilities from a Bayesian point of view. Each investor may have his or her own subjective ideas of the probability of different stock movements. The no-arbitrage pricing theory is attractive, because in the words of Hull (from his text): "Investor attitudes are irrelevant to the relation between the price of the derivative and the value of the underlying variable." To be complete, one should add– "given a model of future market states." The states of such a model are tacitly assumed all to ave positive probability of occuring—otherwise, why include it in the model? But the pricing does not depend on what anyone might think about these probabilities. This allows parties with widely different opinions about what might happen to agree upon a price. You may purchase a call option thinking that the stock price is very likely to rise; the seller might think it more likely to fall. But this does not matter for calculating the price.