642:621: Mathematical Finance: Summary of Lecture 2, September 6

1. The no-arbitrage delivery price of a forward contract. In the first approach we use the mathematical formalism developed in the first lecture. We shall start by asking a more general question. Suppose at time $t = 0$ you enter a contract with party b to purchase a unit of asset at time T for price K. What should be the price of this contract? We will assume that the asset price is $S(t)$; the basic market will consist of this asset and a risk free bank account with continuously compounded return at interest rate r. The price vector is thus

$$A(t) = \begin{pmatrix} e^{rt} \\ S(t) \end{pmatrix}.$$

The pay-off of the contract is $S(T) - K$, as explained in the last item of the notes for Lecture 1.

Consider the portfolio $\triangle^* = (-Ke^{-rT} 1)$, that consists of borrowing Ke^{-rT} at the risk free interest rate and owning a share of asset. The value of this portfolio at time T is

$$A^*(T)\triangle = S(T) - K,$$

since at time T we will owe K to the lender of cash and we own a share of asset. This value is exactly equal to the pay-off the proposed contract; for this reason $\triangle$ is called a *replicating portfolio* for the contract. The value of the replicating portfolio at time $t = 0$ is

$$A^*(0)\triangle = \begin{pmatrix} 1 & S(0) \end{pmatrix} \begin{pmatrix} -Ke^{-rT} \\ 1 \end{pmatrix} = S(0) - Ke^{-rT}.$$

The claim is that this value must be the price of the contract at time $t = 0$ if there is to be no arbitrage. The short justification is that both the contract and the portfolion $\triangle$ have the same pay-off. Therefore they must be worth the same at time $t = 0$; otherwise by trading in the contract and the portfolio simultaneously, one could realize a riskless profit.

To see this mathematically, let's use the notation $C(t)$ for the price(value) of the contract and add the contract to the market. There are only two relevant times to consider the price; the price $C(0)$ at time $t = 0$, and the price $C(T)$ at the delivery time. The value $C(T)$ must equal the pay-off, so $C(T) = S(T) - K$. We consider now the augmented price vector

$$\tilde{A}(t) = \begin{pmatrix} e^{rt} \\ S(t) \\ C(t) \end{pmatrix}.$$

Assume now that $C(0) < S(0) - Ke^{-rT}$. We will show that there is an arbitrage. Because in this situation the contract price is lower than the value of the replicating portfolio, the arbitrage consists of going long in a contract and short in the replicating portfolio. This is the augmented portfolio

$$\tilde{\triangle} = \begin{pmatrix} Ke^{-rT} \\ -1 \\ 1 \end{pmatrix}.$$

(Thus, in this portfolio, one borrows a unit of asset, invests Ke^{-rT} at the risk free rate, and owns a contract.) Its value at time 0 is $\tilde{A}^*(0)\tilde{\triangle} = C(0) - (S(0) - Ke^{-rT}) < 0$. However, its value at time T is

$$\tilde{A}^*(T)\tilde{\triangle} \left(e^{rT}\, S(T)\, C(T)\right) \begin{pmatrix} Ke^{-rT} \\ -1 \\ 1 \end{pmatrix} = K - S(T) + C(T) = 0.$$

Thus $\tilde{\triangle}$ is an arbitrage in the mathematical sense defined in the Lecture 1 notes.

Similarly one can show that $C(0) > S(0) - Ke^{-rT}$ leads to an arbitrage opportunity. You are asked to do this in a homework problem in the first problem set.

The only price that does not lead to arbitrage is $C(0) = S(0) - Ke^{-rT}$.

The result we have obtained can now be used to deduce the no-arbitrage delivery price of a forward contract, because in a forward contract, no money is exchanged at time 0—the value of the contract to both parties at time 0 is zero. Thus, we need to have $C(0) = 0$, and because $C(0) = S(0) - Ke^{-rT}$, we derive

$$K = S(0)e^{rT}.$$

This is called the forward price.

The arbitrage argument we give is essentially that given in Hull, but fancied up with our notation. One can arrive at the forward price more directly, as in Hull. Suppose that delivery price $K < S(0)e^{rT}$. Then borrow a unit of asset at time 0, sell it and invest the proceeds $S(0)$ at the risk free interest rate. At the same time, enter the forward contract which requires no money. Then at time T, the initial investment yield $S(0)e^{rT}$. Take K of this, buy the asset using the contract and return it to the lender. Since $K < S(0)e^{rT}$, you will have gained $S(0)e^{rT} - K$ from these transactions and hence will have managed an arbitrage. Observe that this argument requires that one be able to borrow the asset. Similar reasoning shows that one can also make an arbitrage if $K > S(0)e^{rT}$.

No arbitrage pricing and replication. The logic of pricing by applying no arbitrage when there is a replicating portfolio extends to the general one period/finite state model. Recall in this model we are given a price vector $A(0)$ for the prices of p assets or securities at time 0, state space $\Omega + \{\omega_1, \dots, \omega_m\}$, and price vectors $A(1)(\omega_1), \dots, A(1)(\omega_m)$ for the prices at time 1 for each of the states. A contingent claim is a function $V(\omega)$, $\omega \in \Omega$, that gives the pay-off for each possible state. A replicating portfolio for V is then a portfolio π such that its value at time 1 matches the value of V for **every** state: that is, $V(\omega) = A^*(1)(\omega)\pi$ for every $\omega \in \Omega$. If the contingent claim is added to the market as another security, an arbitrage opportunity will arise unless the price $V(0)$ of the security at time T is the value of the replicating portfolio at time 0. That is, $V(0) = A^*(0)\pi$.

Application to the binomial model. We continue the study of the binomial model with one bond and one stock. Recall that for this model:

$$A(0) = \begin{pmatrix} 1 \\ S(0) \end{pmatrix}, \qquad A(1)(\omega_1) = \begin{pmatrix} 1+\bar{r} \\ dS(0) \end{pmatrix}, \qquad A(1)(\omega_2) = \begin{pmatrix} 1+\bar{r} \\ uS(0) \end{pmatrix}.$$

Here $0 < d < u$. S denotes the price of the stock and, just to repeat for emphasis, $S(1)(\omega_1) = dS(0)$ and $S(1)(\omega_2) = uS(0)$. We know that this model will admit no arbitrage if and only if $d < 1 + \bar{r} < u$ and so we assume this condition.

Now let V be a contingent claim. To find a replicating portfolio we need to determine a 2-vector π such that $A^*(1)(\omega_1)\pi = V(\omega_1)$ and $A^*(1)(\omega_2) = V(\omega_2)$. These two equations are equivalent to the equation

$$\begin{pmatrix} 1+\bar{r} & dS(0) \\ 1+\bar{r} & uS(0) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} V(\omega_1) \\ V(\omega_2) \end{pmatrix}.$$

This is easily solved:

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{(1+\bar{r})S(0)(u-d)} \begin{pmatrix} uS(0) & -dS(0) \\ -(1+\bar{r}) & 1+\bar{r} \end{pmatrix}. \tag{1}$$

For the future we record here a very important consequence called the **delta hedging formula**. The number of units to invest in the stock for the replicating portfolio is

$$\pi_2 = \frac{V(\omega_2) - V(\omega_1)}{S(0)(u-d)} = \frac{V(\omega_2) - V(\omega_1)}{S(1)(\omega_2) - S(1)(\omega_1)}. \tag{2}$$

We get the price of contingent claim V at time 0 using (1):

$$V_0 = A^*(0)\pi = \begin{pmatrix} 1 & S(0) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{1+\bar{r}} \left[\frac{u-(1+\bar{r})}{u-d} V(\omega_1) + \frac{(1+\bar{r})-d}{u-d} V(\omega_2) \right]. \tag{3}$$

We will re-derive and re-interpret this formula in the next lecture. It is very important.

We discuss briefly the financial significance of the delta hedging formula in (2). Suppose we sell the claim V at time 0. Then at time 0 we collect V_0, but we are obligated to pay out $V(\omega)$ at time 1 if the market is in state ω. The replicating portfolio tell us how to invest the V_0 we collect in order that we have $V(\omega)$ at time 1, whatever ω is. We should buy the number of shares of stock given by the delta hedging formula. And since $V_0 = (1 \ S(0))\pi = \pi_1 + \pi_2 S(0)$, we invest the remainder $\pi_1 = V(0) - \pi_2 S(0)$ at the risk free interest rate (which, if $\pi_1 < 0$, means we borrow at the risk free interest rate). In short, the replicating portfolio tells the short position how to invest to meet its contractual obligation.