642:621: Mathematical Finance: Summary of Lecture 1, September 1

Financial Markets: A financial market is a market trading financial assets—stocks, bonds, foreign currency, etc. We distinguish between basic assets—those involved with actual ownership of wealth and promise of earnings–and **contingent claims**, also called derivatives. Contingent claims are contracts for future exchanges of money and assets whose payoffs to the parties entering the contract are *contingent* upon, that is, are *derived* from, the values of basic assets. A contingent claim whose payoff depends wholly on the value of an asset A is *written on* A; in this case A is called the *underlying asset* or simply the *underlying*. Contingent claims may be written on commodities rather than financial assets, and in this case it is necessary to include the commodity in the financial market.

The defining feature of financial markets is risk; we do not know and cannot predict exactly the future prices of financial assets. Thus, the return on investments in financial assets is uncertain. We refer to this uncertainty as risk. We will use the word "risk" only in this general sense. There are serious theories attempting to define quantitative measures of risk and to analyze their behavior; these will not be treated.

Mathematical elements of a market model. Here we establish the basic elements of a market model and the notation we shall use for them. We imagine a market of p financial assets, which we label from 1 to p. In class, I restricted the modelling to basic or underlying assets, and in practice, modelling does start with these. But for the general set-up, we want to allow contingent claims as tradable assets also, and so we shall not imagine any restrictions.

A model will contain the following elements:

1. A set of trading times $\mathcal{T}$. We shall generally take $\mathcal{T} = \{0, 1, \ldots, T\}$, the discrete-time, finite horizon case, or $\mathcal{T} = [0, T]$, the continuous-time, finite horizon case. We usually think of the time $t = 0$ as the present, when we enter the market. When $\mathcal{T} = \{0, 1, \ldots, T\}$, we say the model is a T-period model.

2. A set Ω. The elements ω of this set label all the possible future states of the market admitted by the model. It is at this point that we incorporate risk. We do not know what the future state of the market—its prices and factors affecting its prices—will be; but it is the job of the model to list the possibilites. Connoiseurs of graduate level of probability theory, seeing Ω, will look next for a probability measure on subsets of Ω, but we do not introduce that yet! We will

begin the course with the study of the finite state case: $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is a finite set. As I said, we do not yet put probabilities on these outcome as part of the model, but there is a probabilistic idea lurking in the background; we assume that every one of the states in Ω has a positive probability to occur—it does not make sense to put a state in the model if it can't possibly happen! Uncountably infinite Ω will be necessary later on to support continuous time models, and then, for reasons we hope will become apparent, a probability measure will be necessary to proceed.

3. Prices functions. For each asset i, $1 \leq i \leq p$, the model specifies a function $S_i(t)(\omega)$, $t \in \mathcal{T}$, and $\omega \in \Omega$, that gives the price of asset i at time t if the market is in state ω. Since we think of time $t = 0$ as the present, when we can observe the prices, we shall assume that $S_i(0)$ is deterministic, in the sense that it does not depend on ω. We will gather the prices of the assets in a vector-valued function
$$A(t)(\omega) = \begin{pmatrix} S_1(t)(\omega) \\ \vdots \\ S_p(t)(\omega) \end{pmatrix}.$$
(The reason that we do not use S to denote this vector is that we often want to reserve S for denoting the price of a stock in models with one stock and one bond.)

Our models will often include one asset which is riskless. This is usually called "the bond," and it is assumed to return a known rate of interest. We let $R(t)$ be the return at time t of \$1 invested in the bond at time zero. Then, in continuous-time models with continuous compounding of interest at rate r, $R(t) = e^{rt}$. In discrete-time models, we denote the return in one period by
$$\frac{R(t+1)}{R(t)} = 1 + \bar{r},$$
and so $R(t) = (1 + \bar{r})^t$. The rates r and $\bar{r}$ are called the risk-free interest rates.

In this modeling framework, a contingent claim will be represented by a payoff function $V(\omega)$. Conversely, we allow any function V on Ω to be thought of as a contingent claim. Of course we want to analyze contingent claims that are funtions of the behavior of underlying assets, and so depend on ω through the prices of these assets. However, to keep the framework general, we allow any function on Ω.

Example. We image a one period model with one riskless asset (say, a bond) with value $R(t)$ one risky asset (say, a stock) with price $S(t)$, and two states. For this, we take $\mathcal{T} = \{0, 1\}$, $\Omega = \{\omega_1, \omega_2\}$, $R(0) = 1$, $R(1) = 1 + \bar{r}$, and

$$A(t) = \begin{pmatrix} R(t) \\ S(t) \end{pmatrix}, \qquad t \in \{0, 1\}.$$

We assume $S(0)$ is a known price. We let d denote the return of the risky asset at time 1 if ω_1 is the state and u the return if ω_2 is the state; without loss of generality, we can assume $0 < d < u$. Thus

$$A(0) = \begin{pmatrix} 1 \\ S(0) \end{pmatrix}, \qquad A(1)(\omega_1) = \begin{pmatrix} 1+\bar{r} \\ dS(0) \end{pmatrix}, \qquad A(1)(\omega_2) = \begin{pmatrix} 1+\bar{r} \\ uS(0) \end{pmatrix}.$$

Portfolios and portfolio processes. A portfolio is a list of the amounts an investor holds of each asset in the market. If there are p assets total, we represent a portfolio by a vector in $\mathbb{R}^p$:

$$\triangle = \begin{pmatrix} \triangle_1 \\ \vdots \\ \triangle_p \end{pmatrix},$$

$\triangle_i$ being the number of units of asset i held. We shall begin by assuming that any vector in $\mathbb{R}^p$ qualifies as a possible portfolio. This means that we can hold fractional parts of any asset. A negative $\triangle_i$ is interpreted to mean that the investor has borrowed $|\triangle_i|$ units of asset i. If one of the assets on the market is the risk free bond we discussed above, this means that in our models we can borrow money at the same risk free interest rate at which we can invest money. So this assumption is somewhat unrealistic. However, the idea is to start with the simplest models that allow straightforward analysis.

The value of a portfolio $\triangle$ at time t and state ω is the inner product of the vector $A(t)(\omega)$ and $\triangle$: (using A^* for the transpose of A)

$$A^*(t)(\omega)\triangle = (S_1(t), \ldots, S_p(t)) \begin{pmatrix} \triangle_1 \\ \vdots \\ \triangle_p \end{pmatrix} = \sum_{i=1}^{p} \triangle_i S_i(t)(\omega).$$

In multi-period models, we can update the portfolio at each trading time. This will require models with a portfolio process, $\triangle(t)$, $t \in \mathcal{T}$, in which the portfolio can depend on also on what happens in the market. This shall be made more precise later.

Arbitrage: An arbitrage is a portfolio that earns a riskless profit simply by cross trading several assets in a market. To say that the portfolio is riskless means that a profit, or at least no loss, is achieved whatever the state of the market turns out to be. Arbitrage is the proverbial free lunch, and the basic assumption that is made of markets is that they do not admit arbitrage. Indeed, economists often define arbitrage as simultaneous transactions in different markets that lead to a guaranteed profit, reflecting their opinion that arbitrage can only arise from imbalances between different markets that are not in good communication with each other. We shall we see that the no-arbitrage assumption will, under the right conditions, impose unique prices on contingent claims.

Mathematical formulation for the one-period, finite state model. Consider a model with with a finite number of states $\Omega = \{\omega_1, \ldots, \omega_m\}$, p assets, and one period. So we have a vector $A(0)$ of known prices at time $t = 0$, and the vectors $A(1)(\omega_1), \ldots, A(1)(\omega_m)$ listing the possible price vectors at time $t = 1$. A portfolio $\triangle$ is an arbitrage if either of the following conditions holds (recall the inner product formula for the value of a portfolio):

$$A^*(0)\triangle < 0 \quad \text{and } A^*(\omega_i)\triangle \geq 0 \text{ for every } 1 \leq i \leq m.$$
$$A^*(0)\triangle \geq 0 \quad A^*(\omega_i)\triangle \geq 0 \text{ for every } 1 \leq i \leq m \text{ and } A^*(\omega_j)\triangle > 0 \text{ for some } j.$$

The owner of such a portfolio begins at time $t = 0$ with zero or negative wealth, but ends up with non-negative or positive wealth whatever the state of the market.

No arbitrage in the two asset, two state, one-period model. This section refers to the simple model stated as an example above. For this model, there is no-arbitrage if and only if $d < 1 + \bar{r} < u$. To explain this (partly) in words, suppose to the contrary $1 + \bar{r} \leq d < u$; then the risky asset, the stock, returns at least as much or more than the bond for all market states. It follows that by borrowing at the risk free interest rate and investing in the stock, one makes a risk free profit. To do this mathematically, consider the portfolio

$$\triangle = \begin{pmatrix} -S(0) \\ 1 \end{pmatrix}.$$

Verify that the value of this portfolio at time $t = 0$ is $A^*(0)\triangle = 0$, and if $1+\bar{r} \leq d < u$, then $A^*(1)(\omega_1)\triangle \geq 0$ and $A^*(1)(\omega_2)\triangle > 0$. So there is arbitrage. It is left as an exercise to complete the argument and show no arbitrage if and only if $d < 1+\bar{r} < u$.

Frictions: *Friction* is the term for impediments to constraints on the ease and the returns of financial transactions or contracts or to the flow of information about assets. It is a term, in short, for any unwelcome intrusions of reality on our modeling assumptions! For example, investors cannot generally both invest and borrow at the same risk free interest rate, as we are assuming. This is a friction. Other frictions that we will assume away, even if we do not always mention it explicitly, will be: transaction costs and tax costs, restrictions on borrowing, liquidity constraints, bid-ask spreads. We will also assume: all investors are price takers; an investor can borrow any amount of any asset, sell it, and use the proceeds; all information is available to all investors.

Forward contracts. In a forward contract, one party, call it a, agrees at time zero to purchase from another party b an agreed upon amount of a commodity or asset for a predetermined price K per unit and a predetermined time T. Party a is said to have the long position in the contract. Party b is said to have the short position. The contract obligates each party to fulfill the agreement at time T, no matter what the state of the market is. The price K is called the delivery price and the time T the delivery time. Let the market price of the commodity on which the contract is written be denoted $S(t)(\omega)$; it is called the spot price. We will often drop the ω in our discussion and write simply $S(t)$. (But remember the ω is there!) In a forward contract no money or goods are exchanged at time zero, only at the delivery time. The payoff per unit of commodity to party a is $S(T)(\omega) - K$. This is because, a purchases each unit of commodity for price K, but on the market that unit is worth $S(T)$. So, if the spot price is higher, a could realize an immediate gain at time T by selling the commodity bought for price K.

A party may want to take a long position in a forward contract to protect against the risk of a rise in price of a commodity or asset that they need to buy at a later date. Short positions protect against price decreases.