Problem Solutions. Mathematical Finance; Fall, 2005

1. Suppose $K > S(0)e^{rT}$ on the forward contract. An investor can arbitrage by taking the short position in the contract, borrowing $S(0)$ at the riskless interest rate and using the money to buy a unit of asset. At time T, the investor has the unit of asset and receives K for it. The investor must also pay off the loan, which requires $S(0)e^{RT}$, leaving a profit of $K - S(0)e^{rT}$.

2. Note that the value of the income stream at time T is Ie^{rT}, since its value at time 0 is I.

We can argue in the style of the solution to problem 1. First suppose that the delivery price K is strictly larger than $(S(0) - I)e^{rT}$. Then, as in problem 1, an investor could borrow $S(0)$ at time 0, buy a unit of security, and short a forward contract. Then at time T, he receives K for the security from the other party to the contract and also has Ie^{rT} from the income stream. He must pay $S(0)e^{rT}$ to clear his loan. So he has a profit of $K + Ie^{rT} - S(0)e^{rT}$, which is strictly positive.

Now suppose $K < (S(0) - I)e^{rT}$. This time an investor can go long in a forward contract, borrow a unit of security, sell it for $S(0)$ and invest that $S(0)$ at the risk free rate. She is obligated to pay the party from whom she borrows the security the income stream which the security guarantees. This is equivalent to a payment to the lender of Ie^{rT} at time T. At time T she also pays K for the security, to return to the lender. So the total payments at time T are $K + Ie^{rT}$. However, the investment of $S(0)$ at the risk free rate returns $S(0)e^{rT}$, and so she makes a profit $S(0)e^{rT} - Ie^{rT} - K > 0$.

3. If we examine the argument in problem 2 in the case that $K < (S(0) - I)e^{rT}$ we see that in the arbitrage scheme the investor is obligated to pay the equivalent of Ie^{rT} to the party lending the security. If $I = -U$ where $U > 0$, this would be equivalent to receiving the equivalent of Ue^{rT} from the lender. This would require being able to borrow the asset a lender who was willing to pay the storage costs gratis. As this is unlikely, $K < (S(0) - I)e^{rT}$ cannot be ruled out by no arbitrage.

However, the argument ruling out $K > (S(0) + U)e^{rT}$ does work. In this case, an investor borrows $S(0)$, buys a unit of asset and shorts a forward contract. At time T he must return $S(0)e^{rT}$ to the bank. His total storage costs payments are worth Ue^{rT} at time T. He gets K for the sale of the asset at time T and so realizes a profit of $K - (S(0) + U)e^{rT}$ per unit asset.