Problem Solutions. Mathematical Finance; Fall, 2005

4. We deal with the one-period, two-state model:

$$A(0) = \begin{pmatrix} 1 \\ S(0) \end{pmatrix}, \qquad A(1)(\omega_1) = \begin{pmatrix} 1+\bar{r} \\ dS(0) \end{pmatrix}, \qquad A(1)(\omega_2) = \begin{pmatrix} 1+\bar{r} \\ uS(0) \end{pmatrix}.$$

It is assumed that $d < 1+\bar{r} < u$ for no arbitrage.

Let $S(0) = 4$, $d = 1/2$, $u = 2$, and $r = 1/4$. We will study a European call option with strike price $K = 5$. So if the market is in state ω_2 the price of the stock at time 1 is $S(1)(\omega_2) = 8$; otherwise $S(1)(\omega_1) = 2$.

a) Using the pricing formulas developed in class, show that the no arbitrage price of the call option is 1.2, and give explicitly the replicating portfolio.

The risk neutral probabilities are

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1}{2} \qquad \tilde{q} = \frac{1}{2}.$$

The payoffs of the option are $V(\omega_1) = \max\{S(\omega_1) - K, 0\} = \max\{2-5, 0\} = 0$ and $V(\omega_2) = \max\{S(\omega_2) - 5, 0\} = 3$. Therefore the price of the option is

$$V(0) = \frac{1}{1+r}\left[\tilde{q}V(\omega_1) + \tilde{p}V(\omega_2)\right] = \frac{1}{5/4}\left[\frac{3}{2}\right] = 1.2$$

At time 0, the number of shares of stock tp buy for the replicating portfolio is

$$\pi_2(0) = \frac{V(\omega_2) - V(\omega_1)}{S(\omega_2) - S(\omega_1)} = \frac{3}{6} = \frac{1}{2}.$$

The amount of money to invest at the risk free rate is $\pi_1(0) = V(0) - \pi_2(0)S(0) = 1.2 - 2 = -0.8$. The replicating portfolio thus requires borrowing 0.8 dollars at the risk free rate and purchasing (1/2) shares of stock.

b) This part is a numerical example on the no-arbitrage price 1.2 for the option. It is exercise 1.2 of Shreve, volume 1. An agent starts with an initial wealth $X_0 = 0$. At time 0 she buys $\triangle_2$ shares of stock and $\triangle_3$ options. It is assumed that any fractions of shares and options can be purchased or borrowed so there is no restriction on the values of $\triangle_2$ and $\triangle_3$. To finance this, the agent must take a cash position at the riskless rate of $\triangle_1 = -S(0)\triangle_2 - (1.2)\triangle_3 = -4\triangle_2 - (1.2)\triangle_3$ so that her initial wealth is 0. Determine $X(1)(\omega)$, the agents wealth at time 1, from this portfolio, for $\omega = \omega_1$ and $\omega = \omega_2$. Show that if ω_1 and ω_2 can happen with positive probability (our standing assumption), one cannot choose $\triangle_2$ and $\triangle_3$ so as to make an arbitrage.

Since the initial value of the portfolio is zero, to obtain an arbitrage one must choose $\triangle_2$ and $\triangle_3$ so that the payoffs in both states are non-negative and at least one is positive. The payoff of the agents portfolio in state 1 is

$$\triangle_1(1.25) + \triangle_2 S(\omega_1) + \triangle_3 V(\omega_1) = 1.25(-4\triangle_2 - (1.2)\triangle_3) + \triangle_2 2 = -3\triangle_2 - (1.5)\triangle_3.$$

The payoff in state 2 is

$$\triangle_1(1.25)+\triangle_2 S(\omega_2)+\triangle_3 V(\omega_2) = 1.25(-4\triangle_2-(1.2)\triangle_3)+\triangle_2 8+\triangle_3 3 = 3\triangle_2+(1.5)\triangle_3.$$

Since these payoffs are either both zero or are of opposite sign, it is not possible to have arbitrage.

c) This is exercise 1.6 from volume 1 of Shreve. A bank is long this option and so has paid 1.2 for it. The bank would like to earn 25% on this 1.2 in one period without investing more money. At time 1 they will get the pay-off for the option. Can they establish a portfolio in the stock and bond to insure that the option plus the portfolio value at time 1 is worth at least $(1 + .25)(1.2) = 1.5$ whatever the state turns out to be? If yes, how?

The bank needs to establish a portfolio $(\triangle_1, \triangle_2)$ in the stock and bond so that it has initial value 0 and so that in state ω_1 it pays at least 1.5 (since the payoff of the option in that state is zero), and in state ω_2 pays at least -1.5, since the option pays 3. These conditions require

$$\begin{aligned}
\triangle_1 + \triangle_2 S(0) &= \triangle_1 + 4\triangle = 0 \\
(1.25)\triangle_1 + 2\triangle_2 &\geq 1.5 \\
(1.25)\triangle_1 + 8\triangle_2 &\geq -1.5
\end{aligned}$$

By setting $\triangle_1 = 2$, $\triangle_2 = -1/2$, one can satisfy all three equations at once, with equalities in the second two equations.

5. (From Duffy, Chapter 1, 1.14) Consider the general one-period/finite-state model. Assume that there is no arbitrage. Show that there is a unique state-price vector if and only if the market is complete.

Let $\mathcal{A}$ be the matrix of payoffs at time 1, as in the notes. Assume there are p prices and m states, so that $\mathcal{A}$ is a $p \times m$ matrix. Since there is no arbitrage, there exists state-price vector ψ, which solves $A(0) = \mathcal{A}\psi$. Completeness means that for every contingent claim, that is, for m-vector V, there is a portfolio θ such that $V = \mathcal{A}^*\theta$. This can happen only if the rank of $\mathcal{A}^*$ is m. Thus the rank of A is also m, and since A has m columns, it follows that the null space of A consists only of the zero vector. Thus the solution to $A(0) = \mathcal{A}\psi$ is unique.

7. (Duffy, Chapter 1, 1.20.) (Trading constraints in a one-period model.) Claims A and B both have the same pay-off $V(\omega)$ at time 1. Let the price of A be p and the price of B be q. It is assumed that $V(\omega) \geq 0$ for every ω and strictly positive for at least one ω. An arbitrage is a portfolio $\triangle^* = (\alpha, \beta)$ whose initial value $\alpha p + \beta q \leq 0$ but whose final values $(\alpha+\beta)V(\omega)$ are all non-negative and at least one of initial and final values is non-zero. Positive and negative prices p and q are allowed—imagine a security you would pay someone to take! Obviously, if there are no restrictions on borrowing, we must have $p = q$ to avoid arbitrage.

a). Now imagine that we may not borrow either asset. Find the set of all prices (p, q) in the plane that allow no arbitrage portfolios under the borrowing constraint.

We claim that there is no arbitrage under the borrowing constraint if and only if

$$p > 0 \quad \text{and} \quad q > 0.$$

To argue this first note, that (p, q) is an arbitrage under the borrowing constraint if there exists (α, β) with $\alpha \geq 0$ and $\beta \geq 0$, such that either $\alpha p+\beta q < 0$ and $\alpha+\beta \geq 0$ or $\alpha p+\beta q = 0$ and $\alpha+\beta > 0$. The reason for this is that the payoff is $(\alpha+\beta)V(\omega)$ and $V(\omega) \geq 0$ for every ω and $V(\bar{\omega}) > 0$ for some $\bar{\omega}$. Under the trading constraint $\alpha \geq 0$ and $\beta \geq 0$, it follows that $\alpha + \beta \geq 0$; moreover, $\alpha + \beta = 0$ implies that $\alpha = \beta = 0$. The portfolio with $\alpha = \beta = 0$ does not give an arbitrage under any circumstance and so we need only worry about the case that $\alpha + \beta > 0$.

Now suppose $p > 0$ and $q > 0$. Then $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta > 0$ imply that $\alpha p + \beta q > 0$ and so there is no arbitrage.

On the other hand, suppose $p = 0$; then an arbitrage is $\alpha = 1$, $\beta = 0$—asset A is free so by acquiring some at zero cost, we can arbitrage. If $p < 0$, and $q \neq 0$, then $\alpha = 1$, $\beta = -p\alpha/q$ gives an arbitrage satisfying the borrowing constraint, because then $\beta > 0$, $\alpha p + \beta q = 0$ and $\alpha + \beta = 1 - p > 1$. Likewise, $q = 0$ or $q < 0, p \neq 0$ allows arbitrage.

Thus we have proved that there is no arbitrage under the borrowing constraint if and only if $p > 0$, $q > 0$.

b). Suppose that now either asset can be borrowed, but that to borrow asset A there is a cost of δ per unit asset. Now give the set of all (p, q) for which there is no arbitrage.

Remember that if x is an n-vector, $x > 0$ means $x \neq \underline{0}$, and each component of x is non-negative. We interpret the problem by stipulating that (p, q) is an arbitrage if either

$$\alpha \geq 0 \quad \text{and} \quad \begin{pmatrix} -(\alpha p + \beta q) \\ \alpha + \beta \end{pmatrix} > 0,$$

or

$$\alpha < 0 \quad \text{and} \quad \begin{pmatrix} -(\alpha(p-\delta)+\beta q) \\ \alpha+\beta \end{pmatrix} > 0.$$

Indeed, if $\alpha < 0$, then the investor pays an investment cost of $|\alpha|\delta = -\alpha\delta$. The total outlay an investor must make for the portfolio (α, β) is the value of the portfolio, $\alpha p + \beta q$, plus its cost $-\alpha\delta$.

We claim that there is no arbitrage if and only if

$$0 < q \leq p \leq q + \delta.$$

First, note that by part a) there is certainly an arbitrage if $p \leq 0$ or $q \leq 0$, because under the constraint $\alpha \geq 0$, $\beta \geq 0$ there is no borrowing cost.

Second, there is arbitrage if $q > p$. Indeed, in this case let $\alpha = 1$ and $\beta = -1$. Then $\alpha p + \beta q = p - q < 0$, but $\alpha + \beta = 0$, giving an arbitrage.

Third there is arbitrage if $p > q + \delta$; in this case, letting $\alpha = -1$ and $\beta = 1$, one finds $\alpha(p-\delta) + \beta q = q + \delta - p < 0$, yet $\alpha + \beta = 0$.

These three cases eliminate everything but (p, q) such that $0 < q \leq p \leq q + \delta$. If (p, q) satisfies this constraint, there can be no arbitrage with $\alpha \geq 0$. In this case, $\alpha p + \beta q = q(\alpha + \beta) + \alpha(p-q) \geq q(\alpha + \beta)$, so if $\alpha + \beta > 0$, $\alpha p + \beta q > 0$, as well, and if $\alpha + \beta = 0$, $\alpha p + \alpha q \geq 0$. On the other hand if $\alpha < 0$, then, under the constraint, it is again true that

$$\alpha(p-\delta) + \beta q = q(\alpha + \beta) + \alpha(p-\delta-q) \geq q(\alpha + \beta),$$

which excludes arbitrage.

7. This problem is an exercise in applying the binomial tree algorithm in the case of two steps. It is taken from Hull, Chapter 9, problem 9.5. We want to evaluate a one-year European call option with trading allowed at the end of the first 6 months and at the end of the year. The current price is 100 and the strike price is 100. In each period the stock price can either gain 10% or lose 10%. The risk free interest rate is is 8% per annum. What should the price of the call option be? Repeat the exercise for a European put with the same strike price.

In this case, assuming that interest is accrued every six months, $1 + \bar{r} = \sqrt{1.08}$.