4. In class we have developed the one-period binomial model, consisting of a risk free interest rate $\bar{r}$ and a stock with price S. The two state are labeled ω_1 and ω_2 and the model is given by the price vectors:

$$A(0) = \begin{pmatrix} 1 \\ S(0) \end{pmatrix}, \qquad A(1)(\omega_1) = \begin{pmatrix} 1+\bar{r} \\ dS(0) \end{pmatrix}, \qquad A(1)(\omega_2) = \begin{pmatrix} 1+\bar{r} \\ uS(0) \end{pmatrix}.$$

It is assumed that $d < 1+\bar{r} < u$ for no arbitrage.

This problem is a numerical exercise on this problem, taken out of Shreve's volume I. Let $S(0) = 4$, $d = 1/2$, $u = 2$, and $r = 1/4$. We will study a European call option with strike price $K = 5$. So if the market is in state ω_2 the price of the stock at time 1 is $S(1)(\omega_2) = 8$; otherwise $S(1)(\omega_1) = 2$. Thus the holder (long position) of such an option has the right, but not the obligation, to buy 1 share of stock at price 5 at time 1 from the party selling the option. The holder will exercise the option (that is, but the share from the seller) in state ω_2 for a profit of $S(1)(\omega_2) - K = 3$. If the state is instead ω_1 it makes no sense to purchase the stock for a price of 5 and the payoff is 0.

a) Using the pricing formulas developed in class, show that the no arbitrage price of the call option is 1.2, and give explicitly the replicating portfolio.

b) This part is a numerical example on the no-arbitrage price 1.2 for the option. It is exercise 1.2 of Shreve, volume 1. An agent starts with an initial wealth $X_0 = 0$. At time 0 she buys $\triangle_2$ shares of stock and $\triangle_3$ options. It is assumed that any fractions of shares and options can be purchased or borrowed so there is no restriction on the values of $\triangle_2$ and $\triangle_3$. To finance this, the agent must take a cash position at the riskless rate of $\triangle_1 = -S(0)\triangle_2 - (1.2)\triangle_3 = -4\triangle_2 - (1.2)\triangle_3$ so that her initial wealth is 0. Determine $X(1)(\omega)$, the agents wealth at time 1, from this portfolio, for $\omega = \omega_1$ and $\omega = \omega_2$. Show that if ω_1 and ω_2 can happen with positive probability (our standing assumption), one cannot choose $\triangle_2$ and $\triangle_3$ so as to make an arbitrage.

c) This is exercise 1.6 from volume 1 of Shreve. A bank is long this option and so has paid 1.2 for it. The bank would like to earn 25% on this 1.2 in one period without investing more money. At time 1 they will get the pay-off for the option. Can they establish a portfolio in the stock and bond to insure that the option plus the portfolio value at time 1 is worth at least $(1+.25)(1.2) = 1.5$ whatever the state turns out to be? If yes, how?

5. (From Duffy, Chapter 1, 1.14) Consider the general one-period/finite-state model. Assume that there is no arbitrage. Show that there is a unique state-price vector if and only if the market is complete.

6. (From Duffy, Chapter 1, 1.1). Consider a one-period/finite-state model with initial price vector $A(0)$ and matrix $\mathcal{A}$ of price vectors at time 1. It is said to be weakly arbitrage-free if $\mathcal{A}^*\triangle \geq 0$ (meaning every component of this vector is non-negative) implies the initial portfolio value $A^*(0)\triangle \geq 0$. Show that the model is weakly arbitrage-free if there exists a vector $\psi \geq 0$ (i.e., all components non-negative) such that $A(0) = \mathcal{A}\psi$. Hint: Use the general separation theorem stated in the handout on convex sets as Theorem 2.

7. (Duffy, Chapter 1, 1.20.) (Trading constraints in a one-period model.) Claims A and B both have the same pay-off $V(\omega)$ at time 1. Let the price of A be p and the price of B be q. It is assumed that $V(\omega) \geq 0$ for every ω and strictly positive for at least one ω. An arbitrage is a portfolio $\triangle^* = (\alpha, \beta)$ whose initial value $\alpha p + \beta q \leq 0$ but whose final values $(\alpha+\beta)V(\omega)$ are all non-negative and at least one of initial and final values is non-zero. Positive and negative prices p and q are allowed—imagine a security you would pay someone to take! Obviously, if there are no restrictions on borrowing, we must have $p = q$ to avoid arbitrage.

a). Now imagine that we may not borrow either asset. Find the set of all prices (p, q) in the plane that allow no arbitrage portfolios under the borrowing constraint.

b). Suppose that now either asset can be borrowed, but that to borrow asset A there is a cost of δ per unit asset. Now give the set of all (p, q) for which there is no arbitrage.

8. This problem is an exercise in applying the binomial tree algorithm in the case of two steps. It is taken from Hull, Chapter 9, problem 9.5. We want to evaluate a one-year European call option with trading allowed at the end of the first 6 months and at the end of the year. The current price is 100 and the strike price is 100. In each period the stock price can either gain 10% or lose 10%. The risk free interest rate is is 8% per annum. What should the price of the call option be? Repeat the exercise for a European put with the same strike price.

9. In each 2 month period, the price of a stock can either increase by 6% or decrease by 3%. The risk free interest rate is 4% per two month period. What is the price of six month call option at strike price 105 if the current price is 100?

10. (Hull, Chapter 9, 9.7.) Consider the two-period binomial model, assuming no arbitrage. It is not possible to establish and hold a single portfolio in the stock and the option that is riskless for both periods. Explain.

11. In lecture we studied the binomial tree model for a contingent claim with pay-off at the final time T. We found that if the pay-off at time T is $V(T)(\omega)$, where $\omega = (\xi_1, \ldots, \xi_T)$, then to get the value of the option $V(t)(\xi_1, \ldots, \xi_t)$ at $t < T$, one should solve backwards in time the recursive equation

$$V(t)(\xi_1,\ldots,\xi_t) = \frac{1}{1+\bar{r}}\left[\tilde{q}V(t{+}1)(\xi_1,\ldots,\xi_t,-1) + \tilde{p}V(t{+}1)(\xi_1,\ldots,\xi_t,1)\right], \qquad (1)$$

where $\tilde{q} = \dfrac{u-(1+\bar{r})}{u-d}$, $\tilde{q} = \dfrac{(1+\bar{r})-d}{u-d}$, and $\xi_1, \xi_2, \ldots$ indicate the up and down movements of the stock.

Many options allow early exercise, which is not covered by (2). For example, the American call option at strike price K allows the holder to buy the stock at price K at any time of his or her choosing up to the expiry date T. The value of the option if exercised at T is, as in the European call option, just $\max\{S(T)(\omega)-K, 0\}$, where $\omega = (\xi_1,\ldots,\xi_T)$. At the previous time $T-1$, however, (1) is no longer valid. Equation (1) gives the value if the holder does not exercise at time $T-1$ but waits until T. The holder could also exercise at time $T-1$ for a payoff of $S(T-1)(\xi_1,\ldots,\xi_{T-1})-K$. Clearly the holder should exercise if this is greater than the value obtained by waiting. In other words,

$$\begin{aligned} & V(T{-}1)(\xi_1,\ldots,\xi_{T-1}) \\ &= \max\Big\{S(T{-}1)(\xi_1,\ldots,\xi_{T-1}), \\ &\qquad \frac{1}{1+\bar{r}}\left[\tilde{q}V(T)(\xi_1,\ldots,\xi_{T-1},-1) + \tilde{p}V(T)(\xi_1,\ldots,\xi_{T-1},1)\right]\Big\}. \end{aligned} \qquad (2)$$

The same reasoning applies to earlier times and so by replacing T in (3) by t, we obtain a recursive equation for the value of the American option, and by examining which term gives the maximum we can determine when early exercise is optimal.

Find the value of a two-year American call on a stock in the following circumstances. The option can be exercised at time 0, after 1 year, or at the expiry date of 2 years. The price of the stock at time 0 is 50 and the strike price is 51. Each year the stock can go up 10% or decline 10% and the risk free interest rate is 5%. Find the value of the option at time 0 and intermediate times. Is early exercise ever optimal?