Problems. Mathematical Finance; Fall, 2005

1. Let $S(t)$, $0 \leq t \leq T$, denote the price process of an asset or commodity, and consider a forward contract to purchase one unit at T for price K. In the first lecture, we showed that if $K < S(0)e^{rT}$, there is an arbitrage opportunity. The argument proceed as follows, and assumes it is possible to borrow the asset, sell it, and use the proceeds. Suppose at time zero an investor borrows a unit of asset, sells it for $S(0)$ and invests it at the riskless interest rate r. At the same time the investor enters a long position on a futures contract for delivery price K at time T. At time T the investor must pay K for a unit of asset, which he then returns to the original owner. But meanwhile, the investor as earned $S(0)e^{rT}$ on the risk free investment. If $K < S(0)e^{rT}$, the investors has a profit of $S(0)e^{rT} - K > 0$ dollars.

Use a similar argument to show that there is also an arbitrage opportunity if $K > S(0)e^{rT}$.

2. Let $S(t)$, $0 \leq t \leq T$, denote the price process of a security. Suppose also that ownership of the security entitles the owner to a pre-determined income stream, by which we mean a sequence of payments at predetermined times and amounts—one example is a coupon bearing bond. Let I be the present value at time 0 of this income stream; (*present value* is defined shortly). Show that no-arbitrage that the delivery price for a forward contract on the security with delivery date T should be

$$(S(0) - I)\, e^{rT}.$$

(The present value of an income stream is the amount that would have to be invested at the risk free interest rate to generate exactly the amount of money needed to duplicate the income stream. Thus the present value today of a payment of $i(t)$ at time t units later is $e^{-rt}i(t)$. A sequence of payments $i(t_1), i(t_2), \ldots, i(t_n)$ has present value $\sum_1^n e^{-rt_k} i(t_k)$.)

3. This problem is a follow-up to problem 2. Let $S(t)$, $0 \leq t \leq T$ now be the price of a commodity (oil, copper, etc.). The ownership of the commodity of course does not entitle one to an income stream. Quite the opposite, it may entail a storage expense. Let U be the present value of the storage expense. Thinking of U as the present value of a negative income stream, suggests that the no arbitrage price K on a forward contract at delivery date T should be $(S(0) + U)e^{rT}$. There is a problem with this though. Show that for the no arbitrage argument to prove that $K < (S(0) + U)e^{rT}$ requires that one be able to borrow the commodity and have a third party pay the storage costs. As this does not seem likely, one can not really argue that $K < (S(0) + U)e^{rT}$ is ruled out by no arbitrage. Show on the other hand that the no arbitrage argument does rule out $K > (S(0) + U)e^{rT}$.