**Exercise 4.1, Shreve, volume 2.** In this problem, $M$ is a martingale with respect to a filtration $\{\mathcal{F}_t\}$, and $\triangle$ is an simple process adapted to $\{\mathcal{F}_t\}$: $\triangle(t) = \sum \triangle(t_j)\mathbf{1}_{[t_j,t_{j+1})}(t)$, where $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$ is a partition of $[0,T]$. To avoid any technical issues with existence of expectations, assume there is a constant $K$ such that $\mathbb{P}\left(|\triangle(t)| \le K,\ \forall t \ge 0\right) = 1$; this is a stronger assumption than is actually needed. For $t \in [t_k, t_{k+1})$, $I(t)$ is defined as

$$I(t) = \sum_{j=0}^{k-1} \triangle(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \triangle(t_k)\left[M(t) - M(t_k)\right], \tag{1}$$

and the object is to show that $I$ is a martingale. The purpose of this problem is to indicate that the definition of the stochastic integral extends beyond Brownian motion to general martingales.

Before starting, observe that for any $j$, the martingale property of $M$ and the adaptedness of $\triangle$ imply

$$E\left[\triangle(t_j)[M(t_{j+1}) - M(t_j)] \mid \mathcal{F}_{t_j}\right] = \triangle(t_j)E\left[M(t_{j+1}) - M(t_j) \mid \mathcal{F}_{t_j}\right] = 0 \tag{2}$$

Let $0 \le s < t \le T$. The problem is to show $E[I(t) \mid \mathcal{F}_s] = I(s)$. Let $k$ be the index such that $t_k \le t < t_{k+1}$, and let $i$ be the index such that $t_i \le s < t_{i+1}$. The first case is $i = k$, that is $t_k \le s < t < t_{k+1}$. In this case, the sum of the terms up to $j = k-1$ in (1) is $\mathcal{F}_{t_k}$-measurable and hence is $\mathcal{F}_s$-measurable. Also,

$$E\left[\triangle(t_k)[M(t) - M(t_k)] \mid \mathcal{F}_s\right] = \triangle(t_k)\left[E[M(t) \mid \mathcal{F}_s] - M(t_k)\right] = \triangle(t_k)\left[M(s) - M(t_k)\right], \tag{3}$$

where in the first equality, the $\mathcal{F}_s$ measurability of $\triangle(t_k)$ and $M(t_k)$ is used, and, in the last equality, the martingale property of $M$ is used. Thus, taking expectations in (1),

$$E\left[I(t) \mid \mathcal{F}_s\right] = \sum_{j=0}^{k-1} \triangle(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \triangle(t_k)\left[M(s) - M(t_k)\right] = I(s).$$

Now suppose that $i < k$ and let $j$ be an index such that $i < j \le k$. Then, $\mathcal{F}_s \subset \mathcal{F}_{t_j}$, and by inserting a conditioning on $\mathcal{F}_{t_j}$ via the tower property of conditional expectation, and by using equality (2),

$$E\left[\triangle(t_j)[M(t_{j+1}) - M(t_j)] \mid \mathcal{F}_s\right] = E\left[E[\triangle(t_j)[M(t_{j+1}) - M(t_j)] \mid \mathcal{F}_{t_j}] \mid \mathcal{F}_s\right] = 0. \tag{4}$$

For index $i$ itself, the same type of calculation as in (3) yields

$$E\left[\triangle(t_i)[M(t_{i+1})-M(t_i)] \mid \mathcal{F}_s\right] = \triangle(t_i)\left[M(s)-M(t_i)\right]. \tag{5}$$

Now take the conditional expectation of $I(t)$ using (4) and (5):

$$E\left[I(t) \mid \mathcal{F}_s\right] = \sum_{j=0}^{i-1} \triangle(t_j)\left[M(t_{j+1})-M(t_j)\right] + \triangle(t_i)\left[M(s)-M(t_i)\right] = I(s).$$

This completes the derivation of the martingale property of $I$.

**Shreve, 4.2.** In this problem, $I$ is stochastic integral of a deterministic simple process: $I(t) = \sum_{j=0}^{k-1} \triangle(t_j)\left[W(t_{j+1})-W(t_j)\right] + \triangle(t_k)\left[W(t)-W(t_k)\right]$, for $t_k \le t < t_{k+1}$.

**(i).** We show that the increment $I(t)-I(s)$ is independent of $\mathcal{F}_s$, whenever $s<t$. Let $i$ be such that $t_i \le s < t_{i+1}$, and $k$ such that $t_k \le t < t_{k+1}$. Then, a calculuation shows that

$$I(t)-I(s) = \triangle(t_i)[W(t_{i+1})-W(s)] + \sum_{j=i+1}^{k-1} \triangle(t_j)\left[W(t_{j+1})-W(t_j)\right] + \triangle(t_k)\left[W(t)-W(t_k)\right].$$

(If $i=k$, this reduces to $I(t)-I(s) = \triangle(t_k)[W(t)-W(s)]$.) Since $\triangle$ is deterministic, we see that $I(t)-I(s)$ is a linear combination of increments of $W$ which are all independent of $\mathcal{F}_s$, and hence is independent of $\mathcal{F}_s$.

**(ii).** The expression for $I(t)-I(s)$ in part (i) represents it as a sum of independent zero mean normal random variables. Hence $I(t)-I(s)$ is normal. Its variance is the sum of the variances of its summands, which a calculation shows to be

$$\triangle^2(t_i)(s-t_i) + \sum_{j=i+1}^{k-1} \triangle^2(t_j)(t_{j+1}-t_j) + \triangle^2(t_k)(t-t_k) = \int_s^t \triangle^2(r)\,dr.$$

Or one can derive this immediately from the Itô isometry formula (4.2.6) for the expected value of the square of a stochastic integral, using the fact that, here, $\triangle$ is deterministic.

**(iii).** $I$ is a martingale because for every $s<t$, $I(t)-I(s)$ is a mean zero random variable independent of $\mathcal{F}_s$. Thus $E[I(t)-I(s) \mid \mathcal{F}_s] = E[I(t)-I(s)] = 0$, which implies $E[I(t) \mid \mathcal{F}_s] = I(s)$.

**(iv).** Because $I$ is a normal process, all its second moments are finite and so all conditional expectations involving products of $I$ at various times are defined. Note that for $t>s$,

$$E\left[I(s)(I(t)-I(s)) \mid \mathcal{F}_s\right] = I(s)E\left[I(t)-I(s) \mid \mathcal{F}_s\right] = 0.$$

Thus,

$$\begin{aligned} E\left[I^2(t) \mid \mathcal{F}_s\right] &= E\left[(I(t)-I(s))^2 \,\middle|\, \mathcal{F}_s\right] + 2E\left[I(s)(I(t)-I(s)) \mid \mathcal{F}_s\right] + I^2(s) \\ &= \int_s^t \triangle^2(r)\,dr + I^2(s). \end{aligned}$$

In the last step we used the result of part (ii). It follows easily that $I^2(t) - \int_0^t \triangle^2(r)\,dr$ is a martingale.

**Exercise 4.3, Shreve.** We consider $I$ as defined in problem 4.2, but now we allow $\triangle$ to be random, but adapted.

**(i).** In general, $I(t) - I(s)$ will not be independent of $\mathcal{F}_s$. Consider for example, $s = t_1$ and $t = t_2$. Then $I(t_2) - I(t_1) = \triangle(t_1)\,[W(t_2) - W(t_1)]$. If $\triangle(t_1)$ is an $\mathcal{F}_{t_1}$ measurable random variable, for example suppose $\triangle(t_1) = W(t_1)$, then $I(t_2) - I(t_1)$ is not independent of $\mathcal{F}_{t_1}$ because $\triangle(t_1)$ is determined by $\mathcal{F}_{t_1}$.

**(ii).** In general, $I(t)-I(s)$ will not be normally distributed. Again, taking $s = t_1$, $t = t_2$, and $\triangle(t_1) = W(t_1)$, $I(t_2)-I(t_1) = W(t_1)(W(t_2)-W(t_1))$. A normal random variable satisfies $E[(X-\mu)^4] = 3(\mathrm{Var}(X))^2$, where $\mu = E[X]$. Note that the expected value of $I(t_2)-I(t_1)$ is zero, and using the independent increments property and normality of Brownian motion,

$$\mathrm{Var}\left(I(t_2)-I(t_1)\right) = E[W^2(t_1)]\,E\left[(W(t_2)-W(t_1))^2\right] = t_1(t_2-t_1),$$

but on the other hand

$$E\left[(I(t_2)-I(t_1))^4\right] = E\left[W^4(t_1)\right]\,E\left[(W(t_2)-W(t_1))^4\right] = 3t_1^2(3(t_2-t_1)^2) = 9t_1^2(t_2-t_1)^2.$$

Hence $I(t_2)-I(t_1)$ is not normal.

**(iii).** By Theorem 4.3.1, part (iv), on page 134 of Shreve, $I$ is martingale (as long as $E\left[\int_0^T \triangle^2(u)\,du\right] < \infty$, for all $T > 0$) and so $E\left[I(t) \mid \mathcal{F}_s\right] = I(s)$.

**(iv).** It is still true that

$$E\left[I^2(t) - \int_0^t \triangle^2(u)\,du \mid \mathcal{F}_s\right] = I^2(s) - \int_0^s \triangle^2(u)\,du, \tag{6}$$

(as long as $E\left[\int_0^T \triangle^2(u)\,du\right] < \infty$, for all $T > 0$ ).

We will verify this for $s = t_1$ and $t = t_2$, but the argument generalizes easily to any $0 \le s < t$. Return to the calculation of problem 4.2 part (iv). The derivation of Thus,

$$\begin{aligned} E\left[I^2(t) \mid \mathcal{F}_s\right] &= E\left[(I(t)-I(s))^2 \,\middle|\, \mathcal{F}_s\right] + 2E\left[I(s)(I(t)-I(s)) \mid \mathcal{F}_s\right] + I^2(s) \\ &= E\left[(I(t)-I(s))^2 \,\middle|\, \mathcal{F}_s\right] + I^2(s) \end{aligned}$$

uses only the propery of part (iii) and so is valid for a random, adapted integrand $\triangle$. The fact that $I(t) - I(s) = \triangle(t_1)[W(t_2) - W(t_1)]$ and the independence of $W(t_2) - W(t_1)$ from $\mathcal{F}_s = \mathcal{F}_{t_1}$ imply

$$E\left[(I(t) - I(s))^2 \,\middle|\, \mathcal{F}_{t_1}\right] = \triangle^2(t_1)(t_2 - t_1) = E\left[\int_{t_1}^{t_2} \triangle^2(u)\, du \mid \mathcal{F}_{t_1}\right].$$

(The middle term equals $\int_{t_1}^{t_2} \triangle^2(u)\, du$ and happens to be $\mathcal{F}_{t_1}$-measurable, and so that is why we can re-introduce the conditional expectation in the last term.) Rearranging terms leads to (6).

**Exercise 4.4, Shreve.** (i) For a partition $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ and $t_i^* \stackrel{\triangle}{=} (t_{i+1} + t_i)/2$, define

$$Q_{\Pi/2} = \sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2.$$

For each $j$, $E\left[(W(t_j^*) - W(t_j))^2\right] = t_j^* - t_j = (t_{j+1} - t_j)/2$. Thus

$$E\left[Q_{\Pi/2}\right] = \sum_{j=0}^{n-1} (t_{j+1} - t_j)/2 = \frac{T}{2}.$$

On the other hand,

$$\begin{aligned} E\left[(Q_{\Pi/2} - T/2)^2\right] &= E\left[\left(\sum_{j=0}^{n-1} \left(W(t_j^*) - W(t_j)\right)^2 - (t_j^* - t_j)\right)^2\right] \\ &= \sum_j \sum_k E\left[\left(\left(W(t_j^*) - W(t_j)\right)^2 - (t_j^* - t_j)\right)\left((W(t_k^*) - W(t_k))^2 - (t_k^* - t_k)\right)\right] \end{aligned}$$

However, by independence of increments, all terms in this sum are zero, except those with $k = j$. Also, using the normality of Brownian increments, we know $E\left[\left(W(t_j^*) - W(t_j)\right)^4\right] = 3E\left[\left(W(t_j^*) - W(t_j)\right)^2\right] = 3(t_j^* - t_j)^2$. Using this, one obtains

$$E\left[(Q_{\Pi/2} - T/2)^2\right] = 2\sum_{j=0}^{n-1} (t_j^* - t_j)^2 \leq \|\Pi\|(T/2).$$

Thus $\lim_{\|\Pi\| \to 0} E\left[(Q_{\Pi/2} - T/2)^2\right] = 0$.

**(ii).** We present two ways to do this. First, for a partition $\Pi$, define

$$\bar{Q}_{\Pi/2} = \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j^*)\right)^2.$$

The same proof as in part (i) shows that $\lim_{\|\Pi\|\to 0} E\left[(\bar{Q}_{\Pi/2} - T/2)^2\right] = 0$. Now, by mulitplying out the terms in $Q_{\Pi/2}$ and $\bar{Q}_{\Pi/2}$ and summing up, one can verify that

$$W^2(T) - 2\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1}) - W(t_j)\right) = \bar{Q}_{\Pi/2} - Q_{\Pi/2}.$$

Since $Q_{\Pi/2}$ and $\bar{Q}_{\Pi/2}$ both converge to $T/2$ as $\|\Pi\| \to 0$,

$$W^2(T) - 2\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1}) - W(t_j)\right) \to 0, \quad \text{as } \|\Pi\| \to 0.$$

The second method follows the hint in the problem statement. Write

$$\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1}) - W(t_j)\right) = Q_{\Pi/2} + \sum_{j=0}^{n-1} W(t_j)\left(W(t_j^*) - W(t_j)\right) + W(t_j^*)\left(W(t_{j+1}) - W(t_j^*)\right).$$

The second sum converges to the Itô integral $\int_0^T W(u)\,dW(u) = (W^2(T) - T)/2$, while the first term converges to $T/2$, as $\|\Pi\| \to 0$. Hence

$$\sum_{j=0}^{n-1} W(t_j^*)\left(W(t_{j+1}) - W(t_j)\right) \to 0, \quad \text{as } \|\Pi\| \to W^2(T)/2.$$