Take-Home Problems. Mathematical Finance; Fall, 2005; Due Dec. 16.

1. The object of this problem is to rewrite the price model

$$dS_1(t) = S_1(t)\left[\alpha_1\, dt + \sigma_{11}\, dW_1(t) + \sigma_{12}\, dW_2(t)\right] \tag{1}$$
$$dS_2(t) = S_2(t)\left[\alpha_1\, dt + \sigma_{21}\, dW_1(t) + \sigma_{22}\, dW_2(t)\right] \tag{2}$$

where W_1 and W_2 are independent Brownian motions, in the form

$$dS_1(t) = S_1(t)\left[\alpha_1\, dt + \sigma_1\, dB_1(t)\right] \tag{3}$$
$$dS_2(t) = S_2(t)\left[\alpha_1\, dt + \sigma_2\left(\rho\, dB_1(t) + \sqrt{1-\rho^2}\, dB_2(t)\right)\right], \tag{4}$$

where B_1 and B_2 are independent Brownian motions.

We sketched the construction and proof in lecture and this problem is to fill in the details. It is useful to keep in mind the following facts about jointly normal random variables: (i) if $Z = (X_1, \ldots, X_m)$ is a vector of jointly normal random variables, then for any scalars $\theta_1, \ldots, \theta_m$, $\theta_1 X_1 + \cdots + \theta_m X_m$ is normal; (ii) if $Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ are jointly normal, then $(X_1, \ldots, X_m)$ is independent of $(Y_1, \ldots, Y_n)$ if and only if $\mathrm{Cov}(X_i, Y_j) = 0$ for every $1 \le i \le m$ and $1 \le j \le n$.

Show that if S_1 and S_2 satisfy (1)-(2), and if one defines

$$\sigma_1^2 = \sigma_{11}^2 + \sigma_{12}^2, \quad \sigma_2^2 = \sigma_{21}^2 + \sigma_{22}^2, \quad \rho = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2},$$

and also

$$B_1(t) = \frac{1}{\sigma_1}\left[\sigma_{11} W_1(t) + \sigma_{12} W_2(t)\right]$$
$$B_2(t) = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{1}{\sigma_2}\left(\sigma_{21} W_1(t) + \sigma_{22} W_2(t)\right) - \rho B_1(t)\right],$$

then B_1 and B_2 are independent Brownian motions and S_1, S_2 satisfy equations (3) and (4).

2. This problem shows that more than n Brownian motion noise inputs in a multi-stock Black-Scholes model with constant coefficients are superfluous.

Let W_1, W_2, W_3 be independent Brownian motions and let S_1 and S_2 satisfy

$$dS_1(t) = S_1(t)\left[\alpha_1\, dt + \sigma_{11}\, dW_1(t) + \sigma_{12}\, dW_2(t) + \sigma_{13}\, dW_3(t)\right] \tag{5}$$
$$dS_2(t) = S_2(t)\left[\alpha_1\, dt + \sigma_{21}\, dW_1(t) + \sigma_{22}\, dW_2(t) + \sigma_{23} W_3(t)\right] \tag{6}$$

(i). Define σ_1^2 and σ_2^2, by $\mathrm{Var}(dS_1/S_1) = \sigma_1^2\, dt$ and $\mathrm{Var}(dS_2/S_2) = \sigma_2^2\, dt$. Calculate σ_1, σ_2, and $\rho = \mathrm{Corr}(dS_1/S_1, dS_2/S_2)$ in terms of the coefficients σ_{ij}. (Here, $\mathrm{Corr}(X, Y)$ denotes the correlation of X and Y.)

(ii) Define independent Brownian motions B_1 and B_2 such that the equations for S_1 and S_2 can be rewritten in the form of equations (3) -(4) of problem 1. Sketch a justification of your answer.

3. Consider the m-stock Black-Scholes model, with m driving Brownian motions and constant coefficients:

$$dS_i(t) = \alpha_i S_i(t)\, dt + S_i(t) \sum_1^m \sigma_{ij}\, dW_j(t), \qquad 1 \le i \le m \tag{7}$$

(i). It should be clear now that individually each S_i follows a Black-Scholes model

$$dS_i(t) = \alpha_i S_i(t)\, dt + \sigma_i S_i(t)\, dB_i(t).$$

It should also be clear from the previous problems what the volatility σ_i^2 of S_i is. Write down a formula for σ_i^2.

(ii). Consider a European call option, with expiry T, written on a single stock S_1. Since we have a model with correlated prices, we can ask if the price of this option at times previous to T depends on other stock prices. Show that in fact it does not: the Black-Scholes price derived in the one stock model is the correct price if the volatility σ_1^2 of S_1 is used. (Hint: show that the delta hedging formula of the single stock model provides a replicating portfolio based on trading only in S_1, but you may have your own argument.) (For extra credit, show that this principle extends to any contingent claim written solely on S_1.)

4. In the second midterm we derived a backward recursion for pricing an Asian option in a discrete-time binomial model. The aim of this problem is derive a PDE for the price of an Asian option in the Black-Scholes model.

The assumptions: r is the risk free interest rate and the stock S solves

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t), \tag{8}$$

where we assume that r, α, and σ are constants, and W is a Brownian motion.

An Asian option on S expiring at time T with strike price K, pays out

$$C(T) \triangleq \max\{\frac{1}{T}\int_0^T S(u)\, du, 0\},$$

at time T. We are concerned with deriving a formula for the no arbitrage price of this option at earlier times.

One approach is to use the replicating portfolio idea. Recall: for an adapted process $\triangle$, a wealth process $X(t)$, from the self-financing strategy based on holding $\triangle(t)$ shares of stock at each time t, satisfies the wealth equation

$$dX(t) = rX(t)\,dt + (\alpha - r)\triangle(t)S(t)\,dt + \triangle(t)\sigma S(t)\,dW(t) \tag{9}$$

If one can find a $\triangle$ and an $X(t)$ satisfying (9) such that $X(T) = C(T)$, then $X(t)$ is the no arbitrage price of the option at time t.

The trick to finding an equation for the price is a generalization of the trick used in the binomial model. Define the process

$$Y(t) \triangleq \int_0^t S(u)\,du, \qquad 0 \leq t \leq T.$$

We try to represent the price in the form $X(t) = c(t, S(t), Y(t))$, where $c(t, s, y)$ is a function of (t, s, y), $0 \leq t \leq T$, $s \geq 0$, $y \geq 0$. Find a parial differential equation for c, a terminal condition specifying $C(T, s, y)$, and a portfolio $\triangle$, so that $X(t) = c(t, S(t), Y(t))$ satisfies (9) and $X(T) = C(T)$.

5. Problem 5.10 in Shreve on the chooser option.