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# PU(2) monopoles and relations between four-manifold invariants

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**Abstract**

Motivated by developments in quantum field theory, Witten has conjectured a relation between the Donaldson and Seiberg–Witten invariants of smooth four-manifolds. We describe this conjecture and the program to prove it using a moduli space of PU(2) monopoles. We summarize our generic-parameter transversality and Uhlenbeck compactness results for PU(2) monopoles, along with some of our calculations of Donaldson invariants in terms of Seiberg–Witten invariants. We give a brief overview of issues concerning the gluing theory, focussing on some of the analytical difficulties that are particular to PU(2) monopoles, and its application to the program to prove Witten's conjecture.


*Keywords:* Four-manifolds; Relations between Donaldson and Seiberg–Witten invariants; Moduli spaces of PU(2) monopoles; Non-Abelian monopoles

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## 1. Introduction

The principal objective of our series of articles [13–16] and beyond, for which we provide a brief survey here, is to prove the analogue of the Kotschick–Morgan conjecture for PU(2) monopoles suggested by Pidstrigach and Tyurin [56]. This in turn should lead to a proof of Witten's conjecture concerning the relation between Donaldson and Seiberg–Witten invariants and a deeper understanding of the highly successful role of gauge theory in smooth four-manifold topology. We describe Witten's conjecture below

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and outline the program (see [28,29,37,38,50,51,54,56]), to prove this conjecture using $\mathrm{PU}(2)$ monopoles. While the basic ideas in this program are by now well known, the profound analytical difficulties inherent in attempts to implement it are perhaps much less well known and so we feel it is worthwhile to describe some of these analytical problems here. These analytical difficulties involve the gluing construction of links of lower-level moduli spaces of $\mathrm{U}(1)$ monopoles contained in the Uhlenbeck compactification of the moduli space of $\mathrm{PU}(2)$ monopoles. The question of existence of perturbations for the $\mathrm{PU}(2)$ monopole equations, yielding both useful transversality results and an Uhlenbeck compactification for the perturbed moduli space, is a fairly substantial one in its own right [13]. We describe these transversality and compactness results here, along with some of our calculations of Donaldson invariants in terms of Seiberg–Witten invariants from [14] and a brief overview of issues concerning the gluing theory from [15,16] and its applications.

First, to explain Witten's conjecture we recall that a closed, smooth four-manifold $X$ is said to have *Seiberg–Witten simple type* if the Seiberg–Witten moduli spaces corresponding to nonzero Seiberg–Witten invariants are all zero-dimensional. The manifold $X$ has *Kronheimer–Mrowka simple type* provided the Donaldson invariants corresponding to products $z$ of homology classes in $H_\bullet(X)$ and a generator $x \in H_0(X)$ are related by $D^w_X(x^2 z) = 4D^w_X(z)$. Kronheimer and Mrowka [35] (see also [18]) showed that the Donaldson series of a four-manifold of Kronheimer–Mrowka simple type with $b^1(X) = 0$ and odd $b^+(X) \geqslant 3$ is given by

$$\mathcal{D}^w = \mathrm{e}^{Q/2} \sum_{r=1}^{s} (-1)^{(w^2 + wK_r)/2} a_r \mathrm{e}^{K_r}, \tag{1.1}$$

where $w$ is a line bundle over $X$, $Q$ is the intersection form on $H_2(X;\mathbb{Z})$, the coefficients $a_r$ are nonzero rational numbers, and the $K_r \in H^2(X;\mathbb{Z})$ are the *Kronheimer–Mrowka basic classes*. Let $\mathrm{Spin}^c(X)$ be the set of isomorphism classes of spin$^c$ structures on $X$ and let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and signature of $X$, respectively.

**Conjecture 1.1** (Witten [66]). Suppose $X$ is a closed, oriented four-manifold with $b^1(X) = 0$ and odd $b^+(X) \geqslant 3$, equipped with a homology orientation and a line bundle $w$. Then $X$ has Kronheimer–Mrowka simple type if and only if it has Seiberg–Witten simple type. If $X$ has simple type, then the Kronheimer–Mrowka basic classes are given by

$$\{c_1(W^+_{\mathfrak{s}}):\ \mathfrak{s} \in \mathrm{Spin}^c(X) \text{ such that } SW(\mathfrak{s}) \neq 0\},$$

where $c_1(\mathfrak{s}) := c_1(W^+_{\mathfrak{s}})$ and $W^\pm_{\mathfrak{s}}$ are the spin$^c$ bundles associated to $\mathfrak{s}$ with some choice of Riemannian metric on $X$; furthermore, the Donaldson series for $X$ is given by

$$\mathcal{D}^w = 2^{2+(7e+11\sigma)/4} \mathrm{e}^{Q/2} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{(w^2 + wc_1(\mathfrak{s}))/2} SW(\mathfrak{s}) \mathrm{e}^{c_1(\mathfrak{s})}. \tag{1.2}$$

The conjecture holds for all four-manifolds whose Donaldson and Seiberg–Witten invariants have been independently computed. The mathematical approach to this conjecture uses a moduli space of solutions to the PU(2) monopole equations—which generalize the U(1) monopole equations of Seiberg and Witten—to construct a cobordism between links of the compact moduli spaces of U(1) monopoles of Seiberg–Witten type and the Donaldson moduli space of anti-self-dual connections, which appear as singularities in this larger moduli space. Moreover, this approach should give a precise relation between the Donaldson and Seiberg–Witten invariants even for four-manifolds not of simple type. This is an important point since there are no known examples of four-manifolds with $b^+ > 1$ violating either of the simple type conditions, so we would expect to gain a greater understanding of these conditions from such a general relation.

The moduli space of PU(2) monopoles is noncompact and has an Uhlenbeck compactification similar to that of the moduli space of anti-self-dual connections. The substantial analytical difficulties are due to the contributions of moduli spaces of U(1) monopoles (cobordant to standard Seiberg–Witten moduli spaces) in the lower Uhlenbeck levels—the 'reducibles' at the boundary of the Uhlenbeck compactification. Many of these problems had never been resolved even in the case of Donaldson theory where they arise, albeit in a rather simpler form, in attempts to prove the Kotschick–Morgan conjecture for Donaldson invariants. The Kotschick–Morgan conjecture for Donaldson invariants of four-manifolds $X$ with $b^+(X) = 1$ asserts that the invariants computed using metrics lying in different chambers of the positive cone of $H^2(X;\mathbb{R})/\mathbb{R}^*$ differ by terms depending only on homotopy data [32]. The heart of the problem there lies in describing the links of the reducible connections in the lower Uhlenbeck levels via gluing and then computing integrals of the Donaldson cohomology classes over those links. To date, links of this type in anti-self-dual moduli spaces have been described and their pairings with cohomology classes computed in only a few relatively simple special cases [5–7,11,39,67]: the methods used there fall far short of what is needed to complete the PU(2) monopole program to prove the equivalence between Donaldson and Seiberg–Witten invariants. By assuming the Kotschick–Morgan conjecture, Göttsche has computed the coefficients of the wall-crossing formula in [32] in terms of modular forms by exploiting the presumed homotopy invariance of the coefficients [25]. A related approach to the Witten conjecture has been suggested by Pidstrigach and Tyurin [56]: they proposed a PU(2) monopole analogue of the Kotschick–Morgan conjecture and argue that it can be used to compute the required integrals of analogues of the Donaldson cohomology classes over the links of the lower-level moduli spaces of U(1) monopoles.

In Section 2 we describe the PU(2) monopole equations, the holonomy perturbations we use in order to achieve transversality, and the Uhlenbeck compactification for the perturbed moduli space of PU(2) monopoles. In Section 3 we describe the cohomology classes, the links of the moduli spaces of anti-self-dual connections and Seiberg–Witten monopoles appearing in the top Uhlenbeck level, their orientations, and the relation between the Donaldson and Seiberg–Witten invariants when the moduli spaces of U(1) monopoles appear only in the top Uhlenbeck level. Finally, in Section 4 we describe the Kotschick–Morgan conjecture, its analogue in the case of PU(2) monopoles and how this might be used to prove Witten's conjecture. We also describe the need for gluing, survey some of the results from [15,16] and describe a few of the more prominent difficulties which arise when gluing PU(2) monopoles. Detailed proofs of all our results appear elsewhere [13–16], so we just sketch the main ideas here.

## 2. Holonomy perturbations, transversality, and Uhlenbeck compactness

We consider Hermitian two-plane bundles $E$ over $X$ whose determinant line bundles $\det E$ are isomorphic to a fixed Hermitian line bundle over $X$ endowed with a fixed $C^\infty$, unitary connection. Choose a Riemannian metric on $X$ and let $\mathfrak{s}_0 := (\rho, W)$ be a spin$^c$ structure on $X$, where $\rho: T^*X \to \operatorname{End} W$ is the Clifford map, and the Hermitian four-plane bundle $W = W^+ \oplus W^-$ is endowed with a $C^\infty$ spin$^c$ connection. *The* spin$^c$ *structure* $(\rho, W)$, *the* spin$^c$ *connection on* $W$, *and the Hermitian line bundle together with its connection are fixed once and for all.*

Let $k \geqslant 2$ be an integer and let $\mathcal{A}_E$ be the space of $L^2_k$ connections $A$ on the U(2) bundle $E$ all inducing the fixed determinant connection on $\det E$. Equivalently, following [35, Section 2(i)], we may view $\mathcal{A}_E$ be the space of $L^2_k$ connections $A$ on the SO(3) = PU(2) bundle $\mathfrak{su}(E)$. We shall often pass back and forth between these viewpoints, via the fixed connection on $\det E$, relying on the context to make the distinction clear. Let

$$D_A : L^2_k(W^+ \otimes E) \to L^2_{k-1}(W^- \otimes E)$$

be the corresponding Dirac operators. Given a connection $A$ on $E$ with curvature $F_A \in L^2_{k-1}(\Lambda^2 \otimes \mathfrak{u}(E))$, then $(F_A^+)_0 \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ denotes the traceless part of its self-dual component. Equivalently, if $A$ is a connection on $\mathfrak{su}(E)$ with curvature $F_A \in L^2_{k-1}(\Lambda^2 \otimes \mathfrak{so}(\mathfrak{su}(E)))$, then $\operatorname{ad}^{-1}(F_A^+) \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ is its self-dual component, viewed as a section of $\Lambda^+ \otimes \mathfrak{su}(E)$ via the isomorphism $\operatorname{ad}: \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$.

For an $L^2_k$ section $\Phi$ of $W^+ \otimes E$, let $\Phi^*$ be its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of the Hermitian endomorphism $\Phi \otimes \Phi^*$ of $W^+ \otimes E$ which lies in $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. The spin$^c$ structure $\rho$ defines an isomorphism $\rho^+ : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism $\rho^+ = \rho^+ \otimes \mathrm{id}_{\mathfrak{su}(E)}$ of $\Lambda^+ \otimes \mathfrak{su}(E)$ with $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. Then

$$\begin{aligned} &(F_A^+)_0 - (\rho^+)^{-1}(\Phi \otimes \Phi^*)_{00} = 0, \\ &D_A \Phi = 0, \end{aligned} \tag{2.1}$$

are essentially the unperturbed equations considered in [50,51,54,56] for a pair $(A, \Phi)$ consisting of a fixed-determinant connection $A$ on $E$ and a section $\Phi$ of $W^+ \otimes E$. (The trace conditions and precise setting vary; Eqs. (2.1) are closer to those of [64,65] than [56].) Equivalently, given a pair $(A, \Phi)$ with $A$ a connection on $\mathfrak{su}(E)$, Eqs. (2.1) take the same form except that $(F_A^+)_0$ is replaced by $\operatorname{ad}^{-1}(F_A^+)$ or simply by $F_A^+$, with the isomorphism $\operatorname{ad}: \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$ being implicit.

In this section we briefly describe the holonomy perturbations of these equations which we introduced in [13]: these perturbations allow us to prove transversality for the moduli space of solutions, away from points where the connection is reducible or the spinor vanishes identically, and to prove the existence of an Uhlenbeck compactification for this perturbed moduli space.

Donaldson's proof of the connected sum theorem for his polynomial invariants [10, Theorem B] makes use of certain 'extended anti-self-dual equations' [10, Eq. (4.24)] to which the Freed–Uhlenbeck generic metrics theorem does not apply [10, Section 4(v)]. To obtain transversality for the zero locus of these extended equations, he employs holonomy perturbations which give gauge-equivariant $C^\infty$ maps $\mathcal{A}_E^* \to L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(E))$ [8, Section 2], [10, pp. 282–287]. These perturbations are continuous across the Uhlenbeck boundary and yield transversality not only for the top stratum, but also for all lower strata and for all intersections of the geometric representatives defining the Donaldson invariants.

In [13] we describe a generalization of Donaldson's idea which we use to prove transversality for the moduli space of solutions to a perturbed version of the PU(2) monopole equations (2.1). Unfortunately, in the case of the moduli space of PU(2) monopoles, the analysis is considerably more intricate. In Donaldson's application, some important features ensure that the requisite analysis is relatively tractable: (i) reducible connections can be excluded from the compactification of the extended moduli spaces [10, p. 283], (ii) the cohomology groups for the elliptic complex of his extended equations have simple weak semi-continuity properties with respect to Uhlenbeck limits [10, Proposition 4.33], and (iii) the perturbed zero-locus is cut out of a finite-dimensional manifold [10, p. 281, Lemma 4.35 and Corollary 4.38]. For the development of Donaldson's method for PU(2) monopoles described here and in detail in [13], none of these simplifying features hold and so the corresponding transversality argument is rather complicated. Indeed, one can see from Proposition 7.1.32 in [11] that because of the Dirac operator, the behavior of the cokernels of the linearization of the PU(2) monopole equations can be quite involved under Uhlenbeck limits. The method we describe below uses an infinite sequence of perturbing sections defined on the infinite-dimensional configuration space of pairs; when restricted to small enough open balls in the configuration space, away from reducibles, only finitely many of these perturbing sections are nonzero and they vanish along the reducibles.

We shall describe these perturbations and their properties only in fairly general terms here, as the full description is lengthy; we refer the interested reader to [13] for a detailed account.

Let $\mathcal{G}_E$ be the Hilbert Lie group of $L^2_{k+1}$ unitary gauge transformations of $E$ with *determinant one*. It is generally convenient to take quotients by a slightly larger symmetry group than $\mathcal{G}_E$ when discussing pairs, so let $S^1_Z$ denote the center of U(2) and set

$$^\circ\mathcal{G}_E := S^1_Z \times_{\{\pm \mathrm{id}_E\}} \mathcal{G}_E,$$

which we may view as the group of $L^2_{k+1}$ unitary gauge transformations of $E$ with *constant determinant*. The stabilizer of a unitary connection on $E$ in $^\circ\mathcal{G}_E$ always contains the center $S^1_Z \subset \mathrm{U}(2)$. We call $A$ *irreducible* if its stabilizer is exactly $S^1_Z$ and *reducible* otherwise. Let $\mathcal{B}_E(X) = \mathcal{A}_E(X)/\mathcal{G}_E$ be the quotient space of $L^2_k$ connections on $E$ with fixed-determinant connection and let $\mathcal{A}^*_E(X)$ and $\mathcal{B}^*_E(X)$ be the subspace of irreducible $L^2_k$ connections and its quotient. As before, we may equivalently view $\mathcal{B}_E(X)$ and $\mathcal{B}^*_E(X)$

as quotients of the spaces of $L^2_k$ connections on $\mathfrak{su}(E)$ by the induced action of $\mathcal{G}_E$ on $\mathfrak{su}(E)$.

We construct gauge-equivariant $C^\infty$ maps

$$\begin{aligned} &\mathcal{A}_E(X) \ni A \mapsto \vec{\tau} \cdot \vec{\mathfrak{m}}(A) \in L^2_{k+1}\big(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E))\big), \\ &\mathcal{A}_E(X) \ni A \mapsto \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) \in L^2_{k+1}\big(X, \operatorname{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E)\big), \end{aligned} \tag{2.2}$$

where $\vec{\tau} = (\tau_{j,l,\alpha})$ is a sequence in $\Omega^0(X, \mathfrak{gl}(\Lambda^+))$ and $\vec{\vartheta} = (\vartheta_{j,l,\alpha})$ is a sequence in $\Omega^1(X, \mathbb{C})$, while $\vec{\mathfrak{m}}(A) = (\mathfrak{m}_{j,l,\alpha}(A))$ is a sequence in $L^2_{k+1}(X, \mathfrak{su}(E))$ of holonomy sections constructed by extending the method of [8,10], and

$$\begin{aligned} \vec{\tau} \cdot \vec{\mathfrak{m}}(A) &:= \sum_{j,l,\alpha} \tau_{j,l,\alpha} \otimes_{\mathbb{R}} \operatorname{ad}\big(\mathfrak{m}_{j,l,\alpha}(A)\big), \\ \vec{\vartheta} \cdot \vec{\mathfrak{m}}(A) &:= \sum_{j,l,\alpha} \rho(\vartheta_{j,l,\alpha}) \otimes_{\mathbb{C}} \mathfrak{m}_{j,l,\alpha}(A). \end{aligned}$$

To construct these maps, we fix a collection of $N_b$ small, disjoint balls $\{4B_j\}_{j=1}^{N_b}$ in $X$, a locally finite open cover $\{U_{j,\alpha}\}_{\alpha=1}^{\infty}$ of each quotient space $\mathcal{B}^*_E(2B_j)$ of irreducible connections over $2B_j$, and three loops $\{\gamma_{j,l,\alpha}\}_{l=1}^3 \subset 2B_j$ such that holonomy around these loops spans $\mathfrak{su}(E)|_{B_j}$ for each connection in $\{U_{j,\alpha}\}$. The sections $\mathfrak{m}_{j,l,\alpha}$ are supported on $\overline{B}_j$ in $X$ and on $L^2_k$ balls containing $U_{j,\alpha}$ in $\mathcal{B}^*_E(2B_j)$, by a suitable choice of cutoff functions on $X$ and $\mathcal{B}^*_E(2B_j)$. The set $\{\mathfrak{m}_{j,l,\alpha}(A)\}_{l=1}^3$ spans $\mathfrak{su}(E)|_{B_j}$ for each point $[A|_{2B_j}] \in U_{j,\alpha}$ with energy $\|F_A\|^2_{L^2(4B_j)} < \varepsilon_0^2/2$, where $\varepsilon_0$ is a certain universal constant [13]. When this (regularized) energy bound is exceeded over a ball $4B_{j'}$, the associated perturbations vanish, ensuring continuity across the Uhlenbeck boundary. The number $N_b$ of balls $B_j$ may be chosen sufficiently large that for every solution $(A, \Phi)$ to the perturbed PU(2) monopole equations (2.4), there is at least one ball $B_{j'}$ whose associated holonomy sections $\{\mathfrak{m}_{j',l,\alpha}(A)\}_{l=1}^3$ span $\mathfrak{su}(E)|_{B_{j'}}$. We use the small-time heat kernel for the Neumann Laplacians $d_A^* d_A$ on $L^2(2B_j, \mathfrak{su}(E))$ to ensure that the sections $\mathfrak{m}_{j,l,\alpha}(A)$ are in $L^2_{k+1}$ when $A|_{2B_j}$ is in $L^2_k$.

By construction, the maps $\vec{\tau} \cdot \vec{\mathfrak{m}}$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}$ of (2.2) are uniformly $C^s$-bounded over $\mathcal{A}^*_E(X)$, when $\mathcal{A}^*_E(X)$ is endowed with its $L^2_k$ metric, *provided* $k \geqslant 3$ and which we shall therefore assume for the remainder of the article. Moreover, they are continuous with respect to Uhlenbeck limits, just as are those of [10]. Suppose $\{A_\beta\}$ is a sequence in $\mathcal{A}_E(X)$ which converges to an Uhlenbeck limit $(A, \boldsymbol{x})$ in $\mathcal{A}_{E_{-\ell}}(X) \times \operatorname{Sym}^\ell(X)$, where $E_{-\ell}$ is a Hermitian two-plane bundle over $X$ such that

$$\det(E_{-\ell}) = \det E \quad \text{and} \quad c_2(E_{-\ell}) = c_2(E) - \ell, \text{ with } \ell \in \mathbb{Z}_{\geqslant 0}.$$

The sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(A_\beta)$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A_\beta)$ then converge in $L^2_{k+1}(X)$ to a section $\vec{\tau} \cdot \vec{\mathfrak{m}}(A, \boldsymbol{x})$ of $\mathfrak{gl}(\Lambda^+) \otimes \mathfrak{so}(\mathfrak{su}(E_{-\ell}))$ and a section $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(A, \boldsymbol{x})$ of $\operatorname{Hom}(W^+, W^-) \otimes \mathfrak{sl}(E_{-\ell})$, respectively. For each $\ell \geqslant 0$, the maps of (2.2) extend continuously to gauge-equivariant maps

$$\begin{aligned} &\mathcal{A}_{E_{-\ell}}(X) \times \operatorname{Sym}^\ell(X) \to L^2_{k+1}\big(X, \mathfrak{gl}(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{so}(\mathfrak{su}(E_{-\ell}))\big), \\ &\mathcal{A}_{E_{-\ell}}(X) \times \operatorname{Sym}^\ell(X) \to L^2_{k+1}\big(X, \operatorname{Hom}(W^+, W^-) \otimes_{\mathbb{C}} \mathfrak{sl}(E_{-\ell})\big), \end{aligned} \tag{2.3}$$

given by $(A,\boldsymbol{x}) \mapsto \vec{\tau}\cdot\vec{\mathfrak{m}}(A,\boldsymbol{x})$ and $(A,\boldsymbol{x}) \mapsto \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A,\boldsymbol{x})$, respectively, which are $C^\infty$ on each $C^\infty$ stratum determined by $\mathrm{Sym}^\ell(X)$.

The parameters $\vec{\tau}$ and $\vec{\vartheta}$ vary in the Banach spaces of $\ell^1_\delta(\mathbb{A})$ sequences in $C^r(X,\mathfrak{gl}(\Lambda^+))$ and $C^r(\Lambda^1\otimes\mathbb{C})$, respectively, where $\mathbb{A} = \{(j,l,\alpha)\}$ and $r \geqslant k+1$,

$$\|\vec{\vartheta}\|_{\ell^1_\delta(C^r(X))} := \sum_{j,l,\alpha} \delta_\alpha^{-1}\|\vartheta_{j,l,\alpha}\|_{C^r(X)},$$

and similarly for $\|\vec{\tau}\|_{\ell^1_\delta(C^r(X))}$. The sequence of weights $\delta = (\delta_\alpha)_{\alpha=1}^\infty \in \ell^\infty((0,1])$ may be chosen so that the gauge-equivariant maps of (2.2) are smooth even at reducible connections, where the maps vanish [13].

We call an $L^2_k$ pair $(A,\Phi)$ in the *pre-configuration space*,

$$\widetilde{\mathcal{C}}_{W,E} := \mathcal{A}_E \times L^2_k(X, W^+\otimes E),$$

a $\mathrm{PU}(2)$ *monopole* if it solves

$$\begin{aligned}
&(F_A^+)_0 - \big(\mathrm{id} + \tau_0\otimes \mathrm{id}_{\mathfrak{su}(E)} + \vec{\tau}\cdot\vec{\mathfrak{m}}(A)\big)(\rho^+)^{-1}(\Phi\otimes\Phi^*)_{00} = 0,\\
&D_A\Phi + \rho(\vartheta_0)\Phi + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)\Phi = 0,
\end{aligned} \tag{2.4}$$

where $\tau_0 \in C^r(X,\mathfrak{gl}(\Lambda^+))$ and $\vartheta_0 \in C^r(\Lambda'\otimes\mathbb{C})$. For convenience, we often denote the perturbed Dirac operator $D_A + \rho(\vartheta_0) + \vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ simply by $D_{A,\vec{\vartheta}}$. We let $M_{W,E}$ be the moduli space of solutions cut out of the *configuration space*,

$$\mathcal{C}_{W,E} := \widetilde{\mathcal{C}}_{W,E}/{}^\circ\mathcal{G}_E,$$

by Eqs. (2.4), where $u \in {}^\circ\mathcal{G}_E$ acts by $u(A,\Phi) := (u_*A, u\Phi)$.

We let $\mathcal{C}^{*,0}_{W,E} \subset \mathcal{C}_{W,E}$ be the subspace of pairs $[A,\Phi]$ such that $A$ is irreducible and the section $\Phi$ is not identically zero and set $M^{*,0}_{W,E} = M_{W,E}\cap\mathcal{C}^{*,0}_{W,E}$. Note that we have a canonical inclusion $\mathcal{B}_E \subset \mathcal{C}_{W,E}$ given by $[A] \mapsto [A,0]$ and similarly for the pre-configuration spaces.

The sections $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ vanish at reducible connections $A$ by construction; plainly, the terms in (2.4) involving the perturbations $\vec{\tau}\cdot\vec{\mathfrak{m}}(A)$ and $\vec{\vartheta}\cdot\vec{\mathfrak{m}}(A)$ are zero when $\Phi$ is zero. The holonomy perturbations considered by Donaldson in [10] are inhomogeneous, as he uses the perturbations to kill the cokernels of $d_A^+$ directly. In contrast, the perturbations we consider in (2.4) are homogeneous and we argue indirectly that the cokernels of the linearization vanish away from the reducibles and zero-section solutions.

A careful application of the Agmon–Nirenberg unique continuation theorem [1] to (2.4) ensures that a $\mathrm{PU}(2)$ monopole $(A,\Phi)$ which is irreducible on $X$ gives at least one restriction $A|_{2B_{j'}}$ which is irreducible and whose associated holonomy sections span $\mathfrak{su}(E)|_{B_{j'}}$. The corresponding property for anti-self-dual connections is proved as Lemma 4.3.21 in [11]. The proof of Donaldson and Kronheimer relies on the Agmon–Nirenberg unique continuation theorem for an ordinary differential inequality on a Hilbert space [1, Theorem 2]. We show in [13] that Donaldson and Kronheimer's argument adapts to the case of the $\mathrm{PU}(2)$ monopole equations (2.1) or (2.4), when the initial open set where $(A,\Phi)$ is reducible contains the closed balls $\overline{B}(x_j,R_0)$ supporting holonomy perturbations.

The perturbations $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ then ensure that an element in the cokernel of the linearization of the parametrized version of (2.4), at a point $(A, \Phi, \tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ where $A$ is irreducible and $\Phi \not\equiv 0$, must vanish identically over at least one ball $B_{j'}$ and so must vanish identically over $X$ by the Aronszajn–Cordes unique continuation theorem [2]. Hence, the Sard–Smale theorem yields:

**Theorem 2.1** [13]. *Let $X$ be a closed, oriented, smooth four-manifold with $C^\infty$ Riemannian metric, spin$^c$ structure $(\rho, W)$ with spin$^c$ connection, and a Hermitian line bundle $\det E$ with unitary connection. Then there exists a first-category subset of the space of $C^\infty$ perturbation parameters such that the following holds: For each 4-tuple $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ in the complement of this first-category subset, the moduli space $M^{*,0}_{W,E}(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$ is a smooth manifold of the expected dimension*

$$\begin{aligned}\dim M^{*,0}_{W,E} &= -2p_1(\mathfrak{su}(E)) - \tfrac{3}{2}(e(X) + \sigma(X)) \\ &\quad + \tfrac{1}{2}p_1(\mathfrak{su}(E)) + \tfrac{1}{2}(F^2 - \sigma(X)) - 1,\end{aligned}$$

*where $p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E)$ and $F := c_1(W^+) + c_1(E)$.*

**Remark 2.2.** Different approaches to the question of transversality for Eqs. (2.1) with generic perturbation parameters have also been considered by the authors, by Pidstrigach and Tyurin in [56] and by Teleman in [65]: see [13] for further details.

We now turn to the question of compactness of $M_{W,E}$, for the given generic parameters $(\tau_0, \vartheta_0, \vec{\tau}, \vec{\vartheta})$. We say that a sequence of points $[A_\beta, \Phi_\beta]$ in $\mathcal{C}_{W,E}$ *converges* to a point $[A, \Phi, \mathbf{x}]$ in $\mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$ if the following hold:

- There is a sequence of determinant-one, $L^2_{k+1,\mathrm{loc}}$ bundle maps

  $$u_\beta : E|_{X\setminus\{\mathbf{x}\}} \to E_{-\ell}|_{X\setminus\{\mathbf{x}\}}$$

  such that the sequence of monopoles $u_\beta(A_\beta, \Phi_\beta)$ converges to $(A, \Phi)$ in $L^2_{k,\mathrm{loc}}$ over $X \setminus \{\mathbf{x}\}$, and
- The sequence of measures $|F_{A_\beta}|^2$ converges in the weak-* topology on measures to $|F_A|^2 + 8\pi^2 \sum_{x\in\mathbf{x}} \delta(x)$.

We let $M_{W,E_{-\ell}}(\mathbf{x})$ denote the moduli space of pairs $(A, \Phi)$ solving (2.4) with perturbing sections $\vec{\tau} \cdot \vec{\mathfrak{m}}(\cdot, \mathbf{x})$ and $\vec{\vartheta} \cdot \vec{\mathfrak{m}}(\cdot, \mathbf{x})$, let $\boldsymbol{M}_{W,E_{-\ell}}$ denote the moduli space of triples $(A, \Phi, \mathbf{x})$ solving (2.4) for $\ell \geqslant 0$, and let $\boldsymbol{M}_{W,E_{-0}} = M_{W,E}$. We define $\overline{M}_{W,E}$ to be the Uhlenbeck closure of $M_{W,E}$ in the space of ideal PU(2) monopoles,

$$IM_{W,E} := \bigcup_{\ell=0}^{N} \boldsymbol{M}_{W,E_{-\ell}} \subset \bigcup_{\ell=0}^{N} (\mathcal{C}_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X))$$

for any integer $N \geqslant N_p$ where $N_p$ is a sufficiently large constant. Analogues of Bochner formulas used in the proof of compactness for the Seiberg–Witten equations [34,66] provide a universal energy bound for solutions to (2.4), guaranteeing that the constants $N_b$ and $N_p$ exist. By combining the methods used in the proof of compactness for the

Seiberg–Witten moduli space [34] and Uhlenbeck compactness for the moduli space of anti-self-dual equations [11] we obtain:

**Theorem 2.3** [13]. *Let $X$ be a closed, oriented, smooth four-manifold with $C^\infty$ Riemannian metric, spin$^c$ structure $(\rho, W)$ with spin$^c$ connection, and a Hermitian two-plane bundle $E$ with unitary connection on* $\det E$. *Then there is a positive integer $N_p$, depending at most on the curvatures of the fixed connections on $W$ and* $\det E$ *together with $c_2(E)$, such that for all $N \geqslant N_p$ the topological space $\overline{M}_{W,E}$ is compact, second-countable, Hausdorff, and is given by the closure of $M_{W,E}$ in $\bigcup_{\ell=0}^{N} M_{W,E_{-\ell}}$.*

**Remark 2.4.** The existence of an Uhlenbeck compactification for the moduli space of solutions to the unperturbed PU(2) monopole equations (2.1) was announced by Pidstrigach [54] and an argument was outlined in [56]. A similar argument for Eqs. (2.1) was outlined by Okonek and Teleman in [51]. Theorem 2.3 yields the standard Uhlenbeck compactification for the system (2.1) and for the perturbations of (2.1) described in [56]. A proof of Uhlenbeck compactness for (2.1) (and for certain perturbations of these equations) is also given in [65].

We use the term *(Uhlenbeck) level* to describe the spaces $M_{W,E_{-\ell}}$ for different values of $\ell \geqslant 0$, with $M_{W,E}$ comprising the *top (Uhlenbeck) level*. The space $\mathrm{Sym}^\ell(X)$ is smoothly stratified, the strata being enumerated by partitions of $\ell$. If $\Sigma \subset \mathrm{Sym}^\ell(X)$ is a smooth stratum, we define

$$M_{W,E_{-\ell}}|_\Sigma := \{[A,\Phi,\mathbf{x}] \in M_{W,E_{-\ell}}\colon \mathbf{x} \in \Sigma\}.$$

The proof of Theorem 2.1 shows, more generally, that for each $\ell \geqslant 0$ the moduli spaces

$$M^{*,0}_{W,E_{-\ell}}|_\Sigma := M_{W,E_{-\ell}}|_\Sigma \cap M^{*,0}_{W,E_{-\ell}}$$

are smooth and of the expected dimension, and over the complement in $\Sigma$ of a first-category subset, the projection $M^{*,0}_{W,E_{-\ell}}|_\Sigma \to \Sigma$ is a fiber bundle. See [13] for the general statement. In the more familiar case of the Uhlenbeck closure of the moduli space of solutions to the unperturbed PU(2) monopole equations (2.1), the spaces $M_{W,E_{-\ell}}$ would be replaced by the products $M_{W,E_{-\ell}} \times \mathrm{Sym}^\ell(X)$. In general, though, the spaces $M_{W,E_{-\ell}}$ are not products due to the slight dependence of the lower-level analogues of Eqs. (2.4) on the points $\mathbf{x} \in \mathrm{Sym}^\ell(X)$. A similar phenomenon is encountered in [10, Sections 4(iv)–(vi)] for the case of the extended anti-self-dual equations.

While the description of the holonomy perturbations outlined above may appear fairly complicated at first glance in practice, they do not present any major difficulties beyond those that would be encountered if simpler perturbations not involving the bundle $\mathfrak{su}(E)$ (such as the Riemannian metric on $X$ or the connection on $\det W^+$) were sufficient to achieve transversality [14–16]. We note that related transversality and compactness issues have been recently considered in approaches to defining Gromov–Witten invariants for general symplectic manifolds [41,58,59].

## 3. Cohomology and cobordisms

The moduli space $M_{W,E}$ contains singularities: it is a smoothly stratified space, with strata diffeomorphic to the moduli space of anti-self-dual connections on $\mathfrak{su}(E)$ and to moduli spaces of U(1) monopoles (which are in turn cobordant to moduli spaces of Seiberg–Witten monopoles). The space $M^{*,0}_{W,E}$ therefore gives a cobordism between the links of these two types of singularities. In this section, we introduce cohomology classes on $M^{*,0}_{W,E}$ and define the links of these singularities.

### *3.1. Singularities*

We see from Theorem 2.1 that the moduli space $M^{*,0}_{W,E}$ of PU(2) monopoles $[A,\Phi]$, where $A$ is not reducible and $\Phi \not\equiv 0$, forms a smooth manifold. We now describe the subspaces where $A$ is reducible or $\Phi \equiv 0$.

Let $M^{\mathrm{asd}}_E \subset M_{W,E}$ denote the subspace of points $[A,\Phi]$ where $\Phi \equiv 0$; we refer to pairs representing points in $M^{\mathrm{asd}}_E$ as *zero-section pairs*. Equivalently, we may view $M^{\mathrm{asd}}_E \subset \mathcal{B}_E$ as the moduli space of fixed-determinant connections $A$ on $E$ solving the *anti-self-dual equation*,

$$(F^+_A)_0 = 0, \tag{3.1}$$

or simply $F^+_A = 0$, if $\mathcal{B}_E$ is viewed as the quotient space of connections $A$ on $\mathfrak{su}(E)$.

Suppose we have a reduction of the U(2) bundle $E$ given as an (ordered) direct sum of line bundles,

$$E = L_1 \oplus L_2.$$

Note that gauge transformations of $E$ (in ${}^{\circ}\mathcal{G}_E = S^1_Z \times_{\{\pm \mathrm{id}_E\}} \mathcal{G}_E$) which interchange the line bundles $L_1$ and $L_2$ only exist if $L_1 = L_2$. We let $M^{\mathrm{red}}_{W,E,L_1} \subset M_{W,E}$ denote the subspace of points $[A,\Phi]$ with $\mathrm{Stab}_{A,\Phi} = S^1_{L_2}$, where $S^1_{L_2} = S^1$ acts by constant multiplication on the line bundle $L_2$. We refer to pairs representing points in $M^{\mathrm{red}}_{W,E,L_1}$ as *reducible pairs*: they have the form $(A_1 \oplus A_2, \Phi_1)$, where $A_1$ is a unitary connection on $L_1$ and $A_2 = A_e \otimes A_1^*$ is the corresponding connection on $L_2 = (\det E) \otimes L_1^*$, where $A_e$ is the fixed connection on $\det E$, while $\Phi_1$ is a section of $W^+ \otimes L_1$. The pair $(A_1, \Phi_1)$ is a solution to the U(1) *monopole equations*,

$$\begin{aligned} &F^+_{A_1} - \tfrac{1}{2}(\mathrm{id} + \tau_0)(\Phi_1 \otimes \Phi_1^*)_0 - \tfrac{1}{2}F^+_{A_e} = 0, \\ &D_{A_1}\Phi_1 = 0. \end{aligned} \tag{3.2}$$

The moduli space of solutions to (3.2), which parameterizes $M^{\mathrm{red}}_{W,E,L_1}$, is smooth and of the expected dimension for generic $\tau_0$ away from the zero-section solutions (see [14]) and is cobordant to the standard Seiberg–Witten moduli space $M^{\mathrm{sw}}_{W\otimes L_1}$ associated to the spin$^c$ structure $(\rho, W \otimes L_1)$ (as defined, for example, in [44]).

**Proposition 3.1** [14]. *Let $X$ be a closed, oriented, smooth four-manifold with $b^+(X) \geqslant 1$ and generic Riemannian metric. Suppose the pair $(A,\Phi)$ on $(E, W^+ \otimes E)$ represents* *a point* $[A,\Phi]\in M_{W,E}$ *with nontrivial stabilizer* $\mathrm{Stab}_{A,\Phi}$. *Then one of the following, mutually exclusive situations holds*:

(1) *The pair* $(A,\Phi)$ *is a zero-section pair* ($\Phi\equiv 0$) *and the connection* $A$ *is irreducible. The pair* $(A,0)$ *has stabilizer* $\mathrm{Stab}_{A,0}=S^1_Z$, *the connection* $A$ *has stabilizer* $\mathrm{Stab}_A=S^1_Z$, *and* $A$ *is projectively anti-self-dual (so* $(F_A^+)_0=0$). *The quotient space of zero-section pairs is identified with the moduli space* $M_E^{\mathrm{asd}}$ *of anti-self-dual connections on* $\mathfrak{su}(E)$.

(2) *The pair* $(A,\Phi)$ *is reducible and* $\Phi\not\equiv 0$. *The bundle* $E$ *splits as* $E=L_1\oplus L_2$, *the pair* $(A,\Phi)$ *has stabilizer* $\mathrm{Stab}_{A,\Phi}=S^1_{L_2}$, *and* $A$ *has stabilizer* $\mathrm{Stab}_A=S^1_{L_1}\times S^1_{L_2}$. *If* $M_E^{\mathrm{asd}}\cap M^{\mathrm{red}}_{W,E,L_1}=\emptyset$, *then* $M^{\mathrm{red}}_{W,E,L_1}$ *is smoothly cobordant to the Seiberg–Witten moduli space* $M^{\mathrm{sw}}_{W\otimes L_1}$.

(3) *The pair* $(A,\Phi)$ *is a reducible, zero-section pair. The connection* $A$ *is projectively flat (so* $(F_A)_0=0$) *and* $\Phi\equiv 0$. *The bundle* $E$ *splits as* $E=L_1\oplus(L_1\otimes N)$, *where* $N$ *is a torsion line bundle, so* $c_1(N)\in\operatorname{Tor}H^2(X;\mathbb{Z})$. *The stabilizer of the pair is* $\mathrm{Stab}_{A,0}=\mathrm{Stab}_A$.

*If* $b^+(X)=0$ *or the Riemannian metric metric on* $X$ *is nongeneric, the pair* $(A,\Phi)$ *can have stabilizer* $\mathrm{Stab}_{A,\Phi}=S^1_{L_1}\times S^1_{L_2}$, *where* $\Phi\equiv 0$ *and* $A$ *is a reducible projectively anti-self-dual, but not projectively flat connection.*

**Remark 3.2.** If $X$ is simply-connected, then the third case only occurs when the connection on $\mathfrak{su}(E)$ induced by $A$ is trivial. The stabilizer of the pair is then $\mathrm{U}(2)$.

The undesirable third case in Proposition 3.1 (see [14]) can be excluded with the aid of a criterion due to Fintushel and Stern [17]:

**Proposition 3.3** [17]. *If* $c\in H^2(X;\mathbb{Z})$ *and* $c \pmod 2\in H^2(X;\mathbb{Z}_2)$ *is not a pullback from* $H^2(K(\pi_1(X),1);\mathbb{Z}_2)$, *then there are no* $\mathrm{SO}(3)$ *bundles* $V\to X$ *with* $w_2(V)=c \pmod 2$ *which admit a flat connection.*

We can choose the class $w_2(\mathfrak{su}(E))=c_1(E)\pmod 2$ so that $\mathfrak{su}(E)$ does not admit a flat connection using the blow-up trick of [45]: If $c\in H^2(X;\mathbb{Z})$ and $e^*$ is the Poincaré dual of the exceptional class of the blow-up $\widehat{X}:=X\#\overline{\mathbb{CP}}^2$, then $c+e^*$ does not admit a flat $\mathrm{SO}(3)$ connection. As the Donaldson polynomials and Seiberg–Witten invariants of $X$ and its blow-up $\widehat{X}$ determine each other, no information is lost in this process [19,20]. Therefore, assuming this third possibility does not occur, the moduli space $M_{W,E}$ has a smooth stratification

$$M_{W,E}=M^{*,0}_{W,E}\cup M_E^{\mathrm{asd}}\cup M^{\mathrm{red}}_{W,E},\quad \text{with } M^{\mathrm{red}}_{W,E}:=\bigcup_{L_1}M^{\mathrm{red}}_{W,E,L_1},\tag{3.3}$$

where the union is over the finitely many line bundles $L_1\in H^2(X;\mathbb{Z})$ for which (i) there is a topological splitting $E=L_1\oplus L_2$, where $L_2=(\det E)\otimes L_1^*$ and recalling that $\det E$ is fixed, and (ii) the moduli space $M^{\mathrm{red}}_{W,E,L_1}$ is nonempty. One can show directly that there are only a finite number of line bundles $L_1$ with $M^{\mathrm{red}}_{W,E,L_1}$ nonempty by repeating the usual argument for the standard Seiberg–Witten moduli spaces [44, Theorem 5.2.4].

*For the remainder of this article* we shall assume that $X$ is equipped with an orientation, a homology orientation, has $b^+(X) > 0$, and is equipped with a generic Riemannian metric. In the case $b^+(X) = 1$, the Donaldson invariants refer to the specific chamber in $H^2(X;\mathbb{R})/\mathbb{R}^*$ defined by the choice of metric. The dimensions of our moduli spaces are then given by

$$\begin{aligned} 2d(\mathfrak{su}(E), F) &:= \dim M^{*,0}_{W,E} = 2d_a(\mathfrak{su}(E)) + 2n_a(\mathfrak{su}(E), F) - 1, \\ 2d_a(\mathfrak{su}(E)) &:= \dim M^{\mathrm{asd}}_E = -2p_1(\mathfrak{su}(E)) - \tfrac{3}{2}(e(X) + \sigma(X)) \\ &= -2p_1(\mathfrak{su}(E)) - 3(1 - b^1(X) + b^+(X)), \\ 2n_a(\mathfrak{su}(E), F) &:= 2\operatorname{Ind}_{\mathbb{C}} D_A = \tfrac{1}{2}p_1(\mathfrak{su}(E)) + \tfrac{1}{2}(F^2 - \sigma(X)), \end{aligned}$$

where $p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E)$ and $F = c_1(W^+) + c_1(E)$, while

$$\begin{aligned} 2d_s(K) &:= \dim M^{\mathrm{sw}}_{W\otimes L_1} = \tfrac{1}{4}\big(K^2 - (2e(X) + 3\sigma(X))\big) \\ &= \tfrac{1}{4}(K^2 - \sigma(X)) - (1 - b^1(X) + b^+(X)), \end{aligned}$$

where $K := c_1(W^+ \otimes L_1)$.

### 3.2. Cobordisms of links via moduli spaces of PU(2) monopoles

The essential idea is to use the moduli space $M^{*,0}_{W,E}$ as a cobordism between the 'links' of $M^{\mathrm{asd}}_E$ and $M^{\mathrm{red}}_{W,E}$. In Section 3.3 we define cohomology classes and their dual geometric representatives on $M^{*,0}_{W,E}$. The pairing of a product of these cohomology classes (or intersection of their dual geometric representatives) with the link of $M^{\mathrm{asd}}_E$ can be expressed as a multiple of the Donaldson polynomial (Lemma 3.17) while the pairing of these classes with the link of $M^{\mathrm{red}}_{W,E}$ gives multiples of the Seiberg–Witten invariants (Theorem 3.23). The intersection of the geometric representatives in $M^{*,0}_{W,E}$ is a family of oriented one-manifolds, whose boundaries should lie in the links of $M^{\mathrm{asd}}_E$ and $M^{\mathrm{red}}_{W,E}$, yielding an equality between these pairings and thus a relationship between the Donaldson and Seiberg–Witten invariants.

Two technical difficulties arise in the above program. The first problem is that $M^{*,0}_{W,E}$ is not compact. Thus the boundaries of the one-manifolds might not lie on these links, but in the lower levels of $\overline{M}_{W,E}$. One can work instead with $\overline{M}^{*,0}_{W,E}$, the subspace of $\overline{M}_{W,E}$ given by triples $[A, \Phi, \mathbf{x}]$ where $\Phi \not\equiv 0$ and $A$ is not reducible. In Section 3.4, we describe the intersection of the closure of the geometric representatives in $\overline{M}_{W,E}$ with the lower strata of $\overline{M}^{*,0}_{W,E}$. This description and a dimension-counting argument show that the one-manifolds given by the intersection of the geometric representatives do not have boundary points in the lower levels of $\overline{M}^{*,0}_{W,E}$.

The second problem is to define links of the singularities $M^{\mathrm{asd}}_E$ and $M^{\mathrm{red}}_{W,E,L_1}$. Eqs. (2.4) cutting out $M_{W,E} \subset \mathcal{C}_{W,E}$ do not vanish transversely along these singularities and so the local topology of $M_{W,E}$ could be quite intricate near $M^{\mathrm{asd}}_E$ and $M^{\mathrm{red}}_{W,E,L_1}$. In Section 3.6 we define a smoothly-stratified, codimension-one subspace $\boldsymbol{L}^{\mathrm{asd}}_{W,E} \subset \overline{M}^{*,0}_{W,E}$ and in Lemma 3.17 we compute the intersection of some geometric representatives with this link. In Section 3.7 we outline our definition [14] of a link $\boldsymbol{L}_{W,E,L_1} \subset M^{*,0}_{W,E}$ of the stratum $M^{\mathrm{red}}_{W,E,L_1}$ in $M_{W,E}$ and describe the intersection of the geometric representatives with this link in Theorem 3.23.

If all the reducibles lie only in the top level of $\overline{M}_{W,E}$, the cobordism $\overline{M}^{*,0}_{W,E}$ yields an explicit formula relating the Donaldson invariant and Seiberg–Witten invariants (Theorem 3.21). In general, however, there will be reducible pairs in the lower levels of $\overline{M}_{W,E}$. The one-manifolds given by the intersection of the geometric representatives can then have boundaries at reducible pairs in the lower levels of $\overline{M}_{W,E}$. The space $\overline{M}^{*,0}_{W,E}$ yields a cobordism between $\boldsymbol{L}^{\mathrm{asd}}_{W,E}$ and the links of all the reducibles, including these lower-level reducibles. The definition of the links of the lower-level reducibles is considerably more involved and is discussed in Section 4.

### 3.3. The cohomology classes

In this subsection we define the cohomology classes on $M^{*,0}_{W,E}$, referring the reader to [14] for detailed description of their dual geometric representatives. Recall that $\widetilde{\mathcal{C}}_{W,E} = \mathcal{A}_E \times \Omega^0(W^+ \otimes E)$ is our pre-configuration space of $L^2_k$ pairs, where we have omitted Sobolev indices as these play no role in the present discussion. Let $\widetilde{\mathcal{C}}^*_{W,E}$ denote the subspace of pairs which are not reducible, let $\widetilde{\mathcal{C}}^0_{W,E}$ denote the subspace of those which are not zero-section pairs, and let $\widetilde{\mathcal{C}}^{*,0}_{W,E}$ denote the subspace of those which are neither zero-section nor reducible pairs. Let $P$ be the principal $\mathrm{U}(2)$ bundle underlying the vector bundle $E$ and define

$$\mathbb{P} := \widetilde{\mathcal{C}}^{*,0}_{W,E} \times_{{}^\circ\mathcal{G}_E} P.$$

The space $\mathbb{P}$ is a principal $\mathrm{U}(2)$ bundle over $\mathcal{C}^{*,0}_{W,E} \times X$. The associated $\mathrm{SO}(3)$ bundle, $\mathbb{P}^{\mathrm{ad}} := \mathbb{P}/S^1_Z$, extends over $\mathcal{C}^*_{W,E}$. Indeed, the space $\mathbb{P}$ is isomorphic to $\mathbb{P}/S^1_Z$ over the zero-section pairs. Over the reducible pairs, the space $\mathbb{P}$ becomes an $\mathrm{SO}(3)$ fiber bundle, but is not principal as the stabilizers of these pairs are not normal subgroups of $\mathrm{U}(2)$.

We define maps from the homology of $X$ to the cohomology of $\mathcal{C}^{*,0}_{W,E}$ via

$$\mu_{c_1} : H_\bullet(X;\mathbb{Q}) \to H^{2-\bullet}(\mathcal{C}^{*,0}_{W,E};\mathbb{Q}), \quad \beta \mapsto c_1(\mathbb{P})/\beta,$$
$$\mu_{p_1} : H_\bullet(X;\mathbb{Q}) \to H^{4-\bullet}(\mathcal{C}^{*}_{W,E};\mathbb{Q}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{P}/S^1_Z)/\beta,$$

where

$$\tfrac{1}{4}p_1(\mathbb{P}/S^1_Z)/\beta = \big(c_2(\mathbb{P}) - \tfrac{1}{4}c_1^2(\mathbb{P})\big)/\beta.$$

Following [11, Definition 5.1.11] we define a universal $\mathrm{SO}(3)$ bundle by

$$\mathbb{P}^{\mathrm{ad}}_E := \mathcal{A}^*_E \times_{\mathcal{G}_E} (P/S^1_Z) \to \mathcal{B}^*_E \times X$$

and set

$$\mu_E : H_\bullet(X;\mathbb{Q}) \to H^{4-\bullet}(\mathcal{B}^*_E;\mathbb{Q}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{P}^{\mathrm{ad}}_E)/\beta.$$

If $\pi : \mathcal{C}^{*,0}_{W,E} \to \mathcal{B}^*_E$ is the projection $[A,\Phi] \mapsto [A]$, we see that $(\pi \times \mathrm{id}_X)^*\mathbb{P}^{\mathrm{ad}}_E = \mathbb{P}^{\mathrm{ad}}$. This implies the following relation between the cohomology classes on $\mathcal{C}^{*,0}_{W,E}$ and $\mathcal{B}^*_E$:

**Lemma 3.4.** *If $\beta \in H_\bullet(X;\mathbb{Q})$, then $\pi^*\mu_E(\beta) = \mu_{p_1}(\beta)$.*

The class $\mu_{c_1}(x)$ is nontrivial on the link of the zero-section pairs [14]. It does not pull back from the quotient space of connections and does not even extend over the subspace $M_E^{\mathrm{asd}} \subset M_{W,E}$.

By analogy with the construction of geometric representatives for cohomology classes in Donaldson theory [6,10,11,35], we define geometric representatives $V(\beta)$ and $W(x)$ to represent $\mu_{p_1}(\beta)$ and $\mu_{c_1}(x)$, respectively. Some features of the definition of these geometric representatives are worth mentioning. For a smooth submanifold $Y \subset X$ representing $\beta \in H_\bullet(X;\mathbb{Q})$, we let $U_Y$ be a 'suitable' neighborhood [35, Section 2]. The representatives $V(\beta)$ are the pullbacks of the usual usual geometric representatives of Donaldson theory [35] from the quotient space of connections $\mathcal{B}_E^*(U_Y \cup_j B_j)$, where $\overline{B}_j$ are the balls supporting the holonomy perturbations. If the energy of a connection $A|_{4B_{j'}}$ is greater than a certain universal bound, the representative $V(\beta)$ is independent of its restriction to $B_{j'}$.

As in [35], we let

$$\mathbb{A}(X) := \mathrm{Sym}\,\big(H_{\mathrm{even}}(X;\mathbb{Q})\big) \otimes \Lambda\big(H_{\mathrm{odd}}(X;\mathbb{Q})\big)$$

be the graded algebra, with $z = \beta_1\beta_2\cdots\beta_r$ having total degree $\deg(z) = \sum_i(4 - i_p)$, when $\beta_p \in H_{i_p}(X;\mathbb{Q})$. We write

$$\mu_{p_1}(z) := \mu_{p_1}(\beta_1) \smile \cdots \smile \mu_{p_1}(\beta_r),$$
$$V(z) := V(\beta_1) \cap \cdots \cap V(\beta_r),$$

for $z = \beta_1\beta_2\cdots\beta_r$, and similarly for $\mu_E(z)$. We write

$$\mu_{c_1}(x^m) := \underbrace{\mu_{c_1}(x) \smile \cdots \smile \mu_{c_1}(x)}_{m \text{ times}} \quad \text{and} \quad W(x^m) := \underbrace{W(x) \cap \cdots \cap W(x)}_{m \text{ times}},$$

for products of the class $\mu_{c_1}(x)$ and its dual $W(x)$.

*3.4. The closure of the geometric representatives*

We now describe the intersection of the geometric representatives with the lower strata of $\overline{M}_{W,E}$. Let $\Sigma \subset \mathrm{Sym}^\ell(X)$ be a smooth stratum. Counting dimensions, one sees that

$$\begin{aligned}\dim M^{*,0}_{W,E_{-\ell}}(\Sigma) &= \dim M^{*,0}_{W,E} - 6\ell + \dim \Sigma \\ &\leqslant \dim M^{*,0}_{W,E} - 2\ell, \quad 0 \leqslant \ell \leqslant N_p,\end{aligned}$$

so the strata $M^{*,0}_{W,E_{-\ell}}(\Sigma)$ (with $\ell \geqslant 1$) of the compactification $\overline{M}_{W,E}$ have codimension at least two less than the top stratum $M^{*,0}_{W,E}$. This would allow the definition of a relative fundamental class (with boundaries given by the links of the zero-section and reducible pairs) if we knew $\overline{M}_{W,E}$ had locally finite topology. We consider intersections of geometric representatives whose total codimension is one less than the dimension of $M^{*,0}_{W,E}$. Thus, if these geometric representatives intersect the lower strata of $\overline{M}_{W,E}$ in sets of the same codimension as their intersection with the top stratum $M^{*,0}_{W,E}$, dimension counting shows that the intersection of these geometric representatives, away from the zero-section and reducible pairs, occurs only in the top stratum.

**Definition 3.5.** The closures of the geometric representatives, $V(\beta)$, $W(x)$, in $\overline{M}_{W,E}$ are denoted by $\overline{V}(\beta)$, $\overline{W}(x)$, respectively. For $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$, a generator $x \in H_0(X)$, and an integer $m \geqslant 0$, we denote

$$\overline{V}(z) := \overline{V}(\beta_1) \cap \cdots \cap \overline{V}(\beta_r) \text{ and } \overline{W}(x^m) := \underbrace{\overline{W}(x) \cap \cdots \cap \overline{W}(x)}_{m \text{ times}}.$$

The description of the intersection of $\overline{V}(\beta)$, $\overline{W}(x)$ with the lower strata given below in Lemma 3.6 is incomplete, as it (i) gives only an inclusion and not an equality and (ii) does not give the multiplicities of components of these intersections occurring in lower levels. A more complete description is given in [16], using 'tubular neighborhood' descriptions of the lower strata in $\overline{M}_{W,E}$ obtained from gluing maps.

For $i = 1, \ldots, \ell$, let $\pi_i : X \times \cdots \times X \to X$ be projection onto the $i$th factor. Let $S^\ell(Y)$ be the projection of $\bigcup_i \pi_i^{-1}(Y)$ to $\mathrm{Sym}^\ell(X)$ under the map $X^\ell \to \mathrm{Sym}^\ell(Y)$ and denote $S_\Sigma(Y) = \mathrm{Sym}^\ell(Y) \cap \Sigma$.

On each space $\boldsymbol{M}^{*,0}_{W,E_{-\ell}}$, there are geometric representatives $V_\ell(\beta)$ and $W_\ell(x)$ defined in exactly the same way as the geometric representatives $V(\beta)$, $W(x)$ on $M^{*,0}_{W,E}$, except that we use bundles $P_{-\ell}$ and $P^{\mathrm{ad}}_{-\ell} := (P_{-\ell})/S^1_Z$ with $c_1(P_{-\ell}) = c_1(P)$ and $c_2(P_{-\ell}) = c_2(P) - \ell$. We then have the following description of the intersection of the extended geometric representatives $\overline{V}(\beta)$, $\overline{W}(x)$ with $\boldsymbol{M}^{*,0}_{W,E_{-\ell}}(\Sigma)$:

**Lemma 3.6.** *For a smooth stratum $\Sigma \subset \mathrm{Sym}^\ell(X)$, let $\pi : \boldsymbol{M}^{*,0}_{W,E_{-\ell}}(\Sigma) \to \Sigma$ be the projection map. Let $x \in H_0(X)$ be a generator, let $\beta \in H_\bullet(X;\mathbb{Q})$ have a smooth representative $Y \subset X$, and let $U_Y$ be a suitable neighborhood of $Y$. Then the following hold*:

(1) $\overline{V}(\beta) \cap \boldsymbol{M}^{*,0}_{W,E_{-\ell}}(\Sigma) \subseteq V_\ell(\beta) \cup \pi^{-1}(S_\Sigma(U_Y))$,

(2) $\overline{W}(x) \cap \boldsymbol{M}^{*,0}_{W,E_{-\ell}}(\Sigma) \subseteq W_\ell(x) \cup \pi^{-1}(S_\Sigma(U_x))$.

*Furthermore, if $\ell = 0$ and $\beta \in H_2(X;\mathbb{Q})$ is a two-dimensional class with $\langle 2L_1 - c_1(E), \beta\rangle \neq 0$, then we have the following reverse inclusions*:

(1) $M^{\mathrm{red}}_{W,E,L_1} \subset \overline{V}(\beta)$,

(2) $M^{\mathrm{red}}_{W,E} \subset \overline{V}(x)$,

(3) $\overline{M}^{\mathrm{asd}}_E \cup M^{\mathrm{red}}_{W,E} \subset \overline{W}(x)$.

**Remark 3.7.**

(1) The intersections of the geometric representatives with the strata of reducible pairs and of zero-section pairs in $\overline{M}_{W,E}$ generally do not have the expected codimensions. Indeed, Lemma 3.6 shows that almost all geometric representatives will contain reducible pairs in the top level.

(2) To get equality in the first assertions (replacing $S_\Sigma(U_Y)$ with $S_\Sigma(Y)$), we use gluing to describe the geometric representatives in an Uhlenbeck neighborhood of the lower level.

One cannot use dimension counting directly at this point as the open subsets $\pi^{-1}(S_\Sigma(U_Y))$ in $M^{*,0}_{W,E_{-\ell}}(\Sigma)$ do not have positive codimension. However, it can be shown that the restrictions of the geometric representatives $V_\ell(\beta)$, $W_\ell(x)$ to $\pi^{-1}(S_\Sigma(U_Y))$ are given by a pullback from $\pi^{-1}(S_\Sigma(Y))$. The intersection of the geometric representatives with $M^{*,0}_{W,E_{-\ell}}(\Sigma)$ may thus be computed by replacing $\pi^{-1}(S_\Sigma(U_Y))$ with $\pi^{-1}(S_\Sigma(Y))$.

We then see from Lemma 3.6 and the transversality results of Section 2 that although the closures $\overline{V}(\beta)$ and $\overline{W}(x)$ do not intersect every stratum of $\overline{M}_{W,E}$ in a set of the expected codimension, they do intersect the strata of $\overline{M}^{*,0}_{W,E}$ in sets of the expected codimension. A dimension-counting argument then yields:

**Corollary 3.8** [14]. *Let $n_{p_1}$ and $n_{c_1}$ be nonnegative integers such that $n_{p_1}+n_{c_1}=d_a+n_a-1$. Let $\beta_1,\dots,\beta_r\in H_\bullet(X;\mathbb{Q})$ be homology classes such that $\sum_i(4-\dim\beta_i)=n_{p_1}$ and let $z=\beta_1\beta_2\cdots\beta_r\in\mathbb{A}(X)$. If the collection $\beta_1,\dots,\beta_r$ does not contain both a zero-dimensional class and a three-dimensional class, then for generic choices of geometric representatives, and appropriate choices of suitable neighborhoods, the intersection*

$$\overline{V}(z)\cap\overline{W}(x^{n_{c_1}})\cap\overline{M}^{*,0}_{W,E}$$

*is a collection of one-dimensional manifolds, disjoint from the lower strata of $\overline{M}^{*,0}_{W,E}$.*

**Remark 3.9.** The condition in Corollary 3.8 about the absence of either three- or zero-dimensional homology classes is necessary because the definition of a suitable neighborhood includes loops which weaken the conclusions one can reach by dimension counting (see [35, p. 593] or [14]).

### *3.5. Orientations and the deformation complex*

The deformation complex for the PU(2) monopole equations (2.4) is given by

$$\Omega^0(\mathfrak{su}(E))\oplus i\mathbb{R}_Z \xrightarrow{d^0_{A,\Phi}} \begin{matrix}\Omega^1(\mathfrak{su}(E))\\ \oplus \\ \Omega^0(W^+\otimes E)\end{matrix} \xrightarrow{d^1_{A,\Phi}} \begin{matrix}\Omega^+(\mathfrak{su}(E))\\ \oplus \\ \Omega^0(W^-\otimes E)\end{matrix} \tag{3.4}$$

where $i\mathbb{R}_Z$ is the Lie algebra of $S^1_Z$. Here, $d^0_{A,\Phi}$ is the differential of the action of the gauge group ${}^\circ\mathcal{G}_E$ at $(A,\Phi)$, while $d^1_{A,\Phi}$ is the linearization of the PU(2) monopole equations (2.4). Let

$$\mathcal{D}_{A,\Phi}:=d^{0,*}_{A,\Phi}+d^1_{A,\Phi}$$

be the 'rolled-up' deformation operator. For any point $[A,\Phi]\in M^{*,0}_{W,E}$, there is an isomorphism, $T_{A,\Phi}M^{*,0}_{W,E}\simeq\operatorname{Ker}\mathcal{D}_{A,\Phi}$. In [14] we prove that $M^{*,0}_{W,E}$ is orientable by showing that the real line bundle $\det\mathcal{D}$ is trivial.

An orientation for $M^{*,0}_{W,E}$ can be specified by choosing a value for a section of $\det\mathcal{D}$ at any point $[A,\Phi]\in\mathcal{C}_{W,E}$. At a zero-section PU(2) monopole $(A,0)$, the deformation complex (3.4) splits into the direct sum of complexes:

$$\Omega^0(\mathfrak{su}(E)) \xrightarrow{d_A} \Omega^1(\mathfrak{su}(E)) \xrightarrow{d_A^+} \Omega^+(\mathfrak{su}(E)),$$
$$\Omega^0(W^+\otimes E) \xrightarrow{D_A} \Omega^0(W^-\otimes E).$$

The first complex is the elliptic deformation complex for the moduli space $M_E^{\mathrm{asd}}$ of anti-self-dual connections and $\mathrm{i}\mathbb{R}_Z$ is in the cokernel of $\mathcal{D}_{A,0}$. Because

$$\det \mathcal{D}_{A,0} \simeq \det(d_A^* + d_A^+) \otimes \det D_A \otimes (\mathrm{i}\mathbb{R}_Z)^*, \tag{3.5}$$

we can specify an orientation for $\det \mathcal{D}$ by specifying one for the anti-self-dual moduli space, using the complex orientation on $\det D_A$, and fixing an orientation for $\mathrm{i}\mathbb{R}_Z$.

**Definition 3.10.** If $w \in H^2(X;\mathbb{Z})$ is an integral lift of $w_2(\mathfrak{su}(E))$ and $\Omega$ is a homology orientation for $X$, let $o(\Omega, w)$ be the corresponding orientation defined in [11, Section 7.1.6] for the moduli space $M_E^{\mathrm{asd}}$ of anti-self-dual connections on $\mathfrak{su}(E)$. Let $O^{\mathrm{asd}}(\Omega, w)$ be the orientation for $\det \mathcal{D}$, and so $M_{W,E}^{*,0}$, defined through the isomorphism (3.5), the orientation $o(\Omega, w)$ for the moduli space $M_E^{\mathrm{asd}}$, the complex orientation for $\det D$ and the fixed orientation for $\mathrm{i}\mathbb{R}_Z$. The moduli space $M_E^{\mathrm{asd}}$ is equipped with the *standard orientation* $o(\Omega, c_1(E))$, if no other orientation is specified.

**Remark 3.11.** Since $p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E)$ and $w_2(\mathfrak{su}(E)) = c_1(E) \pmod 2$, then $p_1(\mathfrak{su}(E)) = w^2 \pmod 4$ if $w$ is an integral lift of $w_2(\mathfrak{su}(E))$. The orientation for $M_E^{\mathrm{asd}}$ is then determined by the addition of $-(p_1(\mathfrak{su}(E)) - w^2)/4$ instantons to the U(2) bundle $\underline{\mathbb{C}} \oplus w$, with corresponding SO(3) bundle $\underline{\mathbb{R}} \oplus w^{-1}$.

As shown in [8], the difference between the orientations $o(\Omega, w')$ and $o(\Omega, w'')$ for $M_E^{\mathrm{asd}}$ is given by

$$\varepsilon(w', w'') = (-1)^{(w'-w'')^2/4}, \tag{3.6}$$

where $w', w'' \in H^2(X;\mathbb{Z})$ are any two integral lifts of $w_2(\mathfrak{su}(E))$.

### *3.6. Geometric representatives and zero-section monopoles*

The stratum $M_E^{\mathrm{asd}} \subset M_{W,E}$ of zero-section pairs is identified with the moduli space of anti-self-dual connections on the SO(3) bundle $\mathfrak{su}(E)$. Because the geometric representatives $V(\beta)$ are pulled back by the map $\mathcal{C}_{W,E}^* \to \mathcal{B}_E^*$ given by $[A,\Phi] \mapsto [A]$, the following computation of the intersection of the geometric representatives with the stratum $M_E^{\mathrm{asd}}$ of zero-section monopoles is clear:

**Lemma 3.12.** *Let $E$ be a Hermitian two-plane bundle over a four-manifold $X$ with $b^+(X) > 0$ and generic Riemannian metric. Choose $c_1(E) \pmod 2$ so that $\mathfrak{su}(E)$ does not admit a flat connection. Let $z \in \mathbb{A}(X)$ have degree $2n_{p_1}$, where $n_{p_1} \geqslant d_a$. For a generic choice of geometric representatives, the intersection of $\overline{V}(z)$ with the strata of zero-section pairs in $\overline{M}_{W,E}$ is a finite number of generic points in $M_E^{\mathrm{asd}}$.*

*If $M^{\text{asd}}_E$ is given its standard orientation then the number of points in this intersection, counted with sign, is given by*

$$\#\big(\overline{V}(z)\cap \overline{M}^{\,\text{asd}}_E\big)=\begin{cases} D_X^{c_1(E)}(z) & \text{if } n_{p_1}=d_a,\\ 0 & \text{if } n_{p_1}>d_a.\end{cases}$$

As we shall see in the following lemma, it is important that the above intersection take place at generic points in $M^{\text{asd}}_E$. A neighborhood of a zero-section pair $[A,0]\in M_{W,E}$ can be described by the following Kuranishi model.

**Lemma 3.13.** *For any point $[A]\in M^{\text{asd}}_E$, there is a smoothly stratified diffeomorphism between a neighborhood of $[A,0]$ in $M_{W,E}$ and a neighborhood of zero in $m^{-1}(0)/S^1_Z$, where $m$ is an $S^1_Z$-equivariant map*

$$m: T_A M^{\text{asd}}_E \oplus \operatorname{Ker} D_{A,\vec{\vartheta}} \to \operatorname{Coker} D_{A,\vec{\vartheta}}.$$

*If* $\operatorname{Ind} D_{A,\vec{\vartheta}}>0$ *then for generic points $[A]\in M^{\text{asd}}_E$, the cokernel of the Dirac operator vanishes for generic perturbations $\vec{\vartheta}$.*

The cokernel of the perturbed Dirac operator $D_{A,\vec{\vartheta}}$ vanishes at generic points $[A]\in M^{\text{asd}}_E$ because the map $A\mapsto D_{A,\vec{\vartheta}}$ from $\widetilde{M}^{\text{asd}}_E$ to the space Fredholm operators, for a given index, is transverse to the jumping line strata. As described in [30], the 'jumping line strata' are the strata of Fredholm operators indexed by the dimension of their cokernels and the top stratum consists of operators with vanishing cokernel. Lemma 3.13 then describes the normal cone to $M^{\text{asd}}_E$ at a generic point $[A,0]$ as a cone on $\mathbb{CP}^{n_a-1}$, where $\operatorname{Ker} D_{A,\vec{\vartheta}}\simeq \mathbb{C}^{n_a}$.

We have described the geometric representative $\overline{V}(\beta)$ near the anti-self-dual moduli space; $\overline{W}(x)$ can be described as follows.

**Lemma 3.14.** *When restricted to the link in the normal cone of $M^{\text{asd}}_E$ in $M_{W,E}$ at a generic point $[A,0]\in M^{\text{asd}}_E$, the geometric representative $\overline{W}(x)$ is Poincaré dual to $2h$, where $h\in H^2((\operatorname{Ker} D_{A,\vec{\vartheta}}\backslash\{0\})/S^1_Z;\mathbb{Z})$ is the positive generator.*

**Remark 3.15.** Lemma 3.14 shows that $W(x)$ will have nontrivial intersection with the normal cone of any generic point in $M^{\text{asd}}_E$. Thus, the closure of $W(x)$ in $M_{W,E}$ will contain all generic points and thus all points in $M^{\text{asd}}_E$.

Let $\overline{M}^{\text{asd}}_E$ denote the closure of $M^{\text{asd}}_E$ in $\overline{M}_{W,E}$; note that this may properly contain the closure $M^{\overline{\text{asd}}}_E$ of $M^{\text{asd}}_E$ in $IM^{\text{asd}}_E$.

**Definition 3.16.** The *link of* $\overline{M}^{\,\text{asd}}_E$ in $\overline{M}_{W,E}$ is given by

$$\boldsymbol{L}^{\text{asd},\varepsilon}_{W,E} := \big\{[A,\Phi,\boldsymbol{x}]\in \overline{M}_{W,E}\colon\ \|\Phi\|^2_{L^2}=\varepsilon^2\big\}.$$

It is a simple matter to show that the map $\Phi\mapsto \|\Phi\|^2_{L^2}$ is continuous on $\overline{M}_{W,E}$ and smooth on each stratum. Thus, for generic values of $\varepsilon>0$, the link $\boldsymbol{L}^{\text{asd},\varepsilon}_{W,E}$ is a smoothly stratified, codimension-one subspace of $\overline{M}_{W,E}$. The intersection of $\boldsymbol{L}^{\mathrm{asd},\varepsilon}_{W,E}$ with an appropriate number of generic geometric representatives is then a finite number of points which can be calculated using Lemmas 3.12 and 3.14.

**Lemma 3.17** [14]. *Let $E$ be a Hermitian two-plane bundle over a four-manifold $X$ with $b^+(X) > 0$ and generic Riemannian metric. Choose $c_1(E)$ (mod 2) so that $\mathfrak{su}(E)$ does not admit a flat connection. Let $n_{p_1}$ and $n_{c_1}$ be nonnegative integers such that $n_{p_1} + n_{c_1} = d_a + n_a - 1$. Suppose $z \in \mathbb{A}(X)$ has degree $2n_{p_1} \geqslant 2d_a$. If $M_{W,E}$ is given the orientation $O^{\mathrm{asd}}(\Omega, c_1(E))$, then there is a positive constant $\varepsilon_0$ such that for generic $\varepsilon < \varepsilon_0$ we have*

$$\#\big(\overline{V}(z) \cap \overline{W}(x^{n_{c_1}}) \cap \boldsymbol{L}^{\mathrm{asd},\varepsilon}_{W,E}\big) = \begin{cases} 2^{n_a-1} D_X^{c_1(E)}(z) & \text{if } n_{p_1} = d_a, \\ 0 & \text{if } n_{p_1} > d_a. \end{cases}$$

*3.7. Links of the strata of reducible monopoles*

To describe the geometric representatives in a neighborhood of the reducible monopoles, $M^{\mathrm{red}}_{W,E,L_1}$, it does not suffice to produce a Kuranishi model at a generic point. Neither of the geometric representatives, $V(\beta)$, $W(x)$ intersects $M^{\mathrm{red}}_{W,E,L_1}$ in a set of the expected codimension so we cannot use them to cut down to a set of generic points as we did with the stratum of zero-section monopoles. Instead, we must give a global description of the link of $M^{\mathrm{red}}_{W,E,L_1}$ in $M^{*,0}_{W,E}$. We may assume without loss of generality that $M^{\mathrm{red}}_{W,E,L_1}$ contains no zero-section solutions.

Even in the case where $M^{\mathrm{red}}_{W,E,L_1}$ is in the top level $M_{W,E}$, the problem of defining a link is nontrivial when the dimension of $M^{\mathrm{red}}_{W,E,L_1}$ is positive. The techniques we employ in [14] follow the ideas of Atiyah and Singer for stabilizing index bundles [3,11]. Related methods have also been used in a variety of recent applications of Gromov and Seiberg–Witten invariants (including those of [4,24,40,41,57,58], for example) which essentially involve 'excess intersection theory' in situations where transversality cannot be achieved by 'generic parameter' arguments via the Sard–Smale theorem.

In this subsection, we sketch our construction of the link of $M^{\mathrm{red}}_{W,E,L_1}$ in $M^{*,0}_{W,E}$ when these reducibles lie in the top level [14]. Let $(A,\Phi)$ represent a point in $M^{\mathrm{red}}_{W,E,L_1}$ and recall that $\mathcal{D}_{A,\Phi} = d^{0,*}_{A,\Phi} + d^1_{A,\Phi}$. Let

$$\boldsymbol{E} := L^2_{k-1}\big(\Lambda^+ \otimes \mathfrak{su}(E)\big) \oplus L^2_{k-1}(W^- \otimes E)$$

and let $\mathfrak{S} : \widetilde{\mathcal{C}}_{W,E} \to \boldsymbol{E}$ be the ${}^\circ\mathcal{G}_E$-equivariant map defined by the PU(2) monopole equations (2.4), so $d^1_{A,\Phi} = (D\mathfrak{S})_{A,\Phi}$. It is convenient to temporarily pass to an $S^1$-equivariant setting, so let

$${}^\circ\mathcal{C}_{W,E} := \widetilde{\mathcal{C}}_{W,E}/\mathcal{G}_E,$$

and note that $\mathcal{C}_{W,E} = {}^\circ\mathcal{C}_{W,E}/S^1_Z = {}^\circ\mathcal{C}_{W,E}/S^1_{L_2}$. We then have

$${}^\circ M_{W,E} := \mathfrak{S}^{-1}(0) \cap {}^\circ\mathcal{C}_{W,E},$$

with quotient $M_{W,E} = {}^\circ M_{W,E}/S^1_Z = {}^\circ M_{W,E}/S^1_{L_2}$. If $[A,\Phi]$ is a point in $M^{\mathrm{red}}_{W,E,L_1}$, the stabilizer $\mathrm{Stab}_{A,\Phi}$ of the pair $(A,\Phi)$ is $S^1_{L_2}$ in ${}^\circ\mathcal{G}_E$ but is trivial in $\mathcal{G}_E$.

If $(A,\Phi)$ represents a point $[A,\Phi]\in M^{\mathrm{red}}_{W,E,L_1}$, the full elliptic deformation complex $d^{\bullet}_{A,\Phi}$ of (3.4) for the PU(2) monopole equations splits into a *tangential deformation complex*, $d^{\bullet,t}_{A,\Phi}$, and *normal deformation complex*, $d^{\bullet,n}_{A,\Phi}$ (see [14]). The tangential deformation complex is isomorphic to the elliptic deformation complex for the U(1) monopole equations (3.2). The rolled-up elliptic deformation complex $\mathcal{D}_{A,\Phi}=d^{0,*}_{A,\Phi}\oplus d^{1}_{A,\Phi}$ also splits, of course, into tangential and normal rolled-up deformation complexes: $\mathcal{D}_{A,\Phi}=\mathcal{D}^{t}_{A,\Phi}\oplus\mathcal{D}^{n}_{A,\Phi}$, with $\operatorname{Coker}\mathcal{D}^{t}_{A,\Phi}=0$ and

$$\operatorname{Ker}\mathcal{D}^{t}_{A,\Phi}\simeq T_{A,\Phi}M^{\mathrm{red}}_{W,E,L_1}\quad\text{and}\quad\operatorname{Ker}\mathcal{D}^{n}_{A,\Phi}\simeq\operatorname{Ker}\mathcal{D}_{A,\Phi}/T_{A,\Phi}M^{\mathrm{red}}_{W,E,L_1}.$$

Let $\Pi_{A,\Phi}$ denote the $L^2$ orthogonal projections onto the subspaces

$$\operatorname{Coker}d^{1}_{A,\Phi}\simeq\operatorname{Coker}\mathcal{D}_{A,\Phi}\simeq\operatorname{Coker}\mathcal{D}^{n}_{A,\Phi}\simeq\operatorname{Coker}d^{1,n}_{A,\Phi},$$

noting that $\operatorname{Coker}d^{0,*}_{A,\Phi}=\operatorname{Ker}d^{0}_{A,\Phi}=0$. The Kuranishi model of a neighborhood in $M_{W,E}$ of a point $[A,\Phi]\in M^{\mathrm{red}}_{W,E,L_1}$ is given by

$$\begin{aligned}&\gamma:\mathcal{O}_{A,\Phi}\subset\operatorname{Ker}\mathcal{D}_{A,\Phi}\to{}^{\circ}\mathcal{C}_{W,E},\\&\varphi:\mathcal{O}_{A,\Phi}\subset\operatorname{Ker}\mathcal{D}_{A,\Phi}\to\operatorname{Coker}\mathcal{D}_{A,\Phi},\end{aligned}\tag{3.7}$$

where $\mathcal{O}_{A,\Phi}$ is an $S^1_{L_2}$ invariant open neighborhood of the origin in $\operatorname{Ker}\mathcal{D}_{A,\Phi}=\operatorname{Ker}\mathcal{D}^{t}_{A,\Phi}\oplus\operatorname{Ker}\mathcal{D}^{n}_{A,\Phi}$, $\gamma$ is an $S^1_{L_2}$-equivariant embedding, and $\varphi$ is a smooth $S^1_{L_2}$-equivariant map. The map $\gamma$ descends to a smoothly stratified diffeomorphism from $\varphi^{-1}(0)/S^1_{L_2}$ onto an open neighborhood of $[A,\Phi]$ in $M_{W,E}$. The obstruction map $\varphi$ is given by $\Pi\circ\mathfrak{S}\circ\gamma$.

Since the construction of the link of $M^{\mathrm{red}}_{W,E,L_1}$ in $M^{*,0}_{W,E}$ is complicated in general, it is helpful to begin by considering some simple special cases. When $M^{\mathrm{red}}_{W,E,L_1}$ is zero-dimensional, links in $M^{*,0}_{W,E}$ of the points of $M^{\mathrm{red}}_{W,E,L_1}$ are defined by the Kuranishi model (3.7): The link of a point $[A,\Phi]$ is simply given by the $S^1_{L_2}$ quotient of the zero-locus of $\varphi$ in an $\varepsilon$-sphere around the origin in $\operatorname{Ker}\mathcal{D}_{A,\Phi}$.

For the remainder of this subsection we assume that $M^{\mathrm{red}}_{W,E,L_1}$ may be positive-dimensional. If $\operatorname{Coker}\mathcal{D}$ vanishes along $M^{\mathrm{red}}_{W,E,L_1}$, then $\operatorname{Ker}\mathcal{D}^n$ is a finite-rank, $S^1_{L_2}$-equivariant vector bundle over $M^{\mathrm{red}}_{W,E,L_1}$ with fibers $\operatorname{Ker}\mathcal{D}^{n}_{A,\Phi}$ over points $[A,\Phi]\in M^{\mathrm{red}}_{W,E,L_1}$. There is an $S^1_{L_2}$-equivariant diffeomorphism $\varphi$ from an open neighborhood $\mathcal{O}$ of the zero-section $M^{\mathrm{red}}_{W,E,L_1}\subset\operatorname{Ker}\mathcal{D}^n$ and an open neighborhood of $M^{\mathrm{red}}_{W,E,L_1}$ in $\mathcal{M}_{W,E}$.

More generally, if the cokernel of $\mathcal{D}_{A,\Phi}$ has constant rank as $[A,\Phi]$ varies in $M^{\mathrm{red}}_{W,E,L_1}$ (that is, no spectral flow occurs), then $\operatorname{Ker}\mathcal{D}^n$ and $\operatorname{Coker}\mathcal{D}$ both define finite-rank, $S^1_{L_2}$-equivariant vector bundles over $M^{\mathrm{red}}_{W,E,L_1}$:

$$\begin{array}{ccc}\operatorname{Ker}\mathcal{D}^n & & \operatorname{Coker}\mathcal{D}\\ & \searrow^{\pi_k}\quad {}^{\pi_c}\swarrow & \\ & M^{\mathrm{red}}_{W,E,L_1} & \end{array}\tag{3.8}$$

Let $2\nu$ be the least positive eigenvalue of the Laplacian $\Delta_{A,\Phi}:=\mathcal{D}_{A,\Phi}\mathcal{D}^{*}_{A,\Phi}$ as $[A,\Phi]$ varies along the compact manifold $M^{\mathrm{red}}_{W,E,L_1}$ and let $\Pi_{\nu;A,\Phi}$ denote the $L^2$ orthogonal projection from $\boldsymbol{E}$ onto the subspace spanned by the eigenvectors of $\Delta_{A,\Phi}$ with eigenvalue less than $\nu$. The vector bundle $\operatorname{Coker}\mathcal{D}$ over $M^{\text{red}}_{W,E,L_1}$ then extends to a vector bundle

$$\Xi_\nu := \operatorname{Ker}(\mathrm{id} - \Pi_\nu) \circ \Delta = \operatorname{Coker}(\mathrm{id} - \Pi_\nu) \circ \mathcal{D}$$

of the same rank over an open neighborhood of $M^{\text{red}}_{W,E,L_1}$ in ${}^\circ\mathcal{C}_{W,E}$. The *obstruction section* $\varphi$ over $\mathcal{O} \subset \operatorname{Ker}\mathcal{D}^n$ of the vector bundle

$$\gamma^*\Xi_\nu \to \operatorname{Ker}\mathcal{D}^n \tag{3.9}$$

is given by $\varphi := \Pi_\nu \circ \mathfrak{S} \circ \gamma$ on $\mathcal{O} \subset \operatorname{Ker}\mathcal{D}^n$, where the $S^1_{L_2}$-equivariant embedding $\gamma : \mathcal{O} \to {}^\circ\mathcal{C}_{W,E}$ gives a diffeomorphism from an open neighborhood $\mathcal{O}$ of the zero-section $M^{\text{red}}_{W,E,L_1}$ in $\operatorname{Ker}\mathcal{D}^n$ onto an open neighborhood of $M^{\text{red}}_{W,E,L_1}$ in the $S^1_{L_2}$ invariant *thickened moduli space*

$${}^\circ M_{W,E,L_1}(\Xi_\nu) := \left((\mathrm{id} - \Pi_\nu) \circ \mathfrak{S}\right)^{-1}(0) \subset {}^\circ\mathcal{C}_{W,E}.$$

Then $\gamma$ descends to a smoothly stratified diffeomorphism from the zero locus

$$\varphi^{-1}(0)/S^1_{L_2} \subset \operatorname{Ker}\mathcal{D}^n/S^1_{L_2} \tag{3.10}$$

containing $M^{\text{red}}_{W,E,L_1}$ onto an open neighborhood of $M^{\text{red}}_{W,E,L_1}$ in $M_{W,E}$. On the complement of the zero-section $M^{\text{red}}_{W,E,L_1} \subset \operatorname{Ker}\mathcal{D}^n$, the $S^1_{L_2}$ quotient of the projection (3.9) given by

$$\gamma^*\Xi_\nu/S^1_{L_2} \to \operatorname{Ker}\mathcal{D}^n/S^1_{L_2}, \tag{3.11}$$

is a vector bundle. The homology class of the zero locus (3.10) of the obstruction map can be calculated from the Euler class of the vector bundle (3.11) or, equivalently, from that of

$$\pi_k^* \operatorname{Coker}\mathcal{D}/S^1_{L_2} \to \operatorname{Ker}\mathcal{D}^n/S^1_{L_2},$$

as is easily seen.

In general, though, one cannot guarantee that $\operatorname{Coker}\mathcal{D}$ will either vanish or have constant rank. Let $\widetilde{M}^{\text{red}}_{W,E,L_1} \subset \widetilde{\mathcal{C}}_{W,E}$ be the pre-image of $M^{\text{red}}_{W,E,L_1}$ under the projection from the pre-configuration space $\widetilde{\mathcal{C}}_{W,E}$ onto the quotient ${}^\circ\mathcal{C}_{W,E} = \widetilde{\mathcal{C}}_{W,E}/\mathcal{G}_E$. Because $M^{\text{red}}_{W,E,L_1}$ is compact, we can construct a finite family of gauge-equivariant 'stabilizing maps' from $\widetilde{M}^{\text{red}}_{W,E,L_1}$ to $\boldsymbol{E}$ such that

- the image $\Xi_{A,\Phi}$ of these maps at $(A,\Phi) \in \widetilde{M}^{\text{red}}_{W,E,L_1}$ spans $\operatorname{Coker}\mathcal{D}_{A,\Phi}$,
- the subspace $\Xi_{A,\Phi} \subset \boldsymbol{E}$ is $S^1_{L_2}$ invariant,
- the dimension of $\Xi_{A,\Phi}$ is constant for all pairs $(A,\Phi) \in \widetilde{M}^{\text{red}}_{W,E,L_1}$.

The subspaces $\Xi_{A,\Phi}$ then fit together to form an $S^1_{L_2}$-equivariant vector bundle $\Xi$ over $M^{\text{red}}_{W,E,L_1}$, which extends to an $S^1_{L_2}$-equivariant vector bundle $\Xi$ over an open neighborhood of $M^{\text{red}}_{W,E,L_1}$ in ${}^\circ\mathcal{C}_{W,E}$. Let $\Pi_{\Xi;A,\Phi}$ denote the $L^2$ orthogonal projection from $\boldsymbol{E}$ onto the subspace $\Xi_{A,\Phi}$. The properties of the stabilizing sections ensure that the space

$$N_{W,E,L_1}(\Xi) := \operatorname{Ker}(\mathrm{id} - \Pi_\Xi) \circ \mathcal{D}^n$$

is a vector bundle over $M^{\text{red}}_{W,E,L_1}$ with fibers which are closed under the $S^1_{L_2}$ action:

$$\begin{array}{ccc} N_{W,E,L_1}(\Xi) & & \Xi \\ & \searrow^{\pi_N} \quad {}^{\pi_\Xi}\swarrow & \\ & M^{\text{red}}_{W,E,L_1} & \end{array} \tag{3.12}$$

The bundle $\Xi$ plays the role of $\Xi_\nu$ while $N_{W,E,L_1}(\Xi)$ plays that of $\operatorname{Ker}\mathcal{D}^n$ in the simpler case (3.8) where the cokernel of $\mathcal{D}$ has constant rank along $M^{\text{red}}_{W,E,L_1}$. In [14] we construct a smooth, $S^1_{L_2}$ invariant thickened moduli space,

$$^\circ M_{W,E,L_1}(\Xi) := \left((\mathrm{id} - \Pi_\Xi)\circ\mathfrak{S}\right)^{-1}(0) \subset {}^\circ\mathcal{C}_{W,E},$$

using the stabilizing bundle $\Xi$. Then $N_{W,E,L_1}(\Xi)$ is the $S^1_{L_2}$-equivariant normal bundle of the smooth submanifold $M^{\text{red}}_{W,E,L_1} \subset \mathcal{M}_{W,E,L_1}(\Xi)$, recalling that $M^{\text{red}}_{W,E,L_1}$ is the fixed-point set of $S^1_{L_2}$.

The equivariant tubular neighborhood theorem provides an $S^1_{L_2}$-equivariant diffeomorphism $\gamma : \mathcal{O} \to {}^\circ\mathcal{C}_{W,E}$ from an open neighborhood $\mathcal{O}$ of the zero-section $M^{\text{red}}_{W,E,L_1} \subset N_{W,E,L_1}(\Xi)$ onto an open neighborhood of the submanifold $M^{\text{red}}_{W,E,L_1} \subset {}^\circ M_{W,E,L_1}(\Xi)$ which covers the identity on $M^{\text{red}}_{W,E,L_1}$. The map $\gamma$ then descends to a smoothly stratified diffeomorphism from the zero locus $\varphi^{-1}(0)/S^1_{L_2}$ in $N_{W,E,L_1}(\Xi)/S^1_{L_2}$ onto an open neighborhood of $M^{\text{red}}_{W,E,L_1}$ in the actual moduli space, $M_{W,E}$, where

$$\varphi := \Pi_\Xi \circ \mathfrak{S} \circ \gamma$$

is a section over $\mathcal{O} \subset N_{W,E,L_1}(\Xi)$ of the $S^1_{L_2}$-equivariant vector bundle

$$\gamma^*\Xi \to N_{W,E,L_1}(\Xi).$$

As in the constant rank case, this descends to a vector bundle

$$\gamma^*\Xi/S^1_{L_2} \to N_{W,E,L_1}(\Xi)/S^1_{L_2}$$

on the complement of the zero section, $M^{\text{red}}_{W,E,L_1} \subset N_{W,E,L_1}(\Xi)/S^1_{L_2}$, whose Euler class may be computed from

$$\pi_N^*\Xi/S^1_{L_2} \to N_{W,E,L_1}(\Xi)/S^1_{L_2}.$$

While the bundle $\gamma^*\Xi$ given by this restriction to the complement of the zero-section is trivial—because it is spanned by the stabilizing sections—the quotient $\gamma^*\Xi/S^1_{L_2}$ has a nontrivial Euler class.

**Definition 3.18.** Let $N^\varepsilon_{W,E,L_1}(\Xi)$ denote the sphere bundle of fiber vectors of length $\varepsilon$ and let

$$\mathbb{P}N_{W,E,L_1}(\Xi) = N^\varepsilon_{W,E,L_1}(\Xi)/S^1_{L_2}.$$

The *link of the stratum* $M^{\text{red}}_{W,E,L_1} \subset M_{W,E}$ of reducible pairs is given by

$$\boldsymbol{L}_{W,E,L_1} := \gamma\left(\varphi^{-1}(0) \cap N^\varepsilon_{W,E,L_1}(\Xi)\right)/S^1_{L_2}$$

and thus

$$[\boldsymbol{L}_{W,E,L_1}] = e(\gamma^* \Xi / S^1_{L_2}) \cap [\mathbb{P}N_{W,E,L_1}(\Xi)]$$

is its homology class.

**Remark 3.19.** The orientation given to $\boldsymbol{L}_{W,E,L_1}$ by the orientation on $M^{\mathrm{red}}_{W,E,L_1}$ from the homology orientation $\Omega$ and the complex structure on the fibers of $N_{W,E,L_1}(\Xi)$ (from the $S^1_{L_2}$ action) is equivalent to the orientation given by $O^{\mathrm{asd}}(\Omega, L_2 \otimes L_1^*)$ (see [14]).

*3.8. Reduction formulas for Donaldson invariants:* U(1) *monopoles in the top Uhlenbeck level*

In this subsection we describe some of our results from [14], where we compute Donaldson invariants in terms of Seiberg–Witten invariants when the U(1) monopoles in $\overline{M}_{W,E}$ lie only in the top level $M_{W,E}$.

**Definition 3.20.** The set of moduli spaces of U(1) monopoles contained in the top level $M_{W,E}$ is enumerated by

$$\mathrm{Red}(W,E) := \Big\{ L_1 \in H^2(X;\mathbb{Z}) \colon\ M^{\mathrm{red}}_{W,E,L_1} \neq \emptyset \text{ and } \big(2L_1 - c_1(E)\big)^2 = p_1\big(\mathfrak{su}(E)\big) \Big\}.$$

The set of moduli spaces of U(1) monopoles contained in the compact space of ideal PU(2) monopoles $IM_{W,E}$ is enumerated by

$$\overline{\mathrm{Red}}(W,E) := \Big\{ L_1 \in H^2(X;\mathbb{Z}) \colon\ M^{\mathrm{red}}_{W,E_{-\ell},L_1} \neq \emptyset \text{ and } \big(2L_1 + c_1(E)\big)^2 = p_1\big(\mathfrak{su}(E_{-\ell})\big) + 4\ell,\ \ell \in \mathbb{Z}_{\geqslant 0} \Big\},$$

where $c_1(E_{-\ell}) = c_1(E)$ and $c_2(E_{-\ell}) = c_2(E) - \ell$.

Note that $2L_1 - c_1(E) = K - F$, where $K = c_1(W^+ \otimes L_1)$ and $F = c_1(W^+) + c_1(E)$. The compactification $\overline{M}_{W,E}$ may be a proper subset of $IM_{W,E}$. If the reducibles in $\overline{M}_{W,E}$ appear only in the top Uhlenbeck level $M_{W,E}$ then $\overline{M}^{*,0}_{W,E}$ serves as a cobordism between the link $\boldsymbol{L}^{\mathrm{asd},\varepsilon}_{W,E}$ of the anti-self-dual moduli space $M^{\mathrm{asd}}_E$ and the links $\boldsymbol{L}_{W,E,L_1}$ of the strata of reducibles $M^{\mathrm{red}}_{W,E,L_1}$. This gives the following formula:

**Theorem 3.21** [14]. *Let $E$ be a Hermitian two-plane bundle over a four-manifold $X$ with $b^+(X) > 0$ and generic Riemannian metric. Choose $c_1(E)$ (mod 2) so that $\mathfrak{su}(E)$ does not admit a flat connection. Suppose $z \in \mathbb{A}(X)$ has degree $2d_a$. If* $\mathrm{Red}(W,E) = \overline{\mathrm{Red}}(W,E)$, *so the reducible* PU(2) *monopoles in $\overline{M}_{W,E}$ appear only in the highest Uhlenbeck level, then*

$$2^{n_a - 1} D^{c_1(E)}_X(z) = - \sum_{L_1 \in \mathrm{Red}(W,E)} (-1)^{L_1^2} \big\langle \mu_{p_1}(z) \smile \mu_{c_1}(x^{n_a - 1}), [\boldsymbol{L}_{W,E,L_1}] \big\rangle. \tag{3.13}$$

The sign $(-1)^{L_1^2}$ in (3.13) comes from the parity change $\varepsilon(c_1(E), L_2 \otimes L_1^*)$ of (3.6), noting that $c_1(E) = L_1 + L_2$.

The restriction of the cohomology classes $\mu_{p_1}(\beta)$ and $\mu_{c_1}(x)$ to $\boldsymbol{L}_{W,E,L_1}$ are computed in [14] in terms of the hyperplane class on $M^{\mathrm{red}}_{W,E,L_1}$ and the generator of the cohomology of the fiber of $\mathbb{P}N_{W,E,L_1}(\Xi)$. The Euler class, $e(\gamma^*\Xi/S^1_{L_2})$, can also be expressed in these terms. From the Atiyah–Singer index theorem for families, one can compute the Segre classes of the bundle $N_{W,E,L_1}(\Xi)$ under the assumption $b^1(X) \leqslant 1$. If $b^1(X) > 1$ the computation is still possible in principle, but becomes unmanageable in practice. To describe the results of these computations, we introduce some standard expressions to describe certain constants arising in our reduction formula:

**Definition 3.22** [27, Section 8.96]. The *Jacobi polynomials* are defined by

$$P_n^{(a,b)}(x) := 2^{-n} \sum_{m=0}^{n} \binom{n+a}{m} \binom{n+b}{n-m} (x-1)^{n-m}(x+1)^m.$$

Functional relations, relations with other special functions, and the generating function for the Jacobi polynomials can be found in [27, pp. 1034, 1035]. Recall that $\mathfrak{s}_0 = (\rho, W)$ is a choice of fixed spin$^c$ structure on $X$. For line bundles $L_1 \in H^2(X;\mathbb{Z})$, we denote $\mathfrak{s}_0 \otimes L_1 := (\rho, W \otimes L_1)$.

**Theorem 3.23** [14]. *Let $E$ be a Hermitian two-plane bundle over a four-manifold $X$ with $b^+(X) > 0$, $b^1(X) \leqslant 1$, and generic Riemannian metric. Choose $c_1(E)$ (mod 2) so that $\mathfrak{su}(E)$ does not admit a flat connection. Let $n_{p_1} + n_{c_1} = d_a + n_a - 1$, where $n_{p_1}$, $n_{c_1}$ are nonnegative integers. For the stratum of reducible solutions $M^{\mathrm{red}}_{W,E,L_1}$ contained in the highest level of $\overline{M}_{W,E}$, a generator $x \in H_0(X;\mathbb{Z})$, classes $\beta_1, \ldots, \beta_{n_{p_1}} \in H_2(X;\mathbb{Q})$, and integers $0 \leqslant m \leqslant n_{p_1}/2$, we have*

$$\begin{aligned} &\langle \mu_{p_1}(\beta_1 \cdots \beta_{n_{p_1}-2m} x^m) \smile \mu_{c_1}(x^{n_{c_1}}), [\boldsymbol{L}_{W,E,L_1}] \rangle \\ &\quad = (-1)^m 2^{-n_{p_1}+d_s} C_{W,E,L_1}(n_{p_1}, n_{c_1}) SW(\mathfrak{s}_0 \otimes L_1) \\ &\qquad \times \prod_{i=0}^{n_{p_1}-2m} \langle 2L_1 - c_1(E), \beta_i \rangle, \end{aligned} \tag{3.14}$$

*where, for $I = n_{p_1} - n_s^A - d_s$ and $J = n_{c_1} - d_s$, the constants $n_s^A$ and $C_{K,F}$ are given by*

$$n_s^A(\mathfrak{su}(E)) := -p_1(\mathfrak{su}(E)) - \tfrac{1}{2}(e(X) + \sigma(X)),$$

$$C_{W,E,L_1}(n_{p_1}, n_{c_1}) := P_{d_s}^{(I,J)}(0) = 2^{-d_s} \sum_{u=0}^{d_s} (-1)^u \binom{n_{c_1}}{u} \binom{n_{p_1} - n_s^A}{d_s - u}.$$

**Remark 3.24.**

(1) Note that $2L_1 - c_1(E) = K - F$, where $K = c_1(W^+ \otimes L_1)$ and $F = c_1(W^+) + c_1(E)$, and that the polynomial $C_{W,E,L_1}(\cdot)$ only depends on the classes $K$ and $F$ (together with the Euler characteristic and signature of $X$).

(2) The constant $n_s^A$ is the index of the elliptic complex on $\Omega^\bullet(L_1 \otimes L_2^*)$ induced by homotoping the normal deformation complex at a reducible pair, determined by the reduction $E = L_1 \oplus L_2$, to a diagonal complex.
(3) If $d_s = 0$ we have $P_0^{(I,J)}(0) = 1$ and so for manifolds of Seiberg–Witten simple type, the constant $C_{W,E,L_1}(n_{p_1}, n_{c_1})$ is not interesting. It should, however, prove useful for understanding the relation between the Donaldson and Seiberg–Witten invariants for any manifolds which are not of simple type.

Combining Theorems 3.21 and 3.23 yields:

**Corollary 3.25** [14]. *Let $E$ be a Hermitian two-plane bundle over a four-manifold $X$ with $b^+(X) > 0$, $b^1(X) \leqslant 1$, and generic Riemannian metric. Choose $c_1(E)$ (mod 2) so that $\mathfrak{su}(E)$ does not admit a flat connection. Let $x \in H_0(X;\mathbb{Z})$ be a generator, let $\beta_1, \ldots, \beta_{d_a} \in H_2(X;\mathbb{Q})$, and suppose*

$$z = \beta_1 \cdots \beta_{d_a - 2m} x^m \in \mathbb{A}(X),$$

*for $0 \leqslant m \leqslant d_a/2$. If $\mathrm{Red}(W,E) = \overline{\mathrm{Red}}(W,E)$, so reducible $\mathrm{PU}(2)$ monopoles in $\overline{M}_{W,E}$ appear only in the highest level $M_{W,E}$, then the following holds:*

$$-2^{n_a-1} D_X^{c_1(E)}(z) = \sum_{L_1 \in \mathrm{Red}(W,E)} (-1)^{L_1^2} (-1)^m 2^{-d_a + d_s(c_1(W^+ \otimes L_1))}$$
$$\times C_{W,E,L_1}(d_a, n_a - 1) SW(\mathfrak{s}_0 \otimes L_1) \prod_{i=0}^{d_a - 2m} \langle 2L_1 - c_1(E), \beta_i \rangle,$$

*where $C_{W,E,L_1}(d_a, n_a - 1)$ is defined in Theorem 3.23. If $X$ has Seiberg–Witten simple type then*

$$D_X^{c_1(E)}(z) = \sum_{L_1 \in \mathrm{Red}(W,E)} (-1)^{L_1^2} (-1)^{m-1} 2^{1-d_a-n_a} SW(\mathfrak{s}_0 \otimes L_1) \prod_{i=0}^{d_a - 2m} \langle 2L_1 - c_1(E), \beta_i \rangle.$$

The formula in Corollary 3.25 differs what one might expect from Eqs. (1.1) and (1.2) as it contains terms of the form

$$\langle 2L_1 - c_1(E), \beta_i \rangle = \langle K - F, \beta_i \rangle,$$

where $K = c_1(W^+ \otimes L_1)$ and $F = c_1(E) + c_1(W^+)$, rather than the terms $\langle K, \beta_i \rangle$. In addition, the power $L_1^2$ of $-1$ does not match the exponent $(w^2 + wK)/2$ given in (1.1) for any obvious choice of line bundle $w$ over $X$.

As shown by our examples in [14], the condition $\mathrm{Red}(W,E) = \overline{\mathrm{Red}}(W,E)$ puts severe restrictions on the class $F$ and the intersections $FK_r$, where the $K_r$ are basic classes. Under these restrictions, combinatorial identities give a cancellation of the factors of $F$ in the formula of Corollary 3.25. One sees from these examples that one should not assume that the terms

$$(-1)^{(w^2 + wK_r)/2} \exp(Q/2) SW(K_r) \mathrm{e}^{K_r}$$

in (1.2) translate directly into values for pairings with the link of the reducible $M^{\mathrm{red}}_{W,E,L_1}$ when $K = c_1(W^+ \otimes L_1)$. In the sum over all links, there can be many cancellations between terms contributed by different links. We illustrate the use of Corollary 3.25 below; see [14] for further examples.

**Example 3.26** [14]. We use Corollary 3.25 to calculate the first nontrivial Donaldson polynomial of the elliptic surface $E(n)$ with Euler characteristic $e(E(n)) = 12n$ and signature $\sigma(E(n)) = -8n$. Let $f \in H^2(E(n);\mathbb{Z})$ denote the fiber class of the elliptic fibration. For suitable perturbations, the only nonempty Seiberg–Witten moduli spaces correspond to spin$^c$ structures with

$$K_r := c_1(W^+ \otimes L_{1,r}) = (n-2-2r)f, \quad r = 0, \ldots, n-2.$$

The Seiberg–Witten invariants of the spin$^c$ structures with these classes are given by (see, for example, [21]):

$$SW(K_r) = (-1)^r \binom{n-2}{r}, \quad r = 0, \ldots, n-2.$$

Because

$$p_1(\mathfrak{su}(E)) = (L_1 - L_2)^2 = (K_r - F)^2,$$

where $E = L_{1,r} \oplus (\det E) \otimes L^*_{1,r}$, we can ensure that all the reducibles are in the same level (and make this the top level) by requiring that $K_r F = 0$. Then $p_1(\mathfrak{su}(E)) = (K_r - F)^2 = F^2$. Since $(1 + b^+(E(n))) = 2n$, we find that

$$\begin{aligned} d_a(\mathfrak{su}(E)) &= -F^2 - \tfrac{3}{2}(2n) = -F^2 - 3n, \\ n_a(\mathfrak{su}(E)) &= \tfrac{1}{4}(2F^2 + 8n) = \tfrac{1}{2}F^2 + 2n. \end{aligned}$$

Thus, to obtain $d_a \geqslant 0$ and $n_a > 0$, we impose the constraint $-4n < F^2 \leqslant -3n$. Note that as $K_r$ is characteristic and $K_r F = 0$, we must have $F^2$ even. Applying Corollary 3.25 with $\beta \in H_2(X;\mathbb{Z})$ we find, after some calculation, that

$$D^F_X(\beta^{n-2j-2m} x^m) = \begin{cases} 0 & \text{if } j > 1 \text{ or } m > 0, \\ -(n-2)!\langle f, \beta\rangle^{n-2} & \text{if } j = m = 0, \end{cases}$$

in agreement with the results of [21,35].

## 4. Gluing PU(2) monopoles and the PU(2) monopole analogue of the Kotschick–Morgan conjecture

The problems involved in computing intersection numbers for the link of a family of lower-level reducibles are similar to those encountered in attempts to prove the Kotschick–Morgan conjecture [32]. In this section we first discuss the Kotschick–Morgan conjecture for Donaldson invariants, describe its analogue for pairings with links of lower-level moduli spaces of U(1) monopoles in the Uhlenbeck compactification of the moduli space of PU(2) monopoles, and outline how a resolution of this analogue should lead in turn to a proof of Witten's conjecture.

### 4.1. *The Kotschick–Morgan conjecture for Donaldson invariants*

The conjecture of Kotschick and Morgan for Donaldson invariants of four-manifolds $X$ with $b^+(X) = 1$ gives a prediction of how the Donaldson invariants vary when the underlying Riemannian metric changes. More precisely, it asserts that the invariants computed using metrics lying in different chambers of the positive cone of $H^2(X;\mathbb{R})/\mathbb{R}^*$ differ by terms depending only homotopy data [32]. The definition of the Donaldson invariants requires a choice of Riemannian metric on $X$ and they are only independent of this choice when $b^+(X) > 1$.

The Donaldson invariants of a manifold with $b^+(X) = 1$ are not independent of the metric because the cobordism formed by taking the moduli space of connections anti-self-dual with respect to elements of a path of metrics may contain reducible anti-self-dual connections. The Donaldson cohomology classes evaluate nontrivially on the links of these reducible connections, so the values of the Donaldson invariant given by the metrics at the ends of this path will differ by the pairing of the top power of the cohomology classes with these links. Directly evaluating such pairings or even showing that they depend only on homotopy data is a difficult problem when the reducible connection lies in a lower level of the Uhlenbeck compactification. The conjecture of [32] asserts that these pairings only depend on homotopy data: this has been verified for reducibles in the levels $M^{\mathrm{asd}}_{E_{-\ell}}(X) \times \mathrm{Sym}^{\ell}(X)$ when $\ell \leqslant 2$ [7,31,32,39,67] and for much higher $\ell$ when $X$ is algebraic [12,23].

Motivated by related work of L. Göttsche on the Kotschick–Morgan conjecture for Donaldson invariants [25] and by Fintushel and Stern on the general blow-up formula [20], Pidstrigach and Tyurin suggested that the conjecture of Witten should then follow by calculations—analogous to those of Göttsche—from the Kotschick–Morgan conjecture for PU(2) monopoles [56]. In the case of PU(2) monopoles there are further complications, not present in Donaldson theory, due in part to the many additional obstructions to gluing PU(2) monopoles.

### 4.2. PU(2) *monopoles: gluing and ungluing*

The cobordism scheme requires the use of analogues of Taubes' gluing maps to parameterize neighborhoods of moduli spaces of U(1) monopoles lying at the Uhlenbeck boundary of the moduli space of PU(2) monopoles and in particular, to construct links of these singularities.

In our articles [15,16] we first construct approximate gluing maps—giving approximate solutions to the PU(2) monopole equations—by grafting anti-self-dual connections from the four-sphere, which are concentrated at the north pole, onto a background PU(2) monopole at distinct points which are allowed to vary. We then show that these approximate gluing maps can be perturbed to give a collection of gluing maps $\gamma_\alpha : \mathcal{N}_\alpha \to \mathcal{C}^{*,0}_{W,E}$ and obstruction maps $\varphi_\alpha : \mathcal{N}_\alpha \to \mathcal{V}_\alpha$ which parameterize open neighborhoods of the ends of the noncompact moduli space of PU(2) monopoles in the following sense: The image $\operatorname{Im}\gamma_\alpha$ of a gluing map is a finite-dimensional submanifold of the configuration space $C^{*,0}_{W,E}$ of pairs of connections and spinors; an open neighborhood $\gamma_\alpha(\varphi_\alpha^{-1}(0))$ in the moduli space $M^{*,0}_{W,E}$ of PU(2) monopoles is then cut out of the gluing map image $\operatorname{Im}\gamma_\alpha$ by an obstruction section of a finite-rank obstruction bundle defined over the gluing parameter data $\mathcal{N}_\alpha$.

A gluing map $\gamma_\alpha$ is constructed by solving the 'infinite-dimensional part' of the PU(2) monopole equations (2.4), essentially obtained by projecting out the eigenspaces corresponding to the finitely many 'small eigenvalues' tending to zero. More precisely, the scheme we are forced to use is a variant of that developed by Donaldson [6,11], where we keep the metric fixed and adapt methods of Taubes [62,63] to allow us to glue in entire moduli spaces of anti-self-dual connections on $S^4$: Donaldson's scheme assumes that the connections are restricted to precompact subsets of their moduli spaces, while the Riemannian metric on $X$ is allowed to vary conformally. The obstruction map $\varphi_\alpha$ is then defined by $\gamma_\alpha$ and the 'finite-dimensional part' of the PU(2) monopole equations (2.4) which cannot be solved directly (due to the small eigenvalues and the resulting growth of Green's operator norms needed to solve the quasi-linear equation by the Banach space fixed-point theorem). These small eigenvalues arise here because neither the background monopole nor the anti-self-dual connections over $S^4$—now viewed as 'zero-section PU(2) monopoles'—are smooth points of their respective moduli spaces in the sense of Kodaira–Spencer. These small-eigenvalue phenomena are reminiscent of those in Taubes' earlier work on gluing anti-self-dual connections [61,63] where they arise when the background connection is trivial. However, for the purposes of differential-topological calculations, the difficulties surrounding them can generally be circumvented by working with connections on SO(3) bundles with nonzero $w_2$ or via blow-up tricks [45]: such a strategy does not work in the case of PU(2) monopoles.

The construction of gluing and obstruction maps for PU(2) monopoles is given in [15], where their existence is established, and the proof that they parameterize the ends of $M_{W,E}$ is completed in [16]. The difficulties in constructing PU(2) monopole gluing maps come from several sources:

- There are always obstructions to gluing coming from the anti-self-dual connections over the four-sphere $S^4$, because of the nonzero cokernel of the Dirac operator $D_A$, and from the background moduli spaces of U(1) monopoles.
- The PU(2) monopole equations, like the Seiberg–Witten equations, are not conformally invariant. Hence, the gluing technology for the conformally invariant anti-self-dual equation developed by Donaldson in [6,11] cannot be used directly for PU(2) monopoles.
- The gluing theory of Taubes [60–63] is difficult to adapt to the case of PU(2) monopoles because the Bochner formula for $d_A^+ d_A^{+,*}$—on which the estimates of [60–63] rely and which is well-behaved when the connection $A$ bubbles—must be used in conjunction with a Bochner formula for $D_A D_A^*$ which is badly behaved when the connection $A$ bubbles.
- In the work of Donaldson [6] and Mrowka [48] on the 'gluing theorem' for anti-self-dual connections, the anti-self-dual connections being glued up are assumed to vary in precompact subsets of their respective moduli spaces. While such restrictions

always simplify the analysis greatly, they cannot be imposed here since we need to ensure that the entire ends of the moduli space of PU(2) monopoles are covered by gluing maps.

The Bochner formulas relevant for Taubes' method are given by

$$2d_A^+d_A^* = \nabla_A^*\nabla_A - 2\{\mathcal{W}^+,\cdot\} + \tfrac{1}{3}R + \{F_A^+,\cdot\},$$
$$D_A D_A^* = \nabla_A^*\nabla_A + \tfrac{1}{4}R + \tfrac{1}{2}\rho\big(F(A^-_{\det W^+})\big) + \rho(F_A^-).$$

The term $F_A^+$ will be uniformly $L^\infty$ bounded while the term $F_A^-$ is only uniformly bounded in $L^2$ and its $L^\infty$ norm tends to infinity as the connection $A$ bubbles. This phenomenon makes it extremely difficult to produce Green's operator estimates which are uniform with respect to a degenerating, approximate PU(2) monopole $(A, \Phi)$ and hence solve Eqs. (2.4) for exact, nearby PU(2) monopoles. These problems are overcome in [15,16] by developing a combination of the gluing methods of Donaldson and Taubes, but the above difficulties make the gluing theory and the construction of links much more involved than it is for either anti-self-dual connections or Seiberg–Witten monopoles (the simplification in the latter case stems from the fact that the Seiberg–Witten moduli spaces are compact [47]). For example, we need estimates not only for the gluing maps but also for their differentials (and their inverses) to prove that the gluing maps are diffeomorphisms and cover the moduli space ends [16].

In [16] we show that (i) the PU(2) monopole gluing maps are 'surjective' in the sense that every PU(2) monopole lies in the image of a gluing map (so it can be 'unglued'), (ii) they are diffeomorphisms onto their images, and (iii) the gluing map images have an invariant characterization in the quotient. The surjectivity property of Taubes' gluing maps for anti-self-dual connections is a special case of a more general gluing result for critical points of the Yang–Mills functional [62, Proposition 8.2]. Like the proof of a particular case of the surjectivity statement for anti-self-dual connection gluing maps given by Donaldson and Kronheimer in [11, Section 7.2], Taubes' argument essentially relies on estimates for the inverse of the differential of the gluing map and the 'method of continuity' to show that a given point lies in the image of a gluing map. Again, the main new difficulty here lies in getting estimates which are uniform with respect to an approximate PU(2) monopole connection which is 'bubbling' (and thus approaching the Uhlenbeck boundary). Our construction in [15,16] shows that open neighborhoods of the lower-level strata of $\overline{M}_{W,E}$ are modeled by zero sets of sections of finite-rank obstruction bundles: this generalizes the description given in Section 3.7 of open neighborhoods of the singular strata in the top level $M_{W,E}$.

### *4.3. General reduction formulas and the* PU(2)*-monopole analogue of the Kotschick–Morgan conjecture*

In this section we sketch some of the ideas underlying our approach to the PU(2)-monopole analogue of the Kotschick–Morgan conjecture.

The first observation one needs in order to appreciate why the PU(2)-monopole program should work is that, as discussed in Section 3 and shown in [14], the intersection

$\overline{V}(z) \cap \overline{W}(x^{n_a-1})$ of geometric representatives is a collection of smooth one-manifolds, with one set of boundaries near the moduli space $M_E^{\mathrm{asd}}$ of anti-self-dual solutions and the other boundaries in neighborhoods of Seiberg–Witten reducible solutions of the form

$$M^{\mathrm{red}}_{W,E_{-\ell},L_1} \times \mathrm{Sym}^\ell(X) \subset IM_{W,E}. \tag{4.1}$$

Because of the obstructions to gluing, it is not clear that all the points of (4.1) are necessarily contained in $\overline{M}_{W,E}$, and so $\overline{M}_{W,E}$ may be a proper subset of $IM_{W,E}$.

In [14] we analyze the intersection of these geometric representatives in a neighborhood of the anti-self dual solutions and reducible PU(2) monopoles in the top Uhlenbeck level (as described here in Section 3). To generalize Theorem 3.21 to the case when there are reducible pairs in the lower levels of $\overline{M}_{W,E}$, we need a precise construction of the links $\boldsymbol{L}_{W,E_{-\ell},L_1}$ of the lower-level reducibles (4.1). In [16] we use the gluing and obstruction maps to construct an open neighborhood $U_{W,E_{-\ell},L_1}$ of the points (4.1) in $\overline{M}_{W,E}$ with a 'piecewise smoothly-stratified boundary'

$$\boldsymbol{L}_{W,E_{-\ell},L_1} := \partial U_{W,E_{-\ell},L_1}.$$

This boundary serves as a link of the reducible solutions (4.1) in the compactified moduli space $\overline{M}_{W,E}$. Because there are obstructions to gluing coming from both the background PU(2) monopoles and the anti-self-dual connections over $S^4$, it is not known if the Uhlenbeck compactification has locally finite topology at points in the lower levels. Although the link given by $\partial U_{W,E_{-\ell},L_1}$ might not have finite topology, its intersection with the geometric representatives of the cohomology classes is a finite set of points, as this intersection takes place in the top stratum (in the top level, away from any reducibles).

The above remarks suggest that one should obtain a 'reduction formula', conjectured by Pidstrigach and Tyurin, expressing the Donaldson invariants in terms of integrals over links of Seiberg–Witten moduli spaces:

**Conjecture 4.1** (Pidstrigach and Tyurin). If $z \in \mathbb{A}(X)$, then

$$2^{n_a-1} D_X^{c_1(E)}(z) = \sum_{L_1 \in \overline{\mathrm{Red}}(W,E)} \overline{V}(z) \cap \overline{W}(x^{n_a-1}) \cap \boldsymbol{L}_{W,E_{-\ell},L_1}, \quad \text{if } \deg z = 2d_a,$$

$$0 = \sum_{L_1 \in \overline{\mathrm{Red}}(W,E)} \overline{V}(z) \cap \overline{W}(x^{n_a-1}) \cap \boldsymbol{L}_{W,E_{-\ell},L_1}, \quad \text{if } \deg z > 2d_a.$$

Note that the level index $\ell$ appearing in the right-hand side the above formulas is determined by the reduction $E_{-\ell} = L_1 \oplus (\det E) \otimes L_1^*$ defined by $L_1$, since $\det E_{-\ell} = \det E$ is fixed and $c_2(E_{-\ell}) = c_2(E) - \ell$.

The second formula, while not directly interesting, could be useful in deriving recursion relations determining the intersections with $\boldsymbol{L}_{W,E_{-\ell},L_1}$. An important step towards proving Witten's conjecture would be to show that the intersection on the right has some universal expression (whose precise form might not be known) in terms of Seiberg–Witten invariants:

**Conjecture 4.2** (Pidstrigach and Tyurin). The pairing on the right-hand side of Conjecture 4.1 is given by a universal formula depending only on $\ell$, $F$, $L_1$, $SW(\mathfrak{s}_0 \otimes L_1)$, the intersection form $Q_X$, and invariants of the homotopy type of $X$.

This is the Pidstrigach–Tyurin version of the 'Kotschick–Morgan conjecture' [32, Conjectures 6.2.1 and 6.2.2]. More specifically, one would like to show that the pairing on right-hand side of Conjecture 4.1 is given by

$$q_X(\ell, F, L_1, Q_X) \cdot SW(\mathfrak{s}_0 \otimes L_1)$$

for some universal polynomial $q_X(\cdot)$, where the dependence on $X$ is just through its homotopy type (although even getting the terms on the right-hand side of Conjecture 4.1 to be divisible by $SW(\mathfrak{s}_0 \otimes L_1)$ is a highly nontrivial problem). Naturally, the ultimate aim is to evaluate these pairings explicitly, following the example of Göttsche in [25] for the $b^+ = 1$ wall-crossing formula, and show that they coincide with the prediction of Witten in the case of simple type. We gave calculations of this type for top level reducibles in Theorem 3.23, when $\ell = 0$, and outline the idea for lower-level reducibles below, when $\ell > 0$.

The calculations are simplest when $M^{\text{red}}_{W,E_{-\ell},L_1}$ is zero-dimensional,

$$M^{\text{red}}_{W,E_{-\ell},L_1} = \{[A_r, \Phi_r]\}_{r=1}^{n},$$

so we sketch the basic idea for this special case below. Note that when $X$ has Seiberg–Witten simple type it may still have positive-dimensional Seiberg–Witten moduli spaces and though the associated Seiberg–Witten invariants will vanish, one cannot *a priori* rule out their contributions to the Donaldson polynomials. Hence, even assuming $X$ has Seiberg–Witten simple type, we still need the thickened moduli spaces of Section 3.7 to show that positive-dimensional Seiberg–Witten moduli spaces do not in fact contribute to the Donaldson invariants.

Let $\{U_r\}_{r=1}^{n}$, be neighborhoods of zero in $H^1_{A_r,\Phi_r}$ for the reducibles $\{[A_r, \Phi_r]\}_{r=1}^{n}$ in the background moduli space $M_{W,E_{-\ell}}$ and let $\mathrm{Gl}(U_r, \Sigma)$ be the gluing data associated with $U_r$ and a (precompact open subset of a) smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$. We can cover a neighborhood of $[A_r, \Phi_r] \times \mathrm{Sym}^{\ell}(X)$ in $\overline{M}_{W,E}$ with the images under the gluing maps

$$\gamma_{r,\Sigma}\big(\varphi^{-1}_{r,\Sigma}(0) \cap \mathrm{Gl}(U_r, \Sigma)\big)$$

of the zero loci of the obstruction sections $\gamma_{r,\Sigma}$. The pairing on the right-hand side of Conjecture 4.1 then takes the form

$$\sum_{r=1}^{n} \overline{V}(z) \cap \overline{W}(x^{n_a - 1}) \cap \boldsymbol{L}_r, \tag{4.2}$$

where $\boldsymbol{L}_r$ is the link of $[A_r, \Phi_r] \times \mathrm{Sym}^{\ell}(X)$ in $\bigcup_{\Sigma} \gamma_{r,\Sigma}(\mathrm{Gl}(U_r, \Sigma))$. If one could show that the pairing $\overline{V}(z) \cap \overline{W}(x^{n_a - 1}) \cap \boldsymbol{L}_r$ were a multiple of $\mathrm{sign}[A_r, \Phi_r]$, with coefficient independent of $r$—that is, independent of the background pair, then the sum (4.2) would be a multiple of

$$SW(\mathfrak{s}_0 \otimes L_1) = \# M^{\mathrm{red}}_{W,E_{-\ell},L_1} = \sum_{r=1}^{n} \operatorname{sign}[A_r, \Phi_r].$$

Independence of the background pair can be shown by direct calculation when $\ell = 1$, much as in [31,32,67]. The fact that the individual pairings may depend on the background pairs is essentially because the gluing maps do not quite 'commute': gluing up the same gluing data in different orders yields slightly different composite gluing maps. Similar difficulties have been encountered in attempts to prove the Kotschick–Morgan conjecture of Donaldson theory [32,46].

In the positive-dimensional case there are additional problems due to 'spectral flow' or 'jumping lines' and this makes it difficult to describe the links of the lower-level moduli space of U(1) monopoles, $M^{\mathrm{red}}_{W,E_{-\ell},L_1} \times \mathrm{Sym}^{\ell}(X)$. In general, there is no global Kuranishi model for $M^{\mathrm{red}}_{W,E_{-\ell},L_1}$ which is defined naturally by small-eigenvalue cutoffs which we can glue up with $S^4$ gluing data to form open neighborhoods in $M_{W,E}$—one encounters 'jumping lines' as the points in a neighborhood of the background moduli space $M_{W,E_{-\ell}}$ vary. (Models which are global with respect to the background Seiberg–Witten moduli space are desirable for the purposes of calculating Euler classes of the obstruction bundles.) As outlined in Section 3, we employ stabilization methods [3,11] to address these problems when they are caused by reducibles in the top level in [14], where no gluing is needed. In the general case, we use gluing to parameterize the links of lower-level reducibles in combination with this stabilization procedure [15,16] when $\ell > 0$ and verify Conjectures 4.1 and 4.2 by direct calculation when $\ell \leqslant 1$.


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