Geometry of the Moduli Space of Self-Dual Connections over the Four-Sphere

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ABSTRACT

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A Riemannian metric on a compact four-manifold induces a natural L^2 metric on the corresponding moduli space of (anti-) self-dual connections on a principal G-bundle P. When the bundle structure group G is $\mathbf{SU}(2)$ and $-c_2(P) = k$, Groisser, Parker and others have found explicit formulas for the components of the L^2 metric on the moduli space $\mathcal{M}_k$ when $k = 1$ and the four-manifold is the sphere $\mathbb{S}^4$ or the complex projective space $\mathbb{CP}^2$. The moduli space $\mathcal{M}_1(\mathbb{S}^4)$ is diffeomorphic to the open five-ball, while $\mathcal{M}_1(\mathbb{CP}^2)$ is diffeomorphic to the open cone over $\mathbb{CP}^2$: these moduli spaces have finite volume and diameter with respect to the L^2 metric.

Donaldson, Groisser, and Parker have conjectured that the moduli space $\mathcal{M}_k$ has finite volume and diameter with respect to the L^2 metric for any integer k. We consider the case where the four-manifold is the sphere $\mathbb{S}^4$ with its standard round metric, the group G is $\mathbf{SU}(2)$, and $k = 2$. We obtain estimates for the components of the L^2 metric on the moduli space $\mathcal{M}_2(\mathbb{S}^4)$ of self-dual $\mathbf{SU}(2)$-connections over the four-sphere, a non-compact 13-dimensional manifold which is homotopic to the Grassman manifold of real 2-planes in $\mathbb{R}^5$. As an application, we show that the space $\mathcal{M}_2(\mathbb{S}^4)$ has finite volume and diameter with respect to the L^2 metric.

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INTRODUCTION

A Riemannian metric on a compact four-manifold M induces a natural L^2 metric on the corresponding moduli space $\mathcal{M}_k^*$ of (anti-) self-dual connections (or k-*instantons*) on a principal $\mathbf{SU}(n)$-bundle with second Chern number $-k$ [G-P1]. This metric has been studied by Groisser and Parker and others, and explicit formulas have been found for the metric components when k is 1, the four-manifold is the sphere $\mathbb{S}^4$ or the complex projective space $\mathbb{CP}^2$, and the bundle structure group is $\mathbf{SU}(2)$ [G], [G-P1, 2], [Hab], [D-M-M], [I]. The space $\mathcal{M}_1^*(\mathbb{S}^4, \mathbf{SU}(2))$ is diffeomorphic to the open five-ball [A-H-S], [Har], and $\mathcal{M}_1^*(\mathbb{CP}^2, \mathbf{SU}(2))$ is diffeomorphic to the open cone over $\mathbb{CP}^2$ [Bu], [D2], [G]. With respect to the L^2 metric, these moduli spaces have finite volume and diameter. These finiteness results were extended by Groisser and Parker to the case where M is any compact, oriented, simply-connected, Riemannian four-manifold with positive definite intersection form [G-P3]. In contrast, relatively little is known about either the geometry or topology of the moduli spaces $\mathcal{M}_k^*$ when $|k| > 1$.

Donaldson and Groisser-Parker have conjectured that the moduli space of multi-instantons $\mathcal{M}_k^*$ has finite volume and diameter with respect to the L^2 metric for any integer k [D4, 5], [G], [G-P2]. One of the motivations for studying the asymptotic behaviour of the L^2 metric arises in physics [G-P3], [O]. Path integrals over the infinite-dimensional space of connections modulo gauge transformations arising in quantum Yang-Mills theory are thought to be well-approximated by integrals over the finite-dimensional moduli spaces with respect to an appropriate quantum measure. The standard definitions of this measure involve the L^2 metric [G-P3]. Another motivation comes from the definition of Donaldson invariants arising in the differential topology of four-manifolds [D6], [D-K], [F-M]. For even-dimensional moduli spaces, the Donaldson polynomial invariants may be expressed — at least formally — as integrals of certain differential forms over the moduli space [D-K], [D5], [Wi]. Donaldson has then posed the problem of showing that these integrals converge and that their values coincide with the polynomial invariants as defined in [D6].

We have obtained estimates for the components of the L^2 metric on the moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ of self-dual $\mathbf{SU}(2)$-connections over the four-sphere. This moduli space is a non-compact 13-dimensional manifold homotopy equivalent to the Grassman manifold $G_2(\mathbb{R}^5)$ [Au-Do], [Har2], [Hat] [S-T]. With regard to its geometry, we have the

Theorem A. *The moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ of self-dual $\mathbf{SU}(2)$-connections over the four-sphere $\mathbb{S}^4$ with its standard round metric g_0 has finite volume and diameter with respect to the naturally induced L^2 metric.*

The result also holds for any metric g on $\mathbb{S}^4$ which is globally conformally equivalent to the standard round metric. We require the metric on $\mathbb{S}^4$ to be conformally equivalent to the standard round metric g_0 in order to apply the ADHM construction, and we need compactness of the base four-manifold M in order to achieve finiteness of volume and diameter of the moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$.

The author is currently investigating the problems of (1) extending these finiteness results to more general four-manifolds and arbitrary values of k, and (2) examining whether the Donaldson invariants may be represented by integrals of differential forms over moduli space.

In order to describe our results in more detail, we recall the definition of the L^2 metric on Yang-Mills moduli spaces [G-P1]. We let (M, g) be a compact, oriented, Riemannian four-manifold and let P be a principal bundle over M with structure group G. The space of smooth connections $\mathcal{A}$ on the G-bundle $P \to M$ is an infinite-dimensional affine space whose tangent space may be identified with the Hilbert space completion of $\Omega^1(M, \operatorname{ad} P)$, with respect to a suitable Sobolev L^2_s norm. Here, $\Omega^p(M, \operatorname{ad} P)$ is the bundle of smooth p-forms on M with values in the associated vector bundle $\operatorname{ad} P = P \times_{\mathrm{ad}} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. We assume that G is a compact, semi-simple Lie group, and that the vector bundles $\Omega^p(M, \operatorname{ad} P)$ have fibre metrics $\langle\ ,\ \rangle$ induced by the Riemannian metric g on M and a constant negative multiple of the Cartan-Killing form on G. The L^2 inner product of such forms defines an inner product on $\mathcal{A}$ which is invariant under the action of the *group of gauge transformations* $\mathcal{G}$ — the group of automorphisms covering the identity of M, and so induces a metric on the infinite-dimensional space $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$, of irreducible connections modulo gauge transformations. For $G = \mathbf{SU}(2)$, the space $\mathcal{B}^*$ is a smooth Hilbert manifold.

The tangent space $T_{[\omega]}\mathcal{B}^*$ is identified with the *horizontal subspace* $\operatorname{Ker} d_\omega^*$ of $T_\omega\mathcal{A}^*$. Let α, β be tangent vectors in $T_\omega\mathcal{A}^*$ and define

$$\mathbf{g}(\alpha, \beta) = \int_M \langle h_\omega\alpha, h_\omega\beta\rangle\sqrt{g}\,dx, \qquad h_\omega = I - d_\omega G_\omega^0 d_\omega^*,$$

where d_ω is the exterior covariant derivative associated with the connection ω, h_ω is the L^2 orthogonal projection onto the horizontal subspace $\operatorname{Ker} d_\omega^*$, and G_ω^0 is the Green's operator corresponding to the Laplacian $\Delta_\omega^0 = d_\omega^* d_\omega$ on $\Omega^0(M, \operatorname{ad} P)$. One obtains a well-defined, Riemannian metric $\mathbf{g}$ on the quotient space $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ [G-P1], [I].

For convenience we assume $G = \mathbf{SU}(2)$ and let P be a principal G-bundle with topology fixed by choosing the second Chern number $c_2(P) = -k$. A connection ω on P whose curvature 2-form F_ω satisfies $*F_\omega = F_\omega$ or $*F_\omega = -F_\omega$, is called *self-dual* or *anti-self-dual*, respectively, with respect to the metric g; the self-dual connections have $k > 0$ and the anti-self-dual connections have $k < 0$. Here, $*$ denotes the Hodge star-operator. The self-dual connections on P are solutions of the *Yang-Mills equations*, $d_\omega^* F_\omega = 0$. Under a variety of different combinations of conditions on (M, g), group G, and topology of P — see for example [A-H-S], [T1], [T2], [F-U], [D-K] — the moduli space $\mathcal{M}_k^*$ of gauge-equivalence classes of self-dual connections is non-empty and is a smooth, finite-dimensional, non-compact submanifold of $\mathcal{B}^*$. The tangent space $T_{[\omega]}\mathcal{M}_k^* = \operatorname{Ker} d_\omega^-/\operatorname{Im} d_\omega = H_\omega^1$ may be identified with the *harmonic space* $\mathbf{H}_\omega^1 = \{\alpha \in \Omega^1(M, \operatorname{ad} P) : d_\omega^*\alpha =$

0 and $d_\omega^- \alpha = 0\}$, where $d_\omega^- \alpha = p_- d_\omega \alpha$, with p_- denoting the projection onto anti-self-dual 2-forms. Exactly analogous statements hold for the moduli space of anti-self-dual connections. Applying an orientation-reversing map to M identifies the moduli space of self-dual connections over M with the moduli space of anti-self-dual connections over $\overline{M}$, where the manifold $\overline{M}$ denotes M with the opposite orientation. In either case, one obtains the L^2 metric $\mathbf{g}$ on $\mathcal{M}_k^*$ by restriction [G-P1], [I].

One of the first difficulties one encounters when attempting to evaluate components of $\mathbf{g}$ is that of finding a suitable basis for the harmonic subspace $\mathbf{H}_w^1$ corresponding to an appropriate choice of local coordinates on the moduli space. When G is $\mathbf{SU}(2)$, the space $\mathcal{M}_1^*(\mathbb{S}^4, \mathbf{SU}(2))$ has a global coordinate system given essentially by the *centre* and *scale* of the self-dual connection and the required horizontal projections may be computed explicitly [D-M-M], [G-P1], [Hab]. However, the space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ has a more complicated topology and a good system of local coordinates is not so readily apparent. Moreover, it no longer seems possible to perform the horizontal projections explicitly and hence obtain a basis for $\mathbf{H}_w^1$. Also, as $|k|$ increases, the topology of the moduli spaces becomes more difficult to identify [Bo-Ma], [Hur].

The moduli spaces $\mathcal{M}_k^*$ have topological compactifications $\overline{\mathcal{M}_k^*}$ as subsets of the space of *ideal connections*, which may be viewed as connections with curvature densities possibly having δ-measure concentrations at up to k points of M [D-K, p. 157]. The ideal boundary connections in $\partial\mathcal{M}_k^*$ are then self-dual connections ω with (normalized) curvature density of mass $k - l$, where $1 \leq l \leq k$, together with 'curvature densities' given by δ-measures of mass k_i at m points of M, where $1 \leq m \leq l$, so that $\sum_{i=0}^m k_i = l$ and the curvature density of the ideal boundary connection has total mass k. Neighbourhoods of the boundary corresponding to l distinct points with δ-measures of mass 1 may be described explicitly with the aid of Taubes' gluing construction [T1, 2], [D3]. Using gluing maps onto neighbourhoods of the l distinct centres, one attaches concentrated self-dual connections with $k_i = 1$ over $\mathbb{S}^4$ to a backround self-dual connection on a bundle $P' \to M$ with $-c_2(P') = k - l$, producing a family of almost self-dual connections on a bundle P with $-c_2(P) = k$. One then perturbs this family slightly to produce a family of self-dual connections on $P \to M$ parametrizing an open neighbourhood of the ideal boundary. A difficulty now arises when one wishes to explicitly parametrize neighbourhoods of the moduli space boundary corresponding to configurations of points in M and associated δ-measures with at least one of the masses greater than 1. One must now attach concentrated self-dual connections over $\mathbb{S}^4$ with $k_i > 1$. Thus, a better understanding of the ends of the moduli space $\mathcal{M}_k^*(\mathbb{S}^4)$ for $k > 1$ would aid our understanding of the ends of $\mathcal{M}_k^*$ of more general four-manifolds — for example, compact, oriented, simply-connected, Riemannian four-manifolds with positive definite intersection form.

To determine whether the Riemannian manifold $(\mathcal{M}_k^*, \mathbf{g})$ has finite diameter and volume, we examine the asymptotic behaviour of the L^2 metric $\mathbf{g}$ as one approaches the boundary of moduli space. When k is 2, M is $\mathbb{S}^4$, and G is $\mathbf{SU}(2)$, our approach to this problem makes use of an explicit parametrization of the moduli space.

When M is $\mathbb{S}^4$ and G is one of the classical Lie groups $\mathbf{O}(n)$, $\mathbf{SU}(n)$, or $\mathbf{Sp}(n)$, then the moduli space of self-dual connections may be parametrized, at least in principle, using the *Atiyah-Drinfeld-Hitchin-Manin* correspondence [A-D-H-M]. This correspondence gives a diffeomorphism from the space of ADHM matrices, consisting of isomorphism classes of solutions to the non-linear ADHM matrix equations, and isomorphism classes of solutions to the non-linear self-duality equations. When G is $\mathbf{SU}(2)$ and $-c_2(P) = k$, the moduli space is an $(8k-3)$-dimensional manifold [A-H-S]. As is well-known, the construction of the connection 1-forms provided by the ADHM map becomes very unwieldy when $|k|$ is greater than 1, and it has proved difficult to study the moduli space for arbitrary values of k by appealing directly to the ADHM correspondence [A].

We employ the parametrization, due to Hartshorne [Har] and Jackiw-Nohl-Rebbi [J-N-R], of the moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ in terms of three distinct points P_i in $\mathbb{S}^4$ and three positive weights λ_i (up to a common rescaling). We let $\tilde{T}_2$ denote the space of unordered pairs (P_i, λ_i), with $P_i \in \mathbb{S}^4$ distinct and $(\lambda_0, \lambda_1, \lambda_2) \sim \nu(\lambda_0, \lambda_1, \lambda_2)$ for $\nu > 0$. The space $\tilde{T}_2$ is a 14-dimensional non-compact manifold, while $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ has dimension 13. There is an induced action of the group of gauge transformations on $\tilde{T}_2$: this action has been interpreted as a motion of the points P_i around the circle in $\mathbb{S}^4$ determined by those points [Har2, 3], [J-N-R]. This gauge equivalence is most easily understood in the context of algebraic geometry. The space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ is connected, with fundamental group $\mathbb{Z}_2$ [Au-Do], [Har2], [Hur].

The parametrization of Hartshorne and Jackiw-Nohl-Rebbi corresponds to a particularly convenient choice of ADHM matrices. Our principal $\mathbf{SU}(2)$-bundles P with $-c_2(P) = 2$ are the pull-backs of the quaternionic Hopf bundle $\mathbb{S}^{11} \to \mathbb{HP}^2$, via suitable classifying maps provided by the ADHM construction from $\mathbb{S}^4 = \mathbb{HP}^1$ into the quaternionic projective space $\mathbb{HP}^2$. For the moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$, we obtain the following

Theorem B. *Let $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ denote the moduli space of self-dual connections on a principal $\mathbf{SU}(2)$-bundle P over the sphere $\mathbb{S}^4$, where $-c_2(P) = 2$ and $\mathbb{S}^4$ has its standard round metric g_0. Let these connections be parametrized by the space $\tilde{T}_2$ of unordered pairs (P_i, λ_i), where $\lambda_0, \lambda_1, \lambda_2$ are positive weight parameters satisfying the scaling condition $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$, and P_0, P_1, P_2 are distinct points in $\mathbb{S}^4$. Let c_i^μ denote the standard inhomogeneous coordinates of the point $P_i \in \mathbb{S}^4 = \mathbb{HP}^1$, for $i = 0,1,2$, $\mu = 0, \ldots, 3$, so that $c_i = a_i$ if $P_i = [a_i, 1]$, lying in the southern hemisphere, or $c_i = b_i$ if $P_i = [1, b_i]$, lying in the northern hemisphere. With respect to this choice of parameters, we have the following estimates for the corresponding components of the L^2 metric $\mathbf{g}$:*

$$\mathbf{g}_{\lambda_i\lambda_i} = \mathbf{g}\left(\frac{\partial\omega}{\partial\lambda_i}, \frac{\partial\omega}{\partial\lambda_i}\right) \le C\left\{1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{j\neq i}\log\left(1 + \frac{1}{|c_i - c_j|}\right)\right\},$$

$$\mathbf{g}_{c_i^\mu c_i^\mu} = \mathbf{g}\left(\frac{\partial\omega}{\partial c_i^\mu}, \frac{\partial\omega}{\partial c_i^\mu}\right) \le C\left\{1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{j\neq i}\log\left(1 + \frac{1}{|c_i - c_j|}\right)\right\},$$

where C is a universal constant independent of moduli parameters, and ω denotes the family of self-dual connections parametrized by $\tilde{T}_2$.

Again, the estimates continue to hold for any metric g on $\mathbb{S}^4$ which is globally conformally equivalent to the standard round metric, the constant C now depending on g: it is just the compactness of $(\mathbb{S}^4, g)$ which is required when deriving the estimates. We need a metric g on $\mathbb{S}^4$ in the conformal class $[g_0]$ in order to apply the ADHM construction. The tangent space $T_{[\omega]}\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ is spanned by the vectors $\partial\omega/\partial\lambda_i$, $\partial\omega/\partial c_i^\mu$, for $i = 0, 1, 2$ and $\mu = 0, \ldots, 3$. Non-diagonal components of $\mathbf{g}$ may be estimated via the Schwarz inequality. By seeking upper bounds on the metric components, we avoid the difficulties associated with explicitly computing the horizontal projections appearing in the definition of the metric components — apparently a difficult problem when $|k|$ is greater than 1.

We now provide an outline of the remaining chapters. In Chapter I we establish our notation and conventions, and we review some of the basic concepts of gauge theory. We describe the moduli space of self-dual connections and define the induced L^2 metric. The ADHM construction produces a family of self-dual connections on a family of principal G-bundles over $\mathbb{S}^4$, parametrized by the space of ADHM matrices. So we review the concept of a family of connections and discuss some related issues concerning infinitesimal deformations and horizontal projections. The ADHM construction defines certain classifying maps from $\mathbb{S}^4$ to a classifying space BG, and so we obtain self-dual connections on G-bundles $P \to \mathbb{S}^4$ by pulling back a canonical connection on a universal G-bundle $EG \to BG$.

In Chapter II we review the correspondence between the moduli space of self-dual $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ and the moduli space of instanton bundles over $\mathbb{CP}^3$, which are stable, holomorphic, rank 2 vector bundles over $\mathbb{CP}^3$ satisfying certain technical conditions. This allows us to relate the picture of $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ in terms of distinct points P_i in $\mathbb{S}^4$ and positive weights λ_i due to Jackiw, Nohl, and Rebbi, with that of Hartshorne, where the instanton bundles $\mathcal{E}$ correspond to curves Y in $\mathbb{CP}^3$. The curve Y associated to a bundle $\mathcal{E}$ arises as the zero-set $(s)_0$ of a global section s of the twisted bundle $\mathcal{E}(1) = \mathcal{E} \otimes \mathcal{O}_{\mathbb{CP}^3}(1)$.

In Chapter III, we describe the ADHM construction of self-dual $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$. As an application, we explicitly construct the sections $s \in H^0(\mathcal{E}(1))$ arising in Hartshorne's study of $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$.

In Chapters IV and V, we come to our main goal, which is to examine the asymptotic behaviour of the L^2 metric $\mathbf{g}$ on $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$. In Chapter IV, we describe in detail the parametrization of the self-dual $\mathbf{SU}(2)$-connection 1-forms for $k = 2$. In Chapter V, we give the local 1-forms representing the tangent vectors to the moduli space $\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$. We avoid the difficulty of computing horizontal projections and obtaining harmonic representatives of the tangent space by instead deriving estimates for the L^2 norms of the tangent vectors and hence the corresponding components of the L^2 metric. With these estimates at hand, we are then able to show that the space $(\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2)), \mathbf{g})$ has finite volume and diameter.

CHAPTER I
MODULI SPACE OF SELF-DUAL CONNECTIONS

We review the standard description of the moduli space of self-dual connections on a principal G-bundle $\pi : P \to M$, where G denotes a compact, semi-simple Lie group with Lie algebra $\mathfrak{g}$, and (M, g) is a compact, connected, oriented, Riemannian, smooth four-manifold without boundary. We define the L^2 metric on the moduli space and discuss related issues of deformation theory and problems arising in the computation of the L^2 metric components. General references for this chapter are [A-B], [A-H-S], [D-K], [F-U], [G-P1, 2], [I], and [L].

§1.1. Preliminaries on Gauge Theory

We recall some aspects of gauge theory which will later prove useful and establish our notation. General references for this section are [Fi], [F-U], [Hu], [K-N], [Mo], [M-V], and [St].

Definition 1.1.1. We define the following infinite-dimensional topological groups:
(i) $\mathrm{Aut}(P) = \{f \in \mathrm{Diff}(P) : f(pa) = f(p)a$ for all $p \in P$, $a \in G$, and $\pi \circ f = \pi\}$;
(ii) $C^\infty_{\mathrm{Ad}}(P, G) = \{\varphi \in C^\infty(P, G) : \varphi(pa) = a^{-1}\varphi(p)a$, for all $p \in P$ and $a \in G\}$, where $C^\infty(P, G)$ denotes the space of smooth maps from P to G;
(iii) $\mathcal{G}(P) = \Gamma(\mathrm{Ad}\, P)$, where $\mathrm{Ad}\, P$ denotes the bundle of groups $P \times_{\mathrm{Ad}} G$ and $\Gamma(\mathrm{Ad}\, P)$ denotes the space of smooth sections of $\mathrm{Ad}\, P$.

As topological groups, $\mathrm{Aut}(P)$, $C^\infty_{\mathrm{Ad}}(P, G)$, and $\mathcal{G}(P)$ are naturally isomorphic and comprise alternative descriptions of the *group of gauge transformations*, which we will usually denote simply by $\mathcal{G}(P)$ — unless we wish to emphasize a particular description [Fi], [Mo]. If $f \in \mathrm{Aut}(P)$, then the corresponding element $\varphi \in C^\infty_{\mathrm{Ad}}(P, G)$ is defined by letting $\varphi(p)$ be the unique element of G such that $f(p) = p\varphi(p)$, for each $p \in P$. Conversely, a map $\varphi \in C^\infty_{\mathrm{Ad}}(P, G)$ determines an element $f \in \mathrm{Aut}(P)$ by setting $f(p) = p\varphi(p)$.

To describe a gauge transformation in terms of a section in $\mathcal{G}(P)$, let $\{O_\alpha\}$ be an open cover of M and let $\sigma_\alpha : O_\alpha \to P$ be a system of local sections, with corresponding transition functions $T_{\beta\alpha} : O_\alpha \cap O_\beta \to G$ defined by

$$\sigma_\beta = \sigma_\alpha T_{\alpha\beta} \qquad \text{on } O_\alpha \cap O_\beta,$$

and satisfying the cocycle condition

$$T_{\alpha\beta} T_{\beta\gamma} T_{\gamma\alpha} = 1 \qquad \text{on } O_\alpha \cap O_\beta \cap O_\gamma.$$

The corresponding local trivialisations $\tau_\alpha : P|_{O_\alpha} \to O_\alpha \times G$, $p \mapsto (\pi(p), \varphi_\alpha(p))$, may be defined by setting $\tau_\alpha(\sigma_\alpha(x)a) = (x, a)$, for $(x, a) \in O_\alpha \times G$. As usual, $\varphi_\alpha(p)\varphi_\beta(p)^{-1} = T_{\alpha\beta}(x)$, for any $p \in \pi^{-1}(x)$, and $\tau_\alpha \circ \tau_\beta^{-1}(x, a) = (x, T_{\alpha\beta}(x)a)$.

If $f \in \mathrm{Aut}(P)$, we obtain induced maps $g_\alpha : O_\alpha \to G$ defined by $f(\sigma_\alpha(x)) = \sigma_\alpha(x) g_\alpha(x)$. Hence,

$$g_\beta = T_{\alpha\beta}^{-1} g_\alpha T_{\alpha\beta} = \mathrm{Ad}(T_{\alpha\beta}^{-1}) g_\alpha \qquad \text{on } O_\alpha \cap O_\beta,$$

and so we may glue the g_α together along overlaps to give a global section $g \in \Gamma(\mathrm{Ad}\, P)$. If $\varphi \in C^\infty_{\mathrm{Ad}}(P, G)$, then the section $g \in \Gamma(\mathrm{Ad}\, P)$ may be defined by setting $g_\alpha = \varphi \circ \sigma_\alpha$. Conversely, a section $g \in \Gamma(\mathrm{Ad}\, P)$ determines elements $f \in \mathrm{Aut}(P)$ and $\varphi \in C^\infty_{\mathrm{Ad}}(P, G)$.

More generally, let $E \to M$, $E' \to M$ be fibre bundles with base M, fibre F, and structure group G. Let $T_{\alpha\beta}$, $T'_{\alpha\beta}$ be the transition functions of the bundles E, E' with respect to the open cover $\{O_\alpha\}$ of M. Then, there exists a fibre bundle isomorphism $f : E \to E'$ if and only if there exist maps $g_\alpha : O_\alpha \to G$ such that

$$T'_{\alpha\beta} = g_\alpha^{-1} T_{\alpha\beta} g_\beta \qquad \text{on } O_\alpha \cap O_\beta,$$

where $f(\sigma_\alpha) = \sigma'_\alpha g_\alpha$ on O_α [St, p. 12], [Hu, p. 61].

Let $\varrho : G \to GL(V)$ be a representation defining a left action of G on a vector space V and let $E = E(\varrho)$ be the associated vector bundle $P \times_\varrho V$. When $V = \mathfrak{g}$ and $\varrho = \mathrm{Ad}$, then $E(\varrho)$ is denoted $\mathrm{ad}\, P$. Where convenient, we let $\mathrm{ad}\, E \simeq \mathrm{ad}\, P$ denote the subbundle of $\mathrm{End}(E)$ whose sections have values $\psi(x) \in \mathfrak{g} \subset \mathrm{End}(E_x)$ for all $x \in M$.

Let $C^\infty_\varrho(P, V)$ denote the space of smooth maps $\Phi : P \to V$ such that $\Phi(pa) = \varrho(a^{-1})p$ for all $a \in G$ and $p \in P$. There is a bijection $C^\infty_\varrho(P, V) \to \Gamma(E)$ defined by $\Phi \to \phi$, with $\phi(x) = [p, \Phi(p)]$ for $x \in M$ and any $p \in \pi^{-1}(x)$, where $[p, \xi] = \{(pa, \varrho(a^{-1})\xi) : a \in G\}$ [Hu, p. 46]. A section $\phi \in \Gamma(E)$ may then be represented locally by maps $\phi_\alpha : O_\alpha \to V$, with $\phi_\alpha = \Phi \circ \sigma_\alpha$, so that

$$\phi_\beta = \varrho(T_{\alpha\beta}^{-1}) \phi_\alpha \qquad \text{on } O_\alpha \cap O_\beta.$$

Hence, E has corresponding transition functions $\varrho(T_{\beta\alpha})$. The action of $\mathrm{Aut}\,(P)$ on P induces an action on the associated vector bundle E: if $g \in \mathcal{G}(P)$ is represented locally by $\sigma_\alpha \mapsto \sigma_\alpha g_\alpha$, then $\phi_\alpha \mapsto \varrho(g_\alpha^{-1})\phi_\alpha$ on O_α.

Definition 1.1.2. We define the following bundle-valued forms:
(i) $\Omega^q(P, \mathfrak{g}) = \Gamma(\wedge^q(T^*P) \otimes \mathfrak{g})$;
(ii) $\Omega^q(M, E) = \Gamma(\wedge^q(T^*M) \otimes E)$;
(iii) $\Omega^q(M, \mathrm{ad}\, P) = \Gamma(\wedge^q(T^*M) \otimes \mathrm{ad}\, P)$.

The action of $\mathrm{Aut}\,(P)$ on P induces a right action $\Omega^q(P, \mathfrak{g}) \times \mathrm{Aut}\,(P) \to \Omega^q(P, \mathfrak{g})$ by pull-back, $(\omega, f) \mapsto f^*\omega$, where we will often denote $f^*\omega$ by ω^f or $f(\omega)$ for convenience. Moreover, there is an induced right action of $\mathrm{Aut}\,(P)$ on the bundle-valued q-forms $\Omega^q(M, E)$ and $\Omega^q(M, \mathrm{ad}\, P)$: the action of $\mathcal{G}(P)$ on $\Omega^q(M, E)$ may be represented locally by $\omega_\alpha \mapsto \varrho(g_\alpha^{-1})\omega_\alpha$, where $\omega_\alpha \in \Omega^q(O_\alpha, V)$ [Mo], [K-N1, p. 75]. Where convenient, we denote the corresponding global form by ω^g or $g(\omega)$.

Let $\mathcal{A}(P) \subset \Omega^1(P, \mathfrak{g})$ denote the affine space of connection 1-forms on P and let $\omega_\alpha = \sigma_\alpha^* \omega \in \Omega^1(O_\alpha, \mathfrak{g})$ be the corresponding local connection 1-forms [K-N1, p. 64-66]. Then

$$\omega_\beta = \mathrm{Ad}(T_{\alpha\beta}^{-1})\omega_\alpha + T_{\alpha\beta}^{-1} dT_{\alpha\beta} \qquad \text{on } O_\alpha \cap O_\beta,$$

where $T_{\alpha\beta}^{-1} dT_{\alpha\beta} = T_{\alpha\beta}^* \theta$, if θ denotes the Maurer-Cartan form of G. If we fix any connection $\omega_0 \in \Omega^q(P, \mathfrak{g})$ and let $\omega \in \Omega^q(P, \mathfrak{g})$, then the difference $w - \omega_0$ defines an element δ of $\Omega^q(M, \mathrm{ad}\, P)$ by setting $\delta_\alpha = \sigma_\alpha^*(\omega - \omega_0) \in \Omega^q(O_\alpha, \mathfrak{g})$ [Mo].

The group $\mathrm{Aut}\,(P)$ induces a right action on $\mathcal{A}(P) \times \mathrm{Aut}\,(P) \to \mathcal{A}(P)$ by pull-back, $(\omega, f) \mapsto f^*\omega$ [Mo]. Denoting $f^*\omega$ by ω', the corresponding action of $\varphi \in C^\infty_{\mathrm{Ad}}(P, G)$ on $\mathcal{A}(P)$ is represented by

$$\omega' = \mathrm{Ad}\,(\varphi^{-1})\omega + \varphi^*\theta \ \in \Omega^1(P, \mathfrak{g}).$$

Denoting $\omega'_\alpha = \sigma_\alpha^* \omega'$, the corresponding action of $g \in \mathcal{G}(P)$ is represented locally by

$$\omega'_\alpha = \mathrm{Ad}(g_\alpha^{-1})\omega_\alpha + g_\alpha^{-1} dg_\alpha \ \in \Omega^1(O_\alpha, \mathfrak{g}),$$

where $g_\alpha^{-1} dg_\alpha = g_\alpha^* \theta$ [Fi, p. 239].

Next we consider the tangent space $T_{\omega_0}\mathcal{A}(P)$ to $\mathcal{A}(P)$ at a connection ω_0. If $\omega(t)$ is a smooth curve in $\mathcal{A}(P)$ with $\omega(0) = \omega_0$, then $\dot\omega(0) \in T_{\omega_0}\mathcal{A}(P)$. If $\omega_\alpha(t) = \sigma_\alpha^* \omega(t)$ are the local connection 1-forms, then

$$\dot\omega_\alpha(0) = \frac{d}{dt}\sigma_\alpha^*\omega(t)\bigg|_{t=0} = \sigma_\alpha^*\dot\omega(0) \in \Omega^q(O_\alpha, \mathfrak{g}).$$

Then $\dot\omega_\beta(0) = \mathrm{Ad}\,(T_{\alpha\beta}^{-1})\dot\omega_\alpha(0)$, and so $\dot\omega(0)$ defines an element of $\Omega^q(M, \mathrm{ad}\, P)$. This gives the standard identification of $T_{\omega_0}\mathcal{A}(P)$ with $\Omega^q(M, \mathrm{ad}\, P)$.

Let $\omega \in \mathcal{A}(P)$ and let $\nabla^\omega : \Omega^0(M, E) \to \Omega^1(M, E)$ be the covariant derivative on an associated vector bundle $E(\varrho)$ [Mo], [K-N1]. The exterior covariant derivative $d_\omega : \Omega^q(M, E) \to \Omega^{q+1}(M, E)$ is defined by requiring that
(i) d_ω is $\mathbb{R}$-linear;
(ii) $d_\omega(\phi \otimes \psi) = \nabla^\omega \phi \wedge \psi + \phi \otimes d\psi$ for all $\phi \in \Gamma(E)$ and $\psi \in \Omega^q(M)$.
Consequently, $d_\omega(\phi \wedge \psi) = d_\omega \phi \wedge \psi + (-1)^q \phi \wedge d\psi$ for all $\phi \in \Omega^q(M, E)$ and $\psi \in \Omega^p(M)$. The formal adjoint, $\delta_\omega : \Omega^q(M, E) \to \Omega^{q-1}(M, E)$, of d_ω with respect to the metric g on M is defined by $\delta_\omega = d_\omega^* = - * d_\omega *$. The connection ∇^ω on E naturally induces a connection on $\mathrm{End}(E)$ by setting [K-N1, p. 124]:

$$(\nabla^\omega \psi)(\phi) = \nabla^\omega(\psi(\phi)) - \psi(\nabla^\omega \phi), \quad \text{for all } \psi \in \Gamma(\mathrm{End}(E)),\ \phi \in \Gamma(E).$$

The curvature $F = F_\omega \in \Omega^2(M, \mathrm{End}(E))$ may be defined invariantly by $d_\omega \circ d_\omega \phi = F_\omega \phi$ for all $\phi \in \Gamma(E)$ and defined locally by

$$F_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha \in \Omega^2(O_\alpha, \mathfrak{g}),$$

with $F_\beta = \mathrm{Ad}\,(T_{\beta\alpha})F_\alpha$ on $O_\alpha \cap O_\beta$, with values in $\mathfrak{g} \subset \mathrm{End}(V)$, so that $\{F_\alpha\}$ defines an element of $\Omega^2(M, \mathrm{End}(E))$, [G-H], [K-N1]. More explicitly, a calculation shows [F-U, p. 31]:

$$\begin{aligned} F_{\alpha,ij} &= \frac{\partial \omega_{\alpha,j}}{\partial x_i} - \frac{\partial \omega_{\alpha,i}}{\partial x_j} + [\omega_{\alpha,i}, \omega_{\alpha,j}], \qquad \text{where} \\ F_\alpha &= \frac{1}{2}\sum_{i,j} F_{\alpha,ij}\, dx_i \wedge dx_j = \sum_{i<j} F_{\alpha,ij}\, dx_i \wedge dx_j. \end{aligned}$$

If $G = \mathbf{SU}(n)$, ϱ is the fundamental representation on $\mathbb{C}^n$, and $E = P \times_\varrho \mathbb{C}^n$, then the *instanton number* of P may be computed by

$$k = -c_2(E)[Mo] = -\frac{1}{8\pi^2}\int_M \mathrm{tr}\,(F_\omega \wedge F_\omega).$$

Moreover, $c_1(E)[M] = 0$, and $\mathbf{SU}(2)$ bundles on any closed, oriented, four-manifold M are classified topologically by $c_2(E) \in H^4(M, \mathbb{Z}) \simeq \mathbb{Z}$ [F-U, p. 179].

§1.2. Moduli Space of Self-dual connections

We review the description of the moduli space of self-dual connections. General references for this section are [K-D], [F-U], [M].

We fix the topology of a principal G-bundle $P \to M$ and denote $\mathcal{A}(P)$ and $\mathcal{G}(P)$ by $\mathcal{A}$ and $\mathcal{G}$, respectively. The metric on $\Omega^q(M, \mathrm{ad}\,P)$ is induced by the Riemannian metric g of M and a constant negative multiple of the Cartan-Killing form K on G: if $X, Y \in \mathfrak{su}(n)$, we have $K(X,Y) = \mathrm{tr}\,(XY)$ for the fundamental representation of $\mathfrak{su}(n)$. We recall that ω is a *Yang-Mills connection* if it satisfies the conformally invariant Yang-Mills equations: $d_\omega F_\omega = 0$ and $d_\omega^* F_\omega = 0$. A connection ω is *self-dual* (SD) (respectively, *anti-self-dual* (ASD)), if it satisfies the conformally invariant equation $*F_\omega = F_\omega$ (respectively, $*F_\omega = -F_\omega$), and consequently is a solution of the Yang-Mills equations [F-U, p. 35].

A G-connection ω is *reducible* if its holonomy group Φ_ω is a proper subgroup of G, and is *irreducible* otherwise [D-K, p. 131]. For any connection ω, if $\Gamma_\omega = \{g \in \mathcal{G} : g(\omega) = \omega\}$ denotes the isotropy group of ω, then Γ_ω is isomorphic to $\{g \in G : ghg^{-1} = h \text{ for all } h \in \Phi_\omega\}$, the centralizer of Φ_ω in G, where we view Γ_ω and Φ_ω as subgroups of $\mathrm{Aut}\,(E_x)$ for some fixed $x \in M$ [D-K, p. 131]. Let $\mathcal{Z} = \mathcal{Z}(\mathcal{G})$ denote the centre of the gauge group $\mathcal{G}$ and let $Z = Z(G)$ denote the centre of G. We recall that $\mathcal{Z} = \Gamma(P \times_{\mathrm{Ad}} Z) \simeq Z$. For example, if $G = \mathbf{SU}(2)$, we have $Z = \{\pm 1\}$ and so $\mathcal{Z} \simeq \{\pm 1\}$.

When $G = \mathbf{SU}(2)$, ϱ is the fundamental representation of $\mathbf{SU}(2)$ on $\mathbb{C}^2$, and $E = P \times_\varrho \mathbb{C}^2$, the following are equivalent [F-U, p. 47]:

(i) The connection ω is reducible;
(ii) The connection d_ω and bundle E split, so that $d_\omega = d_1 \oplus d_2$ on $E = L_1 \oplus L_2$;
(iii) $\Gamma_\omega / \{\pm 1\} \simeq U(1)$;
(iv) $\mathrm{Ker}\,\{d_\omega : \Omega^0(E) \to \Omega^1(E)\} \neq (0)$.

We note that if $H^2(M,\mathbb{Z}) = (0)$, (for example, if $M = \mathbb{S}^4$), then there are no reducible connections [F-U, p. 33]. The affine space of irreducible connections on P is denoted by $\mathcal{A}^*$. Fix a Sobolev index $s \geq 2$ and define the following Hilbert manifolds [F-U, p. 46]:

(i) Let $\mathcal{A}_s = \omega_0 + L^2_s\Omega^1(M, \operatorname{ad} P)$, the affine space $\mathcal{A}_s$ of Sobolev L^2_s-connections, where $\omega_0 \in \mathcal{A}$ is a fixed basepoint connection;

(ii) Let $\varrho : G \to \operatorname{Aut}(V)$ be a faithful representation, so that $\mathcal{G} \subset \Gamma(P \times_\varrho \operatorname{Aut}(V))$ and $\mathcal{G}_{s+1} \subset L^2_{s+1}(P \times_\varrho \operatorname{Aut}(V))$. Then $\mathcal{G}_{s+1}$ is an infinite-dimensional Lie group with Lie algebra $L^2_{s+1}\Omega^0(M, \operatorname{ad} E)$;

(iii) Let $\tilde{\mathcal{G}} \subset \Gamma(P \times_{\mathrm{Ad}} \operatorname{Aut}(\mathfrak{g}))$ and $\tilde{\mathcal{G}}_{s+1} \subset L^2_{s+1}(P \times_{\mathrm{Ad}} \operatorname{Aut}(\mathfrak{g}))$. Then $\tilde{\mathcal{G}}_{s+1} = \mathcal{G}_{s+1}/\mathcal{Z}$ and $\tilde{\mathcal{G}}_{s+1}$ is an infinite-dimensional Lie group with Lie algebra $L^2_{s+1}\Omega^0(M, \operatorname{ad} P)$.

We denote the space of all irreducible Sobolev connections by $\mathcal{A}^*_s$. For $s \geq 2$, there is a smooth action $\mathcal{A}^*_s \times \mathcal{G}_{s+1} \to \mathcal{G}_{s+1}$. Let $\mathcal{B}^*_s$ denote the orbit space $\mathcal{A}^*_s/\tilde{\mathcal{G}}_{s+1}$ endowed with the quotient topology. The topology of the moduli space of self-dual connections $\mathcal{M}^* \subset \mathcal{B}^*_s$ is independent of $s \geq 2$, so we will henceforth omit Sobolev subscripts and denote $\mathcal{A}^*_s$, $\mathcal{B}^*_s$, $\tilde{\mathcal{G}}_{s+1}$, and $\mathcal{G}_{s+1}$, by $\mathcal{A}^*$, $\mathcal{B}^*$, $\tilde{\mathcal{G}}$, and $\mathcal{G}$ respectively.

We recall the construction of local coordinate maps for the principal $\tilde{\mathcal{G}}$-bundle $\pi : \mathcal{A}^* \to \mathcal{B}^*$, $\omega \mapsto [\omega]$ [F-U, p. 48]. If ω is irreducible, then its holonomy group is $\Phi_\omega = G$, and $\Gamma_\omega = Z$. Then $\mathcal{G}/Z = \tilde{\mathcal{G}}$ acts freely on $\mathcal{A}^*$ and $\mathcal{B}^* = \mathcal{A}^*/\tilde{\mathcal{G}}$. Fix a connection $\omega \in \mathcal{A}^*$ and let $\mathcal{G}\cdot\omega$ denote the orbit of the gauge group through ω. We have smooth maps:

$$0 \longrightarrow \mathcal{G} \xrightarrow{\cdot\omega} \mathcal{G}\cdot\omega \xrightarrow{\subset} \mathcal{A}^*$$

Computing differentials gives:

$$0 \longrightarrow T_1\mathcal{G} \xrightarrow{-d_\omega} T_\omega(\mathcal{G}\cdot\omega) \xrightarrow{\subset} T_\omega\mathcal{A}^*$$

where $T_1\mathcal{G} = \Omega^0(M, \operatorname{ad} P)$ is the Lie algebra of $\mathcal{G}$ and $T_\omega\mathcal{A}^* = \Omega^1(M, \operatorname{ad} P)$. In particular,

$$T_\omega = (\mathcal{G}\cdot\omega) = \operatorname{Im}\{d_\omega : \Omega^0(M, \operatorname{ad} P) \to \Omega^1(M, \operatorname{ad} P)\},$$

which we denote by $\mathcal{V}_\omega = \operatorname{Im} d_\omega$, the *vertical subspace* through ω. Then $\operatorname{Ker}\{d^*_\omega : \Omega^1(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{ad} P)\}$ is the L^2-orthogonal complement of $\operatorname{Im} d_\omega$, which we denote by $\operatorname{Ker} d^*_\omega = \mathcal{H}_\omega$, the *horizontal subspace* through ω. This gives the L^2-orthogonal decomposition

$$T_\omega\mathcal{A}^* = \mathcal{H}_\omega \oplus \mathcal{V}_\omega$$

for each $\omega \in \mathcal{A}^*$. The corresponding horizontal and vertical projection operators on $T_\omega\mathcal{A}^*$ are:

$$h_\omega = 1 - d_\omega G^0_\omega d^*_\omega \qquad \text{and} \qquad v_\omega = d_\omega G^0_\omega d^*_\omega,$$

where G^0_ω is the Green's operator for the Laplacian $\Delta^0_\omega = d^*_\omega d_\omega$ on $\Omega^0(M, \mathrm{ad}\, P)$. There is an open neighbourhood $\mathcal{O}_\omega$ of ω in $\mathcal{A}^*$, and a diffeomorphism

$$\Phi : \mathcal{A}^* \supset \mathcal{O}_\omega \longrightarrow \mathcal{H}_\omega \times \mathcal{G}, \qquad \beta \longmapsto (\varphi(\beta), g_\beta)$$

onto an open neighbourhood of $(0, 1) \in \mathcal{H}_\omega \times \mathcal{G}$. The corresponding open neighbourhood $\mathcal{S}_\omega$ of $0 \in \mathcal{H}_\omega$ is a *slice* for the action of $\mathcal{G}$ on $\mathcal{A}^*$ [F-U, p. 49]. The map $g : \mathcal{O}_\omega \to \mathcal{G}$, $\beta \to g_\beta$ is obtained by solving

$$d^*_\omega(g^*_\beta(\beta) - \alpha) = 0,$$

so that $g^*_\beta(\beta) \in \omega + \mathcal{H}_\omega$ for $\omega \in \mathcal{O}_\omega$; the submersion $\varphi : \mathcal{O}_\omega \to \mathcal{S}_\omega$ is then given by $\beta \mapsto g^*_\beta(\beta) - \omega$. In particular, $\mathcal{O}_\omega$ may be chosen to be $\mathcal{G}$-invariant, and (Φ, g) is $\mathcal{G}$-equivariant, in the sense that

$$(\varphi(f^*\beta), g_{f^*\beta}) = (\varphi(\beta), f^{-1} \circ g_\beta) \qquad \text{for } f \in \mathcal{G}.$$

Hence, we get an induced diffeomorphism

$$\varphi : \mathcal{B}^* \supset \mathcal{O}_\omega / \tilde{\mathcal{G}} \longrightarrow \mathcal{S}_\omega \subset \mathcal{H}_\omega,$$

with inverse $\pi : \mathcal{S}_\omega \to \mathcal{O}_\omega/\tilde{\mathcal{G}}$, the restriction of the natural projection $\pi : \mathcal{A}^* \to \mathcal{B}^*$. In particular, $\mathcal{B}^*$ is a smooth Hilbert manifold, $\pi : \mathcal{A}^* \to \mathcal{B}^*$ is a principal $\tilde{\mathcal{G}}$-bundle, and the maps $\varphi : \pi(\mathcal{O}_\omega) \to \mathcal{S}_\omega$ provide a system of local coordinate charts on $\mathcal{B}^*$ [F-U, p. 50], [L, p. 33]. Computing differentials at $[\omega] \in \mathcal{B}^*$ gives an isomorphism $\varphi_* : T_{[\omega]}\mathcal{B}^* \to \mathcal{H}_\omega$, and this provides the standard identification of the tangent space $T_{[\omega]}\mathcal{B}^*$ with $\mathcal{H}_\omega$.

We consider the tangent space to $\mathcal{B}^*$ in more detail and compute the differential $\varphi_{*[\omega]} : T_{[\omega]}\mathcal{B}^* \to \mathcal{H}_\omega$. First, we examine the action of the gauge group on paths of connections and their tangent vectors. A calculation yields [I, p. 16]:

Lemma 1.2.1. *Let $\omega(t)$ be a smooth path of connections in $\mathcal{A}^*$ through $\omega(0) = \omega$, with $\dot\omega(0) \in T_\omega\mathcal{A}^*$. If $g(t)$ is a smooth path of gauge transformations in $\mathcal{G}$ through $g(0) = g$, we obtain a new path $\omega^g(t)$ in $\mathcal{A}^*$ through $\omega^g(0) = \omega^g$. Then*

$$\dot\omega^g(0) = \mathrm{Ad}\,(g^{-1})(\dot\omega(0)) + d_{\omega^g}\left(g^{-1}\dot g(0)\right) \in \Omega^1(M, \mathrm{ad}\, P).$$

Proof. Calculation. □

Lemma 1.2.2. *Let $g \in \mathcal{G}$, $\psi \in \Omega^q(M, \mathrm{ad}\, P)$ and $\omega \in \mathcal{A}^*$. Then*
(i) $g(d_\omega\psi) = d_{g(\omega)}g(\psi)$;
(ii) $g(d^*_\omega\psi) = d^*_{g(\omega)}g(\psi)$;
(iii) $g(\Delta_\omega\psi) = \Delta_{g(\omega)}g(\psi)$.

Proof. Calculation. □

(We recall that $\mathcal{G}$ acts on $\mathcal{A}^*$ by $\omega \mapsto g(\omega) = \omega^g = \mathrm{Ad}\,(g^{-1})\omega + g^{-1}dg$, and on $\Omega^q(M, \mathrm{ad}\, P)$, by $\psi \mapsto g(\psi) = \mathrm{Ad}\,(g^{-1})\psi$.)

Lemma 1.2.3. *Let $g \in \mathcal{G}$, $\alpha \in \Omega^1(M, \mathrm{ad}\, P)$ and $\omega \in \mathcal{A}^*$. Then*

$$h_{g(\omega)} g(\alpha) = g(h_\omega \alpha).$$

Proof. Calculation. □

Let $\omega(t)$ be a path in $\mathcal{A}^*$ through $\omega(0) = \omega$, with $\dot{\omega}(0) \in T_\omega \mathcal{A}^*$. Let $g(t)$ be a path in $\mathcal{G}$ through $g(0) = 1$, with $\dot{g}(0) = \phi$, for some $\in \Omega^0(M, \mathrm{ad}\, P)$. Then $\omega^g(t)$ is a path in $\mathcal{A}^*$ through $\omega^g(0) = \omega$, with tangent vector at ω given by

$$\dot{\omega}^g(0) = \dot{\omega}(0) + d_\omega \phi \ \in T_\omega \mathcal{A}^*.$$

Choosing the path $g(t)$ such that $d_\omega^*(\dot{\omega}^g(t) - \omega) = 0$, ensures $\dot{\omega}^g(t) - \omega \in \mathcal{H}_\omega$ (for small t), so $\varphi(\omega(t))$ is a path in the horizontal slice $\mathcal{S}_\omega$. Moreover,

$$\varphi_*(\dot{\omega}(0)) = \dot{\omega}^g(0) = \dot{\omega}(0) + d_\omega \phi \ \in \mathcal{H}_\omega.$$

Enforcing the horizontality condition, $d_\omega^*(\dot{\omega}^g(0)) = 0$, gives $\phi = -G_\omega^0 d_\omega^* \dot{\beta}(0)$. Hence,

$$\begin{aligned} \dot{\omega}^g(0) &= \dot{\omega}(0) - d_\omega G_\omega^0 d_\omega^* \dot{\omega}(0) \\ &= h_\omega \dot{\omega}(0) \ \in \mathcal{H}_\omega, \end{aligned}$$

and so $\dot{\omega}^g(0)$ is the horizontal projection of $\dot{\omega}(0)$.

Lemma 1.2.4. *The differential of $\varphi : \mathcal{O}_\omega \to \mathcal{S}_\omega$ at $\omega \in \mathcal{A}^*$ is given by the horizontal projection operator*

$$\varphi_{*_\omega} : T_\omega \mathcal{A}^* \longrightarrow \mathcal{H}_\omega, \qquad \alpha \longmapsto h_\omega \alpha.$$

Proof. This follows from the above calculations, since $\varphi_{*_\omega}(\alpha) = h_\omega \alpha$ where $\omega(t)$ is a path of conections with $\omega(0) = \omega$, $\alpha = \dot{\omega}(0)$ and $g(t)$ is a path of gauge transformations chosen as above so that $g(0) = 1$, $\dot{\omega}^g(0) = h_\omega \alpha$. □

Next, we observe that the paths $\omega^g(t)$ through $\omega^g(0) = \omega^g$, corresponding to different choices of $g(t)$, $g(0) = g$, all project to the same path $[\omega^g(t)]$ through $[\omega]$ in $\mathcal{B}^*$, for each $g \in \mathcal{G}$. We have exact sequences,

$$0 \longrightarrow T_1 \mathcal{G} \xrightarrow{-d_{\omega^g}} T_{\omega^g} \mathcal{A}^* \xrightarrow{\pi_*} T_{[\omega]} \mathcal{B}^* \longrightarrow 0,$$

so that the tangent space to $\mathcal{B}^*$ at $[\omega]$ is given by

$$\begin{aligned} T_{[\omega]} \mathcal{B}^* &= \frac{\Omega^1(M, \mathrm{ad}\, P)}{\mathrm{Im}\, \{d_\omega : \Omega^0(M, \mathrm{ad}\, P) \to \Omega^1(M, \mathrm{ad}\, P)\}} \\ &\simeq \mathrm{Ker}\, \{d_\omega^* : \Omega^1(M, \mathrm{ad}\, P) \to \Omega^0(M, \mathrm{ad}\, P)\} = \mathcal{H}_\omega. \end{aligned}$$

We have canonical isomorphisms $T_\omega \mathcal{A}^* / \mathrm{Im}\, d_\omega = T_{\omega^g} \mathcal{A}^* / \mathrm{Im}\, d_{\omega^g}$ and $\mathcal{H}_\omega \simeq \mathcal{H}_{\omega^g}$ for any $g \in \mathcal{G}$. The spaces $T_{[\omega]} \mathcal{B}^* = T_\omega \mathcal{A}^* / \mathrm{Im}\, d_\omega$ and $\mathcal{H}_\omega$ are identifed via the induced isomorphism:

$$\varphi_* : T_{[\omega]} \mathcal{B}^* \longrightarrow \mathcal{H}_\omega, \qquad [\alpha] \longmapsto h_\omega \alpha,$$

where $[\alpha] \in T_{[\omega]} \mathcal{B}^* = T_\omega \mathcal{A}^* / \mathrm{Im}\, d_\omega$.

Finally, we define the *self-dual moduli space* $\mathcal{M}$ by

$$\mathcal{M} = \{[\omega] \in \mathcal{B} : F_w^- \equiv 0\}.$$

We recall the

Proposition 1.2.5. [D-K, p. 138] *If ω is a self-dual G-connection over M, then a neighbourhood of $[\omega]$ in the moduli space $\mathcal{M}$ is modelled on a quotient $f^{-1}(0)/\Gamma_\omega$ where*

$$f : \mathrm{Ker}\, d_\omega^* \longrightarrow \mathrm{Coker} d_\omega^-$$

is a Γ_ω-equivariant map and

$$\begin{aligned}\mathrm{Ker}\, d_\omega^* &= \mathrm{Ker}\,\{d_\omega^* : \Omega^1(M, \mathrm{ad}\, P) \to \Omega^0(M, \mathrm{ad}\, P)\},\\ \mathrm{Coker} d_\omega^- &= \frac{\Omega_-^2(M, \mathrm{ad}\, P)}{\mathrm{Im}\,\{d_\omega^- : \Omega^1(M, \mathrm{ad}\, P) \to \Omega_-^2(M, \mathrm{ad}\, P)\}}.\end{aligned}$$

Here, $d_\omega^- = p_- d_\omega$ and p_- denotes the projection $\Omega^2(M, \mathrm{ad}\, P) \to \Omega_-^2(M, \mathrm{ad}\, P)$ onto anti-self-dual 2-forms, with $p_- = \frac{1}{2}(1 - *)$ in terms of the Hodge star operator.

Indeed, f is induced by the map $\psi(\alpha) = F_{\omega+\alpha}^- = d_\omega^- \alpha + (\alpha \wedge \alpha)^-$ for α in an open ball around $0 \in \mathrm{Ker}\, d_\omega^*$ [D-K, p. 134]. Hence, a neighbourhood of $[\omega]$ in $\mathcal{M}$ has a local model:

$$\{\alpha \in \Omega^1(M, \mathrm{ad}\, P) : d_\omega^* \alpha = 0 \text{ and } d_\omega^- \alpha + (\alpha \wedge \alpha)^- = 0\}/\Gamma_\omega$$

The first equation $d_\omega^* \alpha = 0$ defines the construction of a local slice through the $\mathcal{G}$-orbit; the second equation is the self-duality equation $F_{\omega+\alpha}^- = 0$. The self-duality condition ensures that $d_\omega^- \circ d_\omega = 0$ and so one obtains an *elliptic deformation complex*:

$$0 \longrightarrow \Omega^0(M, \mathrm{ad}\, P) \xrightarrow{d_\omega} \Omega^1(M, \mathrm{ad}\, P) \xrightarrow{d_\omega^-} \Omega_-^2(M, \mathrm{ad}\, P) \longrightarrow 0$$

with associated cohomology groups H_ω^0, H_ω^1, H_ω^2, where

$$\begin{aligned}H_\omega^1 &= \frac{\mathrm{Ker}\,\{d_\omega^- : \Omega^1(M, \mathrm{ad}\, P) \to \Omega_-^2(M, \mathrm{ad}\, P)\}}{\mathrm{Im}\,\{d_\omega : \Omega^0(M, \mathrm{ad}\, P) \to \Omega^1(M, \mathrm{ad}\, P)\}},\\ H_\omega^2 &= \frac{\Omega_-^2(M, \mathrm{ad}\, P)}{\mathrm{Im}\,\{d_\omega^- : \Omega^1(M, \mathrm{ad}\, P) \to \Omega_-^2(M, \mathrm{ad}\, P)\}},\end{aligned}$$

while H_ω^0 is the Lie algebra of Γ_ω. The negative Euler characteristic

$$s = -\dim H_\omega^0 + \dim H_\omega^1 - \dim H_\omega^2$$

gives the *virtual dimension* of the moduli space. By the Hodge theory for this complex we have natural isomorphisms:

$$\begin{aligned}H_\omega^1 &\simeq \mathbf{H}_\omega^1 = \{\alpha \in \Omega^1(M, \mathrm{ad}\, P) : d_\omega^- \alpha = 0 \text{ and } d_\omega^* \alpha = 0\},\\ H_\omega^2 &\simeq \mathbf{H}_\omega^2 = \{\alpha \in \Omega_-^2(M, \mathrm{ad}\, P) : d_\omega^- \alpha = 0\}.\end{aligned}$$

We recall that an irreducible self-dual connection is *regular* if $H_\omega^2 = 0$ and the moduli space $\mathcal{M}^*$ is regular if all its points are regular points [D-K, p. 146]. The moduli space of regular, irreducible connections $\mathcal{M}^{*\prime}$ is a smooth manifold of dimension s. For $G = \mathbf{SU}(2)$, the space $\mathcal{M}^*$ coincides with $\mathcal{M}^{*\prime}$ for generic metrics g on M, and so is a smooth manifold [F-U, p. 61], [D-K, p. 149]. Alternatively, under a certain curvature condition on (M, g), we have $H^2_\omega = 0$ and so the moduli space $\mathcal{M}^*$ will be a smooth manifold [A-H-S]. This curvature condition is satisfied by $\mathbb{S}^4$ with its standard round metric.

Indeed, a computation shows that the tangent space at any regular point $[\omega] \in \mathcal{M}^*$ is given by [G-P1, p. 670]:

$$T_{[\omega]}\mathcal{M}^* = H^1_\omega \simeq \mathbf{H}^1_\omega.$$

The description of $T_{[\omega]}\mathcal{M}^*$ as the cohomology group H^1_ω will be more useful for our purposes, due to the difficulty in finding a suitable basis for the harmonic space $\mathbf{H}^1_\omega$.

§1.3. L^2 Metric on Moduli Space

We describe the construction of the natural Riemannian metric on the quotient space $\mathcal{B}^*$ and the moduli space $\mathcal{M}^* \hookrightarrow \mathcal{B}^*$ of self-dual connections. General references for this section are [G-P1, 2], [I].

The affine space $\mathcal{A}^*$ has a (weak) Riemannian metric via the identification $T_\omega\mathcal{A}^* = \Omega^1(M, \operatorname{ad} P)$ and the L^2-inner product on $\Omega^1(M, \operatorname{ad} P)$:

$$(\alpha_1, \alpha_2) = \int_M \langle \alpha_1, \alpha_2 \rangle \sqrt{g}\, dx \qquad \text{for } \alpha_1, \alpha_2 \in \Omega^1(M, \operatorname{ad} P),$$

where $\langle\ ,\ \rangle$ denotes the fibre metric on $T^*M \otimes \operatorname{ad} P$. The gauge group $\mathcal{G}$ acts isometrically on $\mathcal{A}^*$, preserving the L^2-orthogonal splitting of each tangent space $T_\omega\mathcal{A}^*$. The orbit space $\mathcal{B}^*$ inherits a (weak) Riemannian metric by requiring that the natural projection $\pi : \mathcal{A}^* \to \mathcal{B}^*$ be a Riemannian submersion [C-E, p. 65]. Hence, the differential $\pi_{*\omega} : \mathcal{H}_\omega \to T_{[\omega]}\mathcal{B}^*$ is required to be an isometry, while $\pi_{*\omega}$ is zero on $\mathcal{V}_\omega$. The moduli space $\mathcal{M}^*$ is a smoothly embedded submanifold of $\mathcal{B}^*$, and so there is an induced smooth Riemannian metric on $\mathcal{M}^*$ [G-P1, p. 671].

In order to obtain a more explicit expression for the metric on $\mathcal{B}^*$, choose a representative $\omega \in [\omega]$ and observe that, by definition, the Hilbert space isomorphism

$$\varphi_{*[\omega]} = (\pi_{*\omega})^{-1} : T_{[\omega]}\mathcal{B}^* \longrightarrow \mathcal{H}_\omega \subset \Omega^1(M, \operatorname{ad} P)$$

is an isometry. Hence, $T_{[\omega]}\mathcal{B}^*$ acquires an inner product by pulling back the natural L^2 inner product on $\Omega^1(M, \operatorname{ad} P)$. Then, the L^2 metric on $\mathcal{B}^*$ is given explicitly by:

$$\begin{aligned} \mathbf{g}_{[\omega]}([\alpha_1], [\alpha_2]) &= \varphi^*_{[\omega]}([\alpha_1], [\alpha_2]) \\ &= (\varphi_{*[\omega]}[\alpha_1], \varphi_{*[\omega]}[\alpha_2]) \\ &= (h_\omega\alpha_1, h_\omega\alpha_2). \end{aligned}$$

Observe that for a given $[\omega] \in \mathcal{B}^*$, we may choose different coordinate charts $\varphi : \mathcal{B}^* \supset \pi(\mathcal{S}_{\omega^g}) \to \mathcal{S}_{\omega^g}$, where $\mathcal{S}_{\omega^g}$ are the horizontal slices through $\omega^g \in \mathcal{A}^*$ for each $g \in \mathcal{G}$, and the inner product apparently depends on this choice. However, we have the

Lemma 1.3.1. [I, p. 16] *The above expression for the L^2 inner product on $T_{[\omega]}\mathcal{B}^*$ is independent of the choice of slice neighbourhood, and so we obtain a well-defined (weak) Riemannian metric on $\mathcal{B}^*$.*

Proof. Let $\mathcal{S}_\omega$, $\mathcal{S}_{\omega^g}$ be two slice neighbourhoods with associated local coordinate charts

$$\varphi : \mathcal{B}^* \supset \pi(\mathcal{S}_\omega) \to \mathcal{S}_\omega \subset \mathcal{H}_\omega,$$
$$\varphi' : \mathcal{B}^* \supset \pi(\mathcal{S}_{\omega^g}) \to \mathcal{S}_{\omega^g} \subset \mathcal{H}_{\omega^g}.$$

Choose tangent vectors $[\alpha_i] \in T_{[\omega]}\mathcal{B}^*$, with $\alpha_i \in T_\omega \mathcal{A}^*$. Let $\omega_i(t)$ be paths of connections through $\omega_i(0) = \omega$, with tangent vectors $\dot{\omega}_i(0) = \alpha_i$, for $i = 1, 2$, and let $g(t)$ be any path of gauge transformations through $g(0) = g$. Then $\omega_i^g(t) = g(t)^*\omega_i(t)$ are paths through $\omega_i^g(0) = \omega^g$, and the tangent vectors at ω^g are given by

$$\dot{\omega}_i^g(0) = \mathrm{Ad}\,(g^{-1})\dot{\omega}_i(0) + d_{\omega^g}\left(g^{-1}\dot{g}(0)\right) \qquad \text{for } i = 1, 2.$$

Computing differentials, we get

$$\begin{aligned}
\varphi_{*[\omega]}[\alpha_i] &= h_\omega \dot{\omega}_i(0), \\
\varphi'_{*[\omega]}[\alpha_i] &= h_{\omega^g}\dot{\omega}_i^g(0) \\
&= \mathrm{Ad}\,(g^{-1}) h_\omega \dot{\omega}_i(0) \qquad \text{for } i = 1, 2.
\end{aligned}$$

Now computing inner products, we obtain

$$\begin{aligned}
(\varphi')^*_{[\omega]}([\alpha_1], [\alpha_2]) &= (\varphi'_{*[\omega]}[\alpha_1], \varphi'_{*[\omega]}[\alpha_2]) \\
&= (\mathrm{Ad}\,(g^{-1})h_\omega\alpha_1, \mathrm{Ad}\,(g^{-1})h_\omega\alpha_2) \\
&= (h_\omega\alpha_1, h_\omega\alpha_2), \\
&= \varphi^*_{[\omega]}([\alpha_1], [\alpha_2]),
\end{aligned}$$

making use of the Ad-invariance of the L^2 inner product on $\Omega^q(M, \mathrm{ad}\,P)$. Thus, $\mathbf{g}_{[\omega]}([\alpha_1], [\alpha_2])$ is independent of the choice of slice neighbourhood. □

Hence, the L^2 metric $\mathbf{g}$ as described above is well-defined. We restrict $\mathbf{g}$ to $\mathcal{M}^* \hookrightarrow \mathcal{B}^*$ and this defines a smooth Riemannian metric on the moduli space [G-P1, p. 671].

§1.4. Families of Connections

In our description of the moduli space of self-dual connections on a principal G-bundle P, we assumed that the bundle P was fixed. However, in our discussion of the ADHM construction, we will be required to consider the more general situation — familiar from the Kodaira-Spencer deformation theory [Ko] — of a family of connections defined on a family of principal G-bundles. We review this idea and describe the implications for the definition of the L^2 metric $\mathbf{g}$ on $\mathcal{M}^*$. General references for this section are [D-K], [Ko], [M], [A-J].

Definition 1.4.1. A *family of connections* in a principal G-bundle $P \to M$ parametrized by a topological space T is a G-bundle

$$\underline{P} \longrightarrow M \times T$$

with the property that each slice $P_t = \underline{P}|_{M \times \{t\}}$ is isomorphic to P, together with a connection ω_t in P_t for each t, forming a family of connections $\underline{\omega} = \{\omega_t : t \in T\}$ [D-K, p. 173 & p. 237], [Ko, p. 124 & p. 324]. We will generally assume that T is a smooth manifold.

By analogy with [Ko, p. 61 & p. 192], we say that two families $(\underline{P}, \underline{\omega}, T)$ and $(\underline{P}', \underline{\omega}', T)$ are *equivalent* if there exists a G-bundle map Φ so that the following diagram commutes,

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\Phi} & \underline{P}' \\ \downarrow & & \downarrow \\ M \times T & \longrightarrow & M \times T \end{array}$$

where $\omega_t = \Phi_t^* \omega_t'$ for all $t \in T$, and $\Phi_t = \Phi|_{P \times \{t\}}$.

Remark 1.4.2. It is not necessarily true that $\underline{P} \simeq P \times T$.

It is useful to consider framed families of connections. We recall the definition of a framed connection [D-K, p. 173]:

Definition 1.4.3. Let x_0 be a basepoint in M. A *framed connection* in a G-bundle $P \to M$ is a pair (ω, φ), where ω is a connection and φ is an isomorphism of G-spaces, $\varphi : G \to P_{x_0}$.

The gauge group $\mathcal{G}$ acts naturally on framed connections, and we denote the quotient by

$$\tilde{\mathcal{B}} = \left(\omega \times \mathrm{Hom}_G(G, P_{x_0})\right) / \mathcal{G}.$$

Alternatively, if we fix a framing ϕ and define the *group of based gauge transformations*,

$$\mathcal{G}_0 = \{g \in \mathcal{G} : g(x_0) = 1\},$$

then the isotropy groups $\Gamma_{0,\omega} = \{g \in \mathcal{G}_0 : g(\omega) = \omega\}$ are trivial for all $\omega \in \mathcal{A}$, $\mathcal{G}_0$ acts freely on $\mathcal{A}$, and $\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$. There is a natural map $\tilde{\mathcal{B}} \to \mathcal{B}$ which forgets the framing. Alternatively, $\mathcal{B}$ is the quotient by the residual of the gauge group:

$$\mathcal{B} = \mathcal{A}/\mathcal{G} \simeq (\mathcal{A}/\mathcal{G}_0) / (\mathcal{G}/\mathcal{G}_0) = \tilde{\mathcal{B}} / (\mathcal{G}/\mathcal{G}_0), \qquad \text{where } \mathcal{G}/\mathcal{G}_0 \simeq \mathrm{Aut}\,(P_{x_0}) \simeq G.$$

Restricting to irreducible connections, we recall that $\tilde{\mathcal{G}} = \mathcal{G}/\mathcal{Z}$ acts freely on $\mathcal{A}^*$. We have

$$\mathcal{B}^* = \mathcal{A}^*/\tilde{\mathcal{G}} \simeq \tilde{\mathcal{B}} / \left(\tilde{\mathcal{G}}/\mathcal{G}_0\right), \qquad \text{where } \tilde{\mathcal{G}}/\mathcal{G}_0 \simeq G/Z.$$

Then $\tilde{\mathcal{B}}^* \to \mathcal{B}^*$ is a principal G^{ad}-bundle, where $G^{\mathrm{ad}} = G/Z$. (We note that G^{ad} is isomorphic to the image of G under the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$.) For example, if $G = \mathbf{SU}(2)$, then $Z = \{\pm 1\}$, $G^{\mathrm{ad}} = \mathbf{SO}(3)$, and $\tilde{\mathcal{B}}^* \to \mathcal{B}^*$ is a principal $\mathbf{SO}(3)$-bundle.

We have a *framed family of connections* if there is an isomorphism:

$$\underline{\varphi} : \underline{P}|_{\{x_0\}\times T} \longrightarrow T \times G.$$

Then for each t, the pair (ω_t, φ_t) is a framed connection. We recall the construction of the universal family of framed connections parametrized by $\tilde{\mathcal{B}}$ [D-K, p. 175]. Let $\pi_1 : M \times \mathcal{A} \to M$ be the projection onto the first factor and let $\pi_1^* P \to M \times \mathcal{A}$ be the pull-back bundle, so $\pi_1^* P = P \times \mathcal{A}$. The G bundle $P \times \mathcal{A} \to M \times \mathcal{A}$ carries a tautological family of connections $\underline{\omega}$, in which the connection on $P_\omega = (P \times \mathcal{A})|_{M\times\{A\}}$ is $\pi_1^*\omega$. The group $\mathcal{G}_0$ acts freely on $M \times \mathcal{A}$ as well as on $P \times \mathcal{A}$, by $(p, \omega) \mapsto (pg, g(\omega))$ for $g \in \mathcal{G}_0$, and so there is a quotient bundle

$$\tilde{\mathbb{P}} \longrightarrow M \times \tilde{\mathcal{B}},$$
$$\text{where } \tilde{\mathbb{P}} = (P \times \mathcal{A})/\mathcal{G}_0 = P \times_{\mathcal{G}_0} \mathcal{A}.$$

The family of connections $\underline{\omega}$ and the framing φ are preserved by $\mathcal{G}_0$, so $\tilde{\mathbb{P}}$ carries an inherited family of connections $(\tilde{\mathbb{A}}, \underline{\varphi})$. This is the *universal framed family* in $P \to M$ parametrized by $\tilde{\mathcal{B}}$.

If $(\underline{\omega}, \underline{\varphi})$ is a framed family of connections parametrized by a space T and carried by a bundle $\underline{P} \to M \times T$, there is an associated map

$$f : T \longrightarrow \tilde{\mathcal{B}}, \qquad t \longmapsto [\omega_t, \varphi_t].$$

Conversely, given a map $f : T \to \tilde{\mathcal{B}}$, there is a corresponding pull-back family of connections carried by $(1_M \times f)^*\tilde{\mathbb{P}}$:

$$\begin{array}{ccc}
(1_M \times f)^*\tilde{\mathbb{P}} & \longrightarrow & \tilde{\mathbb{P}} \\
\big\downarrow & & \big\downarrow \\
M \times T & \xrightarrow{1_M \times f} & M \times \tilde{\mathcal{B}}
\end{array}$$

These two constructions are inverses of one another:

Lemma 1.4.4. [D-K, p. 175] *The maps $f : T \to \tilde{\mathcal{B}}$ are in one-to-one correspondence with framed families of connections on M parametrized by T, and this correspondence is obtained by pulling back from the universal framed family, $(\tilde{\mathbb{A}}, \tilde{\mathbb{P}}, \underline{\varphi})$.*

Lemma 1.4.5. [D-K, p. 175] *The homotopy classes of maps $[T, \tilde{\mathcal{B}}]$ parametrize isomorphism classes of pairs $(\underline{P}, \underline{\varphi})$, where*
(i) *$\underline{P} \to M \times T$ is a G-bundle with $P_t \simeq P$ for all t;*
(ii) *$\underline{\varphi} : \underline{P}|_{\{x_0\}\times T} \to G \times T$ is a trivialization.*

Restricting to irreducible connections on $P \to M$, we let $P \times \mathcal{A}^* \to M \times \mathcal{A}^*$ denote the pull-back G-bundle, carrying the tautological family of connections. Now taking quotients by $\mathcal{G}$, we obtain a bundle

$$\mathbb{P}^{\mathrm{ad}} \longrightarrow M \times \mathcal{B}^*,$$
$$\text{where } \mathbb{P}^{\mathrm{ad}} = (P \times \mathcal{A}^*)/\mathcal{G}.$$

Then $\mathbb{P}^{\mathrm{ad}} \to M \times \mathcal{B}^*$ is a principal G^{ad}-bundle. For example, if $G = \mathbf{SU}(2) = \mathbf{Spin}(3)$, then $\mathbb{P}^{\mathrm{ad}} \to M \times \mathcal{B}^*$ is an $\mathbf{SO}(3)$-bundle carrying a family of connections (without framing) for the $\mathbf{SO}(3)$-bundle $P/\{\pm 1\} \to M$ parametrized by $\mathcal{B}^*$. This lifts to an $\mathbf{SU}(2)$-family if and only if the second Stiefel-Whitney class $w_2(\mathbb{P}^{ad})$ in $H^2(M \times \mathcal{B}^*, \mathbb{Z}_2)$ is zero.

Example 1.4.6. Suppose $\underline{\omega}$ is a family of G-connections in a G bundle $P \to M$ carried by a G-bundle $\underline{P} \to M \times T$, where T is a contractible space. Then the identity map $1_T : T \to T$ is homotopic to a constant map $t_0 : T \to \{t_0\} \subset T$, and we obtain the pull-back bundle $(1_M \times t_0)^* \underline{P} = P \times T$, where $P = P_{t_0}$:

$$\begin{array}{ccc} P \times T & \longrightarrow & \underline{P} \\ \downarrow & & \downarrow \\ M \times T & \xrightarrow{1_M \times t_0} & M \times T \end{array}$$

Pulling back by the identity map $1_M \times 1_T$ gives $(1_M \times 1_T)^* \underline{P} = \underline{P}$, and since $1_M \times 1_T$ is homotopic to $1_M \times t_0$, then $\underline{P} \simeq P \times T$ as G bundles [St, p. 49].

A more direct argument that $\underline{P} \simeq P \times T$ when T is contractible, may be given by analogy with [Ko, p. 66 & p. 327]. The essential point is that when T is contractible, we may assume that connections ω_t are connections on a fixed bundle $P \to M$. Indeed, if we denote the above bundle isomorphism by $\Phi : P \times T \to \underline{P}$, then we have isomorphisms $\Phi(\cdot, t) : P \to P_t$, and we may replace the family $\{\omega_t : t \in T\}$ by the equivalent family $\{\Phi(\cdot, t)^* \omega_t : t \in T\}$ on the fixed bundle P.

Remark 1.4.7. If $T \hookrightarrow \mathcal{B}^*$ is contractible and M is spin, then $\mathbb{P}^{\mathrm{ad}}|_{M \times T}$ lifts to an $\mathbf{SU}(2)$-bundle $\mathbb{P}|_{M \times T}$, since $w_2(P)[M]$ will be zero in $H^2(M, \mathbb{Z}_2)$.

Example 1.4.8. Suppose T is the moduli space $\mathcal{M}_k^*(\mathbb{S}^4, \mathbf{SU}(2))$, or the framed moduli space $\tilde{\mathcal{M}}_k^*(\mathbb{S}^4, \mathbf{SU}(2))$, of self-dual connections on an $\mathbf{SU}(2)$-bundle $P \to \mathbb{S}^4$ with topology fixed by $-c_2(P)[M] = k$. When $k = 1$, we have $\mathcal{M}_1^*(\mathbb{S}^4, \mathbf{SU}(2)) \simeq \mathbb{B}^5$, and so T is contractible, with $\underline{P} \simeq P \times T$. The family of connections parametrized by T are connections on a fixed $\mathbf{SU}(2)$-bundle defined by the quaternionic Hopf fibration $1 \to \mathbf{Sp}(1) \to \mathbb{S}^7 \to \mathbb{HP}^1 \to 1$. However, for $k > 1$, we have the fundamental groups [Hur]:

$$\pi_1\left(\tilde{\mathcal{M}}_k^*(\mathbb{S}^4, \mathbf{SU}(2))\right) = \mathbb{Z}_2 \quad \text{for all } k;$$

$$\pi_1\left(\mathcal{M}_k^*(\mathbb{S}^4, \mathbf{SU}(2))\right) = \begin{cases} 0 & \text{if } k \text{ odd;} \\ \mathbb{Z}_2 & \text{if } k \text{ even.} \end{cases}$$

Hence, at least for k even, we should not expect $\underline{P}$ to be isomorphic to $P \times T$. For $k = 2$, we have that $\pi_1\left(\mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))\right) = \mathbb{Z}_2$ and moreover, that this moduli space is homotopy equivalent to the Grassmann manifold $G_2(\mathbb{R}^5)$ [S-T, p. 342], [Au]. Hence, we should not expect that the family of connections parametrized by $T = \mathcal{M}_2^*(\mathbb{S}^4, \mathbf{SU}(2))$ are carried by a bundle $\underline{P} = P \times T$.

§1.5. Infinitesimal Deformations and Horizontal Projections

We now consider infinitesimal deformations of a family of irreducible connections $\omega(t) = \omega_t$ on $P(t) = P_t \to M$, where t varies in a parameter space T. The topology of a principal G-bundle on M, for simple, simply-connected Lie groups G, is fixed by specifying $-c_2(P)[M] = k$, so the G-bundles $P(t)$ are all isomorphic to some fixed G-bundle P [T1, p. 168].

For any two isomorphisms $f_t : P \to P_t$, $h_t : P \to P_t$, we have $h_t^{-1} \circ f_t \in \mathcal{G}(P)$, and consequently, the quotient spaces $\mathcal{B}^*(P)$, $\mathcal{B}^*(P_t)$ may be canonically identified [T2, p. 528], [T3, p. 344]. Suppose $I \hookrightarrow T$ is a small interval around $0 \in \mathbb{R}$. We may choose a smooth family of isomorphisms $f(t) = f_t : P \to P(t)$, $t \in I$, where $P(0) = P$, $f(0) = 1_P$, the identity automorphism of P, and pull back the connection $\omega(t)$ on $P(t) \to M$ to give a connection $\omega^f(t) = f(t)^*\omega(t)$ on the fixed G-bundle $P \to M$. Then $\omega^f(t)$ is a family of connections in the fixed bundle $P \to M$ parametrized by the interval I, with $\omega^f(0) = \omega = \omega(0)$. We may then, as usual, compute the derivative $\dot{\omega}^f(0) \in T_\omega \mathcal{A}^*(P) = \Omega^1(M, \operatorname{ad} P)$.

Lemma 1.5.1. *$h_\omega \dot{\omega}^f(0)$ is independent of the path of isomorphisms $f(t)$, and so $\dot{\omega}^f(0)$ defines an element in $T_{[\omega]}\mathcal{B}^*(P) \simeq \mathcal{H}_\omega$ independent of f.*

Proof. If $k(t)$ is another choice, then $\omega^f(t)$ is gauge equivalent to $\omega^k(t)$, using the gauge transformation $k_t^{-1} \circ f_t \in \mathcal{G}(P)$, and so $\dot{\omega}^f(0)$, $\dot{\omega}^k(0)$ define the same element in the tangent space $T_{[\omega]}\mathcal{B}^*(P) \simeq \mathcal{H}_\omega$. Hence, $h_\omega \dot{\omega}^f(0)$ is independent of f. □

We consider some of the consequences for local calculations. The isomorphism $f(t) : P(0) \to P(t)$ is represented locally by $f_t(\sigma_\alpha(0)) = \sigma_\alpha(t) g_\alpha(t)$, where we choose $\{O_\alpha\}$ to be a fixed open cover of M, the corresponding local sections are $\sigma_\alpha(t) \in \Gamma(O_\alpha, P_t)$, and the mapping transformations $g_\alpha(t) : O_\alpha \to G$ relate the transition functions of P and P_t by:

$$T_{\alpha\beta}(t) = g_\alpha(t)^{-1} T_{\alpha\beta}(0) g_\beta(t) \qquad \text{on } O_\alpha \cap O_\beta.$$

A calculation shows that

$$\omega_\alpha^f(t) = \operatorname{Ad}(g_\alpha(t)^{-1})\omega_\alpha(t) + g_\alpha(t)^{-1} dg_\alpha(t) \ \in \Omega^1(O_\alpha, \mathfrak{g}),$$

relating the local connection 1-forms of $\omega^f(t) \in \mathcal{A}^*(P)$ and $\omega(t) \in \mathcal{A}^*(P_t)$. Computing derivatives, we get:

$$\dot{\omega}_\alpha^f(0) = \operatorname{Ad}\left(g_\alpha(0)^{-1}\right) \dot{\omega}_\alpha(0) + d_{\omega^f(0)}\left(g_\alpha(0)^{-1} \dot{g}_\alpha(0)\right) \ \in \Omega^1(O_\alpha, \mathfrak{g}).$$

Since $f(0) = 1_P$, we have $g_\alpha(0) = 1$, and so

$$\dot{\omega}_\alpha^f(0) = \dot{\omega}_\alpha(0) + d_\omega\left(\dot{g}_\alpha(0)\right) \ \in \Omega^1(O_\alpha, \mathfrak{g}).$$

We cannot immediately take horizontal projections, since $\dot{\omega}_\alpha(0)$, $\dot{g}_\alpha(0)$ do not transform as global sections of $\Omega^1(M, \operatorname{ad} P)$ and $\Omega^0(M, \operatorname{ad} P)$, respectively. Unlike the cases considered earlier — where $f \in \mathrm{Aut}\,(P)$ was a gauge transformation — we now have $f_t \in \mathrm{Iso}(P, P_t)$, and consequently the local maps

$$\dot{g}_\alpha(0) : O_\alpha \to \mathfrak{g}$$

do not necessarily transform correctly to give a global section ϕ in $\Omega^0(M, \mathrm{ad}\, P)$. Indeed, using $g_\beta(t) = T_{\alpha\beta}(0)^{-1} g_\alpha(t) T_{\alpha\beta}(t)$, we may compute $\dot{g}_\beta(0)$:

$$\dot{g}_\beta(0) = \mathrm{Ad}\,(T_{\alpha\beta}(0)^{-1})\dot{g}_\alpha(0) + T_{\alpha\beta}(0)^{-1}\dot{T}_{\alpha\beta}(0).$$

A similar calculation shows that the local 1-forms $\dot{\omega}_\alpha(0)$, do not necessarily transform correctly to give a global section of $\Omega^1(M, \mathrm{ad}\, P)$.

In order to take care of this difficulty, we consider the calculation of horizontal projections in more detail. Suppose we have a local map $\psi_\alpha \in \Omega^q(O'_\alpha, \mathfrak{g})$, where O'_α is a coordinate patch in M and $P|O'_\alpha \simeq O'_\alpha \times G$. If $O_\alpha \Subset O'_\alpha$, we obtain a smooth map by restricting ψ_α to the closed set $\overline{O}_\alpha$. Lifting this restricted map to an element in $\Omega^q(\overline{O}_\alpha, \mathrm{ad}\, P)$ via the trivialization, the local section extends to a global section, denoted $\psi^{\mathrm{ext},\alpha}$ in $\Omega^q(M, \mathrm{ad}\, P)$ [K-N1, p. 58], [St, p. 55].

Lemma 1.5.2. *Let $\{O_\alpha\}$ be a refinement of the cover $\{O'_\alpha\}$ of M, with $O_\alpha \Subset O'_\alpha$ and P locally trivial over O'_α. For each patch O'_α, let $\dot{g}^{\mathrm{ext},\alpha}(0)$ be a global extension of $\dot{g}_\alpha(0)$ on $\overline{O}_\alpha$ to a section in $\Omega^0(M, \mathrm{ad}\, P)$. Define corresponding global sections $\dot{\omega}^{\mathrm{ext},\alpha}(0)$ by setting*

$$\dot{\omega}^{\mathrm{ext},\alpha}(0) = \dot{\omega}^f(0) - d_\omega \dot{g}^{\mathrm{ext},\alpha}(0) \ \in \Omega^1(M, \mathrm{ad}\, P).$$

Then:
(i) *Locally, $\dot{\omega}_\alpha^{\mathrm{ext},\alpha}(0) = \dot{\omega}_\alpha(0)$ on O_α;*
(ii) *Globally,*

$$h_\omega \dot{\omega}^f(0) = h_\omega \dot{\omega}^{\mathrm{ext},\alpha}(0) \qquad \text{for all } \alpha.$$

Proof. By definition, $\dot{g}_\alpha^{\mathrm{ext},\alpha}(0) = \dot{g}_\alpha(0)$ on O_α, and so

$$\begin{aligned} \dot{\omega}_\alpha^{\mathrm{ext},\alpha}(0) &= \dot{\omega}_\alpha^f(0) - d_\omega \dot{g}_\alpha(0) \\ &= \dot{\omega}_\alpha^f(0) \qquad \text{on } O_\alpha, \end{aligned}$$

since $\dot{\omega}^\alpha(0) = \dot{\omega}_\alpha^f(0) - \dot{g}^\alpha(0)$ on O_α and this gives (i). Moreover,

$$\dot{\omega}^f(0) = \dot{\omega}^{\mathrm{ext},\alpha}(0) + d_\omega \dot{g}^{\mathrm{ext},\alpha}(0) \ \in \Omega^1(M, \mathrm{ad}\, P),$$

and taking horizontal projections gives (ii). □

We obtain the following useful consequence:

Lemma 1.5.3. *Let the coverings* $\{O_\alpha\}$ *and* $\{O'_\alpha\}$ *be as in the previous lemma. Then, we have the estimate:*

$$\|h_\omega \dot{\omega}^f(0)\|_{L^2(\Omega^1(O_\alpha, \mathrm{ad}\, P))} \leq \|\dot{\omega}(0)\|_{L^2(\Omega^1(O_\alpha, \mathrm{ad}\, P))}.$$

Proof. Observe that $h_\omega \dot{\omega}^f(0) = h_\omega \dot{\omega}^{\mathrm{ext},\alpha}(0)$ by the previous lemma, and so

$$\begin{aligned} \|h_\omega \dot{\omega}^f(0)\|_{L^2(\Omega^1(O_\alpha, \mathrm{ad}\, P))} &= \|h_\omega \dot{\omega}^{\mathrm{ext},\alpha}(0)\|_{L^2(\Omega^1(O_\alpha, \mathrm{ad}\, P))} \\ &\leq \|\dot{\omega}^{\mathrm{ext},\alpha}(0)\|_{L^2(\Omega^1(O_\alpha, \mathrm{ad}\, P))} \\ &= \|\dot{\omega}_\alpha(0)\|_{L^2(\Omega^1(O_\alpha, \mathfrak{g}))}, \end{aligned}$$

and this gives the desired inequality. □

Remark 1.5.4. The essential point is that we can estimate $h_\omega \dot{\omega}^f(0)$ without any explicit knowledge of the 1-parameter family of isomorphisms $f(t)$.

§1.6. Classifying Maps and Canonical Connections

We review some examples of classifying spaces for G bundles and canonical connections which we will later use in our application of the ADHM construction. General references for this section are [F-U], [Hu], [M-S], and [K-N].

If G is a Lie group, we let BG denote the corresponding classifying space and let EG denote the total space of the universal G-bundle $1 \to G \to EG \to BG \to 1$. For $G = \mathbf{SU}(2) = \mathbf{Sp}(1)$, we have $BSU(2) = \mathbb{HP}^\infty$ and $ESU(2) = \mathbb{S}^\infty$, and if $f : M \to BSU(2)$ is a map, then we obtain a principal $\mathbf{SU}(2)$-bundle on M by pull-back:

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{S}^\infty \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & \mathbb{HP}^\infty \end{array}$$

In particular, all G-bundles $P \to M$ arise this way and two classifying maps are homotopic if and only if they induce isomorphic bundles [F-U], [Hu].

We review the construction of the canonical $\mathbf{SU}(2)$-connection γ_0 on the $\mathbf{SU}(2)$ bundle

$$\mathbb{S}^{4k+3} \longrightarrow \mathbb{HP}^k,$$

where $\mathbb{S}^{4k+3} = \{q \in \mathbb{H}^{k+1} : |q| = 1\}$ [K-N2, p. 6]. We consider $\mathbb{H}^{k+1}$ as a right vector space over $\mathbb{H}$ and define the right quaternionic projective space $\mathbb{HP}^k$ as $\{(q_0, \ldots, q_k) \in \mathbb{H}^{k+1} \setminus \{0\} : (q_0, \ldots, q_k) \sim (q_0 q, \ldots, q_k q),\ q \in \mathbb{H}^*\}$, where $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$ and $k \geq 1$. The symplectic group $\mathbf{Sp}(k)$ may be defined by $\{Q \in \mathbf{GL}(k, \mathbb{H}) : (Qp)^\dagger (Qq) = p^\dagger q\}$, where $p^\dagger q = \sum_{i=0}^k \overline{p}_i q_i$ denotes the standard symplectic scalar product on $\mathbb{H}^k$, for any $k \geq 1$. Then, $\mathbb{HP}^k$ and $\mathbb{S}^{4k+3}$ may be viewed as homogeneous spaces:

$$\begin{aligned} \mathbb{HP}^k &= \mathbf{Sp}(k+1)/\mathbf{Sp}(k) \times \mathbf{Sp}(1), \\ \mathbb{S}^{4k+3} &= \mathbf{Sp}(k+1)/\mathbf{Sp}(k). \end{aligned}$$

We have the following principal bundles over $\mathbb{HP}^k$:
(i) An $\mathbf{Sp}(k) \times \mathbf{Sp}(1)$-bundle $\mathbf{Sp}(k+1) \to \mathbb{HP}^k$, denoted by Q;
(ii) An $\mathbf{Sp}(1)$-bundle $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$, denoted by Q_0.

The bundle $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$ is the quaternionic Hopf fibration, with $\mathbf{Sp}(1)$ acting on $\mathbb{S}^{4k+3}$ by right scalar multiplication.

Let Θ be the Maurer-Cartan form of $\mathbf{Sp}(k+1)$ with values in $\mathfrak{sp}(k+1)$ [K-N1, p. 41]. Let γ be the $\mathfrak{sp}(k)+\mathfrak{sp}(1)$ component of Θ with respect to the decomposition

$$\mathfrak{sp}(k+1) = \mathfrak{sp}(k) + \mathfrak{sp}(1) + \mathfrak{g}(k,1),$$

where $\mathfrak{g}(k,1)$ is the orthogonal complement of $\mathfrak{sp}(k)+\mathfrak{sp}(1)$ in $\mathfrak{sp}(k+1)$ with respect to the Cartan-Killing form of $\mathbf{Sp}(k+1)$. Then, γ defines an $\mathbf{Sp}(k)+\mathbf{Sp}(1)$ connection on $\mathbf{Sp}(k+1) \to \mathbb{HP}^k$ by [K-N1, p. 103], and γ is the *canonical connection* for this bundle.

Let $\pi_0 : \mathbf{Sp}(k+1) \to \mathbf{Sp}(k+1)/\mathbf{Sp}(k) = \mathbb{S}^{4k+3}$ denote the natural projection. Then π_0 defines a bundle map $Q \to Q_0$, and by [K-N1, p. 79], there is a unique connection γ_0 on $Q_0 \to \mathbb{HP}^k$ such that $\gamma = (\pi_0)^*\gamma_0$, and γ_0 is the *canonical* $\mathbf{Sp}(1)$-*connection* for this bundle. With respect to the coordinates $q_0, \ldots, q_k$ on $\mathbb{H}^{k+1}$, we have

$$\gamma_0 = \sum_{i=0}^{k} \overline{q}_i dq_i = q^\dagger dq \ \in \Omega^1(\mathbb{S}^{4k+3}, \mathfrak{sp}(1)),$$

where $q^\dagger$ denotes the quaternionic conjugate transpose of q. If $E_0 \to \mathbb{HP}^k$ is the universal quaternionic line bundle, then Q_0 is the associated principal $\mathbf{Sp}(1)$-bundle, and so E_0 inherits a covariant derivative ∇^{γ_0}. This covariant derivative may be obtained by orthogonal projection of the standard flat connection on the trivial bundle $\mathbb{HP}^k \times \mathbb{H}^{k+1} \to \mathbb{HP}^k$.

Finally, if $f : M \to \mathbb{HP}^k$ is a classifying map inducing an $\mathbf{Sp}(1)$-bundle $P \to M$ by pull-back,

$$\begin{array}{ccc} P & \xrightarrow{f'} & \mathbb{S}^{4k+3} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & \mathbb{HP}^k \end{array}$$

then P inherits a connection $\omega = (f')^*\gamma_0$, by pulling back the canonical $\mathbf{Sp}(1)$-connection on $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$. The ADHM method constructs the classifying maps f. Note that the total space P is

$$P = \{([x,y],p) \in \mathbb{H}P^1 \times \mathbb{S}^{4k+3} : f(x,y) = \pi(p)\},$$

and if $\pi_2 : \mathbb{H}P^1 \times \mathbb{S}^{4k+3} \to \mathbb{S}^{4k+3}$ denotes the projection onto the second factor, then the induced map f' on total spaces is just $f' = \pi_2|_P : P \to \mathbb{S}^{4k+3}$. Then

$$\omega = (f')^*\gamma_0 = (f')^\dagger df' \ \in \Omega^1(P, \mathfrak{g})$$

is the corresponding $\mathbf{SU}(2)$-connection 1-form on P.

CHAPTER II
STABLE VECTOR BUNDLES OF RANK TWO

It will be useful — especially in view of our later discussion of the moduli space $\mathcal{M}_2$ of self-dual $\mathbf{SU}(2)$-connections with $k = 2$ over $\mathbb{S}^4$ — to consider the correspondence between anti-self-dual (ASD) connections and holomorphic vector bundles. This correspondence was employed by Hartshorne to discuss certain families of ASD $\mathbf{SU}(2)$-connections discovered by Jackiw, Nohl, and Rebbi using techniques of algebraic geometry [Har2], [J-N-R]. General references for this chapter are [A], [A-H-S], [A-W], [Har2, 3], [O-S-S], [Wa-We].

§2.1. Atiyah-Ward Correspondence

The Atiyah-Ward correspondence which gives a bijection between the moduli space of ASD $\mathbf{SU}(2)$-connections with second Chern number k over $\mathbb{S}^4$ (endowed with its standard round metric g_0) and the moduli space of *instanton bundles* over $\mathbb{CP}^3$, which are rank 2 holomorphic vector bundles over $\mathbb{CP}^3$ satisfying certain technical conditions. We summarise the main features of this correspondence. General references for this section are [A], [A-H-S], [A-W], [Har2], [Wa-We].

Let $\mathbb{H}$ denote the space of quaternions, with basis $1, i, j, k$ over $\mathbb{R}$, and let $\mathbb{HP}^1 = \mathbb{S}^4$ denote the (left) quaternionic projective space. As left complex vector spaces, we identify $\mathbb{C}^2$ with $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ by $(z_0, z_1) \mapsto z_0 + z_1 j$. More generally, we have the following isomorphisms of left complex vector spaces, $\mathbb{C}^{2n} \to \mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n j$, given by

$$(z_0, z_1, \ldots, z_{2n-2}, z_{2n-1}) \longmapsto (z_0 + z_1 j, \ldots, z_{2n-2} + z_{2n-1} j).$$

In particular, we have a fibre bundle,

$$\pi : \mathbb{CP}^3 \longrightarrow \mathbb{HP}^1, \qquad [z_0, z_1, z_2, z_3] \longmapsto [z_0 + z_1 j, z_2 + z_3 j]$$

with fibre $\mathbb{CP}^1$.

Definition 2.1.1. Let X be a complex algebraic variety. A *real structure* on X is a conjugate-linear map $\sigma_X : X \to X$ with $\sigma_X^2 = 1_X$. The fixed points (if any) of σ_X are the *real points of* X with respect to the real structure σ_X. If there is no ambiguity, we often denote σ_X simply by σ.

Left multiplication by j on $\mathbb{H}^2$ induces a conjugate-linear map,

$$\sigma : \mathbb{C}^4 \longrightarrow \mathbb{C}^4, \qquad (z_0, z_1, z_2, z_3) \longmapsto (-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2),$$

and this induces a map $\sigma : \mathbb{CP}^3 \to \mathbb{CP}^3$. Note that $\sigma^2 = -1_{\mathbb{C}^4}$ on $\mathbb{C}^4$, and so $\mathbb{C}^4$ and $\mathbb{H}^2$ are identified as quaternionic vector spaces. We have $\sigma^2 = 1_{\mathbb{CP}^3}$ on $\mathbb{CP}^3$ and hence σ defines a real structure on $\mathbb{CP}^3$, preserving the fibration $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$. The map $\sigma : \mathbb{CP}^3 \to \mathbb{CP}^3$ has no fixed points, but the fixed lines of σ are precisely the fibres $\pi^{-1}(x) \simeq \mathbb{CP}^1$, $x \in \mathbb{HP}^1$. These are the *real lines* of $\mathbb{CP}^3$ with respect to the real structure σ.

Definition 2.1.2. Let X be a complex algebraic variety with a real structure σ_X and let $\mathcal{E}$ be a vector bundle over X. An anti-linear map $\sigma_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ covering $\sigma_X : X \to X$, or equivalently a bundle isomrphism $\sigma_{\mathcal{E}} : \mathcal{E} \to \sigma^* \overline{\mathcal{E}}$, is a *real structure* if $\sigma_{\mathcal{E}}^2 = 1$ and a *symplectic structure* if $\sigma_{\mathcal{E}}^2 = -1$.

Two real or symplectic structures $\sigma_{\mathcal{E}}$ and $\sigma'_{\mathcal{E}}$ are *equivalent* if $\sigma'_{\mathcal{E}} = \lambda \sigma_{\mathcal{E}}$, for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. If there is no ambiguity, we often denote $\sigma_{\mathcal{E}}$ simply by σ. We recall the fundamental

Theorem 2.1.3. (Atiyah-Ward) *Let $E \to \mathbb{HP}^1$ denote a smooth, complex, rank 2 vector bundle with Hermitian metric h, and let g_0 denote the standard metric on $\mathbb{HP}^1$. There is a natural bijection between*

(i) *Isomorphism classes of g_0-ASD $\mathbf{SU}(2)$-connections on $E \to \mathbb{HP}^1$, with $c_2(E) = k > 0$; and*

(ii) *Isomorphism classes of rank 2 holomorphic vector bundles $\mathcal{E} \to \mathbb{CP}^3$, with $c_2(\mathcal{E}) = k > 0$, such that:*

(a) *$\mathcal{E}$ has a holomorphic bundle isomorphism $b : \mathcal{E} \to \mathcal{E}^*$ covering the identity $1_{\mathbb{CP}^3}$ and defining a symplectic form $b(\ ,\ )$ on $\mathcal{E}$;*

(b) *$\mathcal{E}$ has a conjugate-linear isomorphism $\sigma_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$, covering $\sigma : \mathbb{CP}^3 \to \mathbb{CP}^3$, such that $\sigma_{\mathcal{E}}^2 = -1_{\mathcal{E}}$ and $\sigma_{\mathcal{E}}$ is compatible with the symplectic form, in the sense that $b(\sigma\xi, \sigma\eta) = \overline{b(\xi, \eta)}$;*

(c) *$\mathcal{E}$ is holomorphically trivial when restricted to real lines $L \subset \mathbb{CP}^3$ and the Hermitian form h on $\mathcal{O}(\mathcal{E}|L)$, defined by $h(\xi, \eta) = b(\xi, \sigma\eta)$ for $\xi \in \mathcal{E}_z$, $\eta \in \mathcal{E}_{\sigma z}$, is positive definite.*

Proof. See [A-W, p. 119], [A, p. 49], [A-H-S, p. 441], or [Wa-We, p. 390]. □

The rank 2 holomorphic vector bundles $\mathcal{E}$ on $\mathbb{CP}^3$ arising from ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ as above are known as *instanton bundles*. We describe briefly how one passes from ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ to the associated instanton bundle $\mathcal{E}$ on $\mathbb{CP}^3$ via the Atiyah-Ward correspondence. If ω is an ASD $\mathbf{SU}(2)$-connection on E, let $\mathcal{E} = \pi^* E$ with pull-back connection $\tilde{w} = \pi^* w$. Let $\Omega^p(\mathbb{CP}^3, \mathcal{E})$ be the space of smooth p-forms on $\mathbb{CP}^3$ with values in $\mathcal{E}$ and let $\Omega^{(p,q)}(\mathcal{E})$ be the space of smooth (p,q)-forms on $\mathbb{CP}^3$ with values in $\mathcal{E}$. Then we have the associated covariant derivative

$$d_{\tilde{\omega}} : \Omega^0(\mathbb{CP}^3, \mathcal{E}) \longrightarrow \Omega^1(\mathbb{CP}^3, \mathcal{E}),$$

and using the splitting

$$\Omega^1(\mathbb{CP}^3, \mathcal{E}) = \Omega^{(1,0)}(\mathbb{CP}^3, \mathcal{E}) \oplus \Omega^{(0,1)}(\mathbb{CP}^3, \mathcal{E}),$$

we let $d_{\tilde{\omega}} = \partial_{\tilde{\omega}} + \overline{\partial}_{\tilde{\omega}}$. Then we have

$$\overline{\partial}_{\tilde{\omega}} : \Omega^0(\mathbb{CP}^3, \mathcal{E}) \longrightarrow \Omega^{(0,1)}(\mathbb{CP}^3, \mathcal{E}),$$

and the connection ω is ASD if and only if the curvature $F_{\tilde{\omega}}$ of the pullback connection is a (1,1)-form, so that $F_{\tilde{\omega}} \in \Omega^{(1,1)}(\mathbb{CP}^3, \mathrm{End}(\mathcal{E}))$ and hence, $\overline{\partial}_{\tilde{\omega}}$ defines a holomorphic structure on $\mathcal{E}$ [A, p. 48], [A-H-S, p. 441].

The following slight reformulation of the Atiyah-Ward correspondence shows that the identfication of the moduli space of ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ may be viewed as a problem in algebraic geometry:

Theorem 2.1.4. (Atiyah-Ward) *Let $E \to \mathbb{HP}^1$ denote a smooth, complex, rank 2, Hermitian vector bundle with Hermitian metric h, and let g_0 denote the standard metric on $\mathbb{S}^4$. There is a natural bijection between*

(i) *Isomorphism classes of g_0-ASD $\mathbf{SU}(2)$-connections on $E \to \mathbb{HP}^1$ with $c_2(E) = k > 0$; and*

(ii) *Isomorphism classes of holomorphic rank 2 vector bundles $\mathcal{E} \to \mathbb{CP}^3$ together with a map $\sigma_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ (up to multiplication by $\lambda \in \mathbb{C}^*$, $|\lambda| = 1$), such that:*
   (a) *$c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k > 0$;*
   (b) *$\mathcal{E}$ is stable;*
   (c) *For each $x \in \mathbb{HP}^1$, $\mathcal{E}|\pi^{-1}(x)$ is holomorphically trivial — so $\mathcal{E}$ has no real jumping lines;*
   (d) *The map $\sigma_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is conjugate-linear isomorphism, covering $\sigma : \mathbb{CP}^3 \to \mathbb{CP}^3$, such that $\sigma_{\mathcal{E}}^2 = -1$, and so $\mathcal{E}$ has a symplectic structure.*

Proof. [Har2, p. 3], [A, p. 51], [A-W, p. 119]. □

Definition 2.1.5. We have the following moduli spaces:

(i) The moduli space $\mathcal{N}(0, k)$ of stable holomorphic rank 2 vector bundles over $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$;

(ii) The moduli space $\mathcal{N}_{\mathbb{I}}(0, k)$ of *instanton bundles* over $\mathbb{CP}^3$ satisfying the conditions of the Atiyah-Ward correspondence;

(iii) The moduli space $\mathcal{M}_k$ of g_0-ASD $\mathbf{SU}(2)$-connections on $E \to \mathbb{S}^4$ with $c_2(E) = k$.

The problem of identifying the moduli space $\mathcal{N}_{\mathbb{I}}(0, k)$ of instanton bundles may then be solved in two steps. One first identifies the moduli space $\mathcal{N}(0, k)$ of holomorphic rank 2 stable vector bundles on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. One then identifies the points in $\mathcal{N}(0, k)$ corresponding to bundles with symplectic structure and having no real jumping lines — this gives $\mathcal{N}_{\mathbb{I}}(0, k) \subset \mathcal{N}(0, k)$.

§2.2. Stable Vector Bundles on Complex Projective Space

We describe Hartshorne's approach to the problem of identifying the moduli space $\mathcal{N}(0, k)$ of holomorphic, stable, rank 2 vector bundles over $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = k$, employing the correspondence between rank 2 bundles over $\mathbb{CP}^3$ and certain algebraic curves in $\mathbb{CP}^3$ [O-S-S, p. 90], [Har2, 3].

Definition 2.2.1. A holomorphic, rank 2 vector bundle $\mathcal{E} \to \mathbb{CP}^n$ is *stable* (respectively, *semistable*) if for every line bundle $\mathcal{L} \subset \mathcal{E}$, we have $c_1(\mathcal{L}) < c_1(\mathcal{E})/2$ (respectively, $\leq$).

Let $\mathcal{O}_{\mathbb{CP}^3}(1)$ denote the hyperplane bundle on $\mathbb{CP}^3$ and let $\mathcal{E}(m)$ denote the twisted bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbb{CP}^3}(1)^{\otimes m}$. Then a bundle $\mathcal{E}$ is stable if and only if $\mathcal{E}(m)$ is stable, for any $m \in \mathbb{Z}$ [Har3, p. 241]. Moreover, if $c_1(\mathcal{E}) = 0$ or -1, then $\mathcal{E}$ is stable if and only if $H^0(\mathbb{CP}^3, \mathcal{E}) = 0$ [Har3, p. 241].

By a theorem of Serre, $H^0(\mathbb{CP}^3, \mathcal{E}(m)) \neq 0$ for m sufficiently large, and so one may choose a non-zero global section $s \in \mathcal{E}(m)$ [G-H, p. 700], [O-S-S, p. 9]. Let $Y = (s)_0 \subset \mathbb{CP}^3$ be the zero-set of that section: for sufficiently general s, Y will be an algebraic curve in $\mathbb{CP}^3$ [Har3, p. 234]. If $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$, then [Har2, p. 4]:

$$\mathrm{degree}(Y) = k + m^2 \quad \text{and} \quad \mathrm{genus}(Y) = \max\{(k+m^2)(m-2)+1,\ 0\}.$$

(See also [Har3, p. 236] and [O-S-S, pp. 90-110].) The canonical bundle K_Y on Y is isomorphic to $\mathcal{O}_Y(2m-4)$ and is the restriction of a line bundle on $\mathbb{CP}^3$ [Har2, p. 4]. The precise correspondence between rank 2 bundles over $\mathbb{CP}^3$ and curves Y in $\mathbb{CP}^3$ is given by the following

Theorem 2.2.2. [Har2, p. 4], [Har3, p. 232] *A curve Y in $\mathbb{CP}^3$ is the scheme of zeros $(s)_0$ of a section s of a holomorphic rank 2 vector bundle $\mathcal{E} \to \mathbb{CP}^3$ if and only if Y is a locally complete intersection and K_Y is isomorphic to the restriction to Y of some line bundle on $\mathbb{CP}^3$. More precisely, for any fixed line bundle $\mathcal{L} \to \mathbb{CP}^3$ there is a bijection between (i) and (ii):*

- (i) *The set of triples $\langle \mathcal{E}, s, \varphi \rangle$ modulo the equivalence relation $\sim$, where*
  - (a) *$\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{CP}^3$;*
  - (b) *$s \in H^0(\mathbb{CP}^3, \mathcal{E})$ is a global section whose scheme of zeros $(s)_0$ has codimension 2;*
  - (c) *$\varphi : \det \mathcal{E} \to \mathcal{L}$ is an isomorphism; and $\langle \mathcal{E}, s, \varphi \rangle \sim \langle \mathcal{E}', s', \varphi' \rangle$ if there is an isomorphism $\psi : \mathcal{E} \to \mathcal{E}'$ and $\lambda \in \mathbb{C}^*$ such that*

$$s' = \lambda \psi(s) \qquad \text{and} \qquad \varphi' = \lambda^2 \varphi \circ (\det \psi)^{-1},$$

    where $\det \mathcal{E} = \wedge^2 \mathcal{E}$ and $\det \psi = \wedge^2 \psi$.
- (ii) *The set of pairs $\langle Y, \xi \rangle$, where*
  - (a) *Y is a locally complete intersection curve in $\mathbb{CP}^3$; and*
  - (b) *$\xi : \mathcal{L} \otimes K_{\mathbb{CP}^3} \otimes \mathcal{O}_Y \to K_Y$ is an isomorphism.*

Remark 2.2.3. By a *curve*, we mean a 1-dimensional closed subscheme of $\mathbb{CP}^3$ which may be reducible, disconnected, and may have nilpotent elements.

In particular, one has a criterion for distinct sections of a bundle to have to the same scheme of zeros:

Proposition 2.2.4. [Har3, p. 234] *Let $\mathcal{E}$ be a rank 2 bundle on $\mathbb{CP}^3$ and assume that for every nonzero $s \in H^0(\mathbb{CP}^3, \mathcal{E})$, the scheme of zeros $(s)_0$ has codimension 2. (This will be the case if $H^0(\mathbb{CP}^3, \mathcal{E}(-1)) = 0$.) Then two non-zero sections s, s' have the same scheme of zeros if and only if $s' = \lambda s$ for some $\lambda \in \mathbb{C}^*$.*

We recall the criterion for the scheme of zeros of a section to be non-singular:

Proposition 2.2.5. [Har3, p. 234] *Let $\mathcal{E}$ be a rank 2 bundle on $\mathbb{CP}^3$. If $\mathcal{E}$ is generated by global sections, then for all sufficiently general $s \in H^0(\mathbb{CP}^3, \mathcal{E})$, the scheme of zeros $(s)_0$ will be non-singular.*

By a theorem of Maruyama, the set of stable, holomorphic, rank 2 vector bundles over $\mathbb{CP}^3$ with given Chern classes c_1 and c_2 has a *coarse moduli scheme* $\mathcal{N}(c_1, c_2)$ which is separated and of finite type [Har3, p. 245], [Mu-Fo].

Proposition 2.2.6. [Har3, p. 245] *Let $\mathcal{E}$ be a stable bundle over a non-singular projective variety X. Then $H^1(X, \mathcal{E}nd\,\mathcal{E})$ is naturally isomorphic to the Zariski tangent space of the moduli scheme $\mathcal{N}$ at the point corresponding to $\mathcal{E}$. If $H^2(X, \mathcal{E}nd\,\mathcal{E}) = 0$, then $\mathcal{N}$ is nonsingular at that point and its dimension is equal to* $\dim H^1(X, \mathcal{E}nd\,\mathcal{E})$.

Proposition 2.2.7. [Har3, p. 245] *Let $\mathcal{E}$ be a stable bundle over $\mathbb{CP}^3$, with $c_1\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. Then*

$$\dim H^1(X, \mathcal{E}nd\,\mathcal{E}) - \dim H^2(X, \mathcal{E}nd\,\mathcal{E}) = 8k - 3.$$

If $c_1(\mathcal{E}) = 0$, then $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{CP}^3}$ and $\det \mathcal{E}(1) \simeq \mathcal{O}_{\mathbb{CP}^3}(2)$. One obtains a bijection between

(i) The set of pairs $\langle \mathcal{E}, s\rangle$, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$, and $s \in H^0(\mathbb{CP}^3, \mathcal{E}(m))$ is a non-zero section for some m; and

(ii) The set of curves Y in $\mathbb{CP}^3$, together with a given isomorphism $K_Y \simeq \mathcal{O}_Y(2m - 4)$. Furthermore, the bundle is stable if and only if the curve Y is not contained in any surface of degree $\leq m$ [Har2, p. 4], Har3, p. 241].

The curve Y obtained in this correspondence depends on the section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(m))$, as well as the bundle $\mathcal{E}$. To obtain the moduli space for the bundles $\mathcal{E}$, one needs to eliminate the ambiguity introduced by s.

We next describe Hartshorne's example of the family of holomorphic rank 2 bundles over $\mathbb{CP}^3$. When these are endowed with a symplectic structure, this example corresponds to the Jackiw-Nohl-Rebbi family of ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$.

Example 2.2.8. [Har2, p. 5], [Har3, p. 242 & p. 247] Let $Y = Y_0 \cup \ldots \cup Y_k$ be a disjoint union of $k+1$ lines $Y_i \simeq \mathbb{CP}^1$ in $\mathbb{CP}^3$, with $k \geq 1$. For $\mathbb{CP}^n$, the canonical bundle $K_{\mathbb{CP}^n} = \mathcal{O}_{\mathbb{CP}^n}(-n-1)$ and so the canonical bundle of a line is $K_{\mathbb{CP}^1} = \mathcal{O}_{\mathbb{CP}^1}(-2)$ [G-H, p. 146]. Thus, $K_Y \simeq \mathcal{O}_Y(-2)$, and this isomorphism is determined by $k+1$ non-zero complex numbers $\zeta_0, \ldots, \zeta_k \in \mathbb{C}^*$. Taking $m = 1$, we have $H^0(\mathbb{CP}^3, \mathcal{E}(1)) \neq 0$ [Har3, p. 263] and we obtain a bundle $\mathcal{E}$ for each choice of Y and each choice of isomorphism $K_Y \simeq \mathcal{O}(-2)$. Since degree$(Y) = k+1$, we have $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. For $k \geq 2$, Y is not contained in a plane, therefore $\mathcal{E}(1)$ and hence $\mathcal{E}$ will be stable [Har3, p. 241]. By [Har3, p. 246], $H^2(\mathbb{CP}^3, \mathcal{E}nd\,\mathcal{E}) = 0$ and the moduli space $\mathcal{N}(0,k)$ is non-singular of dimension $8k-3$.

One can then compute the dimension of this family of bundles: the Grassman variety $G(1,3)$ of lines $\mathbb{CP}^1$ in $\mathbb{CP}^3$ has dimension $\dim_{\mathbb{C}} G(1,3) = 4$: hence the choice of $Y \subset \mathbb{CP}^3$ requires $4(k+1)$ complex parameters. The choice of isomorphism $K_Y \simeq \mathcal{O}_Y(-2)$ depends on the $k+1$ complex parameters $\zeta_0, \ldots, \zeta_k$. Hence, the pair $\langle \mathcal{E}, s\rangle$ depends on $5k+5$ parameters. One then needs to subtract $\dim_{\mathbb{C}} H^0(\mathbb{CP}^3, \mathcal{E}(1))$, the number of parameters used in the choice of $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$:

$$\dim_{\mathbb{C}} H^0(\mathbb{CP}^3, \mathcal{E}(1)) = \begin{cases} 5 & \text{if } k = 1; \\ 2 & \text{if } k = 2; \\ 1 & \text{if } k \geq 3. \end{cases}$$

One then finds that the bundles constructed by this method form an irreducible algebraic family parametrized by a non-singular variety T of complex dimension

$$\dim_{\mathbb{C}} T = \begin{cases} 5 & \text{if } k = 1; \\ 13 & \text{if } k = 2; \\ 5k + 4 & \text{if } k \geq 3. \end{cases}$$

Further details may be found in [Har3] — in particular, see [Har3, p. 238].

§2.3. Real Structures and Jumping Lines

The moduli space of instanton bundles $\mathcal{N}_{\mathbb{I}}(0, k)$ is obtained as an open subset of the set of real points of the moduli space $\mathcal{N}(0, k)$ of holomorphic rank 2 stable bundles over $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = k$. In order to make this more precise, we first need to specify the required real structures and recall the concept of a jumping line [Har2], [O-S-S].

Proposition 2.3.1. [Har2, p. 9] *If n is even, then $\mathbb{CP}^n$ has a unique real structure, the standard one given by complex conjugation of the coordinates. If n is odd, then $\mathbb{CP}^n$ has two possible real structures: the standard one, and another, with no real points. In the latter case, $\mathcal{O}_{\mathbb{CP}^3}(1)$ has a symplectic structure and one can choose homogeneous coordinates $z_0, z_1, \ldots, z_n$ such that σ is given by $\sigma(z_0, z_1, \ldots, z_{n-1}, z_n) = (-\overline{z}_1, \overline{z}_0, \ldots, -\overline{z}_n, \overline{z}_{n-1})$.*

For example, the non-standard real structure σ on $\mathbb{CP}^3$ canonically induces a map $\sigma_m : \mathcal{O}_{\mathbb{CP}^3}(m) \to \sigma^* \overline{\mathcal{O}_{\mathbb{CP}^3}(m)}$, with $\sigma_m^2 = (-1)^m$, so that $\mathcal{O}_{\mathbb{CP}^3}(m)$ has a real structure for m even and a symplectic structure for m odd. Also, if a bundle $\mathcal{E} \to \mathbb{CP}^3$ has a real structure, then $\mathcal{E} \otimes \mathcal{O}_{\mathbb{CP}^3}(1)$ has a symplectic structure and vice versa [Har2, p. 9], [A-W], [S-T, p. 341].

We consider $\mathbb{CP}^3$ with its (non-standard) real structure σ. Via the Plücker embedding, $G(1, 3)$ may be viewed as a quadric hypersurface in $\mathbb{CP}^5$. The real structure σ on $\mathbb{CP}^3$ induces a real structure on $G(1, 3)$ and the (standard) real structure on the $\mathbb{CP}^5$ in which it is embedded. In particular, the set of real points of $G(1, 3)$ is $\mathbb{HP}^1$, corresponding precisely with the real lines of $\mathbb{CP}^3$.

For any holomorphic rank 2 vector bundle $\mathcal{E} \to \mathbb{CP}^3$, consider its restriction $\mathcal{E}|L$ to a line $L \subset \mathbb{CP}^3$: then $\mathcal{E}|L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ for some $a, b \in \mathbb{Z}$, and if $c_1(\mathcal{E}) = 0$, then $a + b = 0$. Furthermore, if $\mathcal{E}$ is stable, then $a = b = 0$ for lines L corresponding to an open dense subset of the Grassman variety $G(1, 3)$ and $\mathcal{E}|L$ for such an L will be holomorphically trivial. A line for which $a, b \neq 0$ is called a *jumping line* of $\mathcal{E}$. The set of jumping lines of a given bundle $\mathcal{E}$ corresponds to a divisor $Z \subset G(1, 3)$ of degree $c_2(\mathcal{E})$ [Har2, p. 10].

The problem of identifying the moduli space $\mathcal{N}_{\mathbb{I}}(0, k)$ of instanton bundles over $\mathbb{CP}^3$ may then be solved in the following steps [Har2, p. 10]:

(i) Identify the moduli space $\mathcal{N}(0, k)$ of stable, rank 2 vector bundles on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$;

(ii) Identify the real structure on $\mathcal{N}(0, k)$ induced by σ, and find its real points $\mathcal{N}_{\mathbb{R}}(0, k)$;

(iii) Among the real points $\mathcal{N}_{\mathbb{R}}(0,k)$, identify the points $\mathcal{N}_{\mathbb{R}}^{+}(0,k)$ corresponding to bundles $\mathcal{E}$ with a real structure, and the points $\mathcal{N}_{\mathbb{R}}^{-}(0,k)$ corresponding to bundles with a symplectic structure.

(iv) Among the points $\mathcal{N}_{\mathbb{R}}^{-}(0,k)$, identify those corresponding to bundles whose divisor $Z \subset G(1,3)$ of jumping lines has no real points. This will be the moduli space $\mathcal{N}_{\mathbb{I}}(0,k)$ of instanton bundles and will be an open subset of $\mathcal{N}_{\mathbb{R}}(0,k)$.

In the following sections, we will review Hartshorne's construction of the moduli space $\mathcal{N}_{\mathbb{I}}(0,2)$.

§2.4. Moduli Space of Rank Two Stable Bundles

We review Hartshorne's construction of the moduli space $\mathcal{N}(0,2)$ of stable bundles on $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = 2$ [Har3], [S-T].

If $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$, then $H^0(\mathbb{CP}^3, \mathcal{E}(1)) \neq 0$, while $H^0(\mathbb{CP}^3, \mathcal{E}) = 0$ since $\mathcal{E}$ is stable [Har3, p. 263]. Let $0 \neq s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ and let $Y = (s)_0$ denote the zero set of s. Then Y will be a curve of degree 3 such that $K_Y \simeq \mathcal{O}_Y(-2)$. One has the

Proposition 2.4.1. [Har3, p. 267] *Let Y be a curve of degree 3 in $\mathbb{CP}^3$ such that $K_Y \simeq \mathcal{O}_Y(-2)$. Then Y consists of either 3 skew lines, a single line plus a double line, or a single line of multiplicity 3.*

Lemma 2.4.2. [Har3, p. 268] *If Y denotes any curve of degree 3 in $\mathbb{CP}^3$ and $K_Y \simeq \mathcal{O}_Y(-2)$, then Y is contained in a unique nonsingular quadric surface $Q \subset \mathbb{CP}^3$.*

Recall that any non-singular quadric surface $Q \subset \mathbb{CP}^3$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ [G-H, p. 478]. The curve Y is a divisor of type (3,0) on Q: the divisor class group of Q is $\mathbb{Z} \oplus \mathbb{Z}$, generated by a line in each of the two rulings and the type refers to the class in $\mathbb{Z} \oplus \mathbb{Z}$ [Har3, p. 268].

Lemma 2.4.3. [Har3, p. 268] *Let $\mathcal{E}$ be a holomorphic rank 2 vector bundle on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$, and let $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ be a section with zero set $(s)_0 = Y$. Then the nonsingular quadric surface Q containing Y depends only on $\mathcal{E}$, and not on the choice of s. There is a linear map*

$$H^0(\mathbb{CP}^3, \mathcal{E}(1)) \longrightarrow H^0(\mathcal{O}_Q(3,0)), \qquad s \longmapsto s|Q,$$

so that as s varies in $H^0(\mathbb{CP}^3, \mathcal{E}(1))$, Y cuts out a linear system on Q of type (3,0) and dimension 1.

Lemma 2.4.4. [Har3, p. 269] *With $\mathcal{E}$ and Q as in the previous lemmas, the linear system of curves Y on Q obtained by varying $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ is a linear system without basepoints.*

One then has the important

Corollary 2.4.5. [Har3, p. 270] *For $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$, there exists a section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ whose zero set $(s)_0$ is three skew lines.*

Remark 2.4.6. Thus, the rank 2 vector bundle construction described earlier gives all stable bundles with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$.

The linear system of curves Y of type (3,0) on $Q \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$ is induced by a linear system g_3^1, without basepoints, of degree 3 and dimension 1 on one of the factors $\mathbb{CP}^1$. The general member of the g_3^1 will consist of three distinct points and the corresponding curve Y on Q will be three skew lines [Har3, p. 270].

In summary, a stable bundle $\mathcal{E}$ over $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$ corresponds to a curve Y of degree 3 with $K_Y \simeq \mathcal{O}_Y(-2)$, where Y consists of 3 skew lines in $\mathbb{CP}^3$. A set of 3 skew lines in $\mathbb{CP}^3$ determines a unique nonsingular quadric surface Q in $\mathbb{CP}^3$, where the quadric may be constructed as the union of all other lines which meet each of the given lines [G-H, p. 478]. The points of the $\mathbb{CP}^1$ factors correspond to the lines on the quadric surface Q. As the section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ varies, the curves Y move in a linear system on the same quadric Q. Each divisor Y consists of three lines in one of the two families of lines on Q. This selects one of the two factors $\mathbb{CP}^1$ of Q and the linear system of curves Y then corresponds to a linear system g_3^1 on this $\mathbb{CP}^1$, without basepoints, of degree 3 and dimension 1. Collecting all these observations gives the following

Theorem 2.4.7. [Har2, p. 13], [Har3, p. 270] *A stable holomorphic bundle $\mathcal{E}$ on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 2$ determines*
(i) *A nonsingular quadric surface $Q \subset \mathbb{CP}^3$;*
(ii) *A choice of one of the two factors in the isomorphism $Q \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$;*
(iii) *A linear system g_3^1 of degree 3 and dimension 1 on the selected $\mathbb{CP}^1$, without basepoints. Conversely, any such data arise from a unique such bundle $\mathcal{E}$.*

In particular, the moduli space $\mathcal{N}(0,2)$ has a description as a fibre space [Har2, p. 13], [Har3, p. 271], [S-T, p. 336]. The quadric surfaces in $\mathbb{CP}^3$ are parametrized by $\mathbb{CP}^9$: we let $\Delta \subset \mathbb{CP}^9$ be the subset corresponding to the singular or degenerate quadric surfaces. Then $\mathcal{N}(0,2)$ is fibred over $\mathbb{CP}^9 - \Delta$:

$$\begin{array}{c} \mathcal{N}(0,2) \\ \Big\downarrow {\scriptstyle U \sqcup U} \\ \mathbb{CP}^9 - \Delta \end{array}$$

The fibre is the disjoint union of two copies of the variety U wich parametrizes the set of possible g_3^1 without basepoints on $\mathbb{CP}^1$. A g_3^1 is determined by a 2-dimensional subspace of the 4-dimensional vector space $H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(3))$. Thus, the set of all possible g_3^1 is parametrized by $G(1,3)$ and those without basepoints form an open subset U.

Corollary 2.4.8. [Har3, p. 271] *The moduli space $\mathcal{N}(0,2)$ is an irreducible nonsingular complex variety of dimension 13.*

A stable, holomorphic, rank 2, vector bundle $\mathcal{E}$ on $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$, determines a unique point in $\mathcal{N}(0,2)$. The effect of varying $s \in \mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{E}(1)))$ may be described as follows. Recall that $\mathcal{E}$ determines a quadric Q in $\mathbb{CP}^3$ and a choice of factor $\mathbb{CP}^1$ of Q. The points of $\mathbb{CP}^1$ correspond to lines in Q, which in turn correspond to points of the Grassman variety $G(1,3)$ of lines in $\mathbb{CP}^3$. As $p \in \mathbb{CP}^1$ varies, its image in $G(1,3) \subset \mathbb{CP}^5$ describes a conic γ in $G(1,3)$. The g_3^1 on Q induces a g_3^1 on γ. So as s varies in $\mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{E}(1)))$, the corresponding divisor $D_s = Y_1 + Y_2 + Y_3$ varies in the g_3^1 on the conic $\gamma \subset G(1,3)$ [Har2, p. 14], [Har3, p. 276].

§2.5. Moduli Space of Rank Two Instanton Bundles

We review Hartshorne's description of the open set $\mathcal{N}_{\mathbb{I}}(0,2) \subset \mathcal{N}(0,2)$ parametrizing isomorphism classes of instanton bundles [Har2], [S-T].

Considering the above fibration $\mathcal{N}(0,2) \to \mathbb{CP}^9 - \Delta$, the induced real structure on $\mathbb{CP}^9$ is the standard one and its real points are $\mathbb{RP}^9$. The real points of Δ are given by $\Delta_{\mathbb{R}}$ which is isomorphic to the quotient space $\mathbb{CP}^3/\sigma$, a compact real 6-manifold. Next one considers the fibre $U \sqcup U$. A point in the base $\mathbb{RP}^9 - \Delta_{\mathbb{R}}$ corresponds to a nonsingular quadric Q with a real structure σ. It will be convenient to label the $\mathbb{CP}^1$ factors in the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$ as $\mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta$. Then, $Q \simeq \mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta$ and σ leaves each factor fixed so that each $\mathbb{CP}^1$ has a real structure with one factor — say $\mathbb{CP}^1_\alpha$ — having the standard real structure and the other — say $\mathbb{CP}^1_\beta$ — having the non-standard structure. The $G(1,3)$ of which U is an open subset has the standard structure in one case (as desired) and the non-standard one in the other case. Bundles $\mathcal{E}$ with a symplectic structure (as desired) correspond to the choice of factor $\mathbb{CP}^1_\alpha$ with the standard real structure; those $\mathcal{E}$ with a real structure correspond to the other factor, $\mathbb{CP}^1_\beta$. The required U in the fibre $U \sqcup U$ is then the one which is an open subset of $G(1,3)$ with its standard real structure.

A triple $\langle \mathcal{E}, s, \varphi \rangle$ consisting of a rank 2 bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 0$, a suitably chosen section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$, and an isomorphism $\varphi : \det \mathcal{E}(1) \simeq \mathcal{O}_{\mathbb{CP}^3}(2)$, corresponds to three skew lines $Y_i \subset \mathbb{CP}^3$ and three complex weights $\zeta_i \in \mathbb{C}^*$. If $\mathcal{E}$ is endowed with a real structure, then the lines Y_i are real lines (corresponding to points in $\mathbb{S}^4$) and the weights ζ_i are real. Finally, the requirement that the divisor $Z \subset G(1,3)$ of jumping lines has no real points is equivalent to the requirement that the real weights $\zeta_0, \zeta_1, \zeta_2$ are positive [Har2, p. 13-15], [Wa-We, p. 412]. The moduli space $\mathcal{N}_{\mathbb{I}}(0,2)$ of instanton bundles is the total space of a fibre bundle over $\mathbb{RP}^9 - \Delta_{\mathbb{R}}$, with fibre an open connected real 4-manifold [Har2, p. 13], [S-T, p. 340].

Theorem 2.5.1. [Har2, p. 12] *The moduli space $\mathcal{N}_{\mathbb{I}}(0,2)$ of instanton bundles is a connected, but not simply connected, real 13-dimensional manifold. Every instanton bundle $\mathcal{E} \to \mathbb{CP}^3$ may obtained by the construction of Jackiw-Nohl-Rebbi corresponding to 3 points P_0, P_1, P_2 in $\mathbb{S}^4$ and 3 positive numbers $\zeta_0, \zeta_1, \zeta_2$.*

Remark 2.5.2. The analogous theorem holds for the moduli space $\mathcal{N}_{\mathbb{I}}(0,1) \simeq \mathbb{R}^5$, the instanton bundles then corresponding to 2 points P_0, P_1 in $\mathbb{S}^4$ and 2 positive numbers $\zeta_0, \zeta_1 > 0$.

As observed by Hartshorne, it is difficult to use this method to explicitly construct either the holomorphic bundles $\mathcal{E} \to \mathbb{CP}^3$ or the corresponding complex bundles $E \to \mathbb{S}^4$ and their associated ASD $\mathbf{SU}(2)$-connections [Har2, p. 4]. The construction of the corresponding ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ is outlined in [A-W, p. 122] and described in detail in [Wa-We, p. 398 & 412]. We will instead use the ADHM method to construct bundles $E \to \mathbb{S}^4$ and all ASD $\mathbf{SU}(2)$-connections with $c_2(E) = 2$. We describe this relationship between the two methods in the next chapter.

§2.6. Real Structures on Complex Moduli Spaces

We discuss the parametrization of $\mathcal{N}(0,2)$ in terms of skew lines Y_i in $\mathbb{CP}^3$ and complex weights ζ_i, and the corresponding parametrization of $\mathcal{N}_{\mathbb{I}}(0,2)$ in terms of real skew lines Y_i in $\mathbb{CP}^3$ and positive weights ζ_i; we provide proofs for some assertions whose proofs were omitted earlier. General references for this section are [A-W], [Bu], [Har2, 3], [Wa-We].

The moduli space $\mathcal{N}(0,k)$ parametrizes the set of isomorphism classes $[\mathcal{E}]$ of rank 2, holomorphic, stable vector bundles $\mathcal{E} \to \mathbb{CP}^3$, with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k > 0$. Recall [A-W, p. 120] that the (non-standard) real structure σ on $\mathbb{CP}^3$ induces a real structure $\sigma_{\mathcal{N}}$ on the moduli space $\mathcal{N}(0,k)$:

$$\sigma_{\mathcal{N}} : \mathcal{N}(0,k) \longrightarrow \mathcal{N}(0,k), \qquad [\mathcal{E}] \longmapsto [\sigma^*\overline{\mathcal{E}}].$$

Lemma 2.6.1. *Let $\mathcal{N}_{\mathbb{R}}(0,k)$ be the real points of $\mathcal{N}(0,k)$ with respect to the real structure $\sigma_{\mathcal{N}}$. Then $\mathcal{N}_{\mathbb{R}}(0,k) = \{[\mathcal{E}] : [\mathcal{E}] \in \mathcal{N}(0,k)$ and $\sigma_{\mathcal{E}}$ is a real or symplectic structure on $\mathcal{E}$, with $\sigma_{\mathcal{E}} \sim \sigma'_{\mathcal{E}}$ if and only if $\lambda \in \mathbb{C}$, $|\lambda| = 1\}$.*

Proof. Suppose $[\mathcal{E}] \in \mathcal{N}(0,k)$ and that $\sigma_{\mathcal{E}}$ is a real or symplectic structure. Then $\sigma_{\mathcal{E}} : \mathcal{E} \to \sigma^*\overline{\mathcal{E}}$ is a bundle isomorphism. Hence

$$\sigma_{\mathcal{N}}[\mathcal{E}] = [\sigma^*\overline{\mathcal{E}}] = [\mathcal{E}],$$

and so $[\mathcal{E}] \in \mathcal{N}_{\mathbb{R}}(0,k)$.

Conversely, suppose $[\mathcal{E}] \in \mathcal{N}_{\mathbb{R}}(0,k)$. Then

$$\sigma_{\mathcal{N}}[\mathcal{E}] = [\sigma^*\overline{\mathcal{E}}] = [\mathcal{E}],$$

and so there exists a bundle isomorphism $\sigma_{\mathcal{E}} : \mathcal{E} \to \sigma^*\overline{\mathcal{E}}$. Hence, we obtain a bundle automorphism $\sigma_{\mathcal{E}}^2 : \mathcal{E} \to \mathcal{E}$. The holomorphic vector bundle $\mathcal{E} \to \mathbb{CP}^3$ is stable and therefore simple [O-S-S, p. 172]. Since $\mathcal{E}$ is simple, we have $\mathrm{Aut}\,(\mathcal{E}) = \mathbb{C}^*$ [O-S-S, p. 74]. Hence, $\sigma_{\mathcal{E}}^2 = \mu \cdot 1_{\mathcal{E}}$ for some $\mu \in \mathbb{C}^*$. We claim that $\mu \in \mathbb{R}^*$. Suppose $U \subset \mathbb{CP}^3$ is open, $\mathcal{E}|U$ holomorphically trivial, and $\{z, \sigma z\} \subset U$ for some z. (This will certainly be true for instanton bundles $\mathcal{E}$, since we assume that $\mathcal{E}|L$ is trivial for all real lines $L \subset \mathbb{CP}^3$.) Let $\{\phi_1, \phi_2\} \in \mathcal{O}(\mathcal{E})(U)$ be a local frame. Then,

$$\sigma_{\mathcal{E}} : \mathcal{E}|_U \longrightarrow \mathcal{E}|_{\sigma(U)}, \qquad \begin{aligned} \phi_i(z) &\longmapsto \sum_{j=1}^{2} \lambda_{ji}(\sigma z)\phi_j(\sigma z), \\ \phi_i(\sigma z) &\longmapsto \sum_{j=1}^{2} \lambda_{ji}(z)\phi_j(z), \end{aligned}$$

and consequently, we have

$$\sigma_{\mathcal{E}}^2 : \mathcal{E} \longrightarrow \mathcal{E}, \qquad \begin{aligned} \phi_i(z) &\longmapsto \sum_{j,k=1}^{2} \overline{\lambda}_{ji}(\sigma z)\lambda_{kj}(z)\phi_k(z), \\ \phi_i(\sigma z) &\longmapsto \sum_{j,k=1}^{2} \overline{\lambda}_{ji}(z)\lambda_{kj}(\sigma z)\phi_k(\sigma z). \end{aligned}$$

But $\sigma_{\mathcal{E}}^2 = \mu \cdot 1_{\mathcal{E}}$ and so

$$\begin{aligned} \mu &= \sum_{j=1}^{2} \overline{\lambda}_{j1}(\sigma z)\lambda_{1j}(z) = \sum_{j=1}^{2} \overline{\lambda}_{j1}(z)\lambda_{1j}(\sigma z) \\ &= \sum_{j=1}^{2} \overline{\lambda}_{j2}(\sigma z)\lambda_{2j}(z) = \sum_{j=1}^{2} \overline{\lambda}_{j2}(z)\lambda_{2j}(\sigma z) \end{aligned}$$

Then,

$$4\mu = \sum_{j,k=1}^{2} \overline{\lambda}_{ji}(\sigma z)\lambda_{kj}(z) + \sum_{j,k=1}^{2} \overline{\lambda}_{ji}(z)\lambda_{kj}(\sigma z),$$

and so $\mu \in \mathbb{R}^*$. By replacing $\sigma_{\mathcal{E}}$ with $|\mu|^{-1/2}\sigma_{\mathcal{E}}$, we may assume $\sigma_{\mathcal{E}}^2 = \pm 1_{\mathcal{E}}$, corresponding to a real or a symplectic structure, respectively. □

Remark 2.6.2. (i) The condition $\sigma_{\mathcal{E}}^2 = 1$ or $\sigma_{\mathcal{E}}^2 = -1$ is constant on connected components of $\mathcal{N}_{\mathbb{R}}(0, k)$. Indeed, $\mathcal{N}_{\mathbb{R}}(0, k) = \mathcal{N}_{\mathbb{R}}^{+}(0, k) \sqcup \mathcal{N}_{\mathbb{R}}^{-}(0, k)$, where $\mathcal{N}_{\mathbb{R}}^{-}(0, k)$ represent the required bundles with a symplectic structure and $\mathcal{N}_{\mathbb{R}}^{+}(0, k)$ represent bundles with a real structure.

(ii) The condition that $\mathcal{E}|L$ be trivial for any real line $L \subset \mathbb{CP}^3$ is an open condition. Hence, the moduli space of instanton bundles $\mathcal{N}_{\mathbb{I}}(0, k)$ is an open subset of $\mathcal{N}_{\mathbb{R}}^{-}(0, k)$.

Lemma 2.6.3. *Let $Y \subset \mathbb{CP}^3$ consist of $k+1$ skew lines $Y_0, \ldots, Y_k$ and let K_Y denote the canonical bundle of Y. Then $K_Y \simeq \mathcal{O}_Y(-2)$.*

Proof. From [Har1, p. 182] we have

$$K_Y \simeq K_{\mathbb{CP}^3} \otimes \det N_{Y/\mathbb{CP}^3},$$

where $K_{\mathbb{CP}^3} \simeq \mathcal{O}_{\mathbb{CP}^3}(-4)$, $N_{Y/\mathbb{CP}^3} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^*$ is the normal bundle of $Y \subset \mathbb{CP}^3$, and $\mathcal{I}_Y$ is the ideal sheaf of $Y \subset \mathbb{CP}^3$. By assumption, $Y = \bigcup_{i=0}^k Y_i$, with each line $Y_i \simeq \mathbb{CP}^1$ an intersection of hyperplanes $Y_i = H_i \cap \hat{H}_i$, where $H_i = \{z \in \mathbb{CP}^3 : f_i(z) = 0\}$, $\hat{H}_i = \{z \in \mathbb{CP}^3 : h_i(z) = 0\}$, for $i = 0, \ldots, k$. Let $\{U_0, \ldots, U_k\}$ be an open cover of $\mathbb{CP}^3$ with $Y_i \subset U_i$ and $\mathcal{E}|U_i$ holomorphically trivial. Then

$$\mathcal{I}_Y|U_i = (f_i, h_i)\mathcal{O}_{\mathbb{CP}^3}|U_i,$$

and consequently, as $\mathcal{O}_Y = \mathcal{O}_{\mathbb{CP}^3}/\mathcal{I}_Y$,

$$(\mathcal{I}_Y/\mathcal{I}_Y^2)|U_i = (f_i\mathcal{O}_Y \oplus h_i\mathcal{O}_Y)|U_i.$$

Taking determinants, we have

$$\det(\mathcal{I}_Y/\mathcal{I}_Y^2)|U_i = (f_i \wedge h_i)\mathcal{O}_Y|U_i.$$

An ideal sheaf $\mathcal{I}$ that locally has a single generator is locally free of rank one. If D is the corresponding divisor in $\mathbb{CP}^3$, then $D = \operatorname{supp}(\mathcal{O}_{\mathbb{CP}^3}/\mathcal{I})$ and is an analytic subvariety of $\mathbb{CP}^3$. Then denoting $\mathcal{I}$ by $\mathcal{I}_D$ and $\mathcal{O}_{\mathbb{CP}^3}/\mathcal{I}$ by $\mathcal{O}_D$, we have

$$\mathcal{I}_D = \mathcal{O}_{\mathbb{CP}^3}(-D),$$
$$\mathcal{I}_D/\mathcal{I}_D^2 = \mathcal{I}_D \otimes \mathcal{O}_D,$$

where $\mathcal{O}_{\mathbb{CP}^3}(-D) = \mathcal{O}_{\mathbb{CP}^3}([-D])$ and $[-D]$ denotes the line bundle on $\mathbb{CP}^3$ corresponding to the divisor $-D$. If $U \subset \mathbb{CP}^3$ is an open subset and $D \cap U = \{z \in \mathbb{CP}^3 : f(z) = 0\}$, then $\mathcal{I}_D|U = f\mathcal{O}_{\mathbb{CP}^3}|U$ [G-H, p. 138 & p. 698], [O-S-S, p. 4].

Returning to the proof of the lemma, we have

$$f_i\mathcal{O}_{\mathbb{CP}^3}|U_i = \mathcal{O}_{\mathbb{CP}^3}(-H_i)|U_i$$
$$h_i\mathcal{O}_{\mathbb{CP}^3}|U_i = \mathcal{O}_{\mathbb{CP}^3}(-\hat{H}_i)|U_i,$$

and so

$$f_i\mathcal{O}_Y|U_i = \mathcal{O}_Y(-H_i)|U_i,$$
$$h_i\mathcal{O}_Y|U_i = \mathcal{O}_Y(-\hat{H}_i)|U_i.$$

Hence,

$$\mathcal{I}_Y/\mathcal{I}_Y^2|U_i = \mathcal{O}_Y(-H_i) \oplus \mathcal{O}_Y(-\hat{H}_i)|U_i,$$

and taking determinants, we obtain

$$\begin{aligned}\det(\mathcal{I}_Y/\mathcal{I}_Y^2)|U_i &\simeq \mathcal{O}_Y(-H_i)\otimes\mathcal{O}_Y(-\hat{H}_i)|U_i\\ &\simeq \mathcal{O}_Y(-2)|U_i, \qquad \text{for } i=0,\dots,k.\end{aligned}$$

Therefore, $\det(\mathcal{I}_Y/\mathcal{I}_Y^2)\simeq\mathcal{O}_Y(-2)$, and

$$\det N_{Y/\mathbb{CP}^3} = \det(\mathcal{I}_Y/\mathcal{I}_Y^2)^* = \mathcal{O}_Y(2).$$

The canonical line bundle K_Y is then given by

$$\begin{aligned}K_Y &\simeq K_{\mathbb{CP}^3}\otimes\mathcal{O}_Y\otimes\mathcal{O}_{\mathbb{CP}^3}(2)\\ &\simeq \mathcal{O}_{\mathbb{CP}^3}(-4)\otimes\mathcal{O}_Y\otimes\mathcal{O}_{\mathbb{CP}^3}(2)\\ &\simeq \mathcal{O}_Y(-2),\end{aligned}$$

as required. □

Lemma 2.6.4. *Let $\mathcal{E}\to\mathbb{CP}^3$ be a, holomorphic, rank 2 vector bundle with $c_1(\mathcal{E})=0$ and $c_2(\mathcal{E})=k$. Then $c_1(\mathcal{E}(1))=2$ and $c_2(\mathcal{E}(1))=k+1$.*

Proof. Immediate from the general formulas for $c(\wedge^p\mathcal{E})$ and $c(\mathcal{E}\otimes\mathcal{L})$, where $\mathcal{L}$ is a line bundle over $\mathbb{CP}^3$ [O-S-S, p. 16]. □

Lemma 2.6.5. [Har3, p. 232], [O-S-S, p. 93] *Let the triple $\langle\mathcal{E}(1),s,\varphi\rangle$ be as in the statement of Theorem 1.1.* [Har3, p. 232], *with $c_1(\mathcal{E})=0$, $c_2(\mathcal{E})=k$, line bundle $\mathcal{L}=\mathcal{O}_{\mathbb{CP}^3}(2)$, and fixed isomorphism $\varphi:\det\mathcal{E}(1)\simeq\mathcal{L}$. Then the triple $\langle\mathcal{E}(1),s,\varphi\rangle$ canonically determines an isomorphism $\xi:\mathcal{O}_Y(-2)\simeq K_Y$ and corresponding element $\zeta\in H^0(Y,\mathcal{O}_Y)$.*

Proof. Recall from Theorem 1.1. [Har3, p. 232] that the triple $\langle\mathcal{E}(1),s,\varphi\rangle$ canonically determines an isomorphism $\xi:\mathcal{L}\otimes\mathcal{K}_{\mathbb{CP}^3}\otimes\mathcal{O}_Y\simeq K_Y$. Since $\mathcal{L}=\mathcal{O}_{\mathbb{CP}^3}(2)$ and $\mathcal{K}_{\mathbb{CP}^3}=\mathcal{O}_{\mathbb{CP}^3}(-4)$, we then have $\xi:\mathcal{O}_Y(-2)\simeq K_Y$. From [Har3, p. 232], [O-S-S, p. 90], we have a locally free resolution of $\mathcal{I}_Y$ given by the Koszul complex for s:

$$0\longrightarrow\det\mathcal{E}^*\longrightarrow\mathcal{E}^*\xrightarrow{s}\mathcal{I}_Y\longrightarrow 0$$

The bundle map φ induces an isomorphism $\det\mathcal{E}^*\to\mathcal{L}^*$, and so we have an exact sequence

$$0\longrightarrow\mathcal{L}^*\longrightarrow\mathcal{E}^*\xrightarrow{s}\mathcal{I}_Y\longrightarrow 0$$

Since $\mathcal{L}^*=\mathcal{O}_{\mathbb{CP}^3}(-2)$, this global extension of $\mathcal{E}^*$ determines an element of $\mathrm{Ext}^1(\mathbb{CP}^3;\mathcal{I}_Y,\mathcal{O}_{\mathbb{CP}^3}(-2))$ [G-H, p. 725]. Proceeding as in [O-S-S, p. 97] or [Har3, p. 233], we obtain a canonical isomorphism

$$\mathrm{Ext}^1(\mathbb{CP}^3;\mathcal{I}_Y,\mathcal{O}_{\mathbb{CP}^3}(-2))\simeq H^0(Y,\mathcal{O}_Y),$$

as required. □

Lemma 2.6.6. *Let $\mathcal{E} \to \mathbb{CP}^3$ be a stable, holomorphic, rank 2 vector bundle with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. Let $\mathcal{E}$ be endowed with a real or symplectic structure $\sigma_{\mathcal{E}}$ and fix an isomorphism $\det \mathcal{E}(1) \simeq \mathcal{O}_{\mathbb{CP}^3}(2)$. Suppose $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ and that $\langle \mathcal{E}, s\rangle$ corresponds to $k+1$ skew lines $Y = Y_0 \cup \cdots \cup Y_k \subset \mathbb{CP}^3$ and a choice of isomorphism $\xi : \mathcal{O}_Y(-2) \simeq K_Y$ specifed by $\zeta_0, \ldots, \zeta_k \in \mathbb{C}^*$. Then:*

(i) *The lines Y_i are real lines;*

(ii) *The weights ζ_i are real.*

Proof. (i) Let $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ with $s^{-1}(0) = Y$. We have an isomorphism $\sigma_{\mathcal{E}} : \mathcal{E} \to \sigma^* \overline{\mathcal{E}}$, and so the pair $\langle \sigma^* \overline{\mathcal{E}}, \sigma^* \overline{s}\rangle$ determines Y also. Hence,

$$Y = (\overline{s} \circ \sigma)^{-1}(0) = (s \circ \sigma)^{-1}(0) = \sigma^{-1}(Y)$$

and so $\sigma(Y) = Y$. Thus Y is preserved by σ and so consists of $k+1$ real lines. This proves (i).

(ii) Recall that the real structure σ on $\mathbb{CP}^3$ induces a real or symplectic structure on any object that is functorially associated with $\mathbb{CP}^3$ [Har2, p. 8]. For example, a real structure is induced on $\mathcal{N}(0, k)$ by requiring that bundles $\mathcal{E}$ and $\sigma^* \overline{\mathcal{E}}$ be isomorphic, the isomorphism being given by $\sigma_{\mathcal{E}} : \mathcal{E} \to \sigma^* \overline{\mathcal{E}}$. Hence, the bundles $\mathcal{E}$ and $\sigma^* \overline{\mathcal{E}}$ determine the same isomorphism $\xi : \mathcal{O}_Y(-2) \simeq K_Y$, these isomorphisms being parametrized by $H^0(Y, \mathcal{O}_Y) = \mathbb{C}^{k+1}$. The space $H^0(Y, \mathcal{O}_Y)$ inherits a real structure, so that $H^0(Y, \mathcal{O}_Y) = \mathbb{R}^{k+1}$ and hence the weights ζ_i are real. ☐

Lemma 2.6.7. *With the hypotheses of the previous lemma, assume further that $\mathcal{E}$ is an instanton bundle with $c_2(\mathcal{E}) = 2$. Then the weights ζ_i are positive.*

Proof. See [Har2, p. 12], [Wa-We, p. 413]. ☐

CHAPTER III
ATIYAH-DRINFELD-HITCHIN-MANIN CONSTRUCTION

We outline the ADHM monad construction of rank 2 instanton bundles over the complex projective space $\mathbb{CP}^3$ and the corresponding complex vector bundles over the four-sphere $\mathbb{S}^4$ with anti-self-dual $\mathbf{SU}(2)$-connections. We then have two methods of constructing these holomorphic rank 2 vector bundles over $\mathbb{CP}^3$, namely the method of curves and the ADHM monad construction, and so we describe the explicit correspondence between these two methods. General references for this chapter are [A], [A-D-H-M], [B-H], [Bo-Ma], [D-K], [D-M,1-4], [G-H], [O-S-S], [Ra2], [Sa].

§3.1. ADHM Construction of Instanton Bundles

We recall the construction of instanton bundles $\mathcal{E} \to \mathbb{CP}^3$ corresponding to ASD $\mathbf{SU}(2)$-connections on a smooth, complex, Hermitian, rank 2 bundle $E \to \mathbb{S}^4$ with topology fixed by $c_2(E) = k$.

Definition 3.1.1. Let X be a compact, complex manifold. A *monad* over X is a complex

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \longrightarrow 0$$

of holomorphic vector bundles over X such that $\beta\alpha = 0$. The holomorphic vector bundle $\mathcal{E} = \operatorname{Ker}\beta/\operatorname{Im}\alpha$ over X is called the *cohomology of the monad.*

Lemma 3.1.2. [O-S-S, p. 240] *If $\mathcal{E} \to X$ is the cohomology of a monad as above, then the rank* $\operatorname{rk}\mathcal{E}$ *and total Chern class* $c(\mathcal{E})$ *are given by*

$$\begin{aligned} \operatorname{rk}\mathcal{E} &= \operatorname{rk}\mathcal{B} - \operatorname{rk}\mathcal{A} - \operatorname{rk}\mathcal{C}, \\ c(\mathcal{E}) &= c(\mathcal{B})c(\mathcal{A})^{-1}c(\mathcal{C})^{-1}. \end{aligned}$$

We next describe Horrocks' monad construction of rank 2 instanton bundles on $\mathbb{CP}^3$ [A], [Wa-We]. The monad construction may be described in terms of the following data [A, p. 59], [Wa-We, p. 415]:

Data 3.1.3. The linear algebra data for the monad construction of instanton bundles $\mathcal{E} \to \mathbb{CP}^3$ corresponding to ASD $\mathbf{SU}(2)$-connections on a smooth complex rank 2 bundle $E \to \mathbb{S}^4$ with $c_2(E) = k$ is given by the following:

(i) A map $\sigma : \mathbb{C}^4 \to \mathbb{C}^4$ defined by $\sigma(z_0, z_1, z_2, z_3) = (-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2)$.

(ii) A complex vector space W, with $\dim_{\mathbb{C}} W = k$ and a conjugate-linear map $\sigma_W : W \to W$, $\sigma_W^2 = 1$. (When there is no ambiguity, σ_W is denoted by σ.)

(iii) A complex vector space V, with $\dim_{\mathbb{C}} V = 2k + 2$, with a symplectic form b and conjugate-linear map $\sigma_V : V \to V$, so that $\sigma_V^2 = -1$, and satisfying:
   - (a) the form b is compatible with σ_V, in the sense that $b(\sigma u, \sigma v) = \overline{b(u, v)}$;
   - (b) the induced Hermitian form $h(u, v) = b(u, \sigma v)$ is required to be positive definite. (When there is no ambiguity, σ_V is denoted by σ.)

(iv) A linear map $A(z) : W \to V$, depending linearly on $z = (z_0, z_1, z_2, z_3)$, so $A(z) = \sum_{\alpha=0}^{3} A_\alpha z_\alpha$, where $A_\alpha : W \to V$ are constant linear maps satisfying:
   (a) (Rank or non-degeneracy condition.) For all $z \neq 0$, $\dim_{\mathbb{C}} \operatorname{Im} A(z) = k$;
   (b) (Isotropy condition.) For all $z \neq 0$, $\operatorname{Im} A(z)$ is an isotropic subspace of V, so that $\operatorname{Im} A(z) \subset (\operatorname{Im} A(z))^\circ$;
   (c) (Compatibility with σ.) For all $z \in \mathbb{C}^4$, $w \in W$, then $\sigma A(z) w = A(\sigma z)\sigma w$.

(If U is any subspace of V, then $U^\circ = \{v \in V : b(u,v) = 0 \text{ for all } u \in U\}$ denotes the polar subspace corresponding to U.)

The symplectic form $b : V \otimes V \to \mathbb{C}$ induces an isomorphism $b : V \to V^*$ given by $v \mapsto b(v) = b(\cdot, v)$. Since $A^*(z) : V^* \to W^*$, we obtain a map $A^*(z)b : V \to W^*$ defined by

$$(A^*(z)b(v))\,(w) = b(A(z)w, v) \qquad \text{for } v \in V,\ w \in W.$$

We then have the corresponding monad:

$$0 \longrightarrow W(-1) \xrightarrow{A} \underline{V} \xrightarrow{A^* b} W^*(1) \longrightarrow 0$$

where $\underline{V}$ denotes the trivial bundle $V \times \mathbb{CP}^3 \to \mathbb{CP}^3$. We use the symplectic form b to define a conjugate-linear isomorphism $b : V \to V^*$, $v \mapsto b(\cdot, v)$. The bundle $\mathcal{E} \to \mathbb{CP}^3$ is then defined as $\operatorname{Ker}(A^* b)/\operatorname{Im} A$, with fibres

$$\mathcal{E}_z = \operatorname{Ker}(A^*(z)b)/\operatorname{Im} A(z) = (\operatorname{Im} A(z))^\circ/\operatorname{Im} A(z) \qquad \text{for } z \in \mathbb{CP}^3.$$

If we let $\mathcal{O}_{\mathbb{CP}^3}(-1)$ denote the tautological line bundle on $\mathbb{CP}^3$ and recall that $V \simeq \mathbb{C}^{2k+2}$, $W \simeq \mathbb{C}^k$, then we see that the above monad is equivalent to

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{CP}^3}(-1) \xrightarrow{A} \mathcal{O}^{2k+2}_{\mathbb{CP}^3} \xrightarrow{A^* b} \otimes\, \mathcal{O}^k_{\mathbb{CP}^3}(1) \longrightarrow 0$$

One can then verify that the bundle $\mathcal{E}$ constructed from this data is indeed an instanton bundle [A], [Wa-We].

To explicitly construct $\mathcal{E}$, one chooses bases on $\mathbb{C}^4$, W, and V, and one defines maps σ, σ_W, σ_V, and b as follows:

(i) The map $\sigma : \mathbb{C}^4 \to \mathbb{C}^4$ corresponds to left multiplication $j : \mathbb{H}^2 \to \mathbb{H}^2$.

(ii) Let $w_1, \ldots, w_k$ be a real basis for W, so that the map $\sigma_W : W \to W$ is complex conjugation after identifying W with $\mathbb{C}^k$ and recalling that $\sigma w_m = w_m$, for $m = 1, \ldots, k$. Then the vector space W with $\dim_{\mathbb{C}} W = k$, may be viewed as the complexification of the real vector space $W_{\mathbb{R}}$ left fixed by σ, so that $W = \mathbb{C} \otimes W_{\mathbb{R}}$.

(iii) Let $\{v_0, \ldots, v_k, \sigma v_0, \ldots, \sigma v_k\}$ be an orthogonal basis for V with respect to the Hermitian inner product. Then the complex vector space V with $\dim_{\mathbb{C}} V = 2k+2$ may be viewed as a left quaternion vector space with $\dim_{\mathbb{H}} V = k+1$, with left multiplication by j on $\mathbb{H}^{k+1}$ corresponding to $\sigma_V : V \to V$. The symplectic form on V is now represented by a $(2k+2) \times (2k+2)$ complex matrix,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $k \times k$ identity matrix. Then if u, v are row vectors in $\mathbb{C}^{2k+2}$, we have $b(u,v) = uJv^t$. The Hermitian form $h(u,v) = uv^\dagger$, the standard positive definite Hermitian scalar product on $\mathbb{C}^{2k+2}$. The quaternionic basis $\{v_0+\sigma v_0, \ldots, v_k+\sigma v_k\}$ is orthogonal with respect to the standard quaternionic scalar product $q(\cdot,\cdot)$ on $\mathbb{H}^{k+1}$ given by $q(\xi,\eta) = \xi\eta^\dagger$. Note that $q(\cdot,\cdot) = h(\cdot,\cdot) + b(\cdot,\cdot)j$ on $\mathbb{H}^{k+1} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1}j$.

(iv) After identifying V with $\mathbb{C}^{2k+2}$, the map $\sigma_V : V \to V$ is given by $\sigma v = J\bar{v}$. In matrix form, the isotropy condition is equivalent to

$$A(z)^t JA(z) = 0 \qquad \text{for all } z \in \mathbb{C}^4.$$

The matrix representation of $A(z) = \sum_{\alpha=0}^{3} A_\alpha z_\alpha$ is then defined by

$$A(z)w_m = \sum_{\alpha=0}^{3}\sum_{n=0}^{k} z_\alpha A^\alpha_{mn} v_n + z_\alpha A^\alpha_{m,k+1+n}\sigma v_n \qquad \text{for } m = 1, \ldots, k.$$

Then A_α may be written as a block matrix $A_\alpha = (A'_\alpha, A''_\alpha)$ for $\alpha = 0, \ldots, 3$, where A'_α, A''_α are complex $k \times (k+1)$ matrices. The reality condition $\sigma A(z)w = A(\sigma z)\sigma w$ now becomes

$$\begin{array}{llll} A'_1 & = -\overline{A''_0}, & A'_3 & = -\overline{A''_2}, \\ A''_1 & = \overline{A'_0}, & A''_3 & = \overline{A'_2}. \end{array}$$

Alternatively, we may write the reality condition as $J\overline{A(z)} = A(\sigma z)$, and in terms of the matrices $A_0, \ldots, A_3$, this becomes

$$\begin{array}{llll} J\overline{A_0} & = A_1, & J\overline{A_2} = A_3, \\ J\overline{A_1} & = -A_0, & J\overline{A_3} = -A_2, \end{array}$$

and decomposing the matrices A_α into blocks (A'_α, A''_α), we obtain the previous matrix conditions. Letting $C = A'_0 + A''_0 j$, $D = A'_2 + A''_2 j$, we observe that

$$A(z) = z_0A_0 + z_1A_1 + z_2A_2 + z_3A_3 = xC + yD = A(x,y),$$

where $x = z_0 + z_1 j$, $y = z_2 + z_3 j$. This gives the convenient $k \times (k+1)$ quaternion matrix representation of A.

The quaternion formulation may be seen a little more directly as follows. We have $\mathbb{C}^4 \otimes_{\mathbb{C}} W \simeq \mathbb{H}^2 \otimes_{\mathbb{R}} W_{\mathbb{R}}$, and the induced map σ on $\mathbb{C}^4 \otimes_{\mathbb{C}} W$ corresponds to left multiplication by j on the left quaternion vector space $\mathbb{H}^2 \otimes_{\mathbb{R}} W_{\mathbb{R}}$. The complex linear map $A : \mathbb{C}^4 \otimes_{\mathbb{C}} W \to V$ may now be viewed as a map

$$A : \mathbb{H}^2 \otimes_{\mathbb{R}} W_{\mathbb{R}} \longrightarrow V,$$

and compatibility of $A(z)$ with σ is equivalent to requiring that A be quaternion linear. If $C = (C_1, \ldots, C_k)^t$ and $D = (D_1, \ldots, D_k)^t$, where C_i and D_i denote the rows of C and D respectively, then

$$\begin{aligned} C_i &= A((1,0) \otimes w_i), \\ D_i &= A((0,1) \otimes w_i) \qquad \text{for } i = 1, \ldots, k. \end{aligned}$$

The non-degeneracy or rank condition on A is equivalent to requiring that the quaternion matrix $A(x,y)$ have rank k for all $(x,y) \neq 0$. The isotropy condition is equivalent to requiring that $A(x,y)A(x,y)^\dagger$ be real for all (x,y), where $A^\dagger$ denotes the quaternionic-conjugate matrix transpose of A. Let $\mathbb{HP}^1 = \mathbb{S}^4$ denote the left quaternionic projective space and recall that $\mathbf{Sp}(1) = \mathbf{SU}(2)$. The $\mathbf{Sp}(1)$-bundle $E \to \mathbb{HP}^1$ is obtained by setting $E_{(x,y)} = (\operatorname{Im} A(x,y))^\perp \subset \mathbb{H}^{k+1}$, for $[x,y] \in \mathbb{HP}^1$, and the ASD $\mathbf{Sp}(1)$-connection is obtained by orthogonal projection from $\underline{\mathbb{H}}^{k+1}$ to E, where $\underline{\mathbb{H}}^{k+1}$ denotes the trivial bundle $\mathbb{HP}^1 \times \mathbb{H}^{k+1} \to \mathbb{HP}^1$. where $\mathbb{HP}^1 = \mathbb{S}^4$ denotes the left quaternionic projective space. We now have a quaternionic monad over $\mathbb{HP}^1$ given by

$$0 \longrightarrow E \longrightarrow \underline{\mathbb{H}}^{k+1} \longrightarrow kL \longrightarrow 0$$

where $L \to \mathbb{HP}^1$ denotes the tautological quaternionic line bundle and $kL = L \oplus \cdots \oplus L$. The $k \times (k+1)$ quaternion matrix A is now required to satisfy the conditions:
(i) $A(x,y)$ has quaternion rank k for all $(x,y) \in \mathbb{H}^2 \setminus (0,0)$;
(ii) $A(x,y)A(x,y)^\dagger$ is real for all $(x,y) \in \mathbb{H}^2$.

In terms of moduli spaces, the main result is the following:

Theorem 3.1.4. [B-H, p. 19], [A] *There is a bijection between*
(i) *Isomorphism classes of instanton bundles $\mathcal{E} \to \mathbb{CP}^3$; and*
(ii) *Isomorphism classes of holomorphic bundles $\mathcal{E} \to \mathbb{CP}^3$ which arise from the Horrocks construction via a linear map $A(z) : W \to V$.*

Remark 3.1.5. Two bundles $\mathcal{E} \sim (A, W, V)$, $\mathcal{E}' \sim (A', W', V')$ arising from the Horrocks construction are isomorphic if and only if there are complex vector space isomorphisms $W \to W'$, $V \to V'$ preserving structures and taking A to A'. Hence, A and A' give isomorphic bundles $\mathcal{E}$ and $\mathcal{E}'$ if and only if

$$A'(z) = QA(z)R, \qquad \text{for } Q \in \mathbf{Sp}(k+1),\ R \in \mathbf{GL}(k, \mathbb{R}).$$

Here, $\mathbf{Sp}(k+1)$ denotes the subgroup of $\mathbf{GL}(2k+2, \mathbb{C})$ preserving the symplectic form b and real form σ_V, and $\mathbf{GL}(k, \mathbb{R})$ denotes the subgroup of $\mathbf{GL}(k, \mathbb{C})$ preserving the real form σ_W [A, p. 62], [A-D-H-M, p. 186], [Wa-We, p. 417], [B-H, p. 19], [D-M4, p. 847].

In particular, if two triples (A, W, V), (A', W', V') are isomorphic as above, we then obtain isomorphisms between the complex vector bundles E, E' over $\mathbb{S}^4$ and their associated principal bundles P, P', sending the connection ω to ω'. So if $f : P \to P'$ is the induced isomorphism of $\mathbf{SU}(2)$ bundles over $\mathbb{S}^4$, then $\omega = f^* w'$.

Let T_k denote the space of quaternionic $k \times (k+1)$ ADHM matrices C, D, such that $A(x,y)$ has rank k and $A(x,y)A(x,y)^\dagger$ is real for all $(x,y) \in \mathbb{H}^2 \setminus (0,0)$, modulo the action of $\mathbf{Sp}(k+1) \times \mathbf{GL}(k, \mathbb{R})$. Then $\dim_{\mathbb{R}} T_k = 8k - 3$ [A, p. 26], coinciding with $\dim_{\mathbb{R}} \mathcal{M}_k$, where $\mathcal{M}_k$ is the moduli space of ASD $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$. In particular, the ADHM construction gives a diffeomorphism $T_k \to \mathcal{M}_k$.

§3.2. Global Sections of Twisted Instanton Bundles

In the next chapter, we construct a bundle $E \to \mathbb{S}^4$ with $-c_2(E) = k$ using the ADHM monad construction with a choice of ADHM matrices corresponding to $k+1$ distinct points P_i in $\mathbb{S}^4$ and $k+1$ positive weights λ_i. When E has an ASD $\mathbf{SU}(2)$-connection, this construction produces a stable, holomorphic, rank 2 vector bundle $\mathcal{E}$ over $\mathbb{CP}^3$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. We exhibit a section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ with zero set $(s)_0$ given precisely by the real lines $Y_i = \pi^{-1}(P_i)$. Fixing a choice of isomorphism $\varphi : \det \mathcal{E}(1) \simeq \mathcal{O}_{\mathbb{CP}^3}(2)$, we see that a choice of complex weights or 'residues' $\zeta_0, \ldots, \zeta_k$ corresponds to a choice of isomorphism $K_Y \simeq \mathcal{O}_Y(-2)$. If the bundle $\mathcal{E}$ has a real or symplectic structure, then the ζ_i are required to be real. Finally, the requirement that $\mathcal{E}$ has no real jumping lines is equivalent to the condition $\zeta_i > 0$ [Wa-We, p. 413]. The bundle $\mathcal{E}$ constructed explicitly from ADHM data corresponding to a choice of points P_i in $\mathbb{S}^4$ and weights $\lambda_i > 0$ gives $\zeta_i = \lambda_i^2$. This shows the relationship between the ADHM monad construction of instanton bundles over $\mathbb{CP}^3$ and Hartshorne's construction of the same bundles using the method of curves [A], [Har2, 3], [Wa-We]. Moreover, we obtain an alternative verification that our ad hoc choice of ADHM matrices gives all ASD $\mathbf{SU}(2)$-connections with $k = 2$ on $\mathbb{S}^4$.

For simplicity, we assume $k = 2$. Let $A(z) = \sum_{\alpha=0}^{3} z_\alpha A_\alpha = xC + yD$, where the A_α are 2×6 complex matrices, and C,D are 2×3 quaternionic matrices. Let $\underline{\mathbb{HP}}^1$ denote the right projective space. For points $\underline{P}_i$ with homogeneous coordinates $[a_i, b_i]$ in $\underline{\mathbb{HP}}^1$, for $i = 0, 1, 2$, the corresponding matrices C, D may be chosen to be

$$C = \begin{pmatrix} b_0 & -b_1 & 0 \\ b_0 & 0 & -b_2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -a_0 & a_1 & 0 \\ -a_0 & 0 & a_2 \end{pmatrix},$$

and so

$$A(x,y) = \begin{pmatrix} xb_0 - ya_0 & -xb_1 + ya_1 & 0 \\ xb_0 - ya_0 & 0 & -xb_2 + ya_2 \end{pmatrix}.$$

Remark 3.2.1. If $[x, y]$ denote homogeneous coordinates for the left projective space $\mathbb{HP}^1$ and $[a, b]$ denote homogeneous coordinates for the right projective space $\underline{\mathbb{HP}}^1$, then $xb - ya = 0$ if and only if $[x, y] = [b^{-1}, a^{-1}] = P \in \mathbb{HP}^1$. We have a map, denoted $\underline{\mathbb{HP}}^1 \to \mathbb{HP}^1$, $\underline{P} \to P$, given by $[a, 1] \mapsto [a, 1]$, $[1, b] \mapsto [1, b]$, so that on overlapping patches $[a, b] \mapsto [b^{-1}, a^{-1}]$.

Lemma 3.2.2. *Suppose $[a, b] \in \underline{\mathbb{HP}}^1$ (right projective space). Then $xb - ya = 0$ or $[x, y] = [b^{-1}, a^{-1}]$, $[x, y] \in \mathbb{HP}^1$ (left projective space) if and only if $z \in H \cap \hat{H}$, where $H = \{z \in \mathbb{CP}^3 : f(z) = 0\}$, $\hat{H} = \{z \in \mathbb{CP}^3 : h(z) = 0\}$,*

$$f(z) = z_0 b' - z_1 \overline{b}'' - z_2 a' + z_3 \overline{a}'',$$
$$h(z) = z_0 b'' + z_1 \overline{b}' - z_2 a'' - z_3 \overline{a}',$$

$z = [z_0, z_1, z_2, z_3]$, with $x = z_0 + z_1 j$, $y = z_2 + z_3 j$, $a = a' + a'' j$, and $b = b' + b'' j$ under the standard identification $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$.

Proof. If $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$ is the standard projection, then $[x, y] = \pi[z_0, z_1, z_2, z_3]$, where $x = z_0 + z_1 j$, $y = z_2 + z_3 j$, and we observe that

$$xb - ya = (z_0, z_1) \begin{pmatrix} b' & b'' \\ -\overline{b}'' & \overline{b}' \end{pmatrix} - (z_2, z_3) \begin{pmatrix} a' & a'' \\ -\overline{a}'' & \overline{a}' \end{pmatrix},$$

under the identification $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. □

Let $a_i = a_i' + a_i'' j$ and $b_i = b_i' + b_i'' j$, for $i = 0, 1, 2$. Then the relationship between the quaternionic matrices C, D and the complex matrices A_α', A_α'' is given by

$$C = A_0' + A_0'' j, \qquad D = A_2' + A_2'' j,$$

so that the corresponding matrices A_α are given by

$$\begin{aligned} A_0 &= (A_0', A_0''), \qquad & A_1 &= (-\overline{A_0''}, \overline{A_0'}), \\ A_2 &= (A_2', A_2''), \qquad & A_3 &= (-\overline{A_2''}, \overline{A_2'}). \end{aligned}$$

Then, the matrix $A(z)$ has the following form:

$$A(z) = \begin{pmatrix} f_0(z) & f_1(z) & 0 & h_0(z) & h_1(z) & 0 \\ f_0(z) & 0 & f_2(z) & h_0(z) & 0 & h_2(z) \end{pmatrix}.$$

Recall that $\mathcal{E} = \operatorname{Ker}(A^t J)/\operatorname{Im} A$, where

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{CP}^3}(-1) \xrightarrow{A} \mathcal{O}^6_{\mathbb{CP}^3} \xrightarrow{A^t J} \mathcal{O}^2_{\mathbb{CP}^3}(1) \longrightarrow 0$$

and so the fibres are given by $\mathcal{E}_z = \operatorname{Ker}(A^t(z)J)/\operatorname{Im} A(z)$, for $z \in \mathbb{CP}^3$. Let $\{e_0, e_1, e_2, \hat{e}_0, \hat{e}_1, \hat{e}_2\}$ be the standard ordered basis of $\mathbb{C}^6$, so that $e_0 = (1, 0, 0, 0, 0, 0)$, etc. We let $H_i, \hat{H}_i \subset \mathbb{CP}^3$ be the associated hyperplanes

$$H_i = \{z \in \mathbb{CP}^3 : f_i(z) = 0\} \quad \text{and} \quad \hat{H}_i = \{z \in \mathbb{CP}^3 : h_i(z) = 0\},$$

and so the lines Y_i are the complete intersections $Y_i = H_i \cap \hat{H}_i$ for $i = 0, 1, 2$.

We define sections $s_i \in H^0(\mathbb{CP}^3, \mathcal{O}^6_{\mathbb{CP}^3}(1))$ by setting

$$s_i(z) = f_i(z) e_i + h_i(z) \hat{e}_i \qquad \text{for } z = [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3.$$

Then, s_0 is a section of $\mathcal{O}^6_{\mathbb{CP}^3}(1)$ with zero set $Y_0 = H_0 \cap \hat{H}_0$. Moreover,

$$A(z) J s_0^t(z) = A(z) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} s_0^t(z) = 0 \qquad z \in \mathbb{CP}^3,$$

so that $s_0(z) \in \operatorname{Ker} A^t(z) J$, and hence s_0 defines a section in $H^0(\mathbb{CP}^3, \mathcal{E}(1))$. Then, $s_0(z) = 0 \in \mathcal{E}_z$ if and only if $z \in Y_0$ or $s_0(z) \in \operatorname{Im} A(z)$, where $\operatorname{Im} A(z)$ denotes the span of the rows of the matrix $A(z)$ in $\mathbb{C}^6$. If $s_0(z) = 0 \in \mathcal{E}_z$ and $z \neq Y_0$, then $f_0(z) \neq 0$ or $h_0(z) \neq 0$, and it follows that either $z \in Y_1 = H_1 \cap \hat{H}_1$, or $z \in Y_2 = H_2 \cap \hat{H}_2$. We observe that the sections $s_i \in H^0(\mathbb{CP}^3, \mathcal{O}^6_{\mathbb{CP}^3}(1))$ all project to the same section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$. We have the

Lemma 3.2.3. *Let P_0, P_1, P_2 be distinct points in $\mathbb{HP}^1$ (left projective space), with homogeneous coordinates given by $\underline{P}_i = [a_i, b_i]$ in $\underline{\mathbb{HP}}^1$ (right projective space). Let $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ be the corresponding section constructed as above. Then $(s)_0 = Y$, where Y consists of three skew lines Y_0, Y_1, Y_2 in $\mathbb{CP}^3$ given by $Y_i = \pi^{-1}(P_i)$.*

Fix an isomorphism $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{CP}^3}$, and hence an isomorphism $\varphi : \det \mathcal{E}(1) \to \mathcal{O}_{\mathbb{CP}^3}(2)$. Note that $e_i, \hat{e}_i \in \mathbb{C}^6$ project to elements in $\mathcal{E}_z$ when $z \in Y_i$ and $f_i(z) = h_i(z) = 0$. Moreover, $e_i \wedge \hat{e}_i = e_i J \hat{e}_i^t = 1$. For z in a neighbourhood of Y_i, choose $\varepsilon_i(z)$, $\hat{\varepsilon}_i(z)$ vanishing along Y_i, so that $\theta_i(z) = e_i + \varepsilon_i(z)$, $\hat{\theta}_i(z) = \hat{e}_i + \hat{\varepsilon}_i(z)$ give local frames for $\mathcal{E}$. Then

$$\begin{aligned} s_i(z) &= f_i(z)e_i + h_i(z)\hat{e}_i = \tilde{f}_i(z)\theta_i(z) + \tilde{h}_i(z)\hat{\theta}_i(z), \\ \wedge^2 s_i(z) &= \tilde{f}_i(z)\tilde{h}_i(z)\theta_i(z) \wedge \hat{\theta}_i(z), \end{aligned}$$

and the $\wedge^2 s_i$ are sections of $\det \mathcal{E} = \mathcal{O}_{\mathbb{CP}^3}(2)$. Fixing a non-zero section $\theta \in H^0(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(2))$, we have

$$\wedge^2 s_i = \zeta_i^{-1}\theta \in H^0(\mathbb{CP}^3, \mathcal{O}^*_{\mathbb{CP}^3}) = \mathbb{C}^*.$$

Indeed, if $(\wedge^2 s_i) = \theta$ and f_i, h_i are replaced by $\lambda_i^{-1} f_i$, $\lambda_i^{-1} h_i$, where $\lambda_i \in \mathbb{C}^*$, we have $\zeta_i = \lambda_i^2$ for $i = 0, 1, 2$. Since $\mathcal{E}^*$ is an instanton bundle, the constants ζ_i are required to be positive, and hence the λ_i are real.

The analogous relationship between the monad construction of rank 2 holomorphic bundles over $\mathbb{CP}^2$ and the construction of bundles corresponding to configurations of points in $\mathbb{CP}^2$ and complex residues at those points is described in [D-K, p. 397].

§3.3. Conics in Complex Projective Space

Recall that the curve $Y = Y_0 \cup Y_1 \cup Y_2 \subset \mathbb{CP}^3$ arises as the zero set of a section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$. As s varies in $\mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{E}(1))) \simeq \mathbb{CP}^1$, then the corresponding points Y_0, Y_1, Y_2 move in a conic $\gamma \subset G(1,3)$, inducing a linear system g_3^1 without basepoints on $\mathbb{CP}^1_\alpha$. General references for this section are [Har, 1-3], [G-H], [J-N-R], [N-T, 1-3], [S-T].

In order to see the consequences for the space $\mathcal{N}_{\mathbb{I}}(0,2)$, it is useful to follow the approach of Singhof and Trautman in constructing this moduli space as the total space of a certain fibre bundle [S-T, p. 336-342]. Fix a complex vector space $U \simeq \mathbb{C}^2$ and let $V = U \times U$, with $\mathbb{CP}^1 = \mathbb{P}(U)$ and $\mathbb{CP}^3 = \mathbb{P}(V)$. Let $Q \subset \mathbb{CP}^3$ be a non-singular quadric surface, so that $Q \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$. For convenience, we use the labelling $\mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta$. Let $g \in \mathbf{GL}(V)$ and let σ denote the Segré embedding

$$\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3, \qquad ([s_0, s_1], [t_0, t_1]) \longmapsto [s_0t_0, s_0t_1, s_1t_0, s_1t_1].$$

Let $\tilde{g} \in \mathbf{PGL}(V)$ be the induced automorphism of $\mathbb{CP}^3$ and let

$$\varphi_g = \tilde{g} \circ \sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$$

be the corresponding embedding. Let $Q_g = \operatorname{Im}\varphi_g \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, with $Q_g \subset \mathbb{CP}^3$, and note that every non-singular quadric arises via this construction [S-T], [G-H].

For convenience, let $G = \mathbf{GL}(V)$ and let $H \subset G$ be the subgroup $H = \mathbf{GL}(U) \times \mathbf{GL}(U)$. On U we have the real forms c, j, where $c^2 = 1_U$ (corresponding to complex conjugation, the standard real structure) and $j^2 = -1_U$ (the non-standard real or symplectic structure). Let $U_{\mathbb{R}}$ denote the real vector space fixed by c, so that $U \simeq U_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. These structures are inherited by $\mathbb{CP}^1$, so $\mathbb{P}(U_{\mathbb{R}}) = \mathbb{RP}^1$ denotes the real points of $\mathbb{CP}^1$ with respect to c. Moreover, U becomes a 1-dimensional left $\mathbb{H}$-vector space by setting $j \cdot u = j(u)$, and V becomes a 2-dimensional left $\mathbb{H}$-vector space using $J = c \oplus j$. Let $G_J = \operatorname{Aut}_{\mathbb{H}}(V) \subset \mathbf{GL}(V)$ be the subgroup of complex automorphisms φ such that $\varphi(Jv) = J\varphi(v)$, and hence $G_J = \mathbf{GL}(2, \mathbb{H})$. Then for any $g \in G_J$, the corresponding quadric Q_g is invariant under J. Let $H_J = H \cap G_J$, and note that

$$H_J = \mathbf{GL}_{\mathbb{R}}(U_{\mathbb{R}}) \times \mathbf{GL}_{\mathbb{H}}(U) \simeq \mathbf{GL}(2, \mathbb{R}) \times \mathbf{GL}(1, \mathbb{H}).$$

As usual, $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$ denotes the standard projection, taking $\mathbb{C}$-lines in V to the corresponding $\mathbb{H}$-lines in V. Then the J-invariant lines in $\mathbb{CP}^3$ are precisely the fibres of π.

Fix a point $t \in \mathbb{CP}^1_\alpha$. Then $\{t\} \times \mathbb{CP}^1_\alpha \subset \mathbb{CP}^1_\alpha \times \mathbb{CP}^1_\beta$, and so we have an induced map

$$\{t\} \times \mathbb{CP}^1_\alpha \xrightarrow{\varphi_g} \mathbb{CP}^3 \xrightarrow{\pi} \mathbb{S}^4, \qquad \langle t, u\rangle \longmapsto \pi\varphi_g(t, u),$$

for any $u \in \mathbb{CP}^1_\beta$. We recall from our discussion of Hartshorne's description of the space $\mathcal{N}_{\mathbb{I}}(0, 2)$, that the factor $\mathbb{CP}^1_\alpha$ had its standard real structure: the corresponding real points in $\mathbb{CP}^1_\alpha$ are given by $\mathbb{S}^1 \simeq \mathbb{RP}^1 \subset \mathbb{CP}^1_\alpha$. If $t \in \mathbb{RP}^1 \subset \mathbb{CP}^1_\alpha$, then φ_g maps $\{t\} \times \mathbb{CP}^1_\beta$ to a real line in $\mathbb{CP}^3$, and the projection π maps this real line to a point $P \in \mathbb{S}^4$. If we choose any fixed $u \in \mathbb{CP}^1_\beta$, we then have an embedding (independent of the choice of u)

$$\mathbb{RP}^1 \hookrightarrow \mathbb{S}^4, \qquad t \longmapsto \pi\varphi_g(t, u)$$

of $\mathbb{S}^1 \simeq \mathbb{RP}^1$ into a circle in $\mathbb{S}^4$ [S-T, p. 342]. So as t varies in $\mathbb{S}^1 \simeq \mathbb{RP}^1 \subset \mathbb{CP}^1_\alpha$, the corresponding point $P_t = \pi\varphi_g(t, u)$ moves in a circle in $\mathbb{S}^4$.

If t varies in $\mathbb{CP}^1$, then the corresponding point $Y_t = \pi\varphi_g(t, u)$ in $G(1, 3)$ (or line in $\mathbb{CP}^3$) moves in the conic γ in $G(1, 3)$ which we described earlier. Recall that the curve $Y = Y_0 \cup Y_1 \cup Y_2 \subset \mathbb{CP}^3$ arises as the zero set of a section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$. As s varies in $\mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{E}(1))) \simeq \mathbb{CP}^1$, then the corresponding points Y_0, Y_1, Y_2 move in the conic $\gamma \subset G(1, 3)$. Consequently, if the lines $Y_i \subset \mathbb{CP}^3$ are real, then the corresponding points P_0, P_1, P_2 — where $P_i = \pi(Y_i) \in \mathbb{S}^4$ — move in a circle in $\mathbb{S}^4$ as s varies in $\mathbb{P}(H^0(\mathbb{CP}^3, \mathcal{E}(1)))_{\mathbb{R}} \simeq \mathbb{RP}^1$. Of course, the rank 2 holomorphic bundle $\mathcal{E}$ is fixed and so the pair $\langle \mathcal{E}, s\rangle$ determines the same equivalence class of ASD $\mathbf{SU}(2)$-connections while s varies as above. This phenomenon was described in [J-N-R] using different techniques.

§3.4. Two Constructions of Instanton Bundles

We have described how one may use the ADHM method to construct complex, rank 2 bundles with ASD $\mathbf{SU}(2)$-connection over $\mathbb{S}^4$, and then stable, holomorphic, rank 2 bundles over $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = k$, corresponding to configurations of points P_i in $\mathbb{S}^4$ and positive weights λ_i. In this section, we provide an outline of how one may construct the ASD $\mathbf{SU}(2)$ connections, if we are given an instanton bundle over $\mathbb{CP}^3$ corresponding to a curve Y in $\mathbb{CP}^3$ consisting of $k+1$ real skew lines. General references for this section are [A-W], [D-K], [Wa-We].

If $\mathcal{E} \to \mathbb{CP}^3$ is a rank 2 instanton bundle corresponding to a curve Y consisting of $k+1$ skew lines Y_i in $\mathbb{CP}^3$, then $\mathcal{E}(1)$ has a section s vanishing only along the curve $Y = Y_0 \cup \ldots \cup Y_k \subset \mathbb{CP}^3$. Hence, s is nowhere vanishing along the complement $Y' = \mathbb{CP}^3 - Y$, and so generates a trivial line bundle on Y'. Noting that $\mathcal{I}_Y|Y' = \mathcal{O}_{\mathbb{CP}^3}|Y'$, the section s induces an exact sequence [O-S-S, p. 93], [G-H, p. 726]:

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^3}|Y' \xrightarrow{s} \mathcal{E}(1)|Y' \longrightarrow \mathcal{O}_{\mathbb{CP}^3}(2)|Y' \longrightarrow 0$$

and so $\mathcal{E}(1)|Y'$ is an extension of $\mathcal{L}_2 = \mathcal{O}_{\mathbb{CP}^3}(2)|Y'$ by $\mathcal{L}_1 = \mathcal{O}_{\mathbb{CP}^3}|Y'$. Such extensions are classified by

$$\begin{aligned}\mathrm{Ext}^1(Y'; \mathcal{L}_2, \mathcal{L}_1) &\simeq H^1(\mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)) \\ &\simeq H^1(Y', \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) \\ &\simeq H^1(Y', \mathcal{O}_{\mathbb{CP}^3}(-2)),\end{aligned}$$

as discussed in [Wa-We, p. 412], [D-K, p. 388], [G-H, p. 725]. Thus, such an extension naturally corresponds to a cocycle $\Gamma \in H^1(Y', \mathcal{O}_{\mathbb{CP}^3}(-2))$, and this cocycle may be readily computed from the transition functions of $\mathcal{E}(1)|Y'$ — see, for example, [A-W, p. 122], [Wa-We, p. 399], [D-K, p. 389], [Gu]. Indeed, the 2×2 transition matrices may be put in a standard upper triangular form and the cocycle Γ may then be identified with the upper-right entry. If

$$\Gamma = \sum_{i=0}^{k} \zeta_i \Gamma_i, \qquad \text{where } \Gamma_i \in H^1(\mathbb{CP}^3 - Y_i, \mathcal{O}_{\mathbb{CP}^3}(-2)),$$

we note that Γ determines a singular potential ϕ [Wa-We, p. 388-409]:

$$\phi(x) = \sum_{i=0}^{k} \frac{\zeta_i}{|x - a_i|^2} \qquad x \in \mathbb{H}.$$

Here, $a_i = \pi(Y_i) \in \mathbb{H} \cup \{\infty\}$, and the reality condition on $\mathcal{E}$ is equivalent to the condition that the weights ζ_i are real and that the lines Y_i are real lines — corresponding to points $a_i \in \mathbb{H} \cup \{\infty\}$. The requirement that $\mathcal{E}$ has no real jumping lines is equivalent to the condition $\phi(x) > 0$ for $x \in \mathbb{H} \cup \{\infty\}$, and so $\zeta_i > 0$ [Wa-We, p. 412]. Finally, the potential ϕ determines a singular, local, anti-self-dual, $\mathbf{SU}(2)$-connection 1-form:

$$\omega(x) = \sum_{i=0}^{k} \frac{\zeta_i}{|x - a_i|^4 \rho} \mathrm{Im}\, \{(x - a_i) d\overline{x}\}, \qquad \text{where } \rho(x) = \sum_{i=0}^{k} \frac{\zeta_i}{|x - a_i|^2},$$

[Wa-We, p. 388-409]. This is the local connection 1-form described in [J-N-R], corresponding to a choice of positive weights λ_i, with $\zeta_i = \lambda_i^2$ above, and points $P_i \in \mathbb{S}^4$, with $P_i = [a_i, 1]$ above.

Conversely, given distinct points $P_i \in \mathbb{S}^4$ and positive weights λ_i, with $\zeta_i = \lambda_i^2$, we may construct an instanton bundle $\mathcal{E}$ via an appropriate choice of ADHM matrices. By explicitly constructing the section $s \in H^0(\mathbb{CP}^3, \mathcal{E}(1))$ and employing the correspondence between rank 2 holomorphic bundles over $\mathbb{CP}^3$ and curves in $\mathbb{CP}^3$ as discussed in [Har2, 3], [O-S-S], and [Wa-We], we recover the original data consisting of distinct points P_i and positive weights λ_i. This gives the correspondence between the two methods of constructing instanton bundles, and moreover, demonstrates that our ad hoc choice of ADHM matrices does indeed yield all anti-self-dual $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ with $k = 2$.

CHAPTER IV
PARAMETRIZATION OF SELF-DUAL CONNECTIONS

We first describe the ADHM construction of all self-dual connection 1-forms on an $\mathbf{SU}(2)$-bundle $P \to \mathbb{S}^4$ with $-c_2(P)[\mathbb{S}^4] = k > 0$. We then discuss the parametrization of all self-dual connection 1-forms with $k = 2$. In the following, the moduli space $\mathcal{M}_k^*(\mathbb{S}^4, \mathbf{SU}(2))$ of self-dual $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$ simply by $\mathcal{M}_k$. General references for chapter are [A], [C-G-F-T], [C-W-S], [D1], [D-K], [Wa-We].

§4.1. Parametrization of Connection One-forms

We review the ADHM construction of the classifying maps $f : \mathbb{S}^4 \to \mathbb{HP}^k$ which provide all self-dual $\mathbf{SU}(2)$-connections on $P \to \mathbb{S}^4$ by pulling back the canonical $\mathbf{SU}(2)$-connection on $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$. General references for section are [A], [Bo-Ma], [R2], [Sa].

We consider self-dual rather than anti-self-dual connections and so all quaternionic vector spaces are now assumed to be right vector spaces, and quaternionic-linear maps act by left matrix multiplication. We now let $\mathbb{HP}^n$ denote the right quaternionic projective space:

$$\mathbb{HP}^n = \{(q_0, \dots, q_n) \in \mathbb{H}^{n+1} \setminus \{0\} : (q_0, \dots, q_n) \sim (q_0 a, \dots, q_n a), a \in \mathbb{H}^*\}.$$

As before, $G_m(\mathbb{H}^n)$ denotes the Grassmann manifold of m-dimensional quaternionic subspaces of $\mathbb{H}^n$. The symplectic group $\mathbf{Sp}(n)$ is defined by:

$$\mathbf{Sp}(n) = \{Q \in \mathbf{GL}(n, \mathbb{H}) : (Qp)^\dagger (Qq) = p^\dagger q\},$$

where $p^\dagger q = \sum_{i=1}^n \overline{p}_i q_i$ denotes the symplectic scalar product on $\mathbb{H}^n$.

Let $A(x, y) = Cx + Dy$, where C and D are $(k + 1) \times k$ quaternionic ADHM matrices chosen so that $A(x, y)$ satisfies the following conditions:
(i) $\mathrm{Rank}_{\mathbb{H}}\, A(x, y) = k$, for all $(x, y) \in \mathbb{H}^2 \setminus (0, 0)$;
(ii) $A^\dagger(x, y)A(x, y)$ is real for all $(x, y) \in \mathbb{H}^2$.

Hence, we obtain a map $A : \mathbb{HP}^1 \to G_k(\mathbb{H}^{k+1})$, and $A(x, y)$ now acts on $\mathbb{H}^{k+1}$ by left matrix multiplication. Define $u : \mathbb{HP}^1 \to \mathbb{HP}^k$ by requiring that:

$$u(x, y)^\dagger A(x, y) = 0,$$
$$u^\dagger(x, y)u(x, y) = 1,$$

for all $(x, y) \in \mathbb{H}^2 \setminus (0, 0)$. The map u is a quaternionic rational map, in the sense that it may appear to have singularities at a finite number of points of $\mathbb{HP}^1$. However, these are always removable — we will see this explicitly for the special cases considered in the next section — and so u may be assumed to be smooth [A, p. 26]. Let $E = u^* E' \to \mathbb{HP}^1$ denote the pull-back of the universal quaternionic line bundle $E' \to \mathbb{HP}^k$, and let $P = u^*\mathbb{S}^{4k+3} \to \mathbb{HP}^1$ denote the pull-back of the universal principal $\mathbf{Sp}(1)$-bundle $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$.

We first compute the transition functions of the bundle $\pi : \mathbb{S}^{4k+3} \to \mathbb{HP}^k$. Let $\{O'_i\}$ denote the standard open cover of $\mathbb{HP}^k$, so that $O'_i = \{[q_0, \ldots, q_k] : q_i \neq 0\}$, $i = 0, \ldots, k$, where $q = (q_0, \ldots, q_k)$ are coordinates on $\mathbb{H}^{k+1}$. Define a system of local sections $\sigma'_i : O'_i \to \mathbb{S}^{4k+3}$, by setting

$$\sigma'_i(q) = \frac{q}{|q|}\frac{\overline{q}_i}{|q_i|} \qquad \text{on } O'_i.$$

The corresponding transition functions $T'_{ij} : O'_i \cap O'_j \to \mathbf{Sp}(1)$ are defined by $\sigma'_j(q) = \sigma'_i(q)T'_{ij}(q)$, so that

$$T'_{ij}(q) = \frac{q_i}{|q_i|}\frac{\overline{q}_j}{|q_j|} \qquad \text{on } O'_i \cap O'_j.$$

Alternatively, we may define local trivializations $\tau_i : \pi^{-1}(O'_i) \to O'_i \times \mathbf{Sp}(1)$, with $\tau_i(p) = (\pi(p), \varphi_i(p))$, where $\pi(p) = [p]$ and

$$\varphi : \pi^{-1}(O'_i) \to \mathbf{Sp}(1), \qquad (p_0, \ldots, p_k) \longmapsto \frac{p_i}{|p_i|}.$$

The transition functions are then given by $T'_{ij}(q) = \varphi_i(q)\varphi_j(q)^{-1}$, just as before.

Let $\{O_i\}$ be the induced open cover of $\mathbb{HP}^1$ obtained by setting $O_i = u^{-1}(O'_i)$ for $i = 0, \ldots, k$, and let $\sigma_i = u^*\sigma'_i : O_i \to P$ be the corresponding local sections:

$$\sigma_i = \frac{u}{|u|}\frac{\overline{u}_i}{|u_i|} \qquad \text{on } O_i.$$

The induced transition functions $T_{ij} = u^*T'_{ij} : O_i \cap O_j \to \mathbf{Sp}(1)$ of P are:

$$T_{ij} = \frac{u_i}{|u_i|}\frac{\overline{u}_j}{|u_j|} \qquad \text{on } O_i \cap O_j.$$

Recall that the total space P is

$$P = \{([x, y], p) \in \mathbb{HP}^1 \times \mathbb{S}^{4k+3} : u(x, y) = \pi(p)\},$$

and if $\pi_2 : \mathbb{HP}^1 \times \mathbb{S}^{4k+3} \to \mathbb{S}^{4k+3}$ denotes the projection onto the second factor, then the induced map u' on total spaces is given by $u' = \pi_2|_P : P \to \mathbb{S}^{4k+3}$.

If $\gamma \in \Omega^1(\mathbb{S}^{4k+3}, \mathfrak{g})$ denotes the canonical $\mathbf{Sp}(1)$-connection of $\mathbb{S}^{4k+3} \to \mathbb{HP}^k$, let ω denote the pull-back connection on $P \to \mathbb{S}^4$. Then:

$$\begin{aligned}
\gamma(q) &= q^\dagger dq, \qquad q \in \mathbb{H}^{k+1}, \\
\omega &= (u')^*\gamma \ \in \Omega^1(P, \mathfrak{g}), \\
\omega_i &= \sigma_i^*\omega = \sigma_i^\dagger d\sigma_i \ \in \Omega^1(O_i, \mathfrak{g}).
\end{aligned}$$

We have the corresponding local curvature 2-forms:

$$\begin{aligned}
F_i &= d\omega_i + \omega_i \wedge \omega_i \\
&= d\sigma_i^\dagger \wedge d\sigma_i + \sigma_i^\dagger d\sigma_i \wedge \sigma_i^\dagger d\sigma_i \ \in \Omega^2(O_i, \mathfrak{g}).
\end{aligned}$$

Remark 4.1.1. We recall that ADHM matrices A and A' give isomorphic $\mathbf{Sp}(1)$-bundles P, P' with connections ω, ω' if and only if $A' = QAR$, where $Q \in \mathbf{Sp}(k+1)$, $R \in \mathbf{GL}(k, \mathbb{R})$ [A, p. 62].

Remark 4.1.2. The conformal transformations of $\mathbb{S}^4$, $\mathrm{Conf}_0(\mathbb{S}^4) = \mathbf{SO}(5,1) \simeq \mathbf{SL}(2,\mathbb{H})/\{\pm 1\}$, lift to isomorphisms of the total space P. If the conformal map φ acts on $\mathbb{HP}^1$ by $\varphi : [x,y] \mapsto [ax+by, cx+dy]$, then $\varphi^* A(x,y) = (Ca+Dc)x + (Cb+Dd)y$, so that $C \mapsto Ca + Dc$ and $D \mapsto Cb + Dd$. Hence, given any fixed self-dual connection with ADHM matrices C, D, we may generate a family of self-dual connections parametrized by $\mathbf{SO}(5,1)$. Indeed, the moduli space $\mathcal{M}_1$ is diffeomorphic to $\mathbf{SO}(5,1)/\mathbf{SO}(5)$ and may be generated by pulling back the canonical connection via conformal maps [A-H-S], [D-M-M], [Hab], [Har2], [G-P1], [F-U].

Coordinate patches for the sphere $\mathbb{HP}^1$ (or $\mathbb{H} \cup \{\infty\}$) are given by $O_s = \{[x_0, x_1] : x_1 \neq 0\}$ covering the south pole $[0,1]$ (or 0), and $O_n = \{[x_0, x_1] : x \neq 0\}$ covering the north pole $[1,0]$ (or ∞), with the standard local coordinate maps:

$$\psi_s : O_s \to \mathbb{H}, \qquad [x_0, x_1] \mapsto x = x_0 x_1^{-1},$$
$$\psi_n : O_n \to \mathbb{H}, \qquad [x_0, x_1] \mapsto y = x_1 x_0^{-1}.$$

We let $O_{si} = O_s \cap O_i$, $O_{ni} = O_n \cap O_i$ denote the refined covering of $\mathbb{HP}^1$ with corresponding coordinate maps $\psi_{si} = \psi_s | O_{si}$, $\psi_{ni} = \psi_n | O_{ni}$, respectively.

Example 4.1.3. We describe the more explicit standard form of the ADHM matrix equations, and corresponding local connection 1-forms [A], [C-W-S], [C-G-F-T], [Sa]. By making use of gauge equivalence, the matrices C and D can be chosen to have the following form:

$$C = \begin{pmatrix} 0 \\ -I \end{pmatrix}, \qquad \text{and} \qquad D = \begin{pmatrix} \lambda \\ B \end{pmatrix},$$

where B is a $k \times k$ quaternionic matrix and λ is a $k \times 1$ quaternionic vector. Then in local coordinates, the map $A : O_s \to G_k(\mathbb{H}^{k+1})$ becomes

$$A(x) = \begin{pmatrix} \lambda \\ B - xI \end{pmatrix}, \qquad x \in \mathbb{H}.$$

In terms of B and λ, the conditions on A are now:
(i) B is symmetric and $B^\dagger B + \lambda^\dagger \lambda$ is real;
(ii) $\mathrm{Rank}_{\mathbb{H}} \begin{pmatrix} \lambda \\ B - x \end{pmatrix} = k$, for all $x \in \mathbb{H}$.

We choose the section $\sigma_s : O_s \to P$ to be

$$\sigma_s(x) = \frac{1}{\sqrt{\rho(x)}} \begin{pmatrix} -1 \\ U(x) \end{pmatrix},$$

where $\rho_s(x) = 1 + |U(x)|^2$, and setting $U(x) = \left\{\lambda(B - xI)^{-1}\right\}^\dagger$. Hence,

$$\omega_s(x) = \sigma_s^\dagger(x) d\sigma_s(x) = \frac{\mathrm{Im}\, \left\{U^\dagger(x) dU(x)\right\}}{1 + |U(x)|^2} \in \Omega^1(O_s, \mathfrak{g}),$$

where σ_s, and hence ω_s, may have singularities. In particular, every self-dual connection ω on a principal $\mathbf{SU}(2)$-bundle $P \to \mathbb{S}^4$ with $-c_2(P)[\mathbb{S}^4] = k$ arises from the parameters λ, B satisfying (i) and (ii), the local connection 1-form ω_s being given by the above formula. The connections defined by (λ, B) and (λ', B') are gauge equivalent if and only if $\lambda' = a\lambda T$, $B' = T^{-1} B T$ with $a \in \mathbf{Sp}(1)$ and $T \in \mathbf{O}(k, \mathbb{R})$ [A, p. 26].

Example 4.1.5. If $k = 1$, the corresponding local connection 1-form is

$$\omega_s = \frac{\lambda^2}{|x-b|^2(\lambda^2 + |x-b|^2)} \text{Im } \left\{ (\overline{x-b}) dx \right\},$$

where $\lambda > 0$ and $b \in \mathbb{H}$, so that ω_s is specified by 5 real parameters. Setting $\lambda = 1$ and $b = 0$, we obtain the canonical $\mathbf{Sp}(1)$-connection on the universal quaternionic line bundle $L \to \mathbb{HP}^1$, with local connection and curvature forms:

$$\begin{aligned} \omega_s &= \frac{1}{|x|^2(1+|x|^2)} \text{Im } \{\overline{x} dx\} \qquad \text{and} \qquad \omega_n = \frac{1}{(1+|y|^2)} \text{Im } \{d\overline{y}\}, \\ F_s &= \frac{dx \wedge d\overline{x}}{(1+|x|^2)} \qquad \text{and} \qquad F_n = \frac{dy \wedge d\overline{y}}{(1+|y|^2)}. \end{aligned}$$

The Chern-Weil formula gives $-c_2(L)[\mathbb{S}^4] = 1$. By reversing the orientation on $\mathbb{HP}^1$, we obtain a quaternionic line bundle $L \to \mathbb{HP}^1$ with $-c_2(L)[\mathbb{S}^4] = -1$.

Remark 4.1.6. For the case $k = 2$, the pair (λ, B) provides 20 real parameters, but the corresponding formulas for the local connection 1-forms are much less convenient for computational purposes.

Example 4.1.7. Let $b_1, \ldots, b_k$ be distinct centres in $\mathbb{H} \subset \mathbb{S}^4$ and let $\lambda_1, \ldots, \lambda_k$ in $(0, \infty)$ be positive scales. By choosing $B = \operatorname{diag}(b_1, \ldots, b_k)$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$, with $\lambda_i \in (0, \infty)$, we obtain the relatively simple *t'Hooft multi-instanton* solution [A], [C-W-S], [J-N-R]:

$$\omega_s = \sum_{i=1}^{k} \frac{\lambda_i^2}{|x-b_i|^4 \rho} \text{Im } \left\{ (\overline{x-b_i}) dx \right\}, \qquad \text{where } \rho(x) = 1 + \sum_{i=1}^{k} \frac{\lambda_i^2}{|x-b_i|^2}.$$

Unfortunately, this only gives a $5k$-parameter family of self-dual $\mathbf{SU}(2)$-connections and hence does not give a parametrization of $\mathcal{M}_k$ for $k > 1$ since the moduli space $\mathcal{M}_k$ has dimension $8k - 3$ [A-H-S].

Example 4.1.8. The set of t'Hooft multi-instantons described above is not closed under the action of the conformal group, and we may generate new self-dual connections by applying conformal transformations to $\mathbb{S}^4 = \mathbb{H} \cup \{\infty\}$. A $(5k+4)$-parameter family of self-dual connections on $\mathbb{S}^4$ for any $k \geq 1$ was discovered in this way by Jackiw, Nohl and Rebbi [J-N-R]. Let $\{a_0, \ldots, a_k\}$ be distinct points in $\mathbb{H} \subset \mathbb{S}^4$, and let $\{\lambda_0, \ldots, \lambda_k\}$ be positive weights in $(0, \infty)$. The corresponding *Jackiw-Nohl-Rebbi* (JNR) local connection 1-form is given by:

$$\omega_s = \sum_{i=0}^{k} \frac{\lambda_i^2}{|x-a_i|^4 \rho} \text{Im } \{ (\overline{x-a_i}) dx \}, \qquad \text{where } \rho(x) = \sum_{i=0}^{k} \frac{\lambda_i^2}{|x-a_i|^2}.$$

The expression for ω_s is clearly homogeneous in the λ_i's and so one parameter is removable by a common rescaling or by setting one of the λ_i's equal to 1. For

$k \geq 3$, one obtains a $(5k+4)$-dimensional family of self-dual connections with $-c_2(P)[\mathbb{S}^4] = k$, so these solutions do not give all self-dual connections when $k \geq 3$ [J-N-R]. Hartshorne has proven that the JNR family of solutions give all self-dual $\mathbf{SU}(2)$-connections when $k = 1$ or 2 [Har2].

The space $\mathcal{M}_1$ is 5-dimensional, and so four of the JNR parameters must be removable by gauge equivalence. The space $\mathcal{M}_2$ is a 13-dimensional manifold [A-H-S], and so one of the JNR parameters must be removable by gauge equivalence. The residual gauge symmetry has been interpreted in [J-N-R] as a motion of the points a_0, a_1, a_2 around the circle in $\mathbb{H} \cup \{\infty\}$ determined by those points [J-N-R]. In the next section, we will discuss the case $k = 2$ in more detail.

§4.2. Parametrization of Connection One-forms with $k = 2$

We recall how the JNR family of self-dual $\mathbf{SU}(2)$-connections may be obtained from the ADHM construction by making a suitable choice of matrices. We will then be able to obtain the non-singular local connection 1-forms using the ADHM construction as a guide. For $k = -c_2 = 2$, the JNR family gives all self-dual $\mathbf{SU}(2)$-connections over $\mathbb{S}^4$. General references for this section are [A-W], [C-G-F-T], [J-N-R], [O], [Sa], [Sc], [Wa-We].

We want to choose ADHM matrices corresponding to the JNR family of self-dual connections on an $\mathbf{SU}(2)$-bundle $P \to \mathbb{S}^4$ with $-c_2(P) = k$. This family is parametrized by the following data: $k+1$ distinct points $\{P_0, \ldots, P_k\}$ in $\mathbb{S}^4$ and $k+1$ positive weights $\{\lambda_0, \ldots, \lambda_k\}$ in $(0, \infty)$.

Let $\underline{\mathbb{HP}}^1 = \{(x, y) \in \mathbb{H}^2 \setminus (0,0) : (x, y) \sim (ax, ay), a \in \mathbb{H}^*\}$ denote the left projective space, while $\mathbb{HP}^1$ denotes the right projective space. (We have a map $\underline{\mathbb{HP}}^1 \to \mathbb{HP}^1$, $[a, b] \to [b^{-1}, a^{-1}]$. If we replace $\mathbb{HP}^1$ by $\mathbb{CP}^1$, then this map reduces to the identity map on $\mathbb{CP}^1$.) The ADHM matrices C and D which give rise to the JNR $\mathbf{SU}(2)$-connections are easily identifed [C-G-F-T, p. 41], [O, p. 413]. We choose C and D to be of the following form:

$$C = \begin{pmatrix} b_0 & b_0 & \cdots & b_0 \\ -b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -b_k \end{pmatrix}, \qquad D = \begin{pmatrix} -a_0 & -a_0 & \cdots & -a_0 \\ a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_k \end{pmatrix},$$

where the (a_i, b_i) are distinct points in $\mathbb{H}^2 \setminus (0,0)$. Hence, we have the corresponding matrix $A(x, y) = Cx + Dy$:

$$A(x, y) = \begin{pmatrix} b_0x - a_0y & b_0x - a_0y & \cdots & b_0x - a_0y \\ -b_1x + a_1y & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -b_kx + a_ky \end{pmatrix}.$$

Then, the corresponding points in $\mathbb{S}^4$ are given by $\underline{P}_i = [a_i, b_i] \in \underline{\mathbb{HP}}^1$, and the weights are given by $\lambda_i = (|a_i|^2 + |b_i|^2)^{-1/2}$. We see that $A^\dagger(x, y)A(x, y)$ is real and $\mathrm{Rank}_{\mathbb{H}} A(x, y) = k$ for all $(x, y) \in \mathbb{H}^2 \setminus (0, 0)$, provided the points P_i are all distinct.

It is more convenient to define a matrix $A(x,y)$ corresponding to parameters κ_i and (α_i, β_i), $i = 0, \ldots, k$, as follows:

$$A(x,y) = \begin{pmatrix} \kappa_0^{-1}(\beta_0 x - \alpha_0 y) & \kappa_0^{-1}(\beta_0 x - \alpha_0 y) & \cdots & \kappa_0^{-1}(\beta_0 x - \alpha_0 y) \\ -\kappa_1(\beta_1 x - \alpha_1 y) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\kappa_k^{-1}(\beta_k x - \alpha_k y) \end{pmatrix}.$$

Then, if we are given $\underline{P}_i = [a_i, b_i] \in \underline{\mathbb{HP}}^1$ and $\lambda_i \in (0, \infty)$, we may choose:

$$\kappa_i = \lambda_i,$$
$$(\alpha_i, \beta_i) = \frac{(a_i, b_i)}{\sqrt{|a_i|^2 + |b_i|^2}} \in \mathbb{S}^7,$$

The matrix $A(x,y)$ will now be determined up to left multiplication by an element of $\mathbf{Sp}(k+1)$: if $(a_i, b_i) \mapsto (q_i a_i, q_i b_i)$, with $q_i \in \mathbb{H}^*$, then $A \mapsto QA$, where $Q = \mathrm{diag}\,(q_0/|q_0|, \ldots, q_k/|q_k|) \in \mathbf{Sp}(k+1)$. Hence, the data $\lambda_0, \ldots, \lambda_k \in (0,\infty)$ and $P_0, \ldots, P_k \in \mathbb{S}_4$ determines A up to isomorphism via the above assignment.

Example 4.2.1. We recover the JNR local connection 1-form given previously, by choosing $(\alpha_i, \beta_i) = (1, a_i)$ and $\kappa_i = \lambda_i$, for $i = 0, \ldots, k$, where the a_i are distinct points in $\mathbb{H}$ and the λ_i are positive weights.

Remark 4.2.2. We recall that if φ in $\mathbf{SL}(2,\mathbb{H})/\{\pm 1\}$ is a conformal transformation of $\mathbb{HP}^1$ represented by $\varphi : [x,y] \to [ax+by, cx+dy]$, then $\varphi^* A(x,y) = C'x + D'y$, where $C' = Ca + Dc$, $D' = Cb + Dd$. In particular, we see that $(a_i, b_i) \mapsto (a_i', b_i') = (a_i c - b_i a, -a_i d + b_i b)$, and so the set of matrices for the JNR solutions is closed under the induced action of $\mathbf{SL}(2,\mathbb{H})$.

The classifying map $u : \mathbb{HP}^1 \to \mathbb{HP}^k$ is defined by the following conditions:

$$u(x,y)^\dagger A(x,y) = 0,$$
$$u(x,y)^\dagger u(x,y) = 1, \qquad \text{for all } (x,y) \in \mathbb{H}^2 \setminus (0,0).$$

Hence,

$$u(x,y) = \frac{1}{\sqrt{\rho(x,y)}} \begin{pmatrix} \kappa_0(\overline{\beta_0 x - \alpha_0 y})^{-1} \\ \kappa_1(\overline{\beta_1 x - \alpha_1 y})^{-1} \\ \vdots \\ \kappa_k(\overline{\beta_k x - \alpha_k y})^{-1} \end{pmatrix}, \qquad \rho(x,y) = \sum_{m=0}^{k} \frac{\kappa_m^2}{|\beta_m x - \alpha_m y|^2}.$$

Denoting $u = [u_0, \ldots, u_k] \in \mathbb{HP}^k$, we have

$$\begin{aligned} u_i(x,y) &= \frac{\kappa_i(\beta_i x - \alpha_i y)}{\sqrt{\rho(x,y)}|\beta_i x - \alpha_i y|^2} \\ &= \frac{\kappa_i(\beta_i x - \alpha_i y)}{|\beta_i x - \alpha_i y|} \left(\kappa_i^2 + \sum_{m \neq i} \kappa_m^2 \frac{|\beta_i x - \alpha_i y|^2}{|\beta_m x - \alpha_m y|^2} \right)^{-1/2}, \end{aligned}$$

for $i = 0, \ldots, k$. Clearly, u has singularities at the points $\{P_0, \ldots, P_k\}$, but we can now see that these are removable [G-H, p. 490]. Define an open cover of $\mathbb{HP}^1$ by choosing $O_i = \mathbb{HP}^1 \setminus \{P_0, \ldots, \hat{P}_i, \ldots, P_k\}$ for $i = 0, \ldots, k$, where the caret denotes an omission. Then, each patch O_i covers the point P_i and each u_i has singularities at P_j for $j = 0, \ldots, k$.

We extend u on the patches O_i, for $i = 0, \ldots, k$, by defining: $\tilde{u}|O_i = u\overline{g}_i$, where g_i is defined by

$$g_i : \pi^{-1}(O_i \setminus \{P_i\}) \longrightarrow \mathbf{Sp}(1), \qquad (x, y) \longmapsto \frac{u_i(x, y)}{|u_i(x, y)|} = \frac{\beta_i x - \alpha_i y}{|\beta_i x - \alpha_i y|},$$

and $\pi : \mathbb{H}^2 \setminus (0, 0) \to \mathbb{HP}^1$ is the natural projection. Then, the extension $\tilde{u}$ is well-defined on overlapping patches, since $[\tilde{u}] = [u\overline{g}_i] = [u\overline{g}_j]$ on $O_i \cap O_j$. Hence, we obtain a smooth extension $\tilde{u} : \mathbb{HP}^1 \to \mathbb{HP}^k$. If there is no ambiguity, we will often denote the extension $\tilde{u}$ simply by u.

Let $O'_i = \{[q_0, \ldots, q_k] \in \mathbb{HP}^k : q_i \neq 0\}$, for $i = 0, \ldots, k$, denote the collection of standard coordinate patches for $\mathbb{HP}^k$. More explicitly, we have for $j \neq i$:

$$\tilde{u}_j|O_i = \frac{\kappa_j(\beta_j x - \alpha_j y)\overline{(\beta_i x - \alpha_i y)}}{|\beta_j x - \alpha_j y|^2} \left(\kappa_i^2 + \sum_{m \neq i} \kappa_m^2 \frac{|\beta_i x - \alpha_i y|^2}{|\beta_m x - \alpha_m y|^2} \right)^{-1/2},$$

which is smooth at P_i — and so smooth on O_i — with singularities at the points P_j, for $j \neq i$. Next, we have:

$$\tilde{u}_i|O_i = \kappa_i \left(\kappa_i^2 + \sum_{m \neq i} \kappa_m^2 \frac{|\beta_i x - \alpha_i y|^2}{|\beta_m x - \alpha_m y|^2} \right)^{-1/2} \qquad \text{and,}$$

$$\tilde{u}_i|O_j = \frac{\kappa_i(\beta_i x - \alpha_i y)\overline{(\beta_j x - \alpha_j y)}}{|\beta_i x - \alpha_i y|^2} \left(\kappa_j^2 + \sum_{m \neq j} \kappa_m^2 \frac{|\beta_j x - \alpha_j y|^2}{|\beta_m x - \alpha_m y|^2} \right)^{-1/2},$$

for $j \neq i$. Then $\tilde{u}_i$ is smooth at P_i and zero at all points P_m, for $m \neq i$. Hence, $\tilde{u}^{-1}(O'_i) = \{[x, y] \in \mathbb{HP}^1 : u_i(x, y) \neq 0\} = O_i$, where we recall that $O_i \subset \mathbb{HP}^1$ was defined by $O_i = \mathbb{HP}^1 \setminus \{P_0, \ldots, \hat{P}_i, \ldots, P_k\}$.

Next, we give the transition functions, $T_{ij} = u^* T'_{ij}$, and local sections, $\sigma_i = u^* \sigma'_i$, of the pullback bundle $P = u^* \mathbb{S}^{4k+3}$. We have

$$\begin{aligned} T_{ij}(x, y) &= \frac{u_i(x, y)}{|u_i(x, y)|} \frac{\overline{u}_j(x, y)}{|u_j(x, y)|} \\ &= \frac{(\beta_i - \alpha_i y)}{|\beta_i - \alpha_i y|} \frac{\overline{(\beta_j - \alpha_j y)}}{|\beta_j - \alpha_j y|} \qquad \text{on } O_i \cap O_j. \end{aligned}$$

The local sections $\sigma_i : O_i \to P$ are

$$\sigma_i(x,y) = u(x,y)\overline{g}_i(x,y) = u(x,y)\frac{\overline{u}_i(x,y)}{|u_i(x,y)|} \qquad \text{on } O_i,$$

and $\sigma_j = \sigma_i T_{ij}$ on overlaps. The local connection 1-forms are then given by

$$\omega_i = \sigma_i^* \omega = \sigma_i^\dagger d\sigma_i \ \in \Omega^1(O_i, \mathfrak{g}).$$

We again note that the effect of replacing (α_i, β_i) by $(q_i\alpha_i, q_i\beta_i)$, for $i = 0, \ldots, k$, where q_i are constant elements of $\mathbf{Sp}(1)$, is simply to replace the transition functions T_{ij} by $q_i T_{ij} q_j^{-1}$ and this corresponds to an $\mathbf{Sp}(1)$-bundle isomorphism. Hence, the the bundle P and connection ω corresponding to the points (α_i, β_i) in $\mathbb{S}^7$, are equivalent to the bundle with connection corresponding to the points $(q_i\alpha_i, q_i\beta_i)$.

With respect to the cover $\{O_{si}, O_{ni}\}$ of $\mathbb{HP}^1$, we have $2k+2$ smooth local connection 1-forms:

$$\omega_{si} = \sigma_{si}^* \omega = \sigma_{si}^\dagger d\sigma_{si} \ \in \Omega^1(O_{si}, \mathfrak{g}),$$
$$\omega_{ni} = \sigma_{ni}^* \omega = \sigma_{ni}^\dagger d\sigma_{ni} \ \in \Omega^1(O_{ni}, \mathfrak{g}),$$

where $\sigma_{si} = \sigma_i | O_{si}$ and $\sigma_{ni} = \sigma_i | O_{ni}$.

Lemma 4.2.3. *The local 1-form ω_{si} is given by:*

$$\begin{aligned}\omega_{si}(x) = {} & \frac{\rho_i}{\rho|\beta_i x - \alpha_i|^2} \mathrm{Im}\ \left\{(\beta_i x - \alpha_i)\overline{\beta_i dx}\right\} \\ & + \sum_{\substack{j=0\\ j\neq i}}^{k} \frac{\kappa_j^2}{\rho|\beta_j x - \alpha_j|^4 |\beta_i x - \alpha_i|^2} (\beta_i x - \alpha_i) \mathrm{Im}\ \left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\} (\overline{\beta_i x - \alpha_i}),\end{aligned}$$

where

$$\rho_i(x) = \sum_{\substack{j=0\\ j\neq i}}^{k} \frac{\kappa_j^2}{|\beta_j x - \alpha_j|^2},$$

and $\rho(x) = \rho(x,1)$. The local 1-form ω_{ni} is given by:

$$\begin{aligned}\omega_{ni}(y) = {} & \frac{-\rho_i}{\rho|\beta_i - \alpha_i y|^2} \mathrm{Im}\ \left\{(\beta_i - \alpha_i y)\overline{\alpha_i dy}\right\} \\ & + \sum_{\substack{j=0\\ j\neq i}}^{k} \frac{\kappa_j^2}{\rho|\beta_j - \alpha_j y|^4 |\beta_i - \alpha_i y|^2} (\beta_i - \alpha_i y) \mathrm{Im}\ \left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\} (\overline{\beta_i - \alpha_i y}),\end{aligned}$$

where

$$\rho_i(y) = \sum_{\substack{j=0\\ j\neq i}}^{k} \frac{\kappa_j^2}{|\beta_j - \alpha_j y|^2},$$

and $\rho(y) = \rho(1, y)$.

Proof. Calculation. $\square$

Let $O' = \mathbb{HP}^1 \setminus \{P_0, \ldots, P_k\}$ and let $O'_s = O_s \cap O'$, $O'_n = O_n \cap O'$, and define local sections $\sigma_s : O'_s \to P$, $\sigma_n : O'_n \to P$ by:

$$\sigma_s(x,y) = u(x,y)\frac{\overline{x}}{|x|} \qquad \text{on } O'_s,$$
$$\sigma_n(x,y) = u(x,y)\frac{\overline{y}}{|y|} \qquad \text{on } O'_n.$$

Let $\omega_s = \sigma_s^*\omega \in \Omega^1(O'_s, \mathfrak{g})$ and $\omega_n = \sigma_n^*\omega \in \Omega^1(O'_n, \mathfrak{g})$ denote the corresponding local connection 1-forms.

Lemma 4.2.4. *The local connection 1-forms ω_s, ω_n are given by:*

$$\omega_s = \sum_{i=0}^{k} \frac{\kappa_i^2}{\rho|\beta_i x - \alpha_i|^4}\mathrm{Im}\left\{(\overline{\beta_i x - \alpha_i})\beta_i dx\right\},$$
$$\text{where } \rho(x) = \rho(x,1)\sum_{i=0}^{k}\frac{\kappa_i^2}{|\beta_i x - \alpha_i|^2},$$
$$\omega_n = \sum_{i=0}^{k} \frac{-\kappa_i^2}{\rho|\beta_i - \alpha_i y|^4}\mathrm{Im}\left\{(\overline{\beta_i - \alpha_i y})\alpha_i dy\right\},$$
$$\text{where } \rho(y) = \rho(1,y)\sum_{i=0}^{k}\frac{\kappa_i^2}{|\beta_i - \alpha_i y|^2}.$$

Proof. Calculation. $\square$

The connection 1-forms ω_s, ω_n have simple real poles at the points $\{P_0, \ldots, P_k\}$. In order to estimate the lengths of tangent vectors in $T\mathcal{M}_2$, we need smooth local connection 1-forms corresponding to a cover of $\mathbb{S}^4$.

Lemma 4.2.5. *The local curvature 2-form $F_s = d\omega_s + \omega_s \wedge \omega_s \in \Omega^1(O'_s, \mathfrak{g})$ corresponding to the parameters κ_i, $(\alpha_i, 1)$ is given by:*

$$\begin{aligned} F_s = &\sum_{i \neq j} \frac{\kappa_i^2\kappa_j^2}{\rho^2|x-\alpha_i|^6|x-\alpha_j|^2}(\overline{x-\alpha_i})dx \wedge d\overline{x}(x-\alpha_i) \\ &+ \sum_{i\neq j}\frac{\kappa_i^2\kappa_j^2}{\rho^2|x-\alpha_i|^4|x-\alpha_j|^4}(\overline{x-\alpha_i})dx\wedge d\overline{x}(x-\alpha_j), \end{aligned}$$

and similarly for $F_n \in \Omega^2(O'_n, \mathfrak{g})$.

Remark 4.2.6. There is no direct relationship between the $k+1$ *weights* λ_i and $k+1$ *points* P_i arising in the JNR family of self-dual connections and the *scales* and *centres* of curvature-density concentration arising in the Donaldson-Uhlenbeck description of neighbourhoods of the boundary of moduli space [D-K, p. 156]

Definition 4.2.7. We let $\tilde{T}_k$ denote the space of unordered pairs (λ_i, P_i), $i = 0, \ldots, k$, where $\lambda_i \in (0, \infty)$, $P_i \in \mathbb{S}^4$, and $(\lambda_0, \ldots, \lambda_k) \sim \nu(\lambda_0, \ldots, \lambda_k)$, for $\nu > 0$.

The spaces $\tilde{T}_k$ parametrize the family of JNR connections. We recall the

Theorem 4.2.8. (Hartshorne) *The spaces $\tilde{T}_k$ parametrize the moduli spaces $\mathcal{M}_k$ of self-dual $\mathbf{SU}(2)$-connections on $\mathbb{S}^4$ for $k = 1, 2$.*

Proof. [Har2, 3], [A-W], [Wa-We]. □

Remark 4.2.9. In particular, when $k = 2$, the induced ADHM map from $\tilde{T}_2$ to $\mathcal{M}_2$ is surjective.

If we normalize the weights λ_i so that $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$ and $\lambda_i \in (0, 1)$, then one approaches the boundary of $\tilde{T}_2$ by letting $P_i \to P_j$ or $\lambda_i \to 0$. For comparison, we recall the description of the Uhlenbeck-Donaldson topological compactification of $\mathcal{M}_k$ [D-K, p. 156]. An *ideal self-dual connection* on an $\mathbf{SU}(2)$-bundle $P \to \mathbb{S}^4$ with $-c_2(P) = k$ is a pair $([\omega], x_1, \ldots, x_l)$, where $[\omega]$ is a point in $\mathcal{M}_{k-l}$ and $(x_1, \ldots, x_l)$ is an unordered l-tuple of points of $\mathbb{S}^4$. The curvature density of $([\omega], x_1, \ldots, x_l)$ is the measure

$$|F_\omega|^2 + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i}.$$

By defining weak convergence of gauge equivalence classes $[\omega_\alpha]$ to an ideal self-dual connection $([\omega], x_1, \ldots, x_l)$ in terms of convergence of the curvature densities as measures, one may topologize the set of ideal self-dual connections $\mathcal{IM}_k$. The moduli space $\mathcal{M}_k$ is embedded as an open subset of of $\mathcal{IM}_k$ and its closure $\overline{\mathcal{M}}_k$ is then compact. The space $\mathcal{IM}_k$ has the following stratification:

$$\mathcal{IM}_k = \mathcal{M}_k \cup \mathcal{M}_{k-1} \times \mathbb{S}^4 \cup \mathcal{M}_{k-2} \times s^2(\mathbb{S}^4) \cup \ldots \cup \mathcal{M}_1 \times s^{k-1}(\mathbb{S}^4) \cup \mathcal{M}_0 \times s^k(\mathbb{S}^4),$$

where $s^i(\mathbb{S}^4)$, and $\mathcal{M}_0$ consists of a single point corresponding to the product connection on the trivial bundle $\mathbb{S}^4 \times \mathbf{Sp}(1) \to \mathbb{S}^4$.

CHAPTER V
ASYMPTOTIC BEHAVIOUR OF THE L^2 METRIC

Using the parametrization of $\mathcal{M}_2$ by the parameter space $\tilde{T}_2$ due to Hartshorne and Jackiw, Nohl, and Rebbi, we compute the corresponding tangent vectors $\partial\omega/\partial t_\alpha$. We then derive estimates for the components of the L^2 metric $\mathbf{g}$, the principal question being their asymptotic behaviour as one approaches the boundary of moduli space. With these estimates at hand, it is then easy to see that the space $\mathcal{M}_2$ has finite volume with respect to its L^2 metric $\mathbf{g}$, and in particular, that the distance to the boundary is finite.

To obtain the required estimates, we first need to choose appropriate local coordinate maps, for the base manifold $\mathbb{S}^4$, and for the parameter space $\tilde{T}_2$.

§5.1. Local Coordinate Patches on Moduli Space

The parameter space $\tilde{T}_k$ is the space of unordered pairs $(\lambda_0, P_0), \ldots, (\lambda_k, P_k)$, with $P_i \in \mathbb{S}^4$ distinct points and weights $\lambda_i \in (0, \infty)$, modulo rescaling. The parameter space T_k denotes the space of ADHM matrices modulo $\mathbf{Sp}(k+1) \times \mathbf{GL}(k, \mathbb{R})$ and the ADHM map provides a diffeomorphism $T_k \to \mathcal{M}_k$. For $k = 2$, we have submersions $\tilde{T}_2 \to T_2$ and $\tilde{T}_2 \to \mathcal{M}_2$.

Given points $\underline{P}_i$ with homogeneous coordinates $[a_i, b_i] \in \underline{\mathbb{HP}}^1$, we select corresponding representatives (α_i, β_i) in $\mathbb{S}^7$. Cover $\underline{\mathbb{HP}}^1$ with the overlapping hemispheres

$$\underline{O}_s(R_s) = \{[a, b] \in \underline{\mathbb{HP}}^1 : b \neq 0,\ |b^{-1}a| < R_s\},$$
$$\underline{O}_n(R_n) = \{[a, b] \in \underline{\mathbb{HP}}^1 : a \neq 0,\ |a^{-1}b| < R_n\},$$

where $1 < R_s, R_n \leq \infty$ and $\underline{O}_s(\infty)$, $\underline{O}_n(\infty)$ are simply denoted by $\underline{O}_s$, $\underline{O}_n$. Usually we take $R_s = R_n = 2$. The equator of $\underline{\mathbb{HP}}^1$ is the sphere $\mathbb{S}^3$ given by $\{[a, b] \in \underline{\mathbb{HP}}^1 : a \neq 0,\ |a^{-1}b| = 1\}$ and similarly for $\mathbb{HP}^1$. We have the standard local coordinate maps

$$\underline{O}_s \ni [a, b] \longmapsto b^{-1}a \in \mathbb{B}(0, R_s),$$
$$\underline{O}_n \ni [a, b] \longmapsto a^{-1}b \in \mathbb{B}(0, R_n),$$

where $\mathbb{B}(c, R)$ denotes the Euclidean ball $\{x \in \mathbb{H} : |x - c| < R\}$. We define sections of the $\mathbf{Sp}(1)$-bundle $\mathbb{S}^7 \to \underline{\mathbb{HP}}^1$ by

$$\underline{O}_s \ni [a, b] \longmapsto \frac{\overline{b}}{|b|} \frac{(a, b)}{\sqrt{|a|^2 + |b|^2}} \in \mathbb{S}^7|\underline{O}_s,$$
$$\underline{O}_n \ni [a, b] \longmapsto \frac{\overline{a}}{|a|} \frac{(a, b)}{\sqrt{|a|^2 + |b|^2}} \in \mathbb{S}^7|\underline{O}_n.$$

Combining these coordinate maps and local sections, we have maps

$$\mathbb{B}(0, R_s) \ni a \longmapsto \frac{(a,1)}{\sqrt{1+|a|^2}} = (\alpha, \beta) \in \mathbb{S}^7|\underline{O}_s,$$
$$\mathbb{B}(0, R_n) \ni b \longmapsto \frac{(1,b)}{\sqrt{1+|b|^2}} = (\alpha, \beta) \in \mathbb{S}^7|\underline{O}_n.$$

We let $\{c_i^\mu : i = 0,1,2,\ \mu = 0,\dots,3\}$ denote local coordinates for the points P_i in $\mathbb{S}^4$, the corresponding tangent vectors in $T\mathcal{M}_2$ being denoted by $\partial\omega/\partial c_i^\mu$, where $c_i^\mu = a_i^\mu$ on the southern hemisphere and $c_i^\mu = b_i^\mu$ on the northern hemisphere.

The positive weights λ_0, λ_1, λ_2 lie in the positive cone in $\mathbb{RP}^2$ and so we may assume for convenience that $(\lambda_0, \lambda_1, \lambda_2)$ lies in the unit sphere $\mathbb{S}^2$. We then obtain local coordinates on the space $\tilde{T}_2$; the map $\tilde{T}_2 \to T_2$ has already been described, and the ADHM map provides a diffeomorphism $T_2 \to \mathcal{M}_2$.

§5.2. Local Coordinate Patches on the Four-Sphere

We consider the definition of the L^2 inner product on $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$ over an appropriate system of coordinate patches. We have assumed that the Riemannian metric g of the four-sphere $\mathbb{S}^4$ is globally conformally equivalent to the standard round metric g_0. Let $d_g(\ ,\ )$ denote the distance function on the four-sphere corresponding to the metric g, and let $d(\ ,\ )$ be the distance function for the round metric g_0. By compactness, we may fix a constant $K_g \geq 1$ such that

$$K_g^{-1} d(Q, Q') \leq d_g(Q, Q') \leq K_g d(Q, Q') \qquad \text{for all } Q, Q' \in \mathbb{S}^4.$$

Our metric component estimates are most conveniently expressed in terms of the weight parameters λ_i and the distances between distinct points P_i, P_j. Since the distance functions d_g, d are equivalent and our estimate calculations are now entirely local, we may for convenience assume at the outset that the four-sphere has its standard round metric g_0 with radius 1.

The $k+1$ distinct points P_i may be covered by disjoint small balls

$$B(P_i, \varepsilon_i) = \{Q \in \mathbb{HP}^1 : d(Q, P_i) < \varepsilon_i\}, \quad \text{where } \varepsilon_i = \frac{1}{2}\min_{j\neq i}\{1, d(P_i, P_j)\}.$$

The coordinate maps ψ_s, ψ_n pull back the sphere metric g to define a metric on the Euclidean balls $\mathbb{B}(0, R_s)$, $\mathbb{B}(0, R_n)$ in $\mathbb{H}$ which is equivalent to the standard flat metric. Hence, if $z = \psi(Q)$, with $\psi = \psi_s$ or ψ_n, then the distance function d satisfies

$$K^{-1}|z - z'| \leq d(Q, Q') \leq K|z - z'|,$$

where K is a constant.

Let $\{O_{si}, O_{ni}\}$ be the open cover of $\mathbb{HP}^1$ described earlier, with $O_{si} = O_s \cap O_i$, $O_{ni} = O_n \cap O_i$, and corresponding local coordinate maps $\psi_{si} = \psi_s|O_{si}$, $\psi_{ni} = \psi_n|O_{ni}$. The patches O_s, O_n are obtained from the four-sphere by deleting the north and south poles respectively, and the patches O_i are obtained by deleting all points P_j with $j \neq i$. We choose a partition of unity $\{\chi_i\}$ on $\mathbb{HP}^1$ subordinate to the cover $\{O_i\}$ so that

(i) $\operatorname{supp} \chi_i \subset B(P_i, K^{-2}\varepsilon_i)$ and $\chi_i \equiv 1$ on $\overline{B}(P_i, \frac{1}{2}K^{-2}\varepsilon_i)$ for $i = 0, \dots, k$;

(ii) $\operatorname{supp} \chi_{k+1} \subset \mathbb{HP}^1 \setminus \bigcup_{i=0}^k B(P_i, \frac{1}{2}K^{-2}\varepsilon_i)$ and $\chi_{k+1} \equiv 1$ on $\mathbb{HP}^1 \setminus \bigcup_{i=0}^k B(P_i, K^{-2}\varepsilon_i)$;

(iii) $\sum_{i=0}^{k+1} \chi_i = 1$.

Suppose the points Q, P_i lie in the southern hemisphere O_s and $\psi_s(Q) = x$, $\psi_s(P_i) = a_i$. If $Q \in \operatorname{supp} \chi_i$, then $x \in \mathbb{B}(a_i, \delta_i)$, where $\delta_i = \frac{1}{2}\min_{j \neq i}\{1, |a_i - a_j|\}$ and $i = 0, \dots, k$. Similarly, if $Q \in \operatorname{supp} \chi'$, then $|x - a_j| \geq \frac{1}{2}K^{-4}\delta_j$ for all $j = 0, \dots, k$. The analogous statements hold if we assume that the points Q, P_i lie in the northern hemisphere O_n. Finally, we let χ_s, χ_n be a partition of unity subordinate to the cover $\{O_s, O_n\}$, so that $\operatorname{supp} \chi_s \subset O_s$, $\operatorname{supp} \chi_n \subset O_n$, and $\chi_s + \chi_n = 1$. We then set $\chi_{si} = \chi_s \chi_i$, $\chi_{ni} = \chi_n \chi_i$, to give the required partition of unity for $\mathbb{HP}^1$.

Given tangent vectors $[\alpha_1], [\alpha_2] \in T_{[\omega]}\mathcal{M}_k$, their inner product with respect to the Riemannian metric $\mathbf{g}$ on $\mathcal{M}_k$ is computed by

$$
\begin{aligned}
&\mathbf{g}_{[\omega]}([\alpha_1], [\alpha_2]) \\
&\quad = \int_{\mathbb{S}^4} \langle h_\omega \alpha_1, h_\omega \alpha_2 \rangle \sqrt{g}\, dx \\
&\quad = \sum_{i=0}^{k+1} \int_{O_{si}} \chi_{si} \langle h_\omega \alpha_1, h_\omega \alpha_2 \rangle \sqrt{g}\, dx + \sum_{i=0}^{k+1} \int_{O_{ni}} \chi_{ni} \langle h_\omega \alpha_1, h_\omega \alpha_2 \rangle \sqrt{g}\, dx.
\end{aligned}
$$

The tangent space $T_{[\omega]}\mathcal{M}_2$ is spanned by the representatives $\partial\omega/\partial\lambda_i$ and $\partial\omega/\partial c_j^\mu$ for $i, j = 0, 1, 2$, and $\mu = 0, \dots, 3$. These representatives are not necessarily horizontal and it appears to be difficult to explicitly compute the horizontal projections when k is greater than 1. Nonetheless, we can still obtain upper bounds for the metric components, and that will be our strategy in the sequel.

§5.3. Tangent Vectors to Moduli Space

We obtain formulas for the local 1-forms in $\Omega^1(\mathbb{S}^4, \operatorname{ad} P)$ representing tangent vectors to the moduli space $\mathcal{M}_2$. In the following lemmas, we record our formulas for the local 1-forms representing the tangent vectors corresponding to the parameters $\kappa_i > 0$ and $(\alpha_i, \beta_i) \in \mathbb{H}^2 \setminus (0, 0)$.

Lemma 5.3.1. *The 1-form $\partial\omega_s/\partial\kappa_i$ is smooth on $\mathbb{H}$:*

$$
\begin{aligned}
\frac{\partial \omega_s}{\partial \kappa_i} &= \frac{2\kappa_i \rho_i}{\rho^2 |\beta_i x - \alpha_i|^4} \operatorname{Im} \left\{ (\overline{\beta_i x - \alpha_i}) \beta_i dx \right\} \\
&\qquad - \sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{2\kappa_i \kappa_j^2}{\rho^2 |\beta_i x - \alpha_i|^2 |\beta_j x - \alpha_j|^4} \operatorname{Im} \left\{ (\overline{\beta_j x - \alpha_j}) \beta_j dx \right\},
\end{aligned}
$$

$$
\text{with } \rho_i(x) = \sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{\kappa_j^2}{|\beta_j x - \alpha_j|^2} \text{ and } \rho = \rho(x, 1).
$$

Proof. Calculation. □

Lemma 5.3.2. *The 1-form $\partial\omega_n/\partial\kappa_i$ is smooth on $\mathbb{H}$:*

$$\begin{aligned}\frac{\partial\omega_n}{\partial\kappa_i} &= -\frac{2\kappa_i\rho_i}{\rho^2|\beta_i-\alpha_i y|^4}\mathrm{Im}\left\{\overline{(\beta_i-\alpha_i y)}\alpha_i dy\right\}\\ &+\sum_{\substack{j=0\\ j\neq i}}^{k}\frac{2\kappa_i\kappa_j^2}{\rho^2|\beta_i-\alpha_i y|^2|\beta_j-\alpha_j y|^4}\mathrm{Im}\left\{\overline{(\beta_j-\alpha_j y)}\alpha_j dy\right\},\\ &\text{with } \rho_i(y)=\sum_{\substack{j=0\\ j\neq i}}^{k}\frac{\kappa_j^2}{|\beta_j-\alpha_j y|^2} \text{ and } \rho=\rho(1,y).\end{aligned}$$

Proof. Calculation. □

Lemma 5.3.3. *The 1-forms $\partial\omega_s/\partial\alpha_i^\mu$ and $\partial\omega_n/\partial\alpha_i^\mu$ are smooth on $\mathbb{H}\setminus\{\beta_i^{-1}\alpha_i\}$, with real poles at $x=\beta_i^{-1}\alpha_i$:*

(i)
$$\begin{aligned}\frac{\partial\omega_s}{\partial\alpha_i^\mu} &= \frac{4\kappa_i^2(\beta_i x-\alpha_i)^\mu}{\rho|\beta_i x-\alpha_i|^6}\mathrm{Im}\left\{\overline{(\beta_i x-\alpha_i)}\beta_i dx\right\}\\ &-\frac{\kappa_i^2}{\rho|\beta_i x-\alpha_i|^4}\mathrm{Im}\left\{\overline{e}_\mu\beta_i dx\right\}\\ &-\sum_{j=0}^{k}\frac{2\kappa_i^2\kappa_j^2(\beta_i x-\alpha_i)^\mu}{\rho^2|\beta_i x-\alpha_i|^4|\beta_j x-\alpha_j|^4}\mathrm{Im}\left\{\overline{(\beta_j x-\alpha_j)}\beta_j dx\right\}.\end{aligned}$$

(ii)
$$\begin{aligned}\frac{\partial\omega_s}{\partial\beta_i^\mu} &= -\frac{4\kappa_i^2\mathrm{Re}\{\overline{(\beta_i x-\alpha_i)}e_\mu x\}}{\rho|\beta_i x-\alpha_i|^6}\mathrm{Im}\left\{\overline{(\beta_i x-\alpha_i)}\beta_i dx\right\}\\ &+\frac{\kappa_i^2}{\rho|\beta_i x-\alpha_i|^4}\mathrm{Im}\left\{(2\beta_i^\mu\overline{x}-\overline{\alpha}_i e_\mu)dx\right\}\\ &+\sum_{j=0}^{k}\frac{2\kappa_i^2\kappa_j^2\mathrm{Re}\{\overline{(\beta_i x-\alpha_i)}e_\mu x\}}{\rho^2|\beta_i x-\alpha_i|^4|\beta_j x-\alpha_j|^4}\mathrm{Im}\left\{\overline{(\beta_j x-\alpha_j)}\beta_j dx\right\}.\end{aligned}$$

Proof. Calculation. □

Lemma 5.3.4. *The 1-forms $\partial\omega_n/\partial\alpha_i^\mu$ and $\partial\omega_n/\partial\alpha_i^\mu$ are smooth on $\mathbb{H}\setminus\{\beta_i^{-1}\alpha_i\}$, with real poles at $y=\alpha_i^{-1}\beta_i$:*

(i)
$$\begin{aligned}\frac{\partial\omega_n}{\partial\alpha_i^\mu} &= -\frac{4\kappa_i^2\mathrm{Re}\{\overline{(\beta_i-\alpha_i y)}e_\mu y\}}{\rho|\beta_i-\alpha_i y|^6}\mathrm{Im}\left\{\overline{(\beta_i-\alpha_i y)}\alpha_i dy\right\}\\ &-\frac{\kappa_i^2}{\rho|\beta_i-\alpha_i y|^4}\mathrm{Im}\left\{(\overline{\beta}_i e_\mu-2\alpha_i^\mu\overline{y})dy\right\}\\ &+\sum_{j=0}^{k}\frac{2\kappa_i^2\kappa_j^2\mathrm{Re}\{\overline{(\beta_i-\alpha_i y)}e_\mu y\}}{\rho^2|\beta_i-\alpha_i y|^4|\beta_j-\alpha_j y|^4}\mathrm{Im}\left\{\overline{(\beta_j-\alpha_j y)}\beta_j dy\right\}.\end{aligned}$$

(ii)
$$\frac{\partial \omega_n}{\partial \beta_i^\mu} = \frac{4\kappa_i^2(\beta_i - \alpha_i y)^\mu}{\rho|\beta_i - \alpha_i y|^6}\mathrm{Im}\,\left\{(\overline{\beta_i - \alpha_i y})\alpha_i dy\right\} - \frac{\kappa_i^2}{\rho|\beta_i - \alpha_i y|^4}\mathrm{Im}\,\{\overline{e}_\mu \alpha_i dy\} - \sum_{j=0}^{k} \frac{2\kappa_i^2\kappa_j^2(\beta_i - \alpha_i y)^\mu}{\rho^2|\beta_i - \alpha_i y|^4|\beta_j - \alpha_j y|^4}\mathrm{Im}\,\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}.$$

Proof. Calculation. $\square$

Lemma 5.3.5. *The 1-form $\partial\omega_{si}/\partial\alpha_i^\mu$ has real poles at points $x = \beta_j^{-1}\alpha_j$, $j \neq i$, but is smooth elsewhere:*

$$\begin{aligned}\frac{\partial \omega_{si}}{\partial \alpha_i^\mu} &= \frac{2\rho_i^2(\beta_i x - \alpha_i)^\mu}{\rho^2|x - \alpha_i|^4}\mathrm{Im}\,\left\{(\beta_i x - \alpha_i)\overline{\beta_i dx}\right\} - \frac{\rho_i}{\rho|\beta_i x - \alpha_i|^2}\mathrm{Im}\,\left\{e_\mu\overline{\beta_i dx}\right\} \\ &+ \sum_{\substack{j=0\\ j\neq i}}^{k}\left\{\frac{2\kappa_j^2\rho_i(\beta_i x - \alpha_i)^\mu}{\rho^2|\beta_i x - \alpha_i|^4|\beta_j x - \alpha_j|^4}(\beta_i x - \alpha_i)\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}(\overline{\beta_i x - \alpha_i})\right. \\ &\quad - \frac{\kappa_j^2}{\rho|\beta_i x - \alpha_i|^2|\beta_j x - \alpha_j|^4}e_\mu\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}(\overline{\beta_i x - \alpha_i}) \\ &\quad \left. - \frac{\kappa_j^2}{\rho|\beta_i x - \alpha_i|^2|\beta_j x - \alpha_j|^4}(\beta_i x - \alpha_i)\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}\overline{e}_\mu\right\}.\end{aligned}$$

Proof. Calculation. $\square$

Lemma 5.3.6. *The 1-form $\partial\omega_{si}/\partial\beta_i^\mu$ has real poles at points $x = \beta_j^{-1}\alpha_j$, $j \neq i$, but is smooth elsewhere:*

$$\begin{aligned}\frac{\partial \omega_{si}}{\partial \beta_i^\mu} &= -\frac{2\rho_i^2\mathrm{Re}\{(\overline{\beta_i x - \alpha_i})e_\mu x\}}{\rho^2|\beta_i x - \alpha_i|^4}\mathrm{Im}\,\left\{(\beta_i x - \alpha_i)\overline{\beta_i dx}\right\} \\ &+ \frac{\rho_i}{\rho|\beta_i x - \alpha_i|^2}\mathrm{Im}\,\left\{e_\mu x\overline{\beta_i dx} + (\beta_i x - \alpha_i)\overline{e_\mu dx}\right\} \\ &+ \sum_{\substack{j=0\\ j\neq i}}^{k}\left\{-\frac{2\kappa_j^2\rho_i\mathrm{Re}\{(\beta_i x - \alpha_i)e_\mu x\}}{\rho^2|\beta_i x - \alpha_i|^4|\beta_j x - \alpha_j|^4}(\beta_i x - \alpha_i)\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}(\overline{\beta_i x - \alpha_i})\right. \\ &\quad + \frac{\kappa_j^2}{\rho|\beta_i x - \alpha_i|^2|\beta_i x - \alpha_j|^4}(e_\mu x)\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}(\overline{\beta_i x - \alpha_i}) \\ &\quad \left. + \frac{\kappa_j^2}{\rho|\beta_i x - \alpha_i|^2|\beta_j x - \alpha_j|^4}(\beta_i x - \alpha_i)\mathrm{Im}\,\left\{(\overline{\beta_j x - \alpha_j})\beta_j dx\right\}\overline{e_\mu x}\right\}.\end{aligned}$$

Proof. Calculation. $\square$

Lemma 5.3.7. *The 1-form $\partial\omega_{ni}/\partial\alpha_i^\mu$ has real poles at points $y = \alpha_j^{-1}\beta_j$, $j \neq i$, but is smooth elsewhere:*

$$\begin{aligned}
\frac{\partial\omega_{ni}}{\partial\alpha_i^\mu} &= -\frac{2\rho_i^2 \mathrm{Re}\{(\overline{\beta_i - \alpha_i y})e_\mu y\}}{\rho^2|\beta_i x - \alpha_i|^4}\mathrm{Im}\left\{(\overline{\beta_i - \alpha_i y})\alpha_i dy\right\} \\
&\quad - \frac{\rho_i}{\rho|\beta_i - \alpha_i y|^2}\mathrm{Im}\left\{(\beta_i - \alpha_i y)\overline{e_\mu dy} - e_\mu y\overline{\alpha_i dy}\right\} \\
&\quad + \sum_{\substack{j=0 \\ j\neq i}}^{k}\left\{-\frac{2\kappa_j^2\rho_i \mathrm{Re}\{(\overline{\beta_i - \alpha_i y})e_\mu y\}}{\rho^2|\beta_i - \alpha_i y|^4|\beta_j - \alpha_j y|^4}(\beta_i - \alpha_i y)\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}(\overline{\beta_i - \alpha_i y})\right. \\
&\qquad + \frac{\kappa_j^2}{\rho|\beta_i - \alpha_i y|^2|\beta_i - \alpha_j y|^4}(e_\mu y)\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}(\overline{\beta_i - \alpha_i y}) \\
&\qquad \left.+ \frac{\kappa_j^2}{\rho|\beta_i - \alpha_i y|^2|\beta_j - \alpha_j y|^4}(\beta_i - \alpha_i y)\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}\overline{e_\mu y}\right\}.
\end{aligned}$$

Proof. Calculation. □

Lemma 5.3.8. *The 1-form $\partial\omega_{ni}/\partial\beta_i^\mu$ has real poles at points $y = \alpha_j^{-1}\beta_j$, $j \neq i$, but is smooth elsewhere:*

$$\begin{aligned}
\frac{\partial\omega_{ni}}{\partial\beta_i^\mu} &= \frac{2\rho_i^2(\beta_i - \alpha_i y)^\mu}{\rho^2|\beta_i - \alpha_i y|^4}\mathrm{Im}\left\{(\beta_i - \alpha_i y)\overline{\alpha_i dy}\right\} - \frac{\rho_i}{\rho|\beta_i - \alpha_i y|^2}\mathrm{Im}\left\{e_\mu\overline{\alpha_i dy}\right\} \\
&\quad + \sum_{\substack{j=0 \\ j\neq i}}^{k}\left\{\frac{2\kappa_j^2\rho_i(\beta_i - \alpha_i y)^\mu}{\rho^2|\beta_i - \alpha_i y|^4|\beta_j - \alpha_j y|^4}(\beta_i - \alpha_i y)\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}(\overline{\beta_i - \alpha_i y})\right. \\
&\qquad - \frac{\kappa_j^2}{\rho|\beta_i - \alpha_i y|^2|\beta_i - \alpha_j y|^4}e_\mu\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}(\overline{\beta_i - \alpha_i y}) \\
&\qquad \left.- \frac{\kappa_j^2}{\rho|\beta_i - \alpha_i y|^2|\beta_j - \alpha_j y|^4}(\beta_i - \alpha_i y)\mathrm{Im}\left\{(\overline{\beta_j - \alpha_j y})\alpha_j dy\right\}\overline{e}_\mu\right\}.
\end{aligned}$$

Proof. Calculation. □

Recall that the points $\underline{P}_i \in \underline{\mathbb{HP}}^1$ may be assigned representatives (α_i, β_i) in $\mathbb{S}^7 \subset \mathbb{H}^2$. If $c_i = a_i$ or b_i in $\mathbb{H}$ denote the inhomogeneous local coordinates for the point P_i, then

$$a_i \longmapsto (\alpha_i, \beta_i) = \frac{(a_i, 1)}{\sqrt{|a_i|^2 + 1}} \qquad \text{in } \mathbb{S}^7|O_s,$$

$$b_i \longmapsto (\alpha_i, \beta_i) = \frac{(1, b_i)}{\sqrt{1 + |b_i|^2}} \qquad \text{in } \mathbb{S}^7|O_n.$$

Then, derivatives of the connection 1-forms with respect to the coordinates $c_i^\mu = a_i^\mu$ are given by:

$$\frac{\partial\omega}{\partial a_i^\mu} = \sum_{\nu=0}^{3}\frac{\partial\omega}{\partial\alpha_i^\nu}\frac{\partial\alpha_i^\nu}{\partial a_i^\mu} + \frac{\partial\omega}{\partial\beta_i^\nu}\frac{\partial\beta_i^\nu}{\partial a_i^\mu}.$$

Note that with respect to our choice of coordinate patches, for $(\alpha_i, \beta_i) \in O_s$ the factors

$$\frac{\partial \alpha_i^\nu}{\partial a_i^\mu} = \frac{a_i^\nu a_i^\mu}{(|a_i|^2+1)^{3/2}} + \frac{\delta_{\mu\nu}}{(|a_i|^2+1)^{1/2}},$$
$$\frac{\partial \beta_i^\nu}{\partial a_i^\mu} = \frac{a_i^\mu \delta_{0\mu}}{(|a_i|^2+1)^{3/2}},$$

are bounded, and similarly for derivatives of $\alpha_i^\nu, \beta_i^\nu$ with respect to $c_i^\mu = b_i^\mu$, where $(\alpha_i, \beta_i) \in O_n$.

In order to estimate the L^2 norms of the tangent vectors in $T\mathcal{M}_2$, we need to examine the behaviour of the corresponding 1-forms as the moduli parameters approach the boundary of $\tilde{T}_2$. It is useful to consider two distinct cases:

Case 1: *Three points approaching each other.* The points P_0, P_1, P_2 all lie in either the southern extended hemisphere $O_s(2)$ or the northern extended hemisphere $O_n(2)$. If the points lie in $O_s(2)$, then our definition of the partition of unity on $\mathbb{HP}^1$ is completed by choosing:

$$\operatorname{supp} \chi_s \subset O_s(8),$$
$$\operatorname{supp} \chi_n \subset O_n(3/8).$$

In the equivalent configuration, where the three points points lie in $O_n(2)$, we simply interchange the support radii.

Case 2: *Two points approaching each other.* The points P_i, P_j lie below the equator of $\mathbb{HP}^1$, and one point P_m lies in $O_n(2) \setminus O_s(2)$. We choose:

$$\operatorname{supp} \chi_s \subset O_s(7/4),$$
$$\operatorname{supp} \chi_n \subset O_n(4/5),$$

and for the equivalent configuration, where one point lies in $O_s(2) \setminus O_n(2)$ and two points lie above the equator, the support radii are interchanged.

These choices for the partition of unity χ_s, χ_n allow us to obtain estimates for the norms of the tangent vectors for all possible configurations of points P_0, P_1, P_2.

§5.4. Estimates of Tangent Vector Norms. I

We derive estimates for the L^2 norms of the tangent vectors $\partial\omega/\partial\lambda_i$ in terms of the weight parameters λ_i and coordinates of the points P_i.

For notational convenience, we assume $i = 0$. We observe that it is enough to consider Case 1 — where three points are approaching each other — since the estimates obtained in Case 2 will certainly be sharper that those obtained in Case 1. We further assume, again for notational convenience, that the three points P_i all lie in the southern hemisphere $O_s(2) \subset \mathbb{HP}^1$. Hence, the points $\underline{P}_i$ have coordinates $[a_i, 1]$, with $|a_i| < 2$ for all i. Moreover, if $x = x_0 x_1^{-1}$ is a local coordinate on the patch $O_s = \{[x_0, x_1] : |x_0 x_1^{-1}| < 8\}$ and $y = x_1 x_0^{-1}$ is a local coordinate on the patch $O_n = \{[x_0, x_1] : |x_1 x_0^{-1}| < 3/8\}$ in $\mathbb{HP}^1$, then $|x| < 8$ and $|y| < 3/8$.

The connection 1-forms are expressed in terms of parameters κ_i and (α_i, β_i). For weights λ_i and points $\underline{P}_i = [a_i, 1]$, we would usually choose

$$\kappa_i = \lambda_i,$$
$$(\alpha_i, \beta_i) = \frac{(a_i, 1)}{\sqrt{1+|a_i|^2}} \qquad \text{for } i = 0, 1, 2.$$

However, it is more convenient to instead express the 1-forms in terms of the following choices for κ_i and (α_i, β_i):

$$\kappa_i = \lambda_i \sqrt{1+|a_i|^2}$$
$$(\alpha_i, \beta_i) = (a_i, 1) \qquad \text{for } i = 0, 1, 2.$$

Both choices of parameters give the same ADHM matrices and so the same family of self-dual connections. We recall that the weights λ_i are always normalized so that $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. We shall occasionally use $\eta_{ij} = |a_i - a_j|$ to denote the Euclidean distance beteen the coordinates a_i and a_j. We use C to denote a generic positive constant, which is frequently updated, but always independent of moduli parameters.

We record the formulas for the required local 1-forms when $k = 2$ in the following lemmas:

Lemma 5.4.1. *The local tangent vector 1-form $\partial\omega_s/\partial\lambda_0$ on $O_s \subset \mathbb{HP}^1$ is given by:*

$$\begin{aligned}
\frac{\partial \omega_s}{\partial \lambda_0} &= \frac{\partial \omega_s}{\partial \kappa_0}\frac{\partial \kappa_0}{\partial \lambda_0} = \sqrt{1+|a_0|^2}\,\frac{\partial \omega_s}{\partial \kappa_0},\\
\frac{\partial \omega_s}{\partial \kappa_0} &= \frac{2\kappa_0\rho_0}{\rho^2|x-a_0|^4}\mathrm{Im}\,\{(\overline{x-a_0})dx\} - \frac{2\kappa_0\kappa_1^2}{\rho^2|x-a_0|^2|x-a_1|^4}\mathrm{Im}\,\{(\overline{x-a_1})dx\}\,,\\
&\quad - \frac{2\kappa_0\kappa_2^2}{\rho^2|x-a_0|^2|x-a_2|^4}\mathrm{Im}\,\{(\overline{x-a_2})dx\}\,,
\end{aligned}$$

where

$$\begin{aligned}
\rho(x) &= \frac{\kappa_0^2}{|x-a_0|^2} + \frac{\kappa_1^2}{|x-a_1|^2} + \frac{\kappa_2^2}{|x-a_2|^2},\\
\rho_0(x) &= \frac{\kappa_1^2}{|x-a_1|^2} + \frac{\kappa_2^2}{|x-a_2|^2}, \qquad x \in \mathbb{H} \simeq O_s.
\end{aligned}$$

Proof. Apply the previous formulas with $k = 2$, weights $\kappa_i = \lambda_i\sqrt{1+|a_i|^2}$, and $(\alpha_i, \beta_i) = (1, a_i)$ for $i = 0, 1, 2$. $\square$

Lemma 5.4.2. *The local tangent vector 1-form $\partial\omega_n/\partial\lambda_0$ on $O_n \subset \mathbb{HP}^1$ is given by:*

$$\begin{aligned}\frac{\partial\omega_n}{\partial\lambda_0} &= \frac{\partial\omega_n}{\partial\kappa_0}\frac{\partial\kappa_0}{\partial\lambda_0} = \sqrt{1+|a_0|^2}\frac{\partial\omega_n}{\partial\kappa_0},\\ \frac{\partial\omega_n}{\partial\kappa_0} &= -\frac{2\kappa_0\rho_0}{\rho^2|1-a_0y|^4}\mathrm{Im}\ \left\{(\overline{1-a_0y})a_0\,dy\right\}\\ &\quad + \frac{2\kappa_0\kappa_1^2}{\rho^2|1-a_0y|^2|1-a_1y|^4}\mathrm{Im}\ \left\{(\overline{1-a_1})a_1\,dy\right\},\\ &\quad + \frac{2\kappa_0\kappa_2^2}{\rho^2|1-a_0y|^2|1-a_2y|^4}\mathrm{Im}\ \left\{(\overline{1-a_2y})a_2\,dy\right\},\end{aligned}$$

where

$$\begin{aligned}\rho(y) &= \frac{\kappa_0^2}{|1-a_0y|^2} + \frac{\kappa_1^2}{|1-a_1y|^2} + \frac{\kappa_2^2}{|1-a_2y|^2}\\ \rho_0(y) &= \frac{\kappa_1^2}{|1-a_1y|^2} + \frac{\kappa_2^2}{|1-a_2y|^2}, \qquad y \in \mathbb{H} \simeq O_n.\end{aligned}$$

Proof. Again, apply the previous formulas with $k=2$, weights $\kappa_i = \lambda_i\sqrt{1+|a_i|^2}$, and $(\alpha_i, \beta_i) = (1, a_i)$ for $i = 0, 1, 2$. $\square$

The corresponding norms are:

$$\left\|\frac{\partial\omega}{\partial\lambda_0}\right\|^2 = \left\|\frac{\partial\omega_s}{\partial\lambda_0}\sqrt{\chi_s}\right\|^2 + \left\|\frac{\partial\omega_n}{\partial\lambda_0}\sqrt{\chi_n}\right\|^2,$$

where

$$\begin{aligned}\left\|\frac{\partial\omega_s}{\partial\lambda_0}\sqrt{\chi_s}\right\|^2 &= \int_{O_s}\chi_s\left|\frac{\partial\omega_s}{\partial\lambda_0}\right|^2\sqrt{g}\,dx,\\ \left\|\frac{\partial\omega_n}{\partial\lambda_0}\sqrt{\chi_n}\right\|^2 &= \int_{O_n}\chi_n\left|\frac{\partial\omega_n}{\partial\lambda_0}\right|^2\sqrt{g}\,dy,\end{aligned}$$

and $|\cdot|$ denotes the fibre metric on $\Omega^1(\mathbb{HP}^1, \mathrm{ad}\,P)$, and g is the metric on $\mathbb{HP}^1$.

For convenience, we record some elementary integration formulas and inequalities that we will require for our estimate calculations.

Lemma 5.4.3. *Let $a, b, c, m, n, R, t, \delta$ be real numbers, with $0 < t < 1$, $0 < \delta \leq 1$, and $R \geq 1$.*

(i) $\displaystyle\int_m^n \frac{r}{a+r^2b}\,dr = \frac{1}{2b}\log\left(\frac{a+n^2b}{a+m^2b}\right);$

(ii) $\displaystyle\int_m^n \frac{r^3}{a+r^2b}\,dr = \frac{1}{2b^2}\left\{(n^2-m^2)b - a\log\left(\frac{a+n^2b}{a+m^2b}\right)\right\};$

(iii) $\displaystyle\frac{1}{1-t}\log\left(\frac{t+R(1-t)}{t}\right) \leq 2 + 2\log\left(\frac{R}{t}\right), \qquad 0 < t < 1;$

(iv) $\displaystyle\frac{1}{1-t}\log\left(\frac{t+R(1-t)}{t+\delta(1-t)}\right) \leq 2R + 2\log\left(\frac{R}{\delta}\right), \qquad 0 < t < 1;$

(v) $$\int \frac{1}{ar^2+br+c}\,dr = \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ar+b}{\sqrt{4ac-b^2}}\right) \qquad \text{if } b^2-4ac<0;$$

(vi) $$\int \frac{r}{ar^2+br+c}\,dr = \frac{1}{2a}\log\left(ar^2+br+c\right) - \frac{b}{2a}\int\frac{1}{ar^2+br+c}.$$

Proof. Calculation. □

Lemma 5.4.4. *Let $a, b \in \mathbb{H}$, $R > 0$, and let $|a-b|$ denote the Euclidean distance between a and b. Then:*

$$\int_{|x|<R} \frac{1}{|x-a|^2|x-b|^2}\,dx \le C\left\{1+\log\left(1+\frac{R}{|a-b|}\right)\right\},$$

where dx denotes the standard Euclidean measure on $\mathbb{H}=\mathbb{R}^4$, and C is a constant independent of a, b, and R.

Proof. The Euclidean ball $\mathbb{B}(0,R)$ is clearly contained in the union of the following two annuli centred at a and b:

$$\mathbb{B}(0,R) \subset \{x : \eta/2 \le |x-a| < R'\}\bigcup\{x : \eta/2 \le |x-b| < R'\},$$

where $R' = R + |a| + |b|$ and $\eta = |a-b|$. Then, our integal may be estimated by:

$$\begin{aligned}
&\int_{|x|<R} \frac{1}{|x-a|^2|x-b|^2}\,dx \\
&\qquad \le \int_{\substack{|x-a|<R' \\ |x-b|\ge\eta/2}} \frac{1}{|x-a|^2|x-b|^2}\,dx + \int_{\substack{|x-b|<R' \\ |x-a|\ge\eta/2}} \frac{1}{|x-a|^2|x-b|^2}\,dx.
\end{aligned}$$

We choose two spherical polar coordinate systems, one centred at $x = a$, with $r = |x-a|$, and the other centred at $x = b$, with $s = |x-b|$. Then $dx = r^3 dr d\theta$ or $dx = s^3 ds d\theta$, where $d\theta$ denotes the standard Riemannian measure on the 3-sphere $\mathbb{S}^3$ of unit radius. In the first integal, we have both $|x-b| \ge \eta/2$ and $|x-b| \ge |r-\eta|$, so that:

$$|x-b|^2 \ge \left(\eta^2/4 + (r-\eta)^2\right)/2.$$

Similarly, in the second integral we have $|x-a|^2 \ge \left(\eta^2/4+(s-\eta)^2\right)/2$. Using these inequalities we obtain:

$$\begin{aligned}
\int_{|x|<R} \frac{1}{|x-a|^2|x-b|^2}\,dx &\le 2\int_0^{R'} \frac{r}{\eta^2/4+(r-\eta)^2}\,dr + 2\int_0^{R'} \frac{s}{\eta^2/4+(s-\eta)^2}\,ds \\
&= 4\int_0^{R'} \frac{r}{5\eta^2/4 - 2\eta r + r^2}\,dr.
\end{aligned}$$

Integrating, we get:

$$\begin{aligned}
\int_{|x|<R} \frac{1}{|x-a|^2|x-b|^2}\,dx &\le 2\log\left(1+\frac{4R'^2}{5\eta^2}-\frac{8R'}{5\eta}\right) + 8\arctan\left(\frac{2R'-2\eta}{\eta}\right) \\
&\quad - 8\arctan(-2) \\
&\le 8\pi + 4\log\left(1+\frac{R}{\eta}\right),
\end{aligned}$$

and this gives the required estimate. □

Lemma 5.4.5. *The norms of the three terms in $\partial\omega_s/\partial\lambda_0$ have the following upper bounds:*

(i)
$$\left\| \frac{\partial \omega_s^{(1)}}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \left\{ 1 + \log\left(\frac{1}{\lambda_0}\right) \right\},$$

(ii)
$$\left\| \frac{\partial \omega_s^{(2)}}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \left\{ 1 + \log\left(1 + \frac{1}{|a_0 - a_1|}\right) \right\},$$

(iii)
$$\left\| \frac{\partial \omega_s^{(3)}}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \left\{ 1 + \log\left(1 + \frac{1}{|a_0 - a_2|}\right) \right\}.$$

Consequently,

$$\left\| \frac{\partial \omega_s}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \left\{ 1 + \log\left(\frac{1}{\lambda_0}\right) + \log\left(1 + \frac{1}{|a_0 - a_1|}\right) + \log\left(1 + \frac{1}{|a_0 - a_2|}\right) \right\}.$$

Proof. We consider each of the three norms in turn, noting that integration over $\operatorname{supp}\chi_s \subset O_s$ in $\mathbb{HP}^1$ corresponds to integration over the Eulidean ball $\{x \in \mathbb{H} : |x| < 8\} \subset \mathbb{H}$. For the first term in $\partial\omega_s/\partial\lambda_0$, we have

$$\left\| \frac{\partial \omega_s^{(1)}}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \int_{|x|<8} \frac{\kappa_0^2 \rho_0^2}{\rho^4 |x - a_0|^6}\, dx.$$

Now using the inequalities $\rho_0(x) \leq \rho(x)$ and

$$\frac{\kappa_0^2}{|x - a_0|^2} \leq \rho(x),$$

we see that

$$\int_{|x|<8} \frac{\kappa_0^2 \rho_0^2}{\rho^4 |x - a_0|^6}\, dx \leq C \int_{|x|<8} \frac{1}{\rho |x - a_0|^4}\, dx.$$

We choose spherical polar coordinates on $\mathbb{H} = \mathbb{R}^4$ with $r = |x - a_0|$, so that $dx = r^3 dr d\theta$. Since $|x| < 8$, $|a_i| < 2$, we have $|x - a_i| < 10$, for $i = 0, 1, 2$, and so we get the following lower bounds for $\rho(x)$:

$$\begin{aligned} \rho(x) &= \frac{\kappa_0^2}{|x - a_0|^2} + \frac{\kappa_1^2}{|x - a_1|^2} + \frac{\kappa_2^2}{|x - a_2|^2}, \\ &\geq C \left(\frac{\kappa_0^2}{|x - a_0|^2} + \kappa_1^2 + \kappa_2^2 \right). \end{aligned}$$

Then, as $\kappa_i = \lambda_i \sqrt{1 + |a_i|^2} \geq \lambda_i$ for all i, we have

$$\rho(x) \geq C \left(\frac{\lambda_0^2}{r^2} + \lambda_1^2 + \lambda_2^2 \right).$$

Returning to our integral estimate, the ball $\{x : |x| < 8\}$ is contained in $\{x : |x - a_0| < 10\}$, and so

$$\begin{aligned}\int_{|x|<8} \frac{1}{\rho|x-a_0|^4}\,dx &\le \int_{|x-a_0|<10} \frac{1}{\rho|x-a_0|^4}\,dx \\ &\le C\int_0^{10} \frac{r^3}{(r^{-2}\lambda_0^2+\lambda_1^2+\lambda_2^2)r^4}\,dr \\ &= C\int_0^{10} \frac{r}{\lambda_0^2+r^2(\lambda_1^2+\lambda_2^2)}\,dr \\ &= \frac{C}{\lambda_1^2+\lambda_2^2}\log\left(\frac{\lambda_0^2+100(\lambda_1^2+\lambda_2^2)}{\lambda_0^2}\right).\end{aligned}$$

Since $\lambda_0^2+\lambda_1^2+\lambda_2^2=1$, we may set $t=\lambda_0^2$, $1-t=\lambda_1^2+\lambda_2^2$, with $0<t<1$, and applying one of our log inequalities, we obtain

$$\frac{1}{\lambda_1^2+\lambda_2^2}\log\left(\frac{\lambda_0^2+100(\lambda_1^2+\lambda_2^2)}{\lambda_0^2}\right) \le C\left\{1+\log\left(\frac{1}{\lambda_0}\right)\right\},$$

and combining the above inequalities gives estimate (i).

Considering the second term, we have

$$\begin{aligned}\left\|\frac{\partial\omega_s^{(2)}}{\partial\lambda_0}\sqrt{\chi_s}\right\|^2 &\le C\int_{|x|<8} \frac{\kappa_0^2\kappa_1^4}{\rho^4|x-a_0|^4|x-a_1|^6}\,dx \\ &\le C\int_{|x-a_0|<8} \frac{1}{\rho|x-a_0|^2|x-a_1|^2}\,dx\end{aligned}$$

where we use the inequalities

$$\frac{\kappa_i^2}{|x-a_i|^2} \le \rho(x) \qquad \text{for } i=0,1.$$

As before, $\kappa_i \ge \lambda_i$ and $|x-a_i|<10$ for $i=0,1,2$. Then, we have

$$\begin{aligned}\rho(x) &\ge C\left(\kappa_0^2+\kappa_1^2+\kappa_2^2\right) \\ &\ge C\left(\lambda_0^2+\lambda_1^2+\lambda_2^2\right) \\ &= C,\end{aligned}$$

since $\lambda_0^2+\lambda_1^2+\lambda_2^2=1$, and so $\rho(x)$ is bounded below by some positive constant. Hence, we see that

$$\begin{aligned}\int_{|x-a_0|<8} \frac{1}{\rho|x-a_0|^2|x-a_1|^2}\,dx &\le C\int_{|x-a_0|<10} \frac{1}{|x-a_0|^2|x-a_1|^2}\,dx \\ &\le C\left\{1+\log\left(1+\frac{1}{|a_0-a_1|}\right)\right\},\end{aligned}$$

by a previous integration lemma. This gives estimate (ii).

Considering the third term, we have:

$$\left\| \frac{\partial \omega_s^{(3)}}{\partial \lambda_0} \sqrt{\chi_s} \right\|^2 \leq C \int_{|x|<8} \frac{\kappa_0^2 \kappa_2^4}{\rho^4 |x-a_0|^4 |x-a_2|^6}\, dx,$$

and so estimate (iii) follows by symmetry. □

Lemma 5.4.6. *The norms of the three terms in $\partial\omega_n/\partial\lambda_0$ are bounded by a constant C, independent of moduli parameters:*

$$\left\| \frac{\partial \omega_n^{(j)}}{\partial \lambda_0} \sqrt{\chi_n} \right\|^2 \leq C, \quad \text{for } j = 1, 2, 3.$$

Consequently,

$$\left\| \frac{\partial \omega_n}{\partial \lambda_0} \sqrt{\chi_n} \right\|^2 \leq C.$$

Proof. We note that integration over $\operatorname{supp} \chi_n \subset O_n$ in $\mathbb{HP}^1$ corresponds to integration over the Eulidean ball $\{y \in \mathbb{H} : |y| < 3/8\} \subset \mathbb{H}$. Then, considering the first term in $\partial\omega_n/\partial\lambda_0$, we have

$$\left\| \frac{\partial \omega_n^{(1)}}{\partial \lambda_0} \sqrt{\chi_n} \right\|^2 \leq C \int_{|y|<3/8} \frac{\kappa_0^2 \rho_0^2 |a_0|^2}{\rho^4 |1-a_0 y|^6}\, dy.$$

Now $|a_0| < 2$, $\rho_0(y) \leq \rho(y)$, and

$$\frac{\kappa_0^2}{|1-a_0 y|^2} \leq \rho(y).$$

Hence, we see

$$\int_{|y|<3/8} \frac{\kappa_0^2 \rho_0^2 |a_0|^2}{\rho^4 |1-a_0 y|^6}\, dy \leq C \int_{|y|<3/8} \frac{1}{\rho |1-a_0 y|^4}\, dy.$$

Next, we observe that

$$\begin{aligned} &|1-a_i y| \geq 1 - |a_i y| > 1/4, \\ &|1-a_i y| \leq 1 + |a_i y| < 7/4 \qquad \text{for } i = 0, 1, 2, \end{aligned}$$

since $|a_i| < 2$ and $|y| < 3/8$. So $|1-a_0 y|^4$ is bounded below by a positive constant, and moreover

$$\begin{aligned} \rho(y) &= \frac{\kappa_0^2}{|1-a_0 y|^2} + \frac{\kappa_1^2}{|1-a_1 y|^2} + \frac{\kappa_2^2}{|1-a_2 y|^2} \\ &\geq C\left(\kappa_0^2 + \kappa_1^2 + \kappa_2^2\right) \\ &\geq C\left(\lambda_0^2 + \lambda_1^2 + \lambda_2^2\right) \\ &= C, \end{aligned}$$

as $\kappa_i \geq \lambda_i$, for $i = 0, 1, 2$, and $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. We now obtain

$$\begin{aligned}\int_{|y|<3/8} \frac{1}{\rho |1 - a_0 y|^4}\, dy &\leq C \int_{|y|<3/8} dy \\ &= C,\end{aligned}$$

and combining these inequalities gives the required estimate.

Considering the second term, we see

$$\begin{aligned}\left\| \frac{\partial \omega_n^{(2)}}{\partial \lambda_0} \sqrt{\chi_n} \right\|^2 &\leq C \int_{|y|<3/8} \frac{\kappa_0^2 \kappa_1^4 |a_1|^2}{\rho^4 |1 - a_0 y|^4 |1 - a_1 y|^6}\, dy \\ &\leq C \int_{|y|<3/8} \frac{1}{\rho |1 - a_0 y|^2 |1 - a_1 y|^2}\, dy \\ &\leq C \int_{|y|<3/8} dy \\ &= C,\end{aligned}$$

by similar arguments. The estimate for the third term is obtained by symmetry. □

Proposition 5.4.7. *For $i = 0, 1, 2$ we have the following upper bounds:*

$$\left\| \frac{\partial \omega}{\partial \lambda_i} \right\|^2 \leq C \left\{ 1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{\substack{j=0 \\ j \neq i}}^{2} \log\left(1 + \frac{1}{|a_i - a_j|}\right) \right\}.$$

Proof. One makes the obvious changes in the estimates obtained when $i = 0$ to get estimates for $i = 1$ and 2. We then combine these inequalities to give the required upper bound. □

§5.5. Estimates of Tangent Vector Norms. II

In this section we estimate the norms of the tangent vectors $\partial\omega/\partial c_i^\mu$. For notational convenience we assume that $i = 0$. As in the previous section, it suffices to consider Case 1, and we may again assume, for notational convenience, that the three points P_i lie in the southern hemisphere $O_s \subset \mathbb{HP}^1$. Hence, the points $\underline{P}_i$ have coordinates $[a_i, 1]$, with $|a_i| < 2$ for $i = 0, 1, 2$.

We just need to estimate the L^2 norms of the local 1-forms

$$\frac{\partial \omega_{si}}{\partial a_i^\mu}, \quad \frac{\partial \omega_s}{\partial a_i^\mu}, \quad , \frac{\partial \omega_n}{\partial a_i^\mu},$$

on the coordinate patches O_{si}, $O_s \setminus \{\underline{P}_i\}$, and O_n, since these patches obviously cover $\mathbb{HP}^1$. Consequently, we have

$$\left\| \frac{\partial \omega}{\partial a_i^\mu} \right\|^2 = \int_{O_s} \left| \frac{\partial \omega_{si}}{\partial a_i^\mu} \right|^2 \chi_s \chi_i \sqrt{g} dx + \int_{O_s} \left| \frac{\partial \omega_s}{\partial a_i^\mu} \right|^2 \chi_s (1 - \chi_i) \sqrt{g} dx$$
$$+ \int_{O_n} \left| \frac{\partial \omega_n}{\partial a_i^\mu} \right|^2 \chi_n \sqrt{g} dy.$$

As in the previous section, the connection 1-forms are parametrized by κ_i and (α_i, β_i), where we choose

$$\kappa_i = \lambda_i \sqrt{1 + |a_i|^2},$$
$$(\alpha_i, \beta_i) = (a_i, 1),$$

and $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. We recall that with this choice of parametrization, we have:

$$\frac{\partial \omega}{\partial a_i^\mu} = \frac{\partial \omega}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial a_i^\mu} + \left. \frac{\partial \omega}{\partial \alpha_i^\mu} \right|_{\alpha_i = a_i, \beta_i = 1}.$$

We obtained the estimates for the L^2 norms of $\partial \omega / \partial \kappa_i$ in the previous section, and so it suffices to consider the L^2 norms of $\partial \omega / \partial \alpha_i^\mu$. We record our formulas for the local 1-forms representing the required tangent vectors in the following series of lemmas.

Lemma 5.5.1. *The local tangent vector 1-form $\partial \omega_{s0} / \partial \alpha_0^\mu$ on O_{s0} is given by*

$$\begin{aligned}
\left. \frac{\partial \omega_{s0}}{\partial \alpha_0^\mu} \right|_{\alpha_i = a_i, \beta_i = 1} = {} & \frac{2\rho_0^2 (x - a_0)^\mu}{\rho^2 |x - a_0|^4} \mathrm{Im}\ \{(x - a_0) d\overline{x}\} - \frac{\rho_0}{\rho |x - a_0|^2} \mathrm{Im}\ \{e_\mu d\overline{x}\} \\
& + \frac{2\kappa_1^2 \rho_0 (x - a_0)^\mu}{\rho^2 |x - a_0|^4 |x - a_1|^4} (x - a_0) \mathrm{Im}\ \{(\overline{x - a_1}) dx\} (\overline{x - a_0}) \\
& - \frac{\kappa_1^2}{\rho |x - a_0|^2 |x - a_1|^4} e_\mu \mathrm{Im}\ \{(\overline{x - a_1}) dx\} (\overline{x - a_0}) \\
& - \frac{\kappa_1^2}{\rho |x - a_0|^2 |x - a_1|^4} (x - a_0) \mathrm{Im}\ \{(\overline{x - a_1}) dx\} \overline{e}_\mu \\
& + \frac{2\kappa_2^2 \rho_0 (x - a_0)^\mu}{\rho^2 |x - a_0|^4 |x - a_2|^4} (x - a_0) \mathrm{Im}\ \{(\overline{x - a_2}) dx\} (\overline{x - a_0}) \\
& - \frac{\kappa_2^2}{\rho |x - a_0|^2 |x - a_2|^4} e_\mu \mathrm{Im}\ \{(\overline{x - a_2}) dx\} (\overline{x - a_0}) \\
& - \frac{\kappa_2^2}{\rho |x - a_0|^2 |x - a_2|^4} (x - a_0) \mathrm{Im}\ \{(\overline{x - a_2}) dx\} \overline{e}_\mu.
\end{aligned}$$

Proof. Apply the previous formulas with $k = 2$, weights $\kappa_i = \lambda_i \sqrt{1 + |a_i|^2}$, and $(\alpha_i, \beta_i) = (1, a_i)$ for $i = 0, 1, 2$. $\square$

Lemma 5.5.2. *The local tangent vector 1-form $\partial\omega_s/\partial\alpha_0^\mu$ on $O_s \setminus \{P_0\}$ is given by*

$$\begin{aligned}
\left.\frac{\partial\omega_s}{\partial\alpha_0^\mu}\right|_{\alpha_i=a_i,\beta_i=1} &= \frac{4\kappa_0^2(x-a_0)^\mu}{\rho|x-a_0|^6}\mathrm{Im}\ \{(\overline{x-a_0})dx\} \\
&\quad - \frac{\kappa_0^2}{\rho|x-a_0|^4}\mathrm{Im}\ \{\overline{e}_\mu dx\} \\
&\quad - \frac{2\kappa_0^4(x-a_0)^\mu}{\rho^2|x-a_0|^8}\mathrm{Im}\ \{(\overline{x-a_0})dx\} \\
&\quad - \frac{2\kappa_0^2\kappa_1^2(x-a_0)^\mu}{\rho^2|x-a_0|^4|x-a_1|^4}\mathrm{Im}\ \{(\overline{x-a_1})dx\} \\
&\quad - \frac{2\kappa_0^2\kappa_2^2(x-a_0)^\mu}{\rho^2|x-a_0|^4|x-a_2|^4}\mathrm{Im}\ \{(\overline{x-a_2})dx\}\,.
\end{aligned}$$

Proof. Again, apply the previous formulas with $k=2$, weights $\kappa_i = \lambda_i\sqrt{1+|a_i|^2}$, and $(\alpha_i,\beta_i) = (1,a_i)$ for $i=0,1,2$. □

Lemma 5.5.3. *The local tangent vector 1-form $\partial\omega_n/\partial\alpha_0^\mu$ on O_n is given by*

$$\begin{aligned}
\left.\frac{\partial\omega_n}{\partial\alpha_0^\mu}\right|_{\alpha_i=a_i,\beta_i=1} &= -\frac{4\kappa_0^2\mathrm{Re}\{(\overline{1-a_0y})e_\mu y\}}{\rho|1-a_0y|^6}\mathrm{Im}\ \left\{(\overline{1-a_0y})dy\right\} \\
&\quad - \frac{\kappa_0^2}{\rho|1-a_0y|^4}\mathrm{Im}\ \{(\overline{e}_\mu - 2a_0^\mu\overline{y})dy\} \\
&\quad + \frac{2\kappa_0^4\mathrm{Re}\{(\overline{1-a_0y})e_\mu y\}}{\rho^2|1-a_0y|^8}\mathrm{Im}\ \left\{(\overline{1-a_0y})dy\right\} \\
&\quad + \frac{2\kappa_0^2\kappa_1^2\mathrm{Re}\{(\overline{1-a_0y})e_\mu y\}}{\rho^2|1-a_0y|^4|1-a_1y|^4}\mathrm{Im}\ \left\{(\overline{1-a_1y})dy\right\} \\
&\quad + \frac{2\kappa_0^2\kappa_2^2\mathrm{Re}\{(\overline{1-a_0y})e_\mu y\}}{\rho^2|1-a_0y|^4|1-a_2y|^4}\mathrm{Im}\ \left\{(\overline{1-a_2y})dy\right\}.
\end{aligned}$$

Proof. Apply the previous formulas with $k=2$, weights $\kappa_i = \lambda_i\sqrt{1+|a_i|^2}$, and $(\alpha_i,\beta_i) = (1,a_i)$, for $i=0,1,2$. □

Lemma 5.5.4. *The norms of the eight terms in $\partial\omega_{s0}/\partial\alpha_0^\mu$ have the following upper bounds:*

(i) $$\int_{O_s}\left|\frac{\partial\omega_{s0}^{(j)}}{\partial\alpha_0^\mu}\right|^2\chi_s\chi_0\sqrt{g}dx \le C\left\{1+\log\left(\frac{1}{\lambda_0}\right)\right\}, \quad \text{for } j=1,2.$$

(ii) $$\int_{O_s}\left|\frac{\partial\omega_{s0}^{(j)}}{\partial\alpha_0^\mu}\right|^2\chi_s\chi_0\sqrt{g}dx \le C, \quad \text{for } j=3,\ldots,8.$$

Consequently,

$$\int_{O_s} \left| \frac{\partial \omega_{s0}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx \le C \left\{ 1 + \log \left(\frac{1}{\lambda_0} \right) \right\}.$$

Proof. Since we are integrating over the support of $\chi_s\chi_0$ in $\mathbb{HP}^1$, we can assume $|x - a_0| < \delta_0$ and $|x| < 8$, for $x \in \mathbb{H}$. Commencing with the first term in $\partial\omega_{s0}/\partial\alpha_0^\mu$, we have

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_{s0}^{(1)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx &\le C \int_{|x-a_0|<\delta_0} \frac{\rho_0^4}{\rho^4 |x-a_0|^4} dx \\
&\le C \int_{|x-a_0|<\delta_0} \frac{\rho_0}{\rho |x-a_0|^4} dx \\
&= C \int_{|x-a_0|<\delta_0} \frac{\rho_0}{(\lambda_0^2 + |x-a_0|^2 \rho_0)|x-a_0|^2} dx.
\end{aligned}$$

Since $\delta_0 = \frac{1}{2}\min\{1, \eta_{01}, \eta_{02}\}$, we have $|x - a_j| \ge \eta_{0j}/2$ for $j \ne 0$ and $|x - a_0| < \delta_0$, where $\eta_{0j} = |a_0 - a_j|$. Then

$$\rho_0(x) = \frac{\lambda_1^2}{|x-a_1|^2} + \frac{\lambda_2^2}{|x-a_2|^2} \le 4 \left(\frac{\lambda_1^2}{\eta_{01}^2} + \frac{\lambda_2^2}{\eta_{02}^2} \right),$$

and so

$$\frac{\rho_0}{\lambda_0^2 + |x-a_0|^2 \rho_0} \le \frac{4(\lambda_1^2/\eta_{01}^2 + \lambda_2^2/\eta_{02}^2)}{\lambda_0^2 + 4|x-a_0|^2(\lambda_1^2/\eta_{01}^2 + \lambda_2^2/\eta_{02}^2)}.$$

We employ spherical polar coordinates on $\mathbb{H}$, so that $dx = r^3 dr d\theta$ and $r = |x - a_0|$. Hence,

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_{s0}^{(1)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx &\le C \int_0^{\delta_0} \frac{r(\lambda_1^2/\eta_{01}^2 + \lambda_2^2/\eta_{02}^2)}{\lambda_0^2 + r^2(\lambda_1^2/\eta_{01}^2 + \lambda_2^2/\eta_{02}^2)} dr \\
&= C \log \left(\frac{\lambda_0^2 + \delta_0^2(\lambda_1^2/\eta_{01}^2 + \lambda_2^2/\eta_{02}^2)}{\lambda_0^2} \right).
\end{aligned}$$

Since $\delta_0 \le \eta_{01}/2, \eta_{02}/2$, we have

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_{s0}^{(1)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx &\le C \log \left(\frac{\lambda_0^2 + (\lambda_1^2 + \lambda_2^2)/4}{\lambda_0^2} \right) \\
&= C \log \left(\frac{1}{\lambda_0} \right),
\end{aligned}$$

since $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. For the second term, we have

$$\int_{O_s} \left| \frac{\partial \omega_{s0}^{(2)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx \le C \int_{|x-a_0|<\delta_0} \frac{\rho_0^2}{\rho^2 |x-a_0|^4} dx$$
$$\le C \int_{|x-a_0|<\delta_0} \frac{\rho_0}{\rho |x-a_0|^4} dx,$$

with the same bound as the first term, and this establishes (i). Examining the third term in $\partial \omega_{s0}/\partial \alpha_0^\mu$, we see

$$\int_{O_s} \left| \frac{\partial \omega_{s0}^{(3)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx \le C \int_{|x-a_0|<\delta_0} \frac{\kappa_1^4 \rho_0^2}{\rho^4 |x-a_0|^2 |x-a_1|^6} dx$$
$$\le C \int_{|x-a_0|<\delta_0} \frac{1}{|x-a_0|^2 |x-a_1|^2} dx.$$

Since $|x-a_1| \ge \eta_{01}/2$ and $\delta_0 \le \eta_{01}/2$, we obtain

$$\int_{O_s} \left| \frac{\partial \omega_{s0}^{(3)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx \le C \int_0^{\delta_0} \frac{r}{\eta_{01}^2} dr$$
$$\le C.$$

For the fourth term, we have

$$\int_{O_s} \left| \frac{\partial \omega_{s0}^{(4)}}{\partial \alpha_0^\mu} \right|^2 \chi_s \chi_0 \sqrt{g} dx \le C \int_{|x-a_0|<\delta_0} \frac{\kappa_1^4}{\rho^2 |x-a_0|^2 |x-a_1|^6} dx$$
$$\le C \int_{|x-a_0|<\delta_0} \frac{1}{|x-a_0|^2 |x-a_1|^2} dx,$$

which is bounded above by the same constant C, and similarly for term five. The upper bounds for terms six, seven, and eight follow immediately by symmetry. □

Lemma 5.5.5. *The norms of the five terms in $\partial \omega_s/\partial \alpha_0^\mu$ have the following upper bound:*

(i) $$\int_{O_s} \left| \frac{\partial \omega_{s}^{(j)}}{\partial \alpha_0^\mu} \right|^2 \chi_s (1-\chi_0) \sqrt{g} dx \le C \left\{ 1 + \log \left(1 + \frac{1}{|a_0 - a_1|} + \frac{1}{|a_0 - a_2|} \right) \right\};$$

for $j = 1, 2, 3$.

(ii) $$\int_{O_s} \left| \frac{\partial \omega_{s}^{(4)}}{\partial \alpha_0^\mu} \right|^2 \chi_s (1-\chi_0) \sqrt{g} dx \le C \left\{ 1 + \log \left(1 + \frac{1}{|a_0 - a_1|} \right) \right\};$$

(iii) $$\int_{O_s} \left| \frac{\partial \omega_{s}^{(5)}}{\partial \alpha_0^\mu} \right|^2 \chi_s (1-\chi_0) \sqrt{g} dx \le C \left\{ 1 + \log \left(1 + \frac{1}{|a_0 - a_2|} \right) \right\}.$$

Consequently,

$$\int_{O_s} \left| \frac{\partial \omega_s}{\partial \alpha_0^\mu} \right|^2 \chi_s(1-\chi_0)\sqrt{g}dx \le \left\{ 1 + \log\left(1 + \frac{1}{|a_0 - a_1|}\right) + \log\left(1 + \frac{1}{|a_0 - a_2|}\right) \right\}.$$

Proof. We note that $|x| < 8$ and $|x - a_0| \ge \frac{1}{2}K^{-4}\delta_0$ since we are only integrating over the support of $\chi_s(1-\chi_0)$. Commencing with the first term in $\partial\omega_s/\partial a_0^\mu$, and employing spherical polar coordinates on $\mathbb{H}$ with $r = |x - a_0|$, $dx = r^3 dr d\theta$, we have

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_s^{(1)}}{\partial \alpha_0^\mu} \right|^2 \chi_s(1-\chi_0)\sqrt{g}dx &\le C \int_{\substack{|x|<8 \\ |x-a_0|\ge \frac{1}{2}K^{-4}\delta_0}} \frac{\kappa_0^4}{\rho^2 |x-a_0|^8}\, dx \\
&\le C \int_{\frac{1}{2}K^{-4}\delta_0 \le |x-a_0| < 10} \frac{1}{|x-a_0|^4}\, dx \\
&\le C \int_{\frac{1}{2}K^{-4}\delta_0}^{10} \frac{1}{r}\, dr \\
&\le C \left\{ 1 + \log\left(\frac{1}{\delta_0}\right) \right\},
\end{aligned}$$

and recalling that $\delta_0 = \frac{1}{2}\min\{1, \eta_{01}, \eta_{02}\}$, we obtain the upper bound in (i). Considering the second term, we have

$$\int_{O_s} \left| \frac{\partial \omega_s^{(2)}}{\partial \alpha_0^\mu} \right|^2 \chi_s(1-\chi_0)\sqrt{g}dx \le C \int_{\substack{|x|<8 \\ |x-a_0|\ge \frac{1}{2}K^{-4}\delta_0}} \frac{\kappa_0^4}{\rho^2 |x-a_0|^8}\, dx,$$

leading to the same upper bound as obtained for the first term. For the third term, we have

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_s^{(3)}}{\partial \alpha_0^\mu} \right|^2 \chi_s(1-\chi_0)\sqrt{g}dx &\le C \int_{\substack{|x|<8 \\ |x-a_0|\ge \frac{1}{2}K^{-4}\delta_0}} \frac{\kappa_0^8}{\rho^4 |x-a_0|^{12}}\, dx \\
&\le C \int_{\frac{1}{2}K^{-4}\delta_0 \le |x-a_0| < 10} \frac{1}{|x-a_0|^4}\, dx,
\end{aligned}$$

which again leads to the same upper bound. For the fourth term, we have

$$\begin{aligned}
\int_{O_s} \left| \frac{\partial \omega_s^{(4)}}{\partial \alpha_0^\mu} \right|^2 \chi_s(1-\chi_0)\sqrt{g}dx &\le C \int_{\substack{|x|<8 \\ |x-a_0|\ge \frac{1}{2}K^{-4}\delta_0}} \frac{\kappa_0^4 \kappa_1^4}{\rho^4 |x-a_0|^6 |x-a_1|^6}\, dx \\
&\le C \int_{|x|<8} \frac{1}{|x-a_0|^2 |x-a_1|^2}\, dx \\
&\le C \left\{ 1 + \log\left(\frac{1}{|a_0 - a_1|}\right) \right\},
\end{aligned}$$

applying a previous integration lemma to obtain the last inequality, and this gives estimate (ii). Estimate (iii) follows by symmetry. $\square$

Lemma 5.5.6. *The norms of the five terms in $\partial\omega_n/\partial\alpha_0^\mu$ are bounded by a constant:*

$$\int_{O_n} \left| \frac{\partial \omega_n^{(j)}}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dx \le C \quad \text{for } j = 1, \dots, 5.$$

Consequently,

$$\int_{O_n} \left| \frac{\partial \omega_n}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dy \le C.$$

Proof. Since we are integrating over the support of χ_n, we have $|y| < 3/8$, and as $|a_i| < 2$, then $|1 - a_i y| > 1/4$ for $i = 0, 1, 2$. Considering the first term in $\partial\omega_n/\partial\alpha_0^\mu$, we have

$$\begin{aligned} \int_{O_n} \left| \frac{\partial \omega_n^{(1)}}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dx &\le C \int_{|y|<3/8} \frac{\kappa_0^4 |y|^2}{\rho^2 |1 - a_0 y|^8} dy \\ &\le C \int_{|y|<3/8} \frac{1}{|1 - a_0 y|^4} dy \\ &\le C. \end{aligned}$$

For the second term, we see that

$$\begin{aligned} \int_{O_n} \left| \frac{\partial \omega_n^{(2)}}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dx &\le C \int_{|y|<3/8} \frac{\kappa_0^4 |\overline{e}_\mu - 2\alpha_0^\mu \overline{y}|^2}{\rho^2 |1 - a_0 y|^8} dy \\ &\le C \int_{|y|<3/8} \frac{1}{|1 - a_0 y|^4} dy, \end{aligned}$$

which is again bounded by a constant C. Similarly, for the third term, we have

$$\begin{aligned} \int_{O_n} \left| \frac{\partial \omega_n^{(3)}}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dx &\le C \int_{|y|<3/8} \frac{\kappa_0^8 |y|^2}{\rho^4 |1 - a_0 y|^{12}} dy \\ &\le C \int_{|y|<3/8} \frac{1}{|1 - a_0 y|^4} dy. \end{aligned}$$

Considering the fourth term, we have

$$\begin{aligned} \int_{O_n} \left| \frac{\partial \omega_n^{(4)}}{\partial \alpha_0^\mu} \right|^2 \chi_n \sqrt{g} dx &\le C \int_{|y|<3/8} \frac{\kappa_0^4 \kappa_1^4 |y|^2}{\rho^4 |1 - a_0 y|^6 |1 - a_1 y|^6} dy \\ &\le C \int_{|y|<3/8} \frac{1}{|1 - a_1 y|^2 |1 - a_0 y|^2} dy, \end{aligned}$$

which is bounded by a constant C, and the same is true for the fifth term by symmetry. $\square$

Proposition 5.5.7. *For $i = 0, 1, 2$ and $\mu = 0, \ldots, 3$, we have the following upper bounds:*

(i) $$\left\| \frac{\partial \omega}{\partial \alpha_i^\mu} \right\|^2 \leq C \left\{ 1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{\substack{j \neq i \\ j=0}}^{2} \log\left(1 + \frac{1}{|a_i - a_j|}\right) \right\};$$

(ii) $$\left\| \frac{\partial \omega}{\partial a_i^\mu} \right\|^2 \leq C \left\{ 1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{\substack{j \neq i \\ j=0}}^{2} \log\left(1 + \frac{1}{|a_i - a_j|}\right) \right\}.$$

Proof. One makes the obvious changes in the previous estimates for the norms of the local 1-forms obtained when $i = 0$ to get estimates for $i = 1$ and 2. We then combine these inequalities to give the upper bound for the L^2 norm of $\partial\omega/\partial\alpha_i^\mu$ in (i). We then use

$$\begin{aligned} \frac{\partial \omega}{\partial a_i^\mu} &= \frac{\partial \omega}{\partial \kappa_i}\frac{\partial \kappa_i}{\partial a_i^\mu} + \left.\frac{\partial \omega}{\partial \alpha_i^\mu}\right|_{\alpha_i = a_i, \beta_i = 1} \\ \frac{\partial \omega}{\partial \lambda_i} &= \sqrt{1 + |a_i|^2}\,\frac{\partial \omega}{\partial \kappa_i}, \\ \frac{\partial \kappa_i}{\partial a_i^\mu} &= \frac{\lambda_i a_i^\mu}{\sqrt{1 + |a_i|^2}}, \end{aligned}$$

and combine the previous estimates for the norms of $\partial\omega/\partial\lambda_i$, to give the estimate (ii). □

§5.6. Diameter and Volume of the Moduli Space

We apply our estimates for the L^2 norms of the tangent vectors to moduli space to obtain upper bounds for the metric components of the L^2 metric $\mathbf{g}$. We then use these bounds to show that the space $(\mathcal{M}_2, \mathbf{g})$ has finite diameter and volume.

Theorem 5.6.1. *Let $\mathcal{M}_2$ denote the moduli space of self-dual connections on a principal $\mathbf{SU}(2)$-bundle P over the sphere $\mathbb{S}^4$, where $-c_2(P)[\mathbb{S}^4] = 2$ and $\mathbb{S}^4$ has its standard round metric g_0. Let these connections be parametrized by the space $\tilde{T}_2$ of unordered pairs (λ_i, P_i), $i = 0, 1, 2$, where $\lambda_0, \lambda_1, \lambda_2$ are positive weight parameters satisfying the scaling condition $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$, and P_0, P_1, P_2 are distinct points in $\mathbb{S}^4 = \mathbb{HP}^1$ (right projective space). Let c_i^μ denote the standard inhomogeneous coordinates of the points $\underline{P}_i = \underline{\mathbb{HP}}^1$ (left projective space), for $i = 0, 1, 2$, $\mu = 0, \ldots, 3$, so that $c_i = a_i$ if $\underline{P}_i = [a_i, 1]$, lying in the southern hemisphere, or $c_i = b_i$ if $\underline{P}_i = [1, b_i]$, lying in the northern hemisphere. With respect* *to this choice of parameters, we have the following estimates for the components of the L^2 metric* $\mathbf{g}$*:*

$$\mathbf{g}_{\lambda_i\lambda_i} = \mathbf{g}\left(\frac{\partial\omega}{\partial\lambda_i}, \frac{\partial\omega}{\partial\lambda_i}\right) \leq C\left\{1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{j\neq i}\log\left(1 + \frac{1}{|c_i - c_j|}\right)\right\},$$

$$\mathbf{g}_{c_i^\mu c_i^\mu} = \mathbf{g}\left(\frac{\partial\omega}{\partial c_i^\mu}, \frac{\partial\omega}{\partial c_i^\mu}\right) \leq C\left\{1 + \log\left(\frac{1}{\lambda_i}\right) + \sum_{j\neq i}\log\left(1 + \frac{1}{|c_i - c_j|}\right)\right\},$$

for $i = 0, 1, 2$ *and* $\mu = 0, \ldots, 3$*, where* C *is a universal constant independent of moduli parameters.*

Proof. If we denote the moduli parameters by $\{t_\alpha\}$, then we have the following estimates for the corresponding diagonal components of $\mathbf{g}$:

$$\mathbf{g}_{\alpha\alpha} = \mathbf{g}\left(\frac{\partial\omega}{\partial t_\alpha}, \frac{\partial\omega}{\partial t_\alpha}\right) = \left(h_\omega\frac{\partial\omega}{\partial t_\alpha}, h_\omega\frac{\partial\omega}{\partial t_\alpha}\right) = \left\|h_\omega\frac{\partial\omega}{\partial t_\alpha}\right\|^2 \leq \left\|\frac{\partial\omega}{\partial t_\alpha}\right\|^2,$$

and consequently these bounds follow immediately from previous estimates. □

Remark 5.6.2. (i) If $\mathbb{S}^4$ has a metric g which is globally conformally equivalent to the standard round metric g_0, then the above estimates continue to hold, although the constant C will certainly depend on g.

(ii) Estimates of the non-diagonal components of $\mathbf{g}$ may be obtained by the Schwarz inequality.

Corollary 5.6.3. *With respect to the L^2 metric* $\mathbf{g}$*, the moduli space* $\mathcal{M}_2$ *has finite diameter and volume.*

Proof. It is clearly enough to show that the space has finite diameter. Fix a base point $\gamma^* = \{(\lambda_0^*, P_0^*), (\lambda_1^*, P_1^*), (\lambda_2^*, P_2^*)\}$ in the parameter space $\tilde{T}_2$ and let $[\omega^*]$ be the corresponding basepoint in $\mathcal{M}_2$. Let $\gamma : (0,1) \to \tilde{T}_2$ be a smooth curve in $\tilde{T}_2$ which extends continously to the boundary. In terms of our choice of parameters, we have $\gamma(t) = (\lambda_i(t), P_j(t))$. Assume that our curve connects the basepoint, $\gamma(1) = \gamma^*$, with a point on the boundary, $\gamma(0) \in \partial\tilde{T}_2$. Let $\omega : (0,1) \to \mathcal{M}_2$ denote the corresponding smooth curve in $\mathcal{M}_2$, so that $w(t)$ approaches the boundary of $\mathcal{M}_2$ as $t \to 0$, and $w(1) = \omega^*$, the basepoint connection. Let $L(\omega)$ denote the length of the curve $\omega : (0,1) \to \mathcal{M}_2$ with respect to the L^2 metric $\mathbf{g}$.

In order to show that $\mathcal{M}_2$ has finite diameter, it is enough to show that minimal-length geodesics in $(\mathcal{M}_2, \mathbf{g})$, connecting the basepoint ω^* with any point on the boundary $\partial\mathcal{M}_2$, have lengths uniformly bounded by some constant. Since

$$\mathbf{g}\left(\frac{d\omega}{dt}, \frac{d\omega}{dt}\right) = \left\|h_\omega\frac{d\omega}{dt}\right\|^2 \leq \left\|\frac{d\omega}{dt}\right\|^2,$$

the curve length $L(\omega)$ may be estimated by

$$L(\omega) = \int_0^1 \mathbf{g}^{1/2}\left(\frac{d\omega}{dt}, \frac{d\omega}{dt}\right) dt \leq \int_0^1 \left\| \frac{d\omega}{dt} \right\| dt.$$

Note that

$$\frac{d\omega}{dt} = \sum_{i=0}^{2} \frac{\partial \omega}{\partial \lambda_i} \frac{d\lambda_i}{dt} + \sum_{i=0}^{2} \sum_{\mu=0}^{3} \frac{\partial \omega}{\partial c_i^\mu} \frac{dc_i^\mu}{dt},$$

and consequently

$$\int_0^1 \left\| \frac{d\omega}{dt} \right\| dt \leq \sum_{i=0}^{2} \int_0^1 \left\| \frac{\partial \omega}{\partial \lambda_i} \right\| \left| \frac{d\lambda_i}{dt} \right| dt + \sum_{i=0}^{2} \sum_{\mu=0}^{3} \int_0^1 \left\| \frac{\partial \omega}{\partial c_i^\mu} \right\| \left| \frac{dc_i^\mu}{dt} \right| dt.$$

Since we may choose curves of non-minimal length connecting the basepoint ω^* and points on the boundary of $\mathcal{M}_2$, we can make the following assumptions concerning the curve $\gamma(t) = \{\lambda_i(t), P_j(t)\}$, and hence $\omega(t)$:

(i) $t \leq \lambda_i(t) \leq 1$, for $0 < t < 1$ and $i = 0, 1, 2$;

(ii) $t \leq |c_i(t) - c_j(t)| \leq 2$, for $0 < t < 1$, $i \neq j$, and $i, j = 0, 1, 2$;

(iii) $|d\lambda_i/dt| \leq N$ and $|dc_i^\mu/dt| \leq N$, for $0 < t < 1$, $i = 0, 1, 2$, and $\mu = 0, \ldots, 3$, and the constant N is the same for all curves γ.

Hence, we obtain

$$\begin{aligned} L(\omega) &\leq C \left\{ \sum_{i=0}^{2} \int_0^1 \left\| \frac{\partial \omega}{\partial \lambda_i} \right\| dt + \sum_{i=0}^{2} \sum_{\mu=0}^{3} \int_0^1 \left\| \frac{\partial \omega}{\partial c_i^\mu} \right\| dt \right\} \\ &\leq C \left\{ \sum_{i=0}^{2} \int_0^1 \left\| \frac{\partial \omega}{\partial \lambda_i} \right\|^2 dt + \sum_{i=0}^{2} \sum_{\mu=0}^{3} \int_0^1 \left\| \frac{\partial \omega}{\partial c_i^\mu} \right\|^2 dt \right\}, \end{aligned}$$

for some constant C independent of the curve γ. Using our bounds for the metric components, we see that

$$\begin{aligned} \left\| \frac{\partial \omega}{\partial \lambda_i} \right\|^2 &\leq C \left\{ 1 + \log\left(\frac{1}{t}\right) \right\}, \\ \left\| \frac{\partial \omega}{\partial c_i^\mu} \right\|^2 &\leq C \left\{ 1 + \log\left(\frac{1}{t}\right) \right\}, \end{aligned}$$

for $i = 0, 1, 2$ and $\mu = 0, \ldots, 3$. Hence, the length of the curve ω is bounded by

$$\begin{aligned} L(\omega) &\leq C \int_0^1 (1 - \log t)\, dt \\ &= C \Big[2t - t \log t \Big]_0^1 \\ &= 2C, \end{aligned}$$

and this completes the proof. $\square$

BIBLIOGRAPHY

[A] Atiyah, M. F., *Geometry of Yang-Mills Fields*, Lezioni Fermiane, Acad. Naz. Lincei Scuola Normale Sup., Pisa 1979.

[A-B] Atiyah, M. F. and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London **A 308**, 523-615 (1982).

[A-D-H-M] Atiyah, M. F., V. G. Drinfel'd, N. J. Hitchin, and Yu. I. Manin, *Construction of instantons*, Phys. Lett. **65A**, 185-187 (1978).

[A-H-S] Atiyah, M. F., N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Royal Soc. London **A 362**, 425-461 (1978).

[A-J] Atiyah, M. F. and J. D. S. Jones, *Topological aspects of Yang-Mills theory*, Commun. Math. Phys. **61**, 97-118 (1978).

[A-W] Atiyah, M. F. and R. S. Ward, *Instantons and algebraic geometry*, Commun. Math. Phys. **55**, 117-124 (1977).

[Au-Do] Aupetit, H. and A. Douady, *Fibrés stables de rang 2 sur* $\mathbb{CP}^3$ *avec* $c_1 = 0$, $c_2 = 2$, in: Les Equations de Yang-Mills, Eds. A. Douady and J.-L. Verdier, Séminaire E.N.S. 1977-78, Astérisque 71-72 (1980), pp. 171-196.

[B-H] Barth, W. and K. Hulek, *Monads and moduli of vector bundles*, Manu. Math. **25**, 323-347 (1978).

[Ber] Berezin, F. A., *Instantons and Grassman Manifolds*, Funk. Analiz. **13**, 75-76 (1978).

[Bey] Beyer, W. H., *Standard Mathematical Tables*, 26th Ed., Chemical Rubber Co. Press, 1981.

[Bo-Tu] Bott, R. and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag: New York, 1982.

[Bo-Ma] Boyer, C. P. and B. M. Mann, *Homology operations on Instantons*, J. Differential Geometry **28**, 423-465 (1988).

[B-H-M-M] Boyer, C. P., J. C. Hurtubise, B. M. Mann, and R. J. Milgram, *The topology of instanton moduli spaces I: the Atiyah-Jones conjecture*, to appear in Ann. Math.

[Br-tD] Bröcker, T. and T. tomDieck, *Representations of Compact Lie Groups*, Springer-Verlag: New York, 1985.

[Bu] Buchdahl, N. P., *Instantons on* $\mathbb{CP}^2$, J. Differential Geometry **24**, 19-52 (1986).

[C-W-S] Christ, N. H., E. J. Weinberg, and N. K. Stanton, *General self-dual Yang-Mills solutions*, Phys. Rev. D. **18**, 2013-2025 (1978).

[C-F-G-T] Corrigan, E., D. B. Fairlie, P. Goddard, and S. Templeton, *A Green function for the general self-dual gauge field*, Nuc. Phys. **B 140**, 31-44 (1978).

[C-G-O-T] Corrigan, E., P. Goddard, H. Osborn, and S. Templeton, *Zeta-function regularisation and multi-instanton determinants*, Nuc. Phys. **B 159**, 469-496 (1979).

[D-M-M] Doi, H., Y. Matsumoto, and T. Matumoto, *An explicit formula of the metric on the moduli space of BPST-instantons over* $\mathbb{S}^4$, in: A Fête of Topology, eds. Y. Matsumoto et al., Academic Press: New York, 1988.

[D-P] D'Hoker, E. and D. H. Phong, *The Geometry of String Perturbation Theory*, Reviews of Modern Physics **60**, American Physical Society: New York, 1988.

[D1] Donaldson, S. K., *Instantons and geometric invariant theory*, Commun. Math. Phys. **93**, 453-460 (1984).

[D2] Donaldson, S. K., *Vector bundles on the flag manifold and the Ward correspondence*, in: Geometry Today, ed. E. Arbarello et al., Birkhäuser: Boston, 1985.

[D3] Donaldson, S. K., *Connections, cohomology and the intersection forms of four manifolds*, J. Differential Geometry **24**, 275-341 (1986).

[D4] Donaldson, S. K., *Compactification and completion of Yang-Mills moduli spaces*, in: Differential Geometry, Lecture Notes in Mathematics **1410**, eds. F. J. Carreras et al., Springer-Verlag: New York, 1989.

[D5] Donaldson, S. K., *Instantons in Yang-Mills theory*, in: Proceedings of the IMA Conference on Geometry and Particle Physics, Oxford 1988, ed. F. Tsou, Oxford University Press: New York, 1990.

[D6] Donaldson, S. K., *Polynomial invariants for smooth 4-manifolds*, Topology **29**, 257-315 (1990).

[D-K] Donaldson, S. K. and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press: New York, 1990.

[D-M1] Drinfel'd, V. G. and Yu. I. Manin, *Self-dual Yang-Mills Fields over a Sphere*, Funk. Analiz. **12**, 78-79 (1978).

[D-M2] Drinfel'd, V. G. and Yu. I. Manin, *A description of instantons*, Commun. Math. Phys. **63**, 177-192 (1978).

[D-M3] Drinfel'd, V. G. and Yu. I. Manin, *Instantons and bundles on* $\mathbb{CP}^3$, Funk. Analiz. **13**, 59-74 (1979).

[D-M4] Drinfel'd, V. G. and Yu. I. Manin, *Yang-Mills, instantons, tensor products of instantons*, Soviet J. Nucl. Phys. **29**, 845-849 (1979).

[Fi] Fischer, A. E., *The internal symmetry group of a connection on a principal fibre bundle with applications to gauge field theories*, Commun. Math. Phys. **113**, 231-262 (1987).

[F-T1] Fischer, A. E. and A. J. Tromba, *On a purely Riemannian proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, Math. Ann. **267**, 311-345 (1984).

[F-T2] Fischer, A. E. and A. J. Tromba, *On the Weil-Petersson metric on Teichmüller space*, Trans. A.M.S. **284**, 319-335 (1984).

[F-G] Freed, D. S. and D. Groisser, *The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group*, Mich. Math. J. **36**, 323-344 (1989).

[F-U] Freed, D. S. and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, 2nd ed., Springer-Verlag: New York, 1991.

[F-M] Friedman, R. and J. W. Morgan, *On the diffeomorphism types of certain algebraic surfaces I, II*, J. Differential Geometry **27**, 297-398 (1988).

[G-H] Griffiths, P. and J. Harris, *Principles of Algebraic Geometry*, Wiley: New York, 1978.

[G-R] Giambiagi, J. J. and K. D. Rothe, *Regular N-instanton fields and singular gauge transformations*, Nuclear Physics **B129**, 111-124 (1977).

[G-P1] Groisser, D. and T. H. Parker, *The Riemannian geometry of the Yang-Mills moduli space*, Commun. Math. Phys. **112**, 663-689 (1987).

[G-P2] Groisser, D. and T. H. Parker, *The geometry of the Yang-Mills moduli space for definite manifolds*, J. Differential Geometry **29**, 499-544 (1989).

[G-P3] Groisser, D. and T. H. Parker, *Semi-classical Yang-Mills theory I: Instantons*, Commun. Math. Phys. **135**, 101-140 (1990).

[Gu] Gunning, R. C., *Lectures on Vector Bundles over Riemann Surfaces*, Princeton University Press: Princeton, 1967.

[Hab] Habermann, L., *On the geometry of the space of $Sp(1)$-instantons with Pontrjagin index 1 on the 4-sphere*, Ann. Global Anal. Geom. **6**, 3-29 (1988).

[Har1] Hartshorne, R., *Algebraic Geometry*, Springer-Verlag: New York, 1977.

[Har2] Hartshorne, R., *Stable vector bundles and instantons*, Commun. Math. Phys. **59**, 1-15 (1978).

[Har3] Hartshorne, R., *Stable vector bundles of rank 2 on* $\mathbb{P}^3$, Math. Ann. **238**, 229-280 (1978)

[Hat] Hattori, A., *Topology of the moduli space of $SU(2)$-instantons with instanton number 2*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **34**, 741-761 (1987).

[H-K-L-R] Hitchin, N. J., A. Karlhede, U. Lindstrom, and M. Rŏcek, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. **108**, 535-589 (1987).

[Hur] Hurtubise, J., *Instantons and jumping lines*, Commun. Math. Phys. **105**, 107-122 (1986).

[Hus] Husemoller, D., *Fibre Bundles*, 2nd ed., Springer-Verlag: New York, 1966.

[I] Itoh, M., *Geometry of anti-self-dual connections and Kuranishi Map*, J. Math. Soc. Japan **40**, 9-33 (1988).

[J-N-R] Jackiw, R., C. Nohl, and C. Rebbi, *Conformal properties of pseudoparticle configurations*, Phys. Rev. D **15**, 1642-1646 (1977).

[K] Kobayashi, S., *Differential Geometry of Complex Vector Bundles*, Princeton University Press: Princeton, 1987.

[K-N] Kobayashi, S. and K. Nomizu, *Foundations of Differential Geometry*, Volumes I & II, Wiley: New York, 1963 and 1968.

[Ko] Kodaira, K., *Complex Manifolds and Deformation of Complex Structures*, Springer-Verlag: New York, 1986.

[La] Lawson, H. B., *The Theory of Gauge Fields in Four Dimensions*, American Mathematical Society: Providence, 1985.

[Le] LeBrun, C., *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63**, 723-743 (1991).

[Mac] Maciocia, A., *Metrics on the moduli spaces of instantons over Euclidean 4-spaces*, Commun. Math. Phys. **135**, 467-482 (1991).

[MC-S] Mamone Capria, M. and S. M. Salamon, *Yang-Mills fields on quaternionic spaces*, Nonlinearity **1**, 517-530 (1988).

[M-M] Marathe, K. B. and G. Martucci, *The geometry of gauge fields*, J. Geometry and Physics **6**, 1-105 (1989).

[Mas] Masur, H., *The extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. J. **43**, 623-635 (1976).

[Mi-St] Milnor, J. and J. Stasheff, *Characteristic Classes*, Princeton University Press: Princeton, 1974.

[Mo] Morgan, J. W., *The Topology of Four-Manifolds*, unpublished manuscript, Columbia University: New York, 1989.

[Mu-Fo] Mumford, D. and J. Fogarty, *Geometric Invariant Theory*, 2nd ed., Springer-Verlag: New York, 1982.

[N-T1] Narasimhan, M. S. and G. Trautman, *Compactification of* $M(0,2)$, in: Vector Bundles on Algebraic Varieties, eds. M. F. Atiyah et al., Oxford University Press, Bombay 1987, pp. 429-443.

[N-T2] Narasimhan, M. S. and G. Trautman, *Compactification of* $M_{\mathbb{P}_3}(0,2)$ *and Poncelet pairs of conics*, Pacific J. Math. **145**, 255-365 (1990).

[N-T3] Narasimhan, M. S. and G. Trautman, *The Picard group of the compactification of* $M_{\mathbb{P}_3}(0,2)$ J. Reine angew. Math. **422**, 21-44 (1991).

[O] Osborn, H., *Semi-classical functional integrals for self-dual gauge fields*, Ann. Phys. **135**, 373-415 (1981).

[O-S-S] Okonek, C., M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhaüser: Boston, 1980.

[R1] Rawnsley, J. H., *On the Atiyah-Hitchin-Drinfeld-Manin vanishing theorem for cohomology groups of instanton bundles*, Math. Ann. **241**, 43-56 (1979).

[R2] Rawnsley, J. H., *Self-dual Yang-Mills fields*, in: Global Analysis, eds. J Marsden and M. Grmela, Lecture Notes in Mathematics **755**, 295-312 (1979).

[Sa] Salamon, S., *Instantons on the 4-sphere*, Rend. Sem. Mat. Univers. Politec. Torino **40**, 1-20 (1982).

[Sc] Schwarz, A. S., *Instantons and fermions in the field of instanton*, Commun. Math. Phys. **64**, 223-268 (1979).

[Se] Seshadri, C. S., *Theory of moduli*, p. 263-304, in Algebraic Geometry, Arcata 1974, Proc. Symp. Pure Math. **29**, ed. R. Hartshorne, A.M.S.: Providence, 1975.

[S-S-U] Sibner, L. M., R. J. Sibner, and K. K. Uhlenbeck, *Solutions to Yang-Mills equations that are not self-dual*, Proc. Natl. Acad. Sci. USA **86**, 8610-8613 (1989).

[Si1] Singer, I. M., *Some remarks on the Gribov ambiguity*, Commun. Math. Phys. **60**, 7-12 (1978).

[Si2] Singer, I. M., *The geometry of the orbit space for non-abelian gauge theories*, Physica Scripta **24**, 817-820 (1981).

[S-T] Singhof, W. and G. Trautman, *On the topology of the moduli space $M(0,2)$ of stable bundles of rank 2 on $\mathbb{P}^3$*, Quart. J. Math. Oxford **41**, 335-358, (1990).

[St] Steenrod, N., *The Topology of Fibre Bundles*, Princeton University Press: Princeton, 1951.

[T1] Taubes, C. H., *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geometry **17**, 139-170 (1982).

[T2] Taubes, C. H., *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geometry **19**, 517-560 (1984).

[T3] Taubes, C. H., *Path-connected Yang-Mills moduli spaces*, J. Differential Geometry **19**, 337-392 (1984).

[T4] Taubes, C. H., *A framework for Morse theory for the Yang-Mills functional*, Invent. Math. **94**, 327-402 (1988).

[T5] Taubes, C. H., *The stable topology of self-dual moduli spaces*, J. Differential Geometry **29**, 162-230 (1989).

[Wa-We] Ward, R. S. and R. O. Wells, *Twistor Geometry and Field Theory*, Cambridge University Press: Cambridge, 1990.

[Wen] Wentworth, R., *The asymptotics of the Arakalov-Green's function and Falting's delta invariant*, Commun. Math. Phys. **137**, 427-459 (1991).

[Wi] Witten, E., *Topological quantum field theory*, Commun. Math. Phys. **117**, 353-386 (1988).

[Wo1] Wolpert, S. A., *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces*, Commun. Math Phys. **112**, 285-315 (1987).

[Wo2] Wolpert, S. A., *Geodesic length functions and the Nielsen problem*, J. Differential Geometry **25**, 275-296 (1987).

[Wo3] Wolpert, S. A., *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85**, 119-145 (1986).

[Wo4] Wolpert, S. A., *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107**, 969-997 (1985).