# A GENERAL SO(3)-MONOPOLE COBORDISM FORMULA RELATING DONALDSON AND SEIBERG-WITTEN INVARIANTS

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Abstract. We prove an analogue of the Kotschick-Morgan conjecture [19] in the context of SO(3) monopoles, obtaining a formula relating the Donaldson and Seiberg-Witten invariants of smooth four-manifolds using the SO(3)-monopole cobordism. Although this article is in a preliminary form, it appears to be of sufficient interest that it is worth now distributing in its present shape.

## 1. Introduction

In this article, a sequel to [14, 15, 16, 9, 10], we complete the first step in the proof of the Witten and Moore-Witten formulas relating the Seiberg-Witten and Donaldson invariants [28], [22]. In particular, we prove that pairings of certain cohomology classes with the links of lower level Seiberg-Witten moduli subspaces of the Uhlenbeck compactification of the moduli space of SO(3) monopoles depends only on the Seiberg-Witten invariant and the homotopy type of a four-manifold $X$, through its Euler characteristic $\chi$ and signature $\sigma$ (see Theorem 11.1). This leads to the following result.

**Theorem 1.1.** *Let $X$ be a closed, connected, oriented smooth four-manifold with $b_1(X) = 0$, odd $b_2^+(X) > 1$. Let $\Lambda, w \in H^2(X;\mathbb{Z})$ obey $w - \Lambda \equiv w_2(X) \pmod 2$. Let $\delta, m$ be non-negative integers for which $m \leq [\delta/2]$, where $[\,\cdot\,]$ denotes the greatest integer function, and $\delta \equiv -w^2 - \frac{3}{4}(\chi+\sigma) \pmod 4$, with $\Lambda$ and $\delta$ obeying $\delta < i(\Lambda)$, where $i(\Lambda) = \Lambda^2 - \frac{1}{4}(\chi+\sigma)$. Then for any $h \in H_2(X;\mathbb{R})$ and generator $x \in H_0(X;\mathbb{Z})$, we have the following expression for the Donaldson invariant:*

$$
\begin{aligned}
D_X^w(h^{\delta-2m}x^m) = &\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+w\cdot c_1(\mathfrak{s}))} SW_X(\mathfrak{s}) \\
&\times \sum_{i=0}^{\min(\ell,[\delta/2]-m)} \left(p_{\delta,\ell,m,i}(c_1(\mathfrak{s})-\Lambda,\Lambda) Q_X^i\right)(h),
\end{aligned} \tag{1.1}
$$

*where $Q_X$ is the intersection form on $H_2(X;\mathbb{R})$, $\ell = \frac{1}{4}(\delta + (c_1(\mathfrak{s})-\Lambda)^2 + \frac{3}{4}(\chi+\sigma))$ and $p_{\delta,\ell,m,i}(\cdot,\cdot)$ is a degree $\delta - 2m - 2i$ homogeneous polynomial with coefficients which are universal functions of*

$$\chi,\ \sigma,\ c_1(\mathfrak{s})^2,\ \Lambda^2,\ c_1(\mathfrak{s})\cdot\Lambda,\ \delta,\ m,\ \ell.$$

Recall that for a 'spin$^u$ structure' $\mathfrak{t}$ on $X$ (see [15] or §2 here), the moduli space, $\mathcal{M}_{\mathfrak{t}}$, of SO(3) monopoles was a smooth manifold away from two types of singular subspaces. The first type of subspace was identified with the moduli space of anti-self-dual connections [15]. The second type of subspace, that of reducible SO(3) monopoles, was identified with the Seiberg-Witten moduli space $M_{\mathfrak{s}}$ where the spin$^u$ structure $\mathfrak{t}$ admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$

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[15]. The Uhlenbeck compactification $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ of $\mathcal{M}_{\mathfrak{t}}/S^1$ provides a cobordism between a link of the moduli space of anti-self-dual connections and links, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, of subspaces of reducible SO(3) monopoles. However, additional reducible SO(3) monopoles, in the form of subspaces,

$$M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X), \tag{1.2}$$

appear in the lower levels of the Uhlenbeck compactification. The intersection number of certain geometric representatives of cohomology classes with the link of the anti-self-dual connections yields a multiple of the Donaldson invariant. The main technical result of this paper, Theorem 11.1, is a formula (which is *not* completely explicit) for the intersection number of these geometric representatives with the link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ of the subspace (1.2) of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

We note that a statement similar to Theorem 11.1 can also be proved by the methods of this article, without the assumptions that $b_1(X) = 0$ or $z = h^{\delta-2m}x^m$, but the resulting expression becomes considerably more complicated. While the precise expression for the intersection number in (11.1) is still unknown, we also prove the 'multiplicity conjecture' in [16, 6, 11] (see Theorem 11.2 here), which holds even without the simplifying assumptions.

1.1. **Outline of the argument.** We have described the program for proving Witten's conjecture relating the Donaldson and Seiberg-Witten invariants for smooth four-manifolds extensively elsewhere, [14, 15, 13, 11]. The main technical difficulty with this program has been computing the intersection pairings (11.1) when the Seiberg-Witten monopoles lie in a lower level of the Uhlenbeck compactification of the moduli space of SO(3) monopoles. Such a computation was completed in [11] in the case when the reducibles lie in the first level; that is, when the Seiberg-Witten stratum has the form $M_{\mathfrak{s}} \times X$. The computation behind Theorems 11.1 and 11.2 is a qualitative understanding of the intersection pairing for the general case: Seiberg-Witten strata of the form $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$. In this article, we give the topological constructions necessary to prove Theorem 11.1; the required analytic results are established in [9, 10].

The gluing theorems, [9, 10], describe neighborhoods of the strata, $M_{\mathfrak{s}}\times\Sigma$, of the Seiberg-Witten monopoles $M_{\mathfrak{s}}\times\mathrm{Sym}^{\ell}(X)$, where $\Sigma$ is a stratum of the symmetric product $\mathrm{Sym}^{\ell}(X)$. Roughly speaking, there is a fiber bundle, $M(\Sigma) \to \mathrm{Gl}(\Sigma) \to M_{\mathfrak{s}}\times\Sigma$, with a well-understood structure group and fiber $M(\Sigma)$ given by a product of moduli spaces of anti-self-dual connections on $S^4$. There is a section of a pseudo-vector bundle $\Upsilon(\Sigma) \to \mathrm{Gl}(\Sigma)$ and the gluing map identifies the zero-locus of this section with a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in the moduli space of SO(3) monopoles.

There are two main sources of the difficulty in computing the intersection pairings (11.1) when $\ell > 1$. The first difficulty is the presence of higher charge moduli spaces of anti-self-dual connections on $S^4$ in the fiber $M(\Sigma)$. The topology of these spaces is in still unknown although the needed information for the case $\ell = 2$ has been worked out in [21]. We do not address this problem here. Instead, we use a pushforward-pullback argument (see §11.2) to isolate the topology of these fibers into 'universal constants' of the type appearing in (11.1). The second difficulty, which we do address here, is the presence of more than one stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$, and the resulting need for more than one gluing map to parameterize a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$.

1.1.1. *The problem of overlaps.* To compute the intersection pairing (11.1) when $\ell \geq 2$, we must understand the overlap of the images of the gluing maps for different strata $\Sigma$ and $\Sigma'$. We approach this problem by first understanding the overlap of the images of the *splicing maps* for different strata. Recall from [9] that the gluing map is defined by a perturbation of a splicing map which is defined, roughly, by using partitions of unity to piece together pairs on $X$ and on $S^4$. Essentially, it is easier to compare two different splicing maps because the splicing maps can be written down explicitly.

Given two strata $\Sigma$ and $\Sigma'$ of $\mathrm{Sym}^\ell(X)$, with $\Sigma \subset \mathrm{cl}\ \Sigma'$, and splicing maps

$$\boldsymbol{\gamma}'_{\Sigma'} : \mathrm{Gl}(\Sigma') \to \bar{\mathcal{C}}_{\mathfrak{t}}, \quad \boldsymbol{\gamma}'_{\Sigma} : \mathrm{Gl}(\Sigma) \to \bar{\mathcal{C}}_{\mathfrak{t}},$$

we achieve the desired comparison by defining a space of 'overlap data', $\mathrm{Gl}(\Sigma, \Sigma')$, and maps

$$\rho^u_{\Sigma,\Sigma'} : \mathrm{Gl}(\Sigma, \Sigma') \to \mathrm{Gl}(\Sigma'), \quad \rho^d_{\Sigma,\Sigma'} : \mathrm{Gl}(\Sigma, \Sigma') \to \mathrm{Gl}(\Sigma),$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Gl}(\Sigma,\Sigma') & \xrightarrow{\rho^u_{\Sigma,\Sigma'}} & \mathrm{Gl}(\Sigma') \\ {\scriptstyle \rho^d_{\Sigma,\Sigma'}}\big\downarrow & & {\scriptstyle \boldsymbol{\gamma}'_{\Sigma'}}\big\downarrow \\ \mathrm{Gl}(\Sigma) & \xrightarrow{\boldsymbol{\gamma}'_{\Sigma}} & \mathcal{C}_{\mathfrak{t}} \end{array} \tag{1.3}$$

commutes and such that

$$\mathrm{Im}(\boldsymbol{\gamma}'_{\Sigma'}) \cap \mathrm{Im}(\boldsymbol{\gamma}'_{\Sigma}) = \mathrm{Im}(\boldsymbol{\gamma}'_{\Sigma'} \circ \rho^u_{\Sigma,\Sigma'}) = \mathrm{Im}(\boldsymbol{\gamma}'_{\Sigma} \circ \rho^d_{\Sigma,\Sigma'}).$$

The diagram (1.3) is used to define the space of *global splicing data* as a pushout of the spaces of local splicing data $\mathrm{Gl}(\Sigma)$. Once this is accomplished, the result of the argument is largely a formal accounting of choices of cohomology classes with compact support and a use of the familiar pullback-pushforward technique.

1.1.2. *The commutativity of splicing maps.* Since the majority of the article is taken up with establishing the diagram (1.3), we now describe in more detail how that is carried out. The two strata $\Sigma$ and $\Sigma'$ are given by partitions of $\ell$. If $\Sigma$ is the lower stratum, that is, $\Sigma \subset \mathrm{cl}\ \Sigma'$, then the partition defining $\Sigma'$ is a refinement of the partition $\ell = \kappa_1 + \cdots + \kappa_r$ defining $\Sigma$. Write $\kappa_i = \kappa_{i,1} + \cdots + \kappa_{i,r_i}$ for this refinement. The overlap data space, $\mathrm{Gl}(\Sigma, \Sigma')$, is, approximately, defined by a fiber bundle, $M(\Sigma') \to \mathrm{Gl}(\Sigma, \Sigma') \to \nu(\Sigma, \Sigma')$, where $\nu(\Sigma, \Sigma') \to \Sigma$ can be thought of as the normal bundle of $\Sigma$ in $\Sigma'$. The principal bundle underlying $\mathrm{Gl}(\Sigma, \Sigma') \to \nu(\Sigma, \Sigma')$ is defined by a pullback of the principal bundle defining $\mathrm{Gl}(\Sigma)$ by the projection map $\nu(\Sigma, \Sigma') \to \nu(\Sigma)$. The 'upwards' overlap map, $\rho^u_{\Sigma,\Sigma'}$, is then defined by an isomorphism between the restriction of the bundle $\mathrm{Gl}(\Sigma')$ to $\nu(\Sigma, \Sigma')$ and $\mathrm{Gl}(\Sigma, \Sigma')$. The 'downwards' overlap map, $\rho^d_{\Sigma,\Sigma'}$, is defined by taking the data given by the fiber $M(\Sigma')$ and the fiber of $\nu(\Sigma, \Sigma') \to \Sigma$ and using it to splice together connections on $S^4$. That is, the fiber of $\nu(\Sigma, \Sigma') \to \Sigma$ can be interpreted as sets of points in $\mathbb{R}^4$, while the fiber $M(\Sigma')$ is defined by connections $A_{i,j}$ on $S^4$ with instanton numbers $\kappa_{i,j}$. Splicing the connections $A_{i,j}$ to the trivial connection on $\mathbb{R}^4$ at the given points in $\mathbb{R}^4$ yields connections $A_i$ with instanton number $\kappa_i$ and thus a point in the fiber $M(\Sigma)$. The downwards overlap map is then defined by applying the above splicing procedure to the fiber of $\mathrm{Gl}(\Sigma, \Sigma') \to \nu(\Sigma, \Sigma') \to \Sigma$ and then using the identification of the principal bundle underlying $\mathrm{Gl}(\Sigma, \Sigma')$ with that of $\mathrm{Gl}(\Sigma)$.

The key technical results in proving the commutativity of diagram (1.3) appear in Lemma 5.11 and Proposition 5.12. These statements can be thought of as providing an associativity result for splicing maps on $S^4$. In a simple case, they assert that if the connection $A'_1$ is obtained from splicing the connections $A_{1,1}$ and $A_{1,2}$ to the trivial connection $\Theta$ at the points $x_1$ and $x_2$ in $\mathbb{R}^4$,

$$A'_1 = \Theta \#_{x_1} A_{1,1} \#_{x_2} A_{1,2},$$

then the connection obtained by splicing $A_1'$ and $A_2$ to the trivial connection at $y_1$ and $y_2$ is equal to the connection obtained by splicing $A_{1,1}$, $A_{1,2}$, and $A_2$ to the trivial connection at $y_1+x_1$, $y_1+x_2$, and $y_2$ respectively:

$$\begin{aligned}\Theta\#_{y_1}\left(\Theta\#_{x_1}A_{1,1}\#_{x_2}A_{1,2}\right)\#_{y_2}A_2 \\ = \Theta\#_{y_1+x_1}A_{1,1}\#_{y_1+x_2}A_{1,2}\#_{y_2}A_2.\end{aligned} \tag{1.4}$$

(Of course, this result only holds for connections which are suitably concentrated relative to the separation of the points in $\mathbb{R}^4$.)

Proposition 5.12 allows us to define the 'spliced-end moduli space', a deformation of the moduli space of anti-self-dual connections on $S^4$. The spliced-end moduli space differs from the actual moduli space of anti-self-dual connections on $S^4$ only near the strata where the background connection is trivial. A neighborhood of these strata in the actual moduli space is given by the gluing map while a neighborhood of these strata in the spliced-end moduli space is given by the splicing maps. Proposition 5.12 is used to ensure that the union of the images of these different splicing maps is a smoothly stratified space.

Replacing the actual moduli space with the spliced-ends moduli space allows us to prove that the diagram (1.3) commutes by using the results of Proposition 6.9, an analogue of Proposition 5.12 for splicing onto pairs on $X$. Because the ends of $M(\Sigma)$ are replaced by the image of a splicing map, the commutativity follows from an expression of the form (1.4). We note that Proposition 5.12 relies on the triviality of the background connection $\Theta$ and the flatness of the metric on $\mathbb{R}^4$. Thus, Proposition 6.9 only gives the commutativity of the diagram (1.3) for 'crude splicing maps' $\boldsymbol{\gamma}_\Sigma''$ which are defined by flattening the background connection and pair on a larger subspace than the standard splicing map. However, once the space of global splicing data is constructed using the crude splicing maps, it is a straightforward matter to construct (see §6.7) isotopies of these crude splicing maps to the standard splicing maps.

1.2. **Outline of the article.** Section 2 reviews notation and definitions from our previous articles in this sequence. In §3, we describe a stratification of $\operatorname{Sym}^\ell(X)$, and define normal bundles of the strata of $\operatorname{Sym}^\ell(X)$. In §4, we show that the projection maps of the normal bundles introduced in §3 satisfy Thom-Mather type equalities on the overlap of their images. In §5, we define the spliced-ends moduli space of anti-self-dual connections on $S^4$ and this is used in §6 to define the space of global splicing data. An analogous construction of an obstruction bundle over the space of global splicing data is carried out in §7 and §8. The cohomological computations are carried out in §9 where we compute the pullbacks of the relevant cohomology classes to the space of global splicing data, in §10 where we define the link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, and in §11 where we actually carry out the needed computations. We discuss the application of the main result of this article, Equation (1.1), to the proof of Witten's conjecture in a separate article [12].

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## 2. Preliminaries

We begin in §2.1 by recalling the definition of the moduli space of SO(3) monopoles and its basic properties [14, 5]. In §2.2 we describe the stratum of zero-section monopoles, or anti-self-dual connections. In §2.3 we discuss the strata of reducible, or Seiberg-Witten monopoles, together with their 'virtual' neighborhoods and normal bundles. In §2.4 we define the cohomology classes which will be paired with the links of the anti-self-dual and Seiberg-Witten moduli spaces. In §2.5 we review the definition of the Donaldson series. Lastly, in §2.6 we describe the basic relation between the pairings with links of the anti-self-dual and Seiberg-Witten moduli spaces provided by the SO(3)-monopole cobordism.

2.1. **The moduli space of SO(3) monopoles.** Throughout this article, $(X, g)$ will denote a closed, connected, oriented, smooth, Riemannian four-manifold.

2.1.1. *Clifford modules.* Let $V$ be a Hermitian vector bundle over $(X, g)$ and let $\rho : T^*X \to \operatorname{End}_{\mathbb{C}}(V)$ be a real-linear map satisfying

$$\rho(\alpha)^2 = -g(\alpha,\alpha)\mathrm{id}_V \quad \text{and} \quad \rho(\alpha)^\dagger = -\rho(\alpha), \quad \alpha \in C^\infty(T^*X). \tag{2.1}$$

The map $\rho$ uniquely extends to a linear isomorphism, $\rho : \Lambda^\bullet(T^*X) \otimes_{\mathbb{R}} \mathbb{C} \to \operatorname{End}_{\mathbb{C}}(V)$, and gives $V$ the structure of a Hermitian Clifford module for the complex Clifford algebra $\mathbb{C}\ell(T^*X)$. There is a splitting $V = V^+ \oplus V^-$, where $V^\pm$ are the $\mp 1$ eigenspaces of $\rho(\mathrm{vol})$. A unitary connection $A$ on $V$ is *spin* if

$$[\nabla_A, \rho(\alpha)] = \rho(\nabla\alpha) \quad \text{on } C^\infty(V), \tag{2.2}$$

for any $\alpha \in C^\infty(T^*X)$, where $\nabla$ is the Levi-Civita connection.

A Hermitian Clifford module $\mathfrak{s} = (\rho, W)$ is a spin$^c$ structure when $W$ has complex rank four; it defines a class

$$c_1(\mathfrak{s}) = c_1(W^+), \tag{2.3}$$

and every class in $H^2(X;\mathbb{Z})$ lifting the second Stiefel-Whitney class, $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, arises this way.

We call a Hermitian Clifford module $\mathfrak{t} = (\rho, V)$ a spin$^u$ structure when $V$ has complex rank eight. Recall that $\mathfrak{g}_V \subset \mathfrak{su}(V)$ is the SO(3) subbundle given by the span of the sections of the bundle $\mathfrak{su}(V)$ which commute with the action of $\mathbb{C}\ell(T^*X)$ on $V$. We obtain a splitting

$$\mathfrak{su}(V^+) \cong \rho(\Lambda^+) \oplus i\rho(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{g}_V \oplus \mathfrak{g}_V, \tag{2.4}$$

and similarly for $\mathfrak{su}(V^-)$. The fibers $V^+_x$ define complex lines whose tensor-product square is $\det(V^+_x)$ and thus a complex line bundle over $X$,

$$\det{}^{\frac{1}{2}}(V^+). \tag{2.5}$$

A spin$^u$ structure $\mathfrak{t}$ thus defines classes,

$$c_1(\mathfrak{t}) = \tfrac{1}{2}c_1(V^+), \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_V), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{g}_V). \tag{2.6}$$

Given $W$, one has an isomorphism $V \cong W \otimes_{\mathbb{C}} E$ of Hermitian Clifford modules, where $E$ is a rank-two Hermitian vector bundle [15, Lemma 2.3]; then

$$\mathfrak{g}_V = \mathfrak{su}(E) \quad \text{and} \quad \det{}^{\frac{1}{2}}(V^+) = \det(W^+) \otimes_{\mathbb{C}} \det(E).$$

2.1.2. SO(3) *monopoles.* We fix a smooth unitary connection $A_\Lambda$ on the line bundle $\det^{\frac{1}{2}}(V^+)$, let $k \geq 2$ be an integer, and let $\mathcal{A}_\mathfrak{t}$ be the affine space of $L^2_k$ spin connections on $V$ which induce the connection $2A_\Lambda$ on $\det(V^+)$. If $A$ is a spin connection on $V$ then it defines an SO(3) connection $\hat{A}$ on the subbundle $\mathfrak{g}_V \subset \mathfrak{su}(V)$ [15, Lemma 2.5]; conversely, every SO(3) connection on $\mathfrak{g}_V$ lifts to a unique spin connection on $V$ inducing the connection $2A_\Lambda$ on $\det(V^+)$ [15, Lemma 2.11].

Let $\mathcal{G}_\mathfrak{t}$ denote the group of $L^2_{k+1}$ unitary automorphisms of $V$ which commute with $\mathbb{C}\ell(T^*X)$ and which have Clifford-determinant one (see [15, Definition 2.6]). Define

$$\tilde{\mathcal{C}}_\mathfrak{t}(X) = \mathcal{A}_\mathfrak{t}(X) \times L^2_k(X, V^+) \quad \text{and} \quad \mathcal{C}_\mathfrak{t} = \tilde{\mathcal{C}}_\mathfrak{t}/\mathcal{G}_\mathfrak{t}. \tag{2.7}$$

The action of $\mathcal{G}_\mathfrak{t}$ on $V$ induces an adjoint action on $\mathrm{End}_\mathbb{C}(V)$, acting as the identity on $\rho(\Lambda^\bullet_\mathbb{C}) \subset \mathrm{End}_\mathbb{C}(V)$ and inducing an adjoint action on $\mathfrak{g}_V \subset \mathrm{End}_\mathbb{C}(V)$ (see [15, Lemma 2.7]). The space $\tilde{\mathcal{C}}_\mathfrak{t}$ and hence $\mathcal{C}_\mathfrak{t}$ carry circle actions induced by scalar multiplication on $V$:

$$S^1 \times V \to V, \quad (e^{i\theta}, \Phi) \mapsto e^{i\theta}\Phi. \tag{2.8}$$

Because this action commutes with that of $\mathcal{G}_\mathfrak{t}$, the action (2.8) also defines an action on $\mathcal{C}_\mathfrak{t}$. Note that $-1 \in S^1$ acts trivially on $\mathcal{C}_\mathfrak{t}$. Let $\mathcal{C}^0_\mathfrak{t} \subset \mathcal{C}_\mathfrak{t}$ be the subspace represented by pairs whose spinor components are not identically zero, let $\mathcal{C}^*_\mathfrak{t} \subset \mathcal{C}_\mathfrak{t}$ be the subspace represented by pairs where the induced SO(3) connections on $\mathfrak{g}_V$ are irreducible, and let $\mathcal{C}^{*,0}_\mathfrak{t} = \mathcal{C}^*_\mathfrak{t} \cap \mathcal{C}^0_\mathfrak{t}$.

We call a pair $(A, \Phi)$ in $\tilde{\mathcal{C}}_\mathfrak{t}$ a SO(3) *monopole* if

$$\begin{aligned} \mathrm{ad}^{-1}(F^+_{\hat{A}}) - \tau\rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A\Phi + \rho(\vartheta) &= 0. \end{aligned} \tag{2.9}$$

Here, $D_A = \rho \circ \nabla_A : C^\infty(X, V^+) \to C^\infty(X, V^-)$ is the Dirac operator; the isomorphism $\mathrm{ad} : \mathfrak{g}_V \to \mathfrak{so}(\mathfrak{g}_V)$ identifies the self-dual curvature $F^+_{\hat{A}}$, a section of $\Lambda^+ \otimes \mathfrak{so}(\mathfrak{g}_V)$, with $\mathrm{ad}^{-1}(F^+_{\hat{A}})$, a section of $\Lambda^+ \otimes \mathfrak{g}_V$; the section $\tau$ of $\mathrm{GL}(\Lambda^+)$ is a perturbation close to the identity; the perturbation $\vartheta$ is a complex one-form close to zero; $\Phi^* \in \mathrm{Hom}(V^+, \mathbb{C})$ is the pointwise Hermitian dual $\langle \cdot, \Phi \rangle$ of $\Phi$; and $(\Phi \otimes \Phi^*)_{00}$ is the component of the section $\Phi \otimes \Phi^*$ of $i\mathfrak{u}(V^+)$ lying in $\rho(\Lambda^+) \otimes \mathfrak{g}_V$ with respect to the splitting $\mathfrak{u}(V^+) = i\underline{\mathbb{R}} \oplus \mathfrak{su}(V^+)$ and decomposition (2.4) of $\mathfrak{su}(V^+)$.

Equation (2.9) is invariant under the action of $\mathcal{G}_\mathfrak{t}$. We let $\mathcal{M}_\mathfrak{t} \subset \mathcal{C}_\mathfrak{t}$ be the subspace represented by pairs satisfying equation (2.9) and write

$$\mathcal{M}^*_\mathfrak{t} = \mathcal{M}_\mathfrak{t} \cap \mathcal{C}^*_\mathfrak{t}, \quad \mathcal{M}^0_\mathfrak{t} = \mathcal{M}_\mathfrak{t} \cap \mathcal{C}^0_\mathfrak{t}, \quad \text{and} \quad \mathcal{M}^{*,0}_\mathfrak{t} = \mathcal{M}_\mathfrak{t} \cap \mathcal{C}^{*,0}_\mathfrak{t}. \tag{2.10}$$

Since equation (2.9) is invariant under the circle action induced by scalar multiplication on $V$, the subspaces (2.10) of $\mathcal{C}_\mathfrak{t}$ are also invariant under this action.

**Theorem 2.1.** [5, Theorem 1.1], [27] *Let $X$ be a closed, oriented, smooth four-manifold and let $V$ be a complex rank-eight, Hermitian vector bundle over $X$. Then for parameters $(\rho, g, \tau, \vartheta)$, which are generic in the sense of* [5], *and $\mathfrak{t} = (\rho, V)$, the space $\mathcal{M}^{*,0}_\mathfrak{t}$ is a smooth manifold of the expected dimension,*

$$\begin{aligned} \dim \mathcal{M}^{*,0}_\mathfrak{t} = d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}), \quad \textit{where } d_a(\mathfrak{t}) &= -2p_1(\mathfrak{t}) - \tfrac{3}{2}(\chi + \sigma), \\ n_a(\mathfrak{t}) &= \tfrac{1}{4}(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma). \end{aligned} \tag{2.11}$$

For the remainder of the article, we assume that the perturbation parameters in (2.9) are chosen as indicated in Theorem 2.1. For each non-negative integer $\ell$, let $\mathfrak{t}_\ell = (\rho, V_\ell)$, where

$$c_1(V_\ell) = c_1(V), \quad p_1(\mathfrak{g}_{V_\ell}) = p_1(\mathfrak{g}_V) + 4\ell, \quad \text{and} \quad w_2(\mathfrak{g}_{V_\ell}) = w_2(\mathfrak{g}_V).$$

We let $\bar{\mathcal{M}}_{\mathfrak{t}}$ denote the closure of $\mathcal{M}_{\mathfrak{t}}$ in the space of ideal monopoles,

$$I\mathcal{M}_{\mathfrak{t}} = \bigsqcup_{\ell=0}^{\infty}(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)), \tag{2.12}$$

with respect to an Uhlenbeck topology [14, Definition 4.19] and call the intersection of $\bar{\mathcal{M}}_{\mathfrak{t}}$ with $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ its *$\ell$-th level.*

**Theorem 2.2.** [14, Theorem 1.1] *Let $X$ be a Riemannian four-manifold with spin$^u$ structure $\mathfrak{t}$. Then there is a positive integer $N$, depending at most on the curvature of the chosen unitary connection on $\det(V^+)$ together with $p_1(\mathfrak{t})$, such that the Uhlenbeck closure $\bar{\mathcal{M}}_{\mathfrak{t}}$ of $\mathcal{M}_{\mathfrak{t}}$ in $\sqcup_{\ell=0}^N(\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$ is a second countable, compact, Hausdorff space. The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ carries a continuous circle action, which restricts to the circle action defined on $\mathcal{M}_{\mathfrak{t}_\ell}$ on each level.*

2.2. **Stratum of anti-self-dual or zero-section solutions.** From equation (2.9), we see that the stratum of $\mathcal{M}_{\mathfrak{t}}$ represented by pairs with zero spinor is identified with

$$\{A \in \mathcal{A}_{\mathfrak{t}} : F^+_{\hat{A}} = 0\}/\mathcal{G}_{\mathfrak{t}} \cong M^w_\kappa(X, g),$$

the moduli space of $g$-anti-self-dual connections on the SO(3) bundle $\mathfrak{g}_V$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and $w \equiv w_2(\mathfrak{t}) \pmod 2$. For a generic Riemannian metric $g$, the space $M^w_\kappa(X, g)$ is a smooth manifold of the expected dimension $-2p_1(\mathfrak{t}) - \frac{3}{2}(\chi + \sigma) = d_a(\mathfrak{t})$.

As explained in [15, §3.4.1], it is desirable to choose $w \pmod 2$ so as to exclude points in $\bar{\mathcal{M}}_{\mathfrak{t}}$ with associated flat SO(3) connections, so we have a *disjoint* union,

$$\bar{\mathcal{M}}_{\mathfrak{t}} \cong \bar{\mathcal{M}}^{*,0}_{\mathfrak{t}} \sqcup \bar{M}^w_\kappa \sqcup \bar{\mathcal{M}}^{\mathrm{red}}_{\mathfrak{t}}, \tag{2.13}$$

where $\bar{\mathcal{M}}^*_{\mathfrak{t}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose associated SO(3) connections are irreducible, $\bar{\mathcal{M}}^0_{\mathfrak{t}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose spinors are not identically zero, $\bar{\mathcal{M}}^{*,0}_{\mathfrak{t}} = \bar{\mathcal{M}}^*_{\mathfrak{t}} \cap \bar{\mathcal{M}}^0_{\mathfrak{t}}$, while $\bar{\mathcal{M}}^{\mathrm{red}}_{\mathfrak{t}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace $\bar{\mathcal{M}}_{\mathfrak{t}} - \bar{\mathcal{M}}^*_{\mathfrak{t}}$ represented by triples whose associated SO(3) connections are reducible. We recall the

**Definition 2.3.** [16, Definition 3.20] A class $v \in H^2(X; \mathbb{Z}/2)$ is *good* if no integral lift of $v$ is torsion.

If $w \pmod 2$ is good, then the union (2.13) is disjoint, as desired. In practice, rather than constraining $w \pmod 2$ itself, we replace $X$ by the blow-up $X\#\overline{\mathbb{CP}}^2$ and $w$ by $w + \mathrm{PD}[e]$ (where $e \in H_2(X; \mathbb{Z})$ is the exceptional class), noting that $w + \mathrm{PD}[e] \pmod 2$ is always good, and define gauge-theoretic invariants of $X$ in terms of moduli spaces on $X\#\overline{\mathbb{CP}}^2$. When $w \pmod 2$ is good, we define [15, Definition 3.7] the link of $\bar{M}^w_\kappa$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ by

$$\mathbf{L}^w_{\mathfrak{t},\kappa} = \{[A, \Phi, \mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}}/S^1 : \|\Phi\|^2_{L^2} = \varepsilon\}, \tag{2.14}$$

where $\varepsilon$ is a small positive constant; for generic $\varepsilon$, the link $\mathbf{L}^w_{\mathfrak{t},\kappa}$ is a smoothly-stratified, codimension-one subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

2.3. **Strata of Seiberg-Witten or reducible solutions.** We call a pair $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ *reducible* if the connection $A$ on $V$ respects a splitting,

$$V = W \oplus W \otimes L = W \otimes (\underline{\mathbb{C}} \oplus L), \tag{2.15}$$

for some spin$^c$ structure $\mathfrak{s} = (\rho, W)$ and complex line bundle $L$, in which case $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. A spin connection $A$ on $V$ is reducible with respect to the splitting (2.15) if and only if $\hat{A}$ is reducible with respect to the splitting $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$, [15, Lemma 2.9]. If $A$ is reducible, we can write $A = B \oplus B \otimes A_L$, where $B$ is a spin connection on $W$ and $A_L$ is a unitary connection on $L$; then $\hat{A} = d_{\mathbb{R}} \oplus A_L$ and $A_L = A_\Lambda \otimes (B^{\det})^*$, where $B^{\det}$ is the connection on $\det(W^+)$ induced by $B$ on $W$.

2.3.1. *Seiberg-Witten monopoles.* Given a spin$^c$ structure $\mathfrak{s} = (\rho, W)$ on $X$, let $\mathcal{A}_\mathfrak{s}$ denote the affine space of $L^2_k$ spin connections on $W$. Let $\mathcal{G}_\mathfrak{s}$ denote the group of $L^2_{k+1}$ unitary automorphisms of $W$, commuting with $\mathbb{C}\ell(T^*X)$, which we identify with $L^2_{k+1}(X, S^1)$. We then define

$$\tilde{\mathcal{C}}_\mathfrak{s} = \mathcal{A}_\mathfrak{s} \times L^2_k(W^+) \quad \text{and} \quad \mathcal{C}_\mathfrak{s} = \tilde{\mathcal{C}}_\mathfrak{s}/\mathcal{G}_\mathfrak{s}, \tag{2.16}$$

where $\mathcal{G}_\mathfrak{s}$ acts on $\tilde{\mathcal{C}}_\mathfrak{s}$ by

$$(s, (B, \Psi)) \mapsto s(B, \Psi) = (B - (s^{-1}ds)\mathrm{id}_W, s\Psi). \tag{2.17}$$

We call a pair $(B, \Psi) \in \tilde{\mathcal{C}}_\mathfrak{s}$ a Seiberg-Witten monopole if

$$\begin{aligned} \mathrm{Tr}(F^+_B) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+(A_\Lambda) = 0, \\ D_B\Psi + \rho(\vartheta)\Psi = 0, \end{aligned} \tag{2.18}$$

where $\mathrm{Tr} : \mathfrak{u}(W^+) \to i\mathbb{R}$ is defined by the trace on $2 \times 2$ complex matrices, $(\Psi \otimes \Psi^*)_0$ is the component of the section $\Psi \otimes \Psi^*$ of $i\mathfrak{u}(W^+)$ contained in $i\mathfrak{su}(W^+)$, $D_B : C^\infty(W^+) \to C^\infty(W^-)$ is the Dirac operator, and $A_\Lambda$ is a unitary connection on a line bundle with first Chern class $\Lambda \in H^2(X; \mathbb{Z})$. The perturbations are chosen so that solutions to equation (2.18) are identified with reducible solutions to (2.9) when $c_1(\mathfrak{t}) = \Lambda$. Let $\tilde{M}_\mathfrak{s} \subset \tilde{\mathcal{C}}_\mathfrak{s}$ be the subspace cut out by equation (2.18) and denote the moduli space of Seiberg-Witten monopoles by $M_\mathfrak{s} = \tilde{M}_\mathfrak{s}/\mathcal{G}_\mathfrak{s}$.

2.3.2. *Seiberg-Witten invariants.* We let $\mathcal{C}^0_\mathfrak{s} \subset \mathcal{C}_\mathfrak{s}$ be the open subspace represented by pairs whose spinor components which are not identically zero and define a complex line bundle over $\mathcal{C}^0_\mathfrak{s} \times X$ by

$$\mathbb{L}_\mathfrak{s} = \tilde{\mathcal{C}}^0_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} \underline{\mathbb{C}}, \tag{2.19}$$

where $\underline{\mathbb{C}} = X \times \mathbb{C}$ and $s \in \mathcal{G}_\mathfrak{s}$ acts on $(B, \Psi) \in \tilde{\mathcal{C}}_\mathfrak{s}$ and $(x, \zeta) \in \underline{\mathbb{C}}$ by

$$((B, \Psi), (x, \zeta)) \mapsto (s(B, \Psi), (x, s(x)^{-1}\zeta)). \tag{2.20}$$

Define

$$\mathbb{A}_2(X) = \mathrm{Sym}\,(H_0(X; \mathbb{R})) \otimes \Lambda^\bullet(H_1(X; \mathbb{R})) \tag{2.21}$$

be the graded algebra, with $z = \beta_1\beta_2 \cdots \beta_r$ having total degree $\deg(z) = \sum_p (2 - i_p)$, when $\beta_p \in H_{i_p}(X; \mathbb{R})$. Then the map,

$$\mu_\mathfrak{s} : H_\bullet(X; \mathbb{R}) \to H^{2-\bullet}(\mathcal{C}^0_\mathfrak{s}; \mathbb{R}), \quad \beta \mapsto c_1(LL_\mathfrak{s})/\beta,$$

extends in the usual way to a homomorphism of graded real algebras,

$$\mu_\mathfrak{s} : \mathbb{A}_2(X) \to H^\bullet(\mathcal{C}^0_\mathfrak{s}; \mathbb{R}).$$

If $x \in H_0(X; \mathbb{Z})$ denotes the positive generator, we set

$$\mu_\mathfrak{s} = c_1(\mathbb{L}_\mathfrak{s})/x \in H^2(\mathcal{C}^0_\mathfrak{s}; \mathbb{Z}). \tag{2.22}$$

Equivalently, $\mu_\mathfrak{s}$ is the first Chern class of the $S^1$ base-point fibration over $\mathcal{C}^0_\mathfrak{s}$. If $b_1(X) = 0$, then $c_1(\mathbb{L}_\mathfrak{s}) = \mu_\mathfrak{s} \times 1$ by [15, Lemma 2.14].

For $b^+_2(X) > 0$ and generic Riemannian metrics on $X$, the space $M_\mathfrak{s}$ contains zero-section pairs if and only if $c_1(\mathfrak{s}) - \Lambda$ is a torsion class by [23, Proposition 6.3.1]. If $M_\mathfrak{s}$ contains no zero-section pairs then, for generic perturbations, it is a compact, oriented, smooth manifold of dimension

$$d_s(\mathfrak{s}) = \dim M_{\mathfrak{s}} = \tfrac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma). \tag{2.23}$$

Let $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ denote the blow-up of $X$ with exceptional class $e \in H_2(\tilde{X};\mathbb{Z})$ and denote its Poincaré dual by $\mathrm{PD}[e] \in H^2(\tilde{X};\mathbb{Z})$. Let $\mathfrak{s}^{\pm} = (\tilde{\rho}, \tilde{W})$ denote the spin$^c$ structure on $\tilde{X}$ with $c_1(\mathfrak{s}^{\pm}) = c_1(\mathfrak{s}) \pm \mathrm{PD}[e]$ obtained by splicing the spin$^c$ structure $\mathfrak{s} = (\rho, W)$ on $X$ with the spin$^c$ structure on $\overline{\mathbb{CP}}^2$ with first Chern class $\pm\mathrm{PD}[e]$. (See [16, §4.5] for an explanation of the relation between spin$^c$ structures on $X$ and $\tilde{X}$.) Now

$$c_1(\mathfrak{s}) \pm \mathrm{PD}[e] - \Lambda \in H^2(\tilde{X};\mathbb{Z})$$

is not a torsion class and so — for $b_2^+(X) > 0$, generic Riemannian metrics on $X$ and related metrics on the connected sum $\tilde{X}$ — the moduli spaces $M_{\mathfrak{s}^{\pm}}(\tilde{X})$ contain no zero-section pairs. Thus, for our choice of generic perturbations, the moduli spaces $M_{\mathfrak{s}^{\pm}}(\tilde{X})$ are compact, oriented, smooth manifolds, both of dimension $\dim M_{\mathfrak{s}}(X)$.

For $b_1(X) = 0$ and odd $b_2^+(X) > 1$, we define the *Seiberg-Witten invariant* by [16, §4.1]

$$SW_X(\mathfrak{s}) = \langle \mu_{\mathfrak{s}^+}^d, [M_{\mathfrak{s}^+}(\tilde{X})]\rangle = \langle \mu_{\mathfrak{s}^-}^d, [M_{\mathfrak{s}^-}(\tilde{X})]\rangle, \tag{2.24}$$

where $2d = d_s(\mathfrak{s}) = d_s(\mathfrak{s}^{\pm})$. When $b_2^+(X) = 1$ the pairing on the right-hand side of definition (2.24) depends on the chamber in the positive cone of $H^2(\tilde{X};\mathbb{R})$ determined by the period point of the Riemannian metric on $\tilde{X}$. The definition of the Seiberg-Witten invariant for this case is also given in [16, §4.1]: we assume that the class $w_2(X) - \Lambda \pmod 2$ is good to avoid technical difficulties involved in chamber specification. Since $w \equiv w_2(X) - \Lambda \pmod 2$, this coincides with the constraint we use to define the Donaldson invariants in §2.5 when $b_2^+(X) = 1$. We refer to [16, Lemma 4.1 & Remark 4.2] for a comparison of the chamber structures required for the definition of Donaldson and Seiberg-Witten invariants when $b_2^+(X) = 1$.

2.3.3. *Reducible* $\mathrm{SO}(3)$ *monopoles.* If $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$ with $V = W \oplus W \otimes L$, then there is an embedding

$$\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad (B, \Psi) \mapsto (B \oplus B \otimes A_\Lambda \otimes B^{\det,*}, \Psi \oplus 0), \tag{2.25}$$

which is gauge-equivariant with respect to the homomorphism

$$\varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}, \quad s \mapsto \mathrm{id}_W \otimes \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}. \tag{2.26}$$

According to [15, Lemma 3.13], the map (2.25) defines a topological embedding $\iota : M_{\mathfrak{s}}^0 \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, where $M_{\mathfrak{s}}^0 = M_{\mathfrak{s}} \cap \mathcal{C}_{\mathfrak{s}}^0$ and an embedding of $M_{\mathfrak{s}}$ if $w_2(\mathfrak{t}) \neq 0$ or $b_1(X) = 0$; its image in $\mathcal{M}_{\mathfrak{t}}$ is represented by pairs which are reducible with respect to the splitting $V = W \oplus W \otimes L$. Henceforth, we shall not distinguish between $M_{\mathfrak{s}}$ and its image in $\mathcal{M}_{\mathfrak{t}}$ under this embedding.

2.3.4. *Circle actions.* When $V = W \oplus W \otimes L$ and $\mathfrak{t} = (\rho, V)$, the space $\tilde{\mathcal{C}}_{\mathfrak{t}}$ inherits a circle action defined by

$$S^1 \times V \to V, \quad (e^{i\theta}, \Psi \oplus \Psi') \mapsto \Psi \oplus e^{i\theta}\Psi', \tag{2.27}$$

where $\Psi \in C^\infty(W)$ and $\Psi' \in C^\infty(W \otimes L)$. With respect to the splitting $V = W \oplus W \otimes L$, the actions (2.27) and (2.8) are related by

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{pmatrix} = e^{i\theta} u, \text{ where } u = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in \mathcal{G}_{\mathfrak{t}}, \tag{2.28}$$

and so, when we pass to the induced circle actions on the quotient $\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}$, the actions (2.27) and (2.8) on $\mathcal{C}_{\mathfrak{t}}$ differ only in their multiplicity. Recall [15, Lemma 3.11] that the image in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ of the map (2.25) contains all pairs which are fixed by the circle action (2.27).

2.3.5. *The virtual normal bundle of the Seiberg-Witten moduli space.* Suppose $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$, with $V = W \oplus W \otimes L$, so we have a topological embedding $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$; we assume $M_{\mathfrak{s}}$ contains no zero-section monopoles. Recall from [15, §3.5] that there exist finite-rank, complex vector bundles,

$$\pi_\Xi : \Xi_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}} \quad \text{and} \quad \pi_N : N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}, \tag{2.29}$$

with $\Xi_{\mathfrak{t},\mathfrak{s}} \cong M_{\mathfrak{s}} \times \mathbb{C}^{r_\Xi}$, called the *obstruction bundles* and *ambient normal bundles* of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, respectively. For a small enough positive radius $\varepsilon$, there are a topological embedding [15, Theorem 3.21] of an open tubular neighborhood,

$$\boldsymbol{\gamma}_{\mathfrak{s}} : N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \hookrightarrow \mathcal{C}_{\mathfrak{t}}, \tag{2.30}$$

and a smooth section $\boldsymbol{\chi}_{\mathfrak{s}}$ of the pulled-back complex vector bundle,

$$\pi_N^* \Xi_{\mathfrak{t},\mathfrak{s}} \to N_{\mathfrak{t},\mathfrak{s}}(\varepsilon), \tag{2.31}$$

such that the tubular map yields a homeomorphism

$$\boldsymbol{\gamma}_{\mathfrak{s}} : \boldsymbol{\chi}_{\mathfrak{s}}^{-1}(0) \cap N_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \cong \mathcal{M}_{\mathfrak{t}} \cap \boldsymbol{\gamma}_{\mathfrak{s}}(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)), \tag{2.32}$$

restricting to a diffeomorphism on the complement of $M_{\mathfrak{s}}$ and identifying $M_{\mathfrak{s}}$ with its image $\iota(M_{\mathfrak{s}}) \subset \mathcal{M}_{\mathfrak{t}}$. The image $\boldsymbol{\gamma}_{\mathfrak{s}}(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon))$ is an open subset of an *ambient moduli space*, $\mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \subset \mathcal{C}_{\mathfrak{t}}$.

Our terminology is loosely motivated by that of [17] and [25], where the goal (translated to our setting) would be to construct a *virtual fundamental class* for $\mathcal{M}_{\mathfrak{t}}$, given by the cap product of the fundamental class of an ambient space containing $\mathcal{M}_{\mathfrak{t}}$ with the Euler class of a vector bundle over this ambient space, where $\mathcal{M}_{\mathfrak{t}}$ is the zero locus of a (possibly non-transversally vanishing) section. Here, $\mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ plays the role of the ambient space and (the pushforward of) $\Xi_{\mathfrak{t},\mathfrak{s}}$ the vector bundle with zero section yielding (an open neighborhood in) $\mathcal{M}_{\mathfrak{t}}$. Then, $N_{\mathfrak{t},\mathfrak{s}}$ is the normal bundle of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$, while $[N_{\mathfrak{t},\mathfrak{s}}] - [\Xi_{\mathfrak{t},\mathfrak{s}}]$ would more properly be called the 'virtual normal bundle' of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, in the language of $K$-theory.

Recall that minus the index of the SO(3)-monopole elliptic deformation complex [15, Equations (2.47) & (3.35)] at a reducible solution can be written as

$$\dim \mathcal{M}_{\mathfrak{t}} = 2n_s(\mathfrak{t},\mathfrak{s}) + d_s(\mathfrak{s}), \tag{2.33}$$

where $d_s(\mathfrak{s})$ is the expected dimension of the Seiberg-Witten moduli space $M_{\mathfrak{s}}$ (see equation (2.23)), while $n_s(\mathfrak{t},\mathfrak{s}) = n_s'(\mathfrak{t},\mathfrak{s}) + n_s''(\mathfrak{t},\mathfrak{s})$ is minus the complex index of the normal deformation operator [15, Equations (3.71) & (3.72)]), with

$$\begin{aligned} n_s'(\mathfrak{t},\mathfrak{s}) &= -(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \tfrac{1}{2}(\chi + \sigma), \\ n_s''(\mathfrak{t},\mathfrak{s}) &= \tfrac{1}{8}(c_1(\mathfrak{s}) - 2c_1(\mathfrak{t}))^2 - \sigma). \end{aligned} \tag{2.34}$$

If $r_\Xi$ is the complex rank of $\Xi_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$, and $r_N(\mathfrak{t},\mathfrak{s})$ is the complex rank of $N_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$, then

$$r_N(\mathfrak{t},\mathfrak{s}) = n_s(\mathfrak{t},\mathfrak{s}) + r_\Xi, \tag{2.35}$$

as we can see from the dimension relation (2.33) and the topological model (2.32).

The map (2.30) is circle equivariant when the circle acts trivially on $M_{\mathfrak{s}}$, by scalar multiplication on the fibers of $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.27) on $\mathcal{C}_{\mathfrak{t}}$. The bundle (2.31) and section $\boldsymbol{\chi}_{\mathfrak{s}}$ are circle equivariant if the circle acts on $N_{\mathfrak{t},\mathfrak{s}}$ and the fibers of $\boldsymbol{\gamma}_{\mathfrak{s}}^* \Xi_{\mathfrak{t},\mathfrak{s}}$ by scalar multiplication.

Let $\tilde{N}_{\mathfrak{t},\mathfrak{s}} \to \tilde{M}_{\mathfrak{s}}$ be the pullback of $N_{\mathfrak{t},\mathfrak{s}}$ by the projection $\tilde{M}_{\mathfrak{s}} \to M_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, so $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ is a $\mathcal{G}_{\mathfrak{s}}$-equivariant bundle, where $\mathcal{G}_{\mathfrak{s}}$ acts on the base $\tilde{M}_{\mathfrak{s}}$ by the usual gauge group action (2.17) and the induced action on the total space,

$$\tilde{N}_{\mathfrak{t},\mathfrak{s}} \subset \tilde{M}_{\mathfrak{s}} \times L^2_k(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L) \subset \tilde{M}_{\mathfrak{s}} \times L^2_k(\Lambda^1 \otimes_{\mathbb{R}} \mathfrak{g}_V) \oplus L^2(V^+), \tag{2.36}$$

via the embedding (2.26) of $\mathcal{G}_{\mathfrak{s}}$ into $\mathcal{G}_{\mathfrak{t}}$ and the splittings $\mathfrak{g}_V \cong \underline{\mathbb{R}} \oplus L$ [15, Lemma 3.10] and $V = W \oplus W \otimes L$. Thus, $s \in \mathcal{G}_{\mathfrak{s}}$ acts by scalar multiplication by $s^{-2}$ on sections of $\Lambda^1 \otimes_{\mathbb{R}} L$ and by $s^{-1}$ on sections of $W^+ \otimes L$ [15, §3.5.4].

For a small enough positive $\varepsilon$, there is a smooth embedding [15, §3.5.4] of the open tubular neighborhood $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$,

$$\tilde{\gamma}_{\mathfrak{s}} : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \to \tilde{\mathcal{C}}_{\mathfrak{t}}, \tag{2.37}$$

which is gauge equivariant with respect to the preceding action of $\mathcal{G}_{\mathfrak{s}}$, and covers the topological embedding (2.30). The map (2.37) is circle equivariant, where the circle acts trivially on $\tilde{M}_{\mathfrak{s}}$, by scalar multiplication on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.27) on $\mathcal{C}_{\mathfrak{t}}$. We note that the map (2.37) is also circle equivariant with respect to the action (2.8) on $\mathcal{C}_{\mathfrak{t}}$, if the circle acts on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ by

$$(e^{i\theta}, (B, \Psi, \beta, \psi)) \mapsto \varrho(e^{i\theta})(B, \Psi, e^{2i\theta}\beta, e^{2i\theta}\psi) = (B, e^{i\theta}\Psi, \beta, e^{i\theta}\psi), \tag{2.38}$$

where $(B, \Psi) \in \tilde{M}_{\mathfrak{s}}$, $(\beta, \psi) \in L^2_k(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L)$, so $(B, \Psi, \beta, \psi) \in \tilde{N}_{\mathfrak{t},\mathfrak{s}}$, and $\varrho : \mathcal{G}_{\mathfrak{s}} \to \mathcal{G}_{\mathfrak{t}}$ is the homomorphism (2.26). This equivariance follows from the relation (2.28) between the actions (2.27) and (2.8).

We note that the Chern character of the bundle $N_{\mathfrak{t},\mathfrak{s}}$ is computed in [15, Theorem 3.29] while the Segre classes of this bundle are computed, under some additional assumptions, in [16, Lemma 4.11].

## 2.4. Cohomology classes on the moduli space of SO(3) monopoles.

The identity (1.1) arises as an equality of pairings of suitable cohomology classes with a link in $\bar{\mathcal{M}}^{*,0}_{\mathfrak{t}}/S^1$ of the anti-self-dual moduli subspace and the links of the Seiberg-Witten moduli subspaces in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. We recall the definitions of these cohomology classes and their dual geometric representatives given in [16, §3].

The first kind of cohomology class is defined on $\mathcal{M}^*_{\mathfrak{t}}/S^1$, via the associated SO(3) bundle,

$$\mathbb{F}_{\mathfrak{t}} = \tilde{\mathcal{C}}^*_{\mathfrak{t}}/S^1 \times_{\mathcal{G}_{\mathfrak{t}}} \mathfrak{g}_V \to \mathcal{C}^*_{\mathfrak{t}}/S^1 \times X. \tag{2.39}$$

The group $\mathcal{G}_{\mathfrak{t}}$ acts diagonally in (2.39), with $\mathcal{G}_{\mathfrak{t}}$ acting on the left on $\mathfrak{g}_V$. We define [16, §3.1]

$$\mu_p : H_\bullet(X; \mathbb{R}) \to H^{4-\bullet}(\mathcal{C}^*_{\mathfrak{t}}/S^1; \mathbb{R}), \quad \beta \mapsto -\tfrac{1}{4}p_1(\mathbb{F}_{\mathfrak{t}})/\beta. \tag{2.40}$$

On restriction to $M^w_\kappa \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, the cohomology classes $\mu_p(\beta)$ coincide with those used in the definition of Donaldson invariants [16, Lemma 3.1]. Define

$$\mathbb{A}(X) = \operatorname{Sym}\left(H_{\text{even}}(X; \mathbb{R})\right) \otimes \Lambda^\bullet(H_{\text{odd}}(X; \mathbb{R})) \tag{2.41}$$

to be the graded algebra, with $z = \beta_1\beta_2 \cdots \beta_r$ having total degree $\deg(z) = \sum_p (4 - i_p)$, when $\beta_p \in H_{i_p}(X; \mathbb{R})$. Then $\mu_p$ extends in the usual way to a homomorphism of graded real algebras,

$$\mu_p : \mathbb{A}(X) \to H^\bullet(\mathcal{C}^*_{\mathfrak{t}}/S^1; \mathbb{R}),$$

which preserves degrees. Next, we define a complex line bundle over $\mathcal{C}^{*,0}_{\mathfrak{t}}/S^1$,

$$\mathbb{L}_{\mathfrak{t}} = \mathcal{C}^{*,0}_{\mathfrak{t}} \times_{(S^1, \times -2)} \mathbb{C}, \tag{2.42}$$

where the circle action is given, for $[A,\Phi]\in\mathcal{C}_{\mathfrak{t}}^{*,0}$ and $\zeta\in\mathbb{C}$, by

$$([A,\Phi],\zeta)\mapsto([A,e^{i\theta}\Phi],e^{2i\theta}\zeta). \tag{2.43}$$

Then we define the second kind of cohomology class on $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ by

$$\mu_c=c_1(\mathbb{L}_{\mathfrak{t}})\in H^2(\mathcal{C}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{R}). \tag{2.44}$$

For monomials $z\in\mathbb{A}(X)$, we constructed [16, §3.2] geometric representatives $\mathcal{V}(z)$ dual to $\mu_p(z)$ (following the discussion in [20]) and $\mathcal{W}$ dual to $\mu_c$, defined on $\mathcal{M}_{\mathfrak{t}}^{*}/S^1$ and $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$, respectively; their closures in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ are denoted by $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ [16, Definition 3.14]. When

$$\deg(z)+2\eta=\dim(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1)-1, \tag{2.45}$$

and $\deg(z)\geq\dim M_\kappa^w$ it follows that [16, §3.3] the intersection

$$\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^\eta\cap\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0}/S^1, \tag{2.46}$$

is an oriented one-manifold (not necessarily connected) whose closure in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ can only intersect $(\bar{\mathcal{M}}_{\mathfrak{t}}-\mathcal{M}_{\mathfrak{t}})/S^1$ at points in $\mathcal{M}_{\mathfrak{t}}^{\mathrm{red}}\cong\cup(M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X))$, where the union is over $\ell\geq0$ and $\mathfrak{s}\in\mathrm{Spin}^c(X)$ [16, Corollary 3.18].

2.5. **Donaldson invariants.** We first recall the definition [20, §2] of the Donaldson series when $b_1(X)=0$ and $b_2^+(X)>1$ is odd, so that, in this case, $\chi+\sigma\equiv0\pmod 4$. See also §3.4.2 in [16], especially for a definition of the Donaldson invariants when $b_2^+(X)=1$. For any choice of $w\in H^2(X;\mathbb{Z})$, the Donaldson invariant is a linear function

$$D_X^w:\mathbb{A}(X)\to\mathbb{R}.$$

Let $\tilde{X}=X\#\overline{\mathbb{CP}}^2$ be the blow-up of $X$ and let $e\in H_2(\tilde{X};\mathbb{Z})$ be the exceptional class, with Poincaré dual $\mathrm{PD}[e]\in H^2(\tilde{X};\mathbb{Z})$. If $z\in\mathbb{A}(X)$ is a monomial, we define $D_X^w(z)=0$ unless

$$\deg(z)\equiv-2w^2-\tfrac{3}{2}(\chi+\sigma)\pmod 8. \tag{2.47}$$

If $\deg(z)$ obeys equation (2.47), we let $\kappa\in\frac{1}{4}\mathbb{Z}$ be defined by

$$\deg(z)=8\kappa-\tfrac{3}{2}\left(\chi+\sigma\right).$$

There exists an SO(3) bundle over $\tilde{X}$ with first Pontrjagin number $-4\kappa-1$ and second Stiefel-Whitney class $w+\mathrm{PD}[e]\pmod 2$. One then defines the Donaldson invariant on monomials by

$$D_X^w(z)=\#\left(\bar{\mathcal{V}}(ze)\cap\bar{M}_{\kappa+1/4}^{w+\mathrm{PD}[e]}(\tilde{X})\right), \tag{2.48}$$

and extends to a real linear function on $\mathbb{A}(X)$. Note that $w+\mathrm{PD}[e]\pmod 2$ is good in the sense of Definition 2.3. If $w'\equiv w\pmod 2$, then [2]

$$D_X^{w'}=(-1)^{\frac{1}{4}(w'-w)^2}D_X^w. \tag{2.49}$$

The Donaldson series is a formal power series,

$$\mathbf{D}_X^w(h)=D_X^w((1+\tfrac{1}{2}x)e^h),\quad h\in H_2(X;\mathbb{R}). \tag{2.50}$$

By equation (2.47), the series $\mathbf{D}_X^w$ is even if

$$-w^2-\tfrac{3}{4}(\chi+\sigma)\equiv0\pmod 2,$$

and odd otherwise. A four-manifold has KM-simple type if for some $w$ and all $z\in\mathbb{A}(X)$,

$$D_X^w(x^2z)=4D_X^w(z).$$

According to [20, Theorem 1.7], when $X$ has KM-simple type the series $\mathbf{D}_X^w(h)$ is an analytic function of $h$ and there are finitely many characteristic cohomology classes $K_1,\dots,K_m$ (the KM-basic classes) and constants $a_1,\dots,a_m$ (independent of $w$) so that

$$\mathbf{D}_X^w(h) = e^{\frac{1}{2}h\cdot h}\sum_{i=1}^r (-1)^{\frac{1}{2}(w^2+K_i\cdot w)} a_i e^{\langle K_i,h\rangle}.$$

Witten's conjecture, [28], then relates the Donaldson and Seiberg-Witten series for four-manifolds of simple type.

When $b_2^+(X)=1$ the pairing on the right-hand side of definition (2.48) depends on the chamber in the positive cone of $H^2(\tilde X;\mathbb{R})$ determined by the period point of the Riemannian metric on $\tilde X$, just as in the case of Seiberg-Witten invariants described in §2.3.2. We refer to §3.4.2 in [16] for a detailed discussion of this case and, as in [16], we assume that the class $w \pmod 2$ is good in order to avoid technical difficulties involved in chamber specification.

2.6. **Links and the cobordism.** Since the endpoints of the components of the one-manifold (2.46) either lie in $M_\kappa^w$ or $M_{\mathfrak{s}}\times \operatorname{Sym}^\ell(X)$, for some $\mathfrak{s}$ and $\ell(\mathfrak{t},\mathfrak{s})\geq 0$ for which $\mathfrak{t}_\ell = \mathfrak{s}\oplus\mathfrak{s}\otimes L$, we have when $w_2(\mathfrak{t})$ is good in the sense of Definition 2.3 that

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w\right) = -\sum_{\mathfrak{s}\in\operatorname{Spin}^c(X)}\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right), \tag{2.51}$$

where $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ is the link of $\bar M_\kappa^w$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ (see [15, Definition 3.7]) and where $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is empty if $\ell(\mathfrak{t},\mathfrak{s})<0$ and is the boundary of an open neighborhood of the Seiberg-Witten stratum $M_{\mathfrak{s}}\times\operatorname{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ if $\ell=\ell(\mathfrak{t},\mathfrak{s})\geq 0$. By construction, the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is a smooth manifold and the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with the geometric representatives is in this top stratum. The precise definition of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is given in [15, Definition 3.22] for $\ell=0$ and in Definition 10.2 for $\ell\geq 1$.

When $\deg(z)=\dim M_\kappa^w$, the intersection of the one-manifold (2.46) with the link $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ is given by [16, Lemma 3.30]

$$2^{1-n_a}\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{n_a-1}\cap\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w\right) = \#\left(\bar{\mathcal{V}}(z)\cap\bar M_\kappa^w\right). \tag{2.52}$$

Applying this identity to the blow-up, $X\#\overline{\mathbb{CP}}^2$, when $n_a(\mathfrak{t})>0$, we recover the Donaldson invariant $D_X^w(z)$ on the right-hand side of (2.52) via definition (2.48).

## 3. Diagonals

In this section, we describe the stratification of the symmetric product $\operatorname{Sym}^\ell(X)$ and the normal bundles of the strata.

3.1. **Definitions.** In this section, we introduce some vocabulary for describing the strata of the symmetric product $\operatorname{Sym}^\ell(X)$.

3.1.1. *Subgroups of the symmetric group.* Let $\mathfrak{S}_\ell$ denote the symmetric group on $\ell$ elements which acts on $X^\ell$ by permutation. We will write partitions of $N_\ell=\{1,\dots,\ell\}$ as $\mathcal{P}=\{P_1,\dots,P_{r(\mathcal{P})}\}$ where $P_i\subset N_\ell$. Each such partition of $N_\ell$ gives a partition of $\ell$: $\ell = |P_1|+\dots|P_{r(\mathcal{P})}|$ which we denote by $|\mathcal{P}|$. The symmetric group $\mathfrak{S}_\ell$ acts on the set of partitions of $N_\ell$, by:

$$(\sigma,\mathcal{P})\to\sigma(\mathcal{P}) := \{\sigma(P_1),\dots,\sigma(P_{r(\mathcal{P})})\}. \tag{3.1}$$

Let $\mathfrak{S}(\mathcal{P}) < \mathfrak{S}_\ell$ be the stabilizer of a partition $\mathcal{P}$ with respect to the action (3.1). Note that $\mathfrak{S}(\mathcal{P})$ does not preserve the *ordering* of the sets $P_1, \ldots, P_{r(\mathcal{P})}$. In addition, let $\Gamma(\mathcal{P})$ be the subgroup of $\mathfrak{S}_\ell$ defined by the image of the inclusion:

$$\prod_{P \in \mathcal{P}} \mathfrak{S}_{|P|} \to \mathfrak{S}_\ell, \tag{3.2}$$

given by $\mathfrak{S}_{|P|}$ acting trivially on the elements of $N_\ell$ not in $P$.

**Lemma 3.1.** *Let $\mathcal{P}$ be a partition of $N_\ell$. Then the subgroup $\Gamma(\mathcal{P})$ of $\mathfrak{S}_\ell$ defined by the image of the inclusion* (3.2) *is the stabilizer of $\mathcal{P}$ under the action* (3.1)*.*

*Proof.* Let $\Gamma'(\mathcal{P})$ be the stabilizer of $\mathcal{P}$ under the action (3.1). There is then an inclusion $\Gamma(\mathcal{P}) \to \Gamma'(\mathcal{P})$. This map is surjective because any $\sigma \in \mathfrak{S}_\ell$ which preserves the subsets $P_1, \ldots, P_{r(\mathcal{P})}$ is a permutation of each of these subsets and thus in $\Gamma(\mathcal{P})$. □

**Lemma 3.2.** *Let $\mathcal{P} = \{P_1, \ldots, P_{r(\mathcal{P})}\}$ be a partition of $N_\ell$. The action* (3.1) *defines an action of $\Gamma(\mathcal{P})$ on the set $\{P_1, \ldots, P_{r(\mathcal{P})}\}$. Let $\mathfrak{S}(|\mathcal{P}|)$ be the subgroup of $\mathfrak{S}_{r(\mathcal{P})}$ which preserves the partition $|P_1|, \ldots, |P_{r(\mathcal{P})}|$. Then $\Gamma(\mathcal{P})$ is a normal subgroup of $\mathfrak{S}(\mathcal{P})$ and there is an isomorphism:*

$$\mathfrak{S}(|\mathcal{P}|) \simeq \mathfrak{S}(\mathcal{P})/\Gamma(\mathcal{P}).$$

*Proof.* Let $t : \mathfrak{S}(\mathcal{P}) \to \mathfrak{S}_{r(\mathcal{P})}$ be the homomorphism defined by the action of $\mathfrak{S}(\mathcal{P})$ on the set $\{P_1, \ldots, P_{r(\mathcal{P})}\}$. The kernel of $t$ is the subgroup of elements preserving the sets $P_1, \ldots, P_{r(\mathcal{P})}$ which is $\Gamma(\mathcal{P})$ by Lemma 3.1, implying $\Gamma(\mathcal{P})$ is a normal subgroup of $\mathfrak{S}(\mathcal{P})$. Because each $\sigma \in \mathfrak{S}(\mathcal{P})$ is a bijection of $N_\ell$, $t(\sigma)$ preserves the partition $|\mathcal{P}|$. Thus, $t(\mathfrak{S}(\mathcal{P})) \subset \mathfrak{S}(|\mathcal{P}|)$. To prove surjectivity, given $\sigma \in \mathfrak{S}(|\mathcal{P}|)$, construct $\tilde{\sigma} \in \mathfrak{S}(\mathcal{P})$ with $t(\tilde{\sigma}) = \sigma$ as follows. Define an order relation on $\{P_1, \ldots, P_{r(\mathcal{P})}\}$ by comparing the smallest elements (with respect to the natural order on $N_\ell$) of elements of $P_i$ and $P_j$. If $a \in N_\ell$ is the $k$-th element of $P_i$ and $\sigma(i) = j$, define $\tilde{\sigma}(a)$ to be the $k$-th element of $P_j$. One then checks $\tilde{\sigma} \in \mathfrak{S}(\mathcal{P})$ and $t(\tilde{\sigma}) = \sigma$. □

3.1.2. *Defining the diagonals.* The strata of $\mathrm{Sym}^\ell(X)$ are defined by the quotients of diagonals in $X^\ell$ by the action of $\mathfrak{S}_\ell$ on $X^\ell$. Explicitly, define:

$$\begin{aligned} \Delta^\circ(X^\ell, \mathcal{P}) &= \{(x_1, \ldots, x_\ell) \in X^\ell : x_i = x_j \text{ if and only if there is } P \in \mathcal{P} \text{ with } i, j \in P\ \}, \\ \Delta(X^\ell, \mathcal{P}) &= \{(x_1, \ldots, x_\ell) \in X^\ell : x_i = x_j \text{ if there is } P \in \mathcal{P} \text{ with } i, j \in P\ \}. \end{aligned} \tag{3.3}$$

The preceding subspaces are related by

$$\mathrm{cl}_{X^\ell} \Delta^\circ(X^\ell, \mathcal{P}) = \Delta(X^\ell, \mathcal{P}). \tag{3.4}$$

We can identify $\Delta^\circ(X^\ell, \mathcal{P})$ with the complement of the big diagonal in $X^{r(\mathcal{P})}$ as follows. Write $\mathcal{P}$ as an ordered collection $P_1, \ldots, P_{r(\mathcal{P})}$. Then define an embedding,

$$\iota_{\mathcal{P}} : X^r \setminus \cup_{i<j} \{x_i = x_j\} \to \Delta^\circ(X^\ell, \mathcal{P}) \subset X^\ell, \tag{3.5}$$

by $\iota(x_1, \ldots, x_r) = (y_1, \ldots, y_\ell)$ where $y_k = x_i$ for $k \in P_i$. The map $\iota_{\mathcal{P}}$ is a diffeomorphism.

**Lemma 3.3.** *Let $\mathcal{P}$ be a partition of $N_\ell$. Then:*

1. *The diagonal $\Delta(X^\ell, \mathcal{P})$ is the fixed point set of the group $\Gamma(\mathcal{P})$,*
2. *If $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then $\mathrm{Stab}_{\mathbf{x}} = \Gamma(\mathcal{P})$,*
3. *If $\sigma \in \mathfrak{S}_\ell$ then $\sigma(\Delta^\circ(X^\ell, \mathcal{P})) \subset \Delta^\circ(X^\ell, \mathcal{P})$ if and only if $\sigma \in \mathfrak{S}(\mathcal{P})$.*

*Proof.* The fixed point set of $\mathfrak{S}_i$ acting on $X^i$ is the small diagonal. Thus, by the definition of $\Gamma(\mathcal{P})$, if $\mathbf{x} = (x_1, \dots, x_\ell)$ is a fixed point for every $\sigma \in \Gamma(\mathcal{P})$, we have $x_i = x_j$ for all $i, j \in P$ for every $P \in \mathcal{P}$, implying the first assertion.

The first assertion implies $\Gamma(\mathcal{P}) \subset \mathrm{Stab}_{\mathbf{x}}$ for $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$. If $\sigma \in \mathrm{Stab}_{\mathbf{x}}$ and $\sigma \notin \Gamma(\mathcal{P})$, then there is $P \in \mathcal{P}$ and $i \in P$ such that $\sigma(i) \notin P$. But $\sigma(\mathbf{x}) = \mathbf{x}$ then implies $x_{\sigma(i)} = x_i$ which contradicts $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$. This implies the second assertion.

By definition, $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$ if and only if $\sigma(\mathbf{x}) \in \Delta^\circ(X, \sigma(\mathcal{P}))$. □

3.1.3. *Strata of the symmetric product.* Let $\pi_{\mathfrak{S}} : X^\ell \to \mathrm{Sym}^\ell(X) = X^\ell/\mathfrak{S}_\ell$ be the projection. Define:

$$\Sigma(X^\ell, \mathcal{P}) = \pi_{\mathfrak{S}}\left(\Delta^\circ(X^\ell, \mathcal{P})\right). \tag{3.6}$$

From item (3) of Lemma 3.3, we have the relation:

$$\Sigma(X^\ell, \mathcal{P}) = \Delta^\circ(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P}). \tag{3.7}$$

The action of $\mathfrak{S}(\mathcal{P})$ on $\Delta^\circ(X^\ell, \mathcal{P})$ is not free as noted in item (2) of Lemma 3.3. From Lemma 3.2, $\mathfrak{S}(|\mathcal{P}|)$ is the natural group acting on $\Delta^\circ(X^\ell, \mathcal{P})$. We see this action as follows.

**Lemma 3.4.** *Choose an ordering $P_1, \dots, P_{r(\mathcal{P})}$ of the partition $\mathcal{P}$ and use the ordering $|P_1|, \dots, |P_{r(\mathcal{P})}|$ of $|\mathcal{P}|$. Then the map $\iota_{\mathcal{P}}$ given in* (3.5) *is equivariant with respect to the actions of $\mathfrak{S}(|\mathcal{P}|)$ on $X^r$ and $\mathfrak{S}(\mathcal{P})$ on $\Delta^\circ(X^\ell, \mathcal{P})$.*

The proof of the following relations between $\Sigma(X^\ell, \mathcal{P})$ and $\Delta^\circ(X^\ell, \mathcal{P})$ is elementary.

**Lemma 3.5.** *Let $\mathcal{P}$ be a partition of $N_\ell$ and let $[\mathcal{P}]$ denote the orbit of $\mathcal{P}$ under $\mathfrak{S}_\ell$. Then,*

1. *If $\mathcal{P}' = \sigma(\mathcal{P})$, then $\Gamma(\mathcal{P}) = \sigma\Gamma(\mathcal{P}')\sigma^{-1}$,*
2. *If $\mathcal{P}' = \sigma(\mathcal{P})$, then $\mathfrak{S}(\mathcal{P}) = \sigma\mathfrak{S}(\mathcal{P}')\sigma^{-1}$,*
3. *$\pi_{\mathfrak{S}}^{-1}\left(\Sigma(X^\ell, \mathcal{P})\right) = \cup_{\mathcal{P}' \in [\mathcal{P}]} \Delta^\circ(X, \mathcal{P}')$.*

3.2. **Incidence relations.** We now describe incidence relations among the diagonals of $X^\ell$ and the resulting relations among the strata $\Sigma$ of $\mathrm{Sym}^\ell(X)$.

**Definition 3.6.** Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $N_\ell$. We say $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$, or $\mathcal{P} < \mathcal{P}'$, if for every $P' \in \mathcal{P}'$ there is $P \in \mathcal{P}$ with $P' \subseteq P$.

**Lemma 3.7.** *If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $N_\ell$ then $\Delta^\circ(X^\ell, \mathcal{P}) \subset \Delta(X, \mathcal{P}')$ if and only $\mathcal{P}'$ is a refinement of $\mathcal{P}$.*

*Proof.* If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ and $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then for every $P' \in \mathcal{P}$ and $i, j \in P'$, there is $P \in \mathcal{P}$ with $i, j \in P' \subset P$. Since $i, j \in P \in \mathcal{P}$, we have $x_i = x_j$ so $\mathbf{x} \in \Delta(X, \mathcal{P}')$ proving one implication.

Conversely, if $\Delta^\circ(X^\ell, \mathcal{P}) \subset \Delta(X, \mathcal{P}')$ and $P' \in \mathcal{P}'$, then because every $\mathbf{x} = (x_1, \dots, x_\ell) \in \Delta^\circ(X^\ell, \mathcal{P})$ is also in $\Delta(X, \mathcal{P}')$ we see that for every $i, j \in P'$, $x_i = x_j$. This implies (by the only if in the definition of $\Delta^\circ(X^\ell, \mathcal{P})$ that there must be $P \in \mathcal{P}$ with $i, j \in P$. Thus, $P' \subset P$ and $\mathcal{P} < \mathcal{P}$. □

Now, we compare the incidence relations with the strata of $\mathrm{Sym}^\ell(X)$. Recall that for a partition $\mathcal{P}$ of $N_\ell$, $[\mathcal{P}]$ denoted the orbit of $\mathcal{P}$ under the action (3.1) of $\mathfrak{S}_\ell$ on the set of partitions of $N_\ell$. For $\mathcal{P}$ and $\mathcal{P}'$ partitions of $N_\ell$, define

$$[\mathcal{P} < \mathcal{P}'] = \{\mathcal{P}'' \in [\mathcal{P}'] : \mathcal{P} < \mathcal{P}''\}. \tag{3.8}$$

**Remark 3.8.** Observe that $\mathfrak{S}(\mathcal{P})$ acts transitively on $[\mathcal{P} < \mathcal{P}']$ by the action (3.1).

The next lemma follows from the definition of $\Sigma(X^\ell, \mathcal{P})$ (see (3.6)) as well as Lemma 3.7 and item (3) in Lemma 3.5.

**Lemma 3.9.** *If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $N_\ell$ then*

$$\Sigma(X^\ell,\mathcal{P}) \subset \mathrm{cl}_{\mathrm{Sym}^\ell(X)}\Sigma(\mathcal{P}')$$

*if and only if there is $\mathcal{P}'' \in [\mathcal{P}]$ with $\mathcal{P}' < \mathcal{P}''$.*

3.3. **Normal bundles.** We now introduce the normal bundles of the strata $\Delta^\circ(X^\ell,\mathcal{P})$ of $X^\ell$.

**Lemma 3.10.** *Let $\mathcal{P}$ be any partition of $N_\ell$. The tangent bundle of the submanifold $\Delta^\circ(X^\ell,\mathcal{P})$ is given by:*

$$T\Delta^\circ(X^\ell,\mathcal{P}) = \{(v_1,\dots,v_\ell) \in TX^\ell : v_i = v_j \text{ if and only if there is } P\in\mathcal{P} \text{ with } i,j\in P\}. \tag{3.9}$$

*If $g$ is any metric on $X$, then the orthogonal complement of $T\Delta^\circ(X^\ell,\mathcal{P})$ with respect to the metric on $X^\ell$ given by the $\ell$-fold product of $g$ is the following normal bundle of $\Delta^\circ(X^\ell,\mathcal{P})$:*

$$\tilde\nu(X^\ell,\mathcal{P}) := \{(v_1,\dots,v_\ell) \in TX^\ell|_{\Delta^\circ(X^\ell,\mathcal{P})} : \sum_{i\in P} v_i = 0 \text{ for all } P\in\mathcal{P}\}. \tag{3.10}$$

*Proof.* Differentiate any path in $\Delta^\circ(X^\ell,\mathcal{P})$ to see the form of $T\Delta^\circ(X^\ell,\mathcal{P})$ appearing in equation (3.9). For $\vec v = (v_1,\dots,v_\ell) \in T\Delta^\circ(X^\ell,\mathcal{P})$, we then write $v_P$ for any $v_i$ with $i\in P$. Then, if $\vec v = (v_1,\dots,v_\ell) \in T\Delta^\circ(X^\ell,\mathcal{P})$ and $\vec w \in \tilde\nu(X^\ell,\mathcal{P})$, the equality

$$\vec v\cdot\vec w = \sum_i v_i\cdot w_i = \sum_{P\in\mathcal{P}} v_P \cdot \left(\sum_{i\in P} w_i\right) = 0,$$

which holds for any product metric on $X^\ell$, implies that the bundle in equation (3.10) is contained in the orthogonal complement of $T\Delta^\circ(X^\ell,\mathcal{P})$. Since the above inclusion must hold for any value of $v_P$, we see that any element of the orthogonal complement of $T\Delta^\circ(X^\ell,\mathcal{P})$ must satisfy the equations defining the bundle in (3.10). □

There is an obvious $\mathfrak{S}(\mathcal{P})$ action on the normal bundle $\tilde\nu(X^\ell,\mathcal{P})$. We will write $\nu(X^\ell,\mathcal{P})$ for the quotient $\tilde\nu(X^\ell,\mathcal{P})/\mathfrak{S}(\mathcal{P})$.

**Lemma 3.11.** *There is an open neighborhood, $\tilde{\mathcal{O}}(X^\ell,\mathcal{P}) \subset \tilde\nu(X^\ell,\mathcal{P})$, of the zero-section and an $\mathfrak{S}(\mathcal{P})$-equivariant exponential map identifying $\tilde{\mathcal{O}}(X^\ell,\mathcal{P})$ with an open neighborhood, $\tilde{\mathcal{U}}(X^\ell,\mathcal{P})$, of $\Delta^\circ(X^\ell,\mathcal{P})$ in $X^\ell$.*

Because the exponential map is $\mathfrak{S}(\mathcal{P})$ equivariant, equation (3.7) implies:

**Lemma 3.12.** *Continue the notation of Lemma 3.11. If $\mathcal{O}(X^\ell,\mathcal{P}) = \tilde{\mathcal{O}}(X^\ell,\mathcal{P})/\mathfrak{S}(\mathcal{P})$, then a neighborhood,*

$$\mathcal{U}(X^\ell,\mathcal{P}) = \tilde{\mathcal{U}}(X^\ell,\mathcal{P})$$

*of $\Sigma(X^\ell,\mathcal{P})$ in $\mathrm{Sym}^\ell(X)$ is homeomorphic to $\mathcal{O}(X^\ell,\mathcal{P})$.*

We now identify the intersection of $\Delta^\circ(X,\mathcal{P}')$ with $\tilde{\mathcal{U}}(X^\ell,\mathcal{P})$.

**Lemma 3.13.** *If $\mathcal{P} < \mathcal{P}'$ are partitions of $N_\ell$ and*

$$\begin{aligned}\tilde\nu&(X^\ell,\mathcal{P},\mathcal{P}')\\ &= \{(v_1,\dots,v_\ell)\in\tilde\nu(X^\ell,\mathcal{P}) : v_i = v_j \text{ if and only if there is } P'\in\mathcal{P}' \text{ with } i,j\in P'\}.\end{aligned} \tag{3.11}$$

*then the intersection of $\Delta^\circ(\mathcal{P}')$ with $\tilde{\mathcal{U}}(X^\ell,\mathcal{P})$ is homeomorphic to $\tilde\nu(X^\ell,\mathcal{P},\mathcal{P}')\cap\tilde{\mathcal{O}}(X^\ell,\mathcal{P})$.*

*Proof.* This follows immediately from the equivariance of the exponential map. □

Thus, a neighborhood of $\Delta^\circ(X^\ell,\mathcal{P}')$ near the lower diagonal $\Delta^\circ(X^\ell,\mathcal{P})$ can be described by the bundle $\tilde{\nu}(X^\ell,\mathcal{P},\mathcal{P}')$. However, the end of the stratum $\Sigma(X^\ell,\mathcal{P}')$ near the lower stratum $\Sigma(X^\ell,\mathcal{P})$ can be more complicated than a quotient of $\tilde{\nu}(X^\ell,\mathcal{P},\mathcal{P}')$ by a subgroup of $\mathfrak{S}_\ell$. Recall that the pre-image of $\Sigma(X^\ell,\mathcal{P})$ is not just $\Delta^\circ(X^\ell,\mathcal{P})$ but rather all the diagonals given by a partition in $[\mathcal{P}]$, the orbit of $\mathcal{P}$ under the action (3.1). Thus, while we can cover $\Sigma(X^\ell,\mathcal{P})$ by $\Delta^\circ(X^\ell,\mathcal{P})$, to account for all the ends of $\Delta^\circ(X^\ell,\mathcal{P}')$ which get mapped to a neighborhood of the lower stratum, $\Sigma(X^\ell,\mathcal{P})$, we must consider all diagonals $\Delta^\circ(X^\ell,\mathcal{P}')$ where $\mathcal{P}''\in[\mathcal{P}']$ and $\mathcal{P}<\mathcal{P}'$. That is, we must consider all partitions $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$, where $[\mathcal{P}<\mathcal{P}']$ is defined in (3.8). The next lemma follows from Lemmas 3.11 and 3.13.

**Lemma 3.14.** *Let $\mathcal{P}<\mathcal{P}'$ be partitions of $N_\ell$. Let $[\mathcal{P}<\mathcal{P}']$ be the set of partitions defined in* (3.8)*. Then a neighborhood of $\Sigma(X^\ell,\mathcal{P})$ in $\Sigma(X^\ell,\mathcal{P}')$ is homeomorphic to a neighborhood of the zero-section in*

$$\nu(X^\ell,\mathcal{P},[\mathcal{P}'])=\left(\sqcup_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\tilde{\nu}(X^\ell,\mathcal{P},\mathcal{P}')\right)/\mathfrak{S}(\mathcal{P}). \tag{3.12}$$

## 4. The Thom-Mather structure of the symmetric product

In this section, we define a partial Thom-Mather structure on the symmetric product. In general, by a partial Thom-Mather structure on a stratified space $Z$ with strata $\Sigma_i$, we mean neighborhoods $U_i$ of $\Sigma_i$ with *tubular neighborhood projections* $\pi_i:U_i\to\Sigma_i$ and *tubular distance functions* $t_i:U_i\to[0,1)$ satisfying $t_i^{-1}(0)=\Sigma_i$ and for $\Sigma_i\subset \mathrm{cl}_Z(\Sigma_j)$ the compatability conditions

$$\begin{aligned}\pi_i\circ\pi_j&=\pi_i,\\ t_i\circ\pi_j&=t_i.\end{aligned} \tag{4.1}$$

If $t$ and $t_r$ are tubular distance functions of a tubular neighborhood $\pi:U\to\Sigma$, then we say $t_r$ is a *rescaling* of $t$ if there is a non-vanishing function $f:\Sigma\to(0,1]$ such that

$$t_r(\cdot)=t(\cdot)f(\pi(\cdot)). \tag{4.2}$$

Note that an actual Thom-Mather structure has a number of additional requirements on the derivatives of the maps $\pi_i$ and $t_i$. While it is well-known that $\mathrm{Sym}^\ell(X)$, being the quotient of the smooth, compact manifold $X^\ell$ by a finite group $\mathfrak{S}_\ell$, is a Thom-Mather stratified space, the precise construction we present here will be useful in following sections.

We begin by defining subspaces of $\mathrm{Sym}^\kappa(\mathbb{R}^4)$ which will appear as the fibers of the normal bundles of §3.3. We construct partial Thom-Mather structures on this subspace in §4.1. To use the normal bundles of §3.3 to construct Thom-Mather structures on $\mathrm{Sym}^\ell(X)$, we first introduce some deformations of the metric in §4.2. We describe the overlap of the exponential maps from these normal bundles in §4.3. Finally, we produce the partial Thom-Mather structure on $\mathrm{Sym}^\ell(X)$ in §4.4.

### 4.1. Diagonals in $\mathbb{R}^4$.

The fiber of the normal bundles in §3.3 are defined by the subspace of mass-centered points in $\mathrm{Sym}^\kappa(\mathbb{R}^4)$. Hence, for any subset $P$ of $N_\ell$, define

$$z_P:\oplus_{i\in P}\mathbb{R}^4\to\mathbb{R}^4,\quad z_P((v_i)_{i\in P})=\sum_i v_i.$$

and

$$Z_P=z_P^{-1}(0)\subset\oplus_{i\in P}\mathbb{R}^4. \tag{4.3}$$

We will write $z_\ell$ for $z_{N_\ell}$. We define the *cone point*

$$c_P\in Z_P \tag{4.4}$$

to be the zero vector in $\oplus_{i\in P}\mathbb{R}^4$. The fiber of $\tilde\nu(X^\ell,\mathcal{P})$ is then given by (compare (3.10))

$$\prod_{P\in\mathcal{P}} Z_P. \tag{4.5}$$

For $\mathcal{P}<\mathcal{P}'$ partitions of $N_\ell$, define a partition of each $P\in\mathcal{P}$ by

$$\mathcal{P}'_P=\{P'\in\mathcal{P}': P'\subseteq P\}. \tag{4.6}$$

If $P\in\mathcal{P}\cap\mathcal{P}'$, then $\mathcal{P}'_P=\{P\}$. If we define

$$\Delta^\circ(Z_P,\mathcal{P}'_P)=\{(v_i)_{i\in P}\in Z_P : v_i=v_j \text{ if and only if there is } P'\in\mathcal{P}'_P \text{ with } i,j\in P'\}, \tag{4.7}$$

then the fiber of the bundle $\tilde\nu(X^\ell,\mathcal{P},\mathcal{P}')$ of equation (3.11) is given by:

$$\prod_{P\in\mathcal{P}}\Delta^\circ(Z_P,\mathcal{P}'_P). \tag{4.8}$$

Note that if $\mathcal{P}'_P$ is given by one set, $P$, as is the case if $P\in\mathcal{P}\cap\mathcal{P}'$, then $\Delta^\circ(Z_P,\mathcal{P}'_P)$ is a single point, given by $|P|$ copies of the zero vector in $\mathbb{R}^4$. The following lemma describes the normal bundle of $\Delta^\circ(Z_P,\mathcal{P}'_P)\subset z_P^{-1}(0)$.

**Lemma 4.1.** *Let $\mathcal{P}'_P$ be a partition of $P$. Then the restriction of the map*

$$e(Z_P,\mathcal{P}'_P):\Delta^\circ(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P} Z_{P'}\to Z_P \tag{4.9}$$

*defined by*

$$\Big((v_i)_{i\in P},(z_j)_{j\in P',P'\in\mathcal{P}'_P}\Big)\to(v_i+z_i)_{i\in P} \tag{4.10}$$

*to a neighborhood $\mathcal{O}(Z_P,\mathcal{P}'_P)$ of the subspace*

$$\Delta^\circ(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}\{c_{P'}\}$$

*is a homeomorphism onto its image.*

*Proof.* We observe the map (4.10) can be inverted by the center of mass map

$$\begin{aligned}(w_i)_{i\in P}&\to\Big((v_i)_{i\in P},(z_i)_{i\in P',P'\in\mathcal{P}'_P}\Big)\\ v_i&=\frac{1}{|P'|}\sum_{j\in P'}w_j,\quad\text{for } i\in P',\\ z_j&=w_j-v_j.\end{aligned} \tag{4.11}$$

(Note that the above map is an inverse because $\sum_{j\in P'}z_j=0$ by the definition of $Z_{P'}$.) However, the map (4.11) only takes values in

$$\Delta(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}Z_{P'}$$

unless we restrict to a sufficiently small neighborhood $\mathcal{O}(Z_P,\mathcal{P}'_P)$. □

Because $Z_P$ is a normal topological space, we can find open neighborhoods $U(Z_P,\mathcal{P}'_P)$ of $\Delta^\circ(Z_P,\mathcal{P}'_P)$ for every partition $\mathcal{P}'_P$ of $P$ such that $U(Z_P,\mathcal{P}'_P)\cap U(Z_P,\mathcal{P}''_P)$ is empty unless $\mathcal{P}'_P<\mathcal{P}''_P$ or $\mathcal{P}''_P<\mathcal{P}'_P$. The normal bundle structure given by the map (4.10) then defines a tubular neighborhood projection,

$$\pi(Z_P,\mathcal{P}'_P):U(Z_P,\mathcal{P}'_P)\to\Delta^\circ(Z_P,\mathcal{P}'_P), \tag{4.12}$$

given by the $v_i$ component of the map (4.11):

$$\begin{aligned}\pi(Z_P,\mathcal{P}'_P)\left((w_i)_{i\in P}\right)&=(v_i)_{i\in P},\\ \text{where for } i\in P'\in\mathcal{P}'_P,\ v_i&=\frac{1}{|P'|}\sum_{j\in P'}w_j.\end{aligned} \tag{4.13}$$

The following lemma shows that the maps $\pi(Z_P,\mathcal{P}'_P)$ satisfy the first equality of (4.1).

**Lemma 4.2.** *If $\mathcal{P}'_P<\mathcal{P}''_P$ are partitions of $P$, then on the intersection $U(Z_P,\mathcal{P}'_P)\cap U(Z_P,\mathcal{P}''_P)$, the equality*

$$\pi(Z_P,\mathcal{P}'_P)\circ\pi(Z_P,\mathcal{P}''_P)=\pi(Z_P,\mathcal{P}'_P) \tag{4.14}$$

*holds.*

*Proof.* The lemma is immediate from the definition (4.13). □

We use the above set of consistent tubular neighborhoods to define a scale function which is constant on the fibers of the maps $\pi(Z_P,\mathcal{P}'_P)$.

**Lemma 4.3.** *There is a smooth function $t(Z_P):Z_P\to[0,\infty)$ and open neighborhoods $U'(Z_P,\mathcal{P}'_P)\Subset U(Z_p,\mathcal{P}'_P)$ of the diagonals $\Delta^\circ(Z_P,\mathcal{P}'_P)$ satisfying*

1. *If $c_P\in Z_P$ is the cone point defined in (4.4), $t(Z_P)^{-1}(0)=c_P$,*
2. *For every partition $\mathcal{P}'_P$ of $P$, $t(Z_P)\circ\pi(Z_P,\mathcal{P}'_P)=t(Z_P)$, on $U'(Z_P,\mathcal{P}'_P)$*
3. *The function is invariant with respect to the permutation action of $\mathfrak{S}_{|P|}$ and the diagonal rotation action of $\mathrm{SO}(4)$ on $Z_P$.*

*Proof.* By induction on $|P|$, we can assume the functions $t(Z_{P'})$ are already defined if $|P'|<|P|$. (For $|P|=1$, $Z_P=\{c_P\}$ and $t(Z_P)$ must then be the zero function.) If

$$\pi_{P'}:\Delta^\circ(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}Z_{P'}\to Z_{P'}$$

is the projection map, then the map $t(Z_P,\mathcal{P}'_P):U(Z_P,\mathcal{P}'_P)\to[0,\infty)$ defined by

$$t(Z_P,\mathcal{P}'_P)=\left(\sum_{P'\in\mathcal{P}'_P}|P'|(t(Z_{P'})\circ\pi_{P'})^2\right)\circ e(Z_P,\mathcal{P}'_P)^{-1},$$

where $e(Z_P,\mathcal{P}'_P)$ is defined in (4.9), is a tubular distance function. By rescaling $t(Z_P,\mathcal{P}'_P)$ we can obtain a tubular distance function $t_r(Z_P,\mathcal{P}'_P):U(Z_P,\mathcal{P}'_p)\to[0,\infty)$ satisfying $t_r(Z_P,\mathcal{P}'_P)^{-1}(0)=\Delta^\circ(Z_P,\mathcal{P}'_P)$, and $t_r(Z_P,\mathcal{P}'_P)^{-1}([0,\frac12))=U'(Z_P,\mathcal{P}'_P)\Subset U(Z_P,\mathcal{P}'_P)$.

Enumerate the partitions $\mathcal{P}'_P$ of $P$ as $\mathcal{P}_1,\dots,\mathcal{P}_r$ such that if $\mathcal{P}_i<\mathcal{P}_j$, then $i<j$. Let $\beta:[0,1]\to[0,1]$ be a smooth function with $\beta([0,\frac14])=1$ and $\beta([\frac12,1])=0$. If we define $\beta_i=\beta\circ t_r(Z_P,\mathcal{P}'_i)$, then $U'(Z_P,\mathcal{P}_i)\subset\beta_i^{-1}(1)$, and $\beta_i$ is compactly supported on $\mathcal{U}(Z_P,\mathcal{P}_i)$.

We construct functions $t_i:Z_P\to[0,\infty)$ using upwards induction on $i$. Begin with the standard scale, $t_0((v_i)_{i\in P})=\sum_i|v_i|^2$. Assume $t_{i-1}$ satisfies

$$t_{i-1}\circ\pi(Z_P,\mathcal{P}_j)=t_{i-1}\quad\text{for all } j<i \tag{4.15}$$

and the invariance conditions in the conclusion of the lemma. We construct $t_i$ by

$$t_i(\cdot)=\beta_i(\cdot)t_{i-1}\circ\pi(Z_P,\mathcal{P}_i)(\cdot)+(1-\beta_i(\cdot))\,t_{i-1}(\cdot). \tag{4.16}$$

Observe that by the inclusion $U'(Z_P,\mathcal{P}_i)\subset\beta_i^{-1}(1)$, $t_i=t_{i-1}\circ\pi(Z_P,\mathcal{P}_i)$ on $U'(Z,\mathcal{P}_i)$ and thus, because $\pi(Z_P,\mathcal{P}_i)\circ\pi(Z_P,\mathcal{P}_i)=\pi(Z_P,\mathcal{P}_i)$, $t_i\circ\pi(Z_P,\mathcal{P}_i)=t_i$ on $U'(Z,\mathcal{P}_i)$ on $U'(Z_P,\mathcal{P}_i)$. By construction, $t_i$ is $\mathrm{SO}(4)$-invariant. We make $t_i$ $\mathfrak{S}(\mathcal{P})$-invariant by defining all $t_{i'}$ for $\mathcal{P}_{i'}$ conjugate to $\mathcal{P}_i$ by the obvious conjugation. Finally, we observe that for $j < i$, the equality

$$\pi(Z_P, \mathcal{P}_j) \circ \pi(Z_P, \mathcal{P}_i) = \pi(Z_P, \mathcal{P}_j)$$

(see (4.14)) and the inductive assumption (4.15) imply that on $U'(Z_P, \mathcal{P}_j)$,

$$t_{i-1} \circ \pi(Z_P, \mathcal{P}_i) = t_{i-1} \circ \pi(Z_P, \mathcal{P}_j) \circ \pi(Z_P, \mathcal{P}_i) = t_{i-1} \circ \pi(Z_P, \mathcal{P}_j) = t_{i-1}.$$

The interpolation between $t_{i-1}$ and $t_{i-1} \circ \pi(Z_P, \mathcal{P}_i)$ in (4.16) thus leaves $t_i$ equal to $t_{i-1}$ on $U'(Z_P, \mathcal{P}_j')$. Hence, for $j < i$,

$$t_i \circ \pi(Z_P, \mathcal{P}_j)|_{U'(Z_P,\mathcal{P}_j)} = t_{i-1}\pi(Z_P, \mathcal{P}_j)|_{U'(Z_P,\mathcal{P}_j)} = t_{i-1}|_{U'(Z_P,\mathcal{P}_j)},$$

completing the induction. □

4.2. **Families of metrics.** A metric $g$ on $X$ determines a product metric on $X^\ell$ and thus exponential maps of the normal bundles of the diagonal. We will use a non-standard diffeomorphism of the normal bundle $\tilde\nu(X^\ell, \mathcal{P})$ with a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$ in $X^\ell$. This diffeomorphism will be defined by varying the metric on $X$ and thus the resulting product metric on $X^\ell$. We begin by noting that it is possible to define an exponential map with a varying metric.

**Lemma 4.4.** *If $g_{\mathcal{P}}$ is a smooth family of metrics on $X$ parameterized by $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then there is an open neighborhood $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \subset \tilde\nu(X^\ell, \mathcal{P})$ of the zero-section and a smooth embedding,*

$$e(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \subset \tilde\nu(X^\ell, \mathcal{P}) \to X^\ell, \tag{4.17}$$

*parameterizing an open neighborhood, $\tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}})$, of $\Delta^\circ(X^\ell, \mathcal{P})$ in $X^\ell$ such that the restriction of the map $e(X^\ell, g_{\mathcal{P}})$ to a fiber $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}})|_{\mathbf{x}}$ is given by the exponential map defined by the $\ell$-fold product of the metric $g_{\mathcal{P},\mathbf{x}}$.*

*Proof.* For any metric $g'$ on $X$, the derivative of the exponential map at a point in the zero-section is the identity. Thus, the derivative of this exponential map with varying product metric is also the identity and this map defines an embedding of a neighborhood of the zero-section. □

The image of the exponential map in (4.17) is a tubular neighborhood. Such a tubular neighborhood, $\tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \subset X^\ell$, defines a projection $\pi(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \to \Delta^\circ(X^\ell, \mathcal{P})$ by the composition of $e(X^\ell, g_{\mathcal{P}})^{-1}$ and the projection $\tilde\nu(X^\ell, \mathcal{P}) \to \Delta^0(X, \mathcal{P})$.

For any smooth n-manifold $M$, let $\mathrm{Fr}_{\mathrm{GL}}(TM)$ denote the $\mathrm{GL}(n)$-bundle of frames of tangent vectors. Define $\mathrm{Fr}_{\mathrm{Gl}}(TX^\ell, \mathcal{P})$ as:

$$\mathrm{Fr}_{\mathrm{GL}}(TX^\ell, \mathcal{P}) = \{(F_1, \dots, F_\ell) \in \mathrm{Fr}(TX^\ell)|_{\Delta^\circ(X^\ell,\mathcal{P})} : F_i = F_j \text{ for all } i, j \in P \in \mathcal{P}\}.$$

If $g_{\mathcal{P}}$ is a family of metrics parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$, then we define $\mathrm{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ to be the subbundle of $\mathrm{Fr}_{\mathrm{GL}}(TX^\ell, \mathcal{P})$ of frames such that if $(F_1, \dots, F_\ell) \in \mathrm{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})|_{\mathbf{x}}$ then $F_i$ is orthonormal with respect to $g_{\mathcal{P},\mathbf{x}}$. The structure group of $\mathrm{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ is

$$G(T, \mathcal{P}) = \mathrm{SO}(4)^{r(\mathcal{P})}, \quad \text{where } \mathcal{P} = \{P_1, \dots, P_{r(\mathcal{P})}\}.$$

Then we have the identity:

$$\tilde\nu(X^\ell, \mathcal{P}) = \mathrm{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T,\mathcal{P})} \prod_{P\in\mathcal{P}} Z_P. \tag{4.18}$$

Then Lemma 3.13 implies the following.

**Lemma 4.5.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Then, the restriction of the exponential map $e(X^\ell, g_{\mathcal{P}})$ to*

$$\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \cap \left( \operatorname{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T,\mathcal{P})} \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P) \right), \tag{4.19}$$

*takes values in $\Delta(X, \mathcal{P}')$.*

Define:

$$G(T, \mathcal{P}'_P) = \prod_{P' \in \mathcal{P}'_P} \mathrm{SO}(4).$$

Then, we have:

**Lemma 4.6.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Let $\tilde{\mathcal{U}}(X, \mathcal{P}, \mathcal{P}') \subset \Delta^\circ(X^\ell, \mathcal{P}')$ be the image of the restriction of the exponential map in (4.19). Let $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ be smooth families of metrics parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$ and $\Delta^\circ(X, \mathcal{P}')$ respectively. Assume that $g_{\mathcal{P}', \mathbf{x}'} = g_{\mathcal{P}, \mathbf{x}}$ if $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{x}') = \mathbf{x}$. Then,*

$$\operatorname{Fr}(TX^\ell, \mathcal{P}', g_{\mathcal{P}'})|_{\tilde{\mathcal{U}}(X,\mathcal{P},\mathcal{P}')} \simeq \operatorname{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T,\mathcal{P})} \prod_{P \in \mathcal{P}} \left( \Delta^\circ(Z_P, \mathcal{P}'_P) \times G(T, \mathcal{P}'_P) \right). \tag{4.20}$$

*Proof.* By Lemma 4.5, a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$ in $\Delta^\circ(X^\ell, \mathcal{P}')$ is diffeomorphic to a neighborhood of the zero-section in

$$\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \cong \operatorname{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T,\mathcal{P})} \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P) \tag{4.21}$$

by the exponential map $e(X^\ell, g_{\mathcal{P}})$. The isomorphism (4.20) is then defined by the obvious parallel translation of the frames in $\operatorname{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ to the point in $\tilde{\mathcal{U}}(X^\ell, \mathcal{P}, \mathcal{P}')$ given by the data in (4.21). □

4.3. **Overlap maps.** Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. To define tubular neighborhoods and projection maps which satisfy the conditions (4.1), we define a space to control the overlap of the tubular neighborhoods $\mathcal{U}(X, \mathcal{P}, g_{\mathcal{P}})$ and $\mathcal{U}(X, \mathcal{P}', g_{\mathcal{P}'})$ as follows.

$$\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) = \operatorname{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T,\mathcal{P})} \prod_{P \in \mathcal{P}} \left( \Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \right). \tag{4.22}$$

We define an open subspace $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}')$ of $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}')$ and maps $\rho^{X,u}_{\mathcal{P},\mathcal{P}'}$ and $\rho^{X,d}_{\mathcal{P},\mathcal{P}'}$ so that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) & \xrightarrow{\rho^{X,u}_{\mathcal{P},\mathcal{P}'}} & \tilde{\mathcal{O}}(X^\ell, \mathcal{P}', g_{\mathcal{P}'}) \\ {\scriptstyle \rho^{X,d}_{\mathcal{P},\mathcal{P}'}} \downarrow & & \downarrow {\scriptstyle e(X^\ell, g_{\mathcal{P}'})} \\ \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) & \xrightarrow{e(X^\ell, g_{\mathcal{P}'})} & X^\ell \end{array} \tag{4.23}$$

commutes and controls the overlaps of the images of the exponential maps $e(X^\ell, g_{\mathcal{P}})$ and $e(X^\ell, g_{\mathcal{P}'})$.

4.3.1. *The downwards overlap map.* The downwards overlap map,

$$\rho^{X,d}_{\mathcal{P},\mathcal{P}'} : \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \to \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \tag{4.24}$$

is given by the fiberwise inclusions of the map of equation (4.9),

$$\Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \to Z_P.$$

Because this map is SO(4) equivariant, it extends to the domain given in (4.24). The open subspace $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ must satisfy

$$\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq (\rho^{X,d}_{\mathcal{P},\mathcal{P}'})^{-1} \left( \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \right) \tag{4.25}$$

for the composition in the diagram (4.23) to be defined.

4.3.2. *The upwards overlap map.* We define the upwards overlap map,

$$\rho^{X,u}_{\mathcal{P},\mathcal{P}'} : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \to \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, g_{\mathcal{P}}) \tag{4.26}$$

as follows. Let

$$\pi_1 : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \to \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \tag{4.27}$$

be the projection implicit in (4.21) and (4.22). We assume the open set $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ satisfies

$$\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq \pi_1^{-1} \left( \tilde{\mathcal{O}}(X^\ell, \mathcal{P} \to \mathcal{P}') \right). \tag{4.28}$$

Then, the composition $e(X^\ell, g_{\mathcal{P}}) \circ \pi_1$ is defined on $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ and takes values in $\Delta^\circ(X^\ell, \mathcal{P}')$.

We now define the map $\rho^{X,u}_{\mathcal{P},\mathcal{P}'}$. We make the assumption on the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ appearing in Lemma 4.6. Given a point in $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$, lying over $\mathbf{y} = (y_P) \in \Delta^\circ(X^\ell, \mathcal{P})$, define a point $\mathbf{y}' = (y_{P'})_{P' \in \mathcal{P}'} \in \Delta^\circ(X^\ell, \mathcal{P}')$ by the composition

$$e(X^\ell, g_{\mathcal{P}}) \circ \pi_1 : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \to \Delta^\circ(X^\ell, \mathcal{P}').$$

Then, for $P' \in \mathcal{P}'_P$ (where $P \in \mathcal{P}$), define $F_{P'}$ to be the parallel translate of the frame $F_P \in \mathrm{Fr}(TX)|_{y_P}$ from $y_P$ to $y'_{P'}$ using the Levi-Civita connection defined by the metric $g_{\mathcal{P},\mathbf{y}}$. Leave the data in $Z_{P'}$ alone. The result is an element of

$$\mathrm{Fr}(TX^\ell, sP', g_{\mathcal{P}'})|_{\mathbf{y}'} \times_{G(T,\mathcal{P}')} \prod_{P' \in \mathcal{P}'} Z_{P'} = \tilde{\nu}(X^\ell, \mathcal{P}', g_{\mathcal{P}'})|_{\mathbf{y}'},$$

thus defining the map $\rho^{X,u}_{\mathcal{P},\mathcal{P}'}$.

4.3.3. *Commuting overlaps maps.* We now define conditions on the metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ which ensure that the diagram (4.23) commutes.The diagram will only commute if we chose the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ in such a way as to eliminate holonomy. To do this, we introduce the notion of a locally flattened metric.

**Definition 4.7.** If $A \subset X^\ell$, then the *support* of $A$ in $X$ is the union of the images of $A$ under the projection maps $X^\ell \to X$.

Let $g_{\mathcal{P}}$ be a smooth family of metrics on $X$ parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$. Let $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}})$ be a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$. The family of metrics $g_{\mathcal{P}}$ is *locally flat with respect to* $\tilde{\mathcal{U}}'$ *at* $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$ if the metric $g_{\mathcal{P},\mathbf{x}}$ is flat on the support of $\tilde{\mathcal{U}}' \cap \pi_{\mathcal{P}}^{-1}(\mathbf{x})$ in $X$. The family of metrics is *locally flat with respect to* $\tilde{\mathcal{U}}'$ if this holds for all $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$.

The following lemma shows that the diagram (4.23) commutes when the metrics are flat in the sense of the preceding definition.

**Lemma 4.8.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Assume that the smooth families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ satisfy*

(1) *The metrics $g_{\mathcal{P}}$ are flat with respect to a neighborhood $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}})$,*
(2) *For $\mathbf{y}' \in \tilde{\mathcal{U}}(X^\ell, \mathcal{P}, g_{\mathcal{P}'}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ with $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}') = \mathbf{y}$, the metrics satisfy $g_{\mathcal{P},\mathbf{y}} = g_{\mathcal{P}',\mathbf{y}'}$.*

*Then there is a neighborhood $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ of the zero-section of the bundle* (4.22) *such that the diagram* (4.23) *commutes when restricted to $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$.*

*Proof.* We observe that for the compositions,

$$e(X^\ell, g_{\mathcal{P}}) \circ \rho^{X,d}_{\mathcal{P},\mathcal{P}'} \quad \text{and} \quad e(X^\ell, g_{\mathcal{P}'}) \circ \rho^{X,u}_{\mathcal{P},\mathcal{P}'}$$

to be defined, the neighborhood $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ must satisfy the inclusion relations (4.25) and (4.28). However, we replace the condition (4.25) with the stronger requirement that

$$\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq (e(X^\ell, g_{\mathcal{P}}) \circ \rho^{X,d}_{\mathcal{P},\mathcal{P}'})^{-1}(\tilde{\mathcal{U}}'). \tag{4.29}$$

On any open subset satisfying the above condition, the composition

$$e(X^\ell, g_{\mathcal{P}}) \circ \rho^{X,d}_{\mathcal{P},\mathcal{P}'}$$

is defined by the exponential map of the metric $g_{\mathcal{P},\mathbf{y}}$ of the vectors obtained by the appropriate adding of the vectors in the fiber of $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}'})$. The composition

$$e(X^\ell, g_{\mathcal{P}'}) \circ \rho^{X,u}_{\mathcal{P},\mathcal{P}'}$$

is defined by first parallel translating the vectors defined by the elements of $Z_{\mathcal{P}'}$ from $\mathbf{y}$ to the nearby points in the support of $\mathbf{y}'$, and then exponentiating. Because, by (4.29) and the first assumption on the metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$, the $g_{\mathcal{P},\mathbf{y}}$ is flat where this parallel translation takes place, the two compositions are equal. □

4.4. **Constructing the Thom-Mather maps.** We now show how to construct the families of metrics which satisfy the conditions of Lemma 4.8 and thus yield projection maps $\pi(X^\ell, g_{\mathcal{P}})$ which satisfy the first condition of (4.1). The construction of tubular distance functions is then quite straightforward.

4.4.1. *The projection maps.* The maps $\pi(X^\ell, g_{\mathcal{P}})$ were defined in the previous section. For $\mathcal{P} < \mathcal{P}'$ partitions of $N_\ell$, the first Thom-Mather property

$$\pi(X^\ell, g_{\mathcal{P}}) \circ \pi(X^\ell, g_{\mathcal{P}'}) = \pi(X^\ell, g_{\mathcal{P}}) \tag{4.30}$$

will follow from the commutativity of the diagram (4.23). Explicitly:

**Lemma 4.9.** *Continue the hypotheses and notation of Lemma 4.8. Then, restricted to $\tilde{\mathcal{U}}' \cap \Delta^\circ(X^\ell, \mathcal{P}')$, the equality* (4.30) *holds.*

*Proof.* The subspace $\tilde{\mathcal{U}}' \cap \Delta^\circ(X^\ell, \mathcal{P}')$ is in the image of the composition $e(X^\ell, g_{\mathcal{P}}) \circ \rho^{X,d}_{\mathcal{P},\mathcal{P}'}$. The equality follows from the observation that the pullback of $\pi(X^\ell, g_{\mathcal{P}'})$ by $\rho^{X,d}_{\mathcal{P},\mathcal{P}'}$ is given by the projection $\pi_1$ of (4.27). □

Thus, we must go about constructing families of metrics $g_{\mathcal{P}}$ for each partition of $N_\ell$ which satisfy the conditions of Lemma 4.8. The first step is showing how to flatten a family of metrics.

**Lemma 4.10.** *If $g_{\mathcal{P}}$ is a family of metrics smoothly parameterized by $\Delta^\circ(X^\ell,\mathcal{P})$, then there is a family of smooth metrics $F(g_{\mathcal{P}})$ smoothly parameterized by $\Delta^\circ(X^\ell,\mathcal{P})$ such that*

1. *$F(g_{\mathcal{P}})$ is locally flat with respect to a tubular neighborhood $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$,*
2. *For every $\mathbf{x}\in\Delta^\circ(X^\ell,\mathcal{P})$, $F(g_{\mathcal{P}})_{\mathbf{x}}$ is equal to $g_{\mathcal{P},\mathbf{x}}$ on the complement of the support of $\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})\cap\pi(X^\ell,g_{\mathcal{P}})^{-1}(\mathbf{x})$,*
3. *$g_{\mathcal{P}}$ and $F(g_{\mathcal{P}})$ are $C^1$ close within support of $\tilde{\mathcal{U}}'$,*
4. *The exponential maps $e(X^\ell,g_{\mathcal{P}})$ and $e(X^\ell,F(g_{\mathcal{P}}))$ are equal.*
5. *If $g_{\mathcal{P}}$ is already flat with respect to $\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$ at $\mathbf{x}\in\Delta^\circ(X^\ell,\mathcal{P})$, then $F(g_{\mathcal{P},\mathbf{x}}) = g_{\mathcal{P},\mathbf{x}}$.*

*Proof.* Let $\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$ be the tubular neighborhood of $\Delta^\circ(X^\ell,\mathcal{P})$ defined by the family of metrics $g_{\mathcal{P}}$, so $e(X^\ell,g_{\mathcal{P}}) : \tilde{\mathcal{O}}(X^\ell,\mathcal{P},g_{\mathcal{P}}) \subset \tilde{\nu}(X^\ell,\mathcal{P}) \simeq \tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$. For $P\in\mathcal{P}$, let $s_P : \Delta^\circ(X^\ell,\mathcal{P})\to[0,\infty)$ be a smooth function such that

$$\{(v_1,\dots,v_\ell)\in\tilde{\nu}(X^\ell,\mathcal{P})|_{\mathbf{x}} : |v_i|\le 4s_P(\mathbf{x}) \quad \text{for all } i\in P \text{ and for all } P\in\mathcal{P}\}\subset\mathcal{O}(\mathcal{P}) \tag{4.31}$$

For $\mathbf{x}\in\Delta^\circ(\mathcal{P})$, let $(F_P)_{P\in\mathcal{P}}$ be frames in $\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}})$ lying over $\mathbf{x}$. Use the frame $F_P$ and the exponential map of the metric $g_{\mathcal{P},\mathbf{x}}$ to identify $B(0,4s_P(\mathbf{x}))\subset\mathbb{R}^4$ with $B_{g_{\mathcal{P},\mathbf{x}}}(x_P,4s_P(\mathbf{x}))$. Let $\delta_{x_P}$ denote the flat metric on $B_{g_{\mathcal{P},\mathbf{x}}}(x_P,4s_P(\mathbf{x}))$ given by the pushforward of the Euclidean metric by the exponential map. Let $\chi:\mathbb{R}\to\mathbb{R}$ satisfy: $\chi(x)=0$ for $x\le\frac{1}{2}$ and $\chi(x)=1$ for $x\ge1$. Then, we define $F(g)_{\mathcal{P},\mathbf{x}}$ by:

$$F(g)_{\mathcal{P},\mathbf{x}} = \begin{cases} g_{\mathcal{P},\mathbf{x}} & \text{on } X\setminus\cup_i B_{g_{\mathcal{P},\mathbf{x}}}(x_P,4s_P(\mathbf{x})),\\ \chi_{4s_P(\mathbf{x})}(|x|)g_{\mathcal{P},\mathbf{x}} + (1-\chi_{4s_P(\mathbf{x})}(|x|))\delta_{x_P} & \text{on } \Omega_{g_{\mathcal{P},\mathbf{x}}}(x_P,2s_P(\mathbf{x}),4s_P(\mathbf{x}))),\\ \delta_{x_P} & \text{on } B_{g_{\mathcal{P},\mathbf{x}}}(x_P,2s_P(\mathbf{x})) \end{cases} \tag{4.32}$$

where $x$ refers to the coordinates given by the $g_{\mathcal{P},\mathbf{x}}$ exponential map.

The resulting family $F(g)_{\mathcal{P}}$ is locally flat with respect to the tubular neighborhood $\tilde{\mathcal{U}}'$ defined by replacing $4s_P(\mathbf{x})$ with $2s_P(\mathbf{x})$ in (4.31).

Because the metric $g_{\mathcal{P},\mathbf{x}}$ has not been changed at $x_P$, the frame bundles do not change. If the metric $g_{\mathcal{P},\mathbf{x}}$ is flat on $B(x_P,4s_P(\mathbf{x}))$, then it is given by the Euclidean metric in exponential coordinates, by [26, Vol. II, Ch. 7, Thm. 11, p. 306]. Thus, $\delta_{x_P}=g_{\mathcal{P},\mathbf{x}}$ and the interpolation in (4.32) does not change the metric. □

We now construct the desired families of metrics.

**Lemma 4.11.** *For each partition $\mathcal{P}$ of $N_\ell$, there is a smooth family of metrics on $X$, $g_{\mathcal{P}}$, defining tubular neighborhoods $\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$ by their exponential maps, such that*

1. *The family $g_{\mathcal{P}}$ is locally flat with respect to a tubular neighborhood $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})\subset\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})$,*
2. *If $\mathcal{P}<\mathcal{P}'$, then for all $\mathbf{x}'\in\Delta^\circ(X^\ell,\mathcal{P}')\cap\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ $g_{\mathcal{P}',\mathbf{x}'}=g_{\mathcal{P},\mathbf{x}}$ where $\mathbf{x}=\pi(X^\ell,g_{\mathcal{P}})(\mathbf{x}')$.*

*Proof.* The proof is by induction on the partitions $\mathcal{P}$.

Fix a metric $g$ on $X$. To every partition $\mathcal{P}$ of $N_\ell$, define an initial family of metrics $g_{\mathcal{P}}$ by $g_{\mathcal{P},\mathbf{x}}=g$.

Let $\mathcal{P}_0$ be the crudest partition. Redefine $g_{\mathcal{P}_0}$ by applying the flattening procedure of Lemma 4.10. Then the redefined family of metrics $g_{\mathcal{P}_0}$ is locally flat with respect to the tubular neighborhood $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}_0,g_{\mathcal{P}_0})$ given in Lemma 4.10.

Assume families of metrics $g_{\mathcal{P}}$ and tubular neighborhoods $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ satisfying the conclusions of the Lemma have been defined for all $\mathcal{P}<\mathcal{P}'$. There are two families of metrics on $X$ parameterized by the intersection

$$\Delta^\circ(X^\ell,\mathcal{P}')\cap\mathcal{U}(X^\ell,\mathcal{P},g_{\mathcal{P}}). \tag{4.33}$$

There is the initial family $g_{\mathcal{P}'}$ and the family $\pi(X^\ell,g_{\mathcal{P}})^*g_{\mathcal{P}}$ which is defined by

$$\pi(X^\ell,g_{\mathcal{P}})^*g_{\mathcal{P},\mathbf{x}'}=g_{\mathcal{P},\mathbf{x}}\quad\text{where}\quad \pi(X^\ell,g_{\mathcal{P}})(\mathbf{x}')=\mathbf{x}.$$

Observe that, by shrinking the tubular neighborhoods if necessary, for $\mathcal{P}_1<\mathcal{P}'$ and $\mathcal{P}_2<\mathcal{P}'$ if the intersection

$$\Delta^\circ(X^\ell,\mathcal{P}')\cap\tilde{\mathcal{U}}(X^\ell,\mathcal{P}_1,g_{\mathcal{P}_1})\cap\tilde{\mathcal{U}}(X^\ell,\mathcal{P}_2,g_{\mathcal{P}_2}) \tag{4.34}$$

is non-empty, then there is a relation $\mathcal{P}_1<\mathcal{P}_2<\mathcal{P}'$. Then, the relations

$$\pi(X^\ell,\mathcal{P}_1)\circ\pi(X^\ell,\mathcal{P}_2)=\pi(X^\ell,\mathcal{P}_1)$$

(which follows by induction and Lemma 4.9) and

$$g_{\mathcal{P}_2}=\pi(X^\ell,\mathcal{P}_1)^*g_{\mathcal{P}_1}$$

imply that the families of metrics

$$\pi(X^\ell,g_{\mathcal{P}_1})^*g_{\mathcal{P}_1}\quad\text{and}\quad\pi(X^\ell,g_{\mathcal{P}_1})^*g_{\mathcal{P}_1} \tag{4.35}$$

are equal on (4.34). Thus we can form a new family of metrics by interpolating between $g_{\mathcal{P}'}$ on

$$\Delta^\circ(X,\mathcal{P}')-\cup_{\mathcal{P}<\mathcal{P}'}\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}}),$$

and the pullback metrics (4.35) . By further shrinking the tubular neighborhoods $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ for $\mathcal{P}<\mathcal{P}'$ and taking a sufficiently small neighborhood $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ we can assume this interpolated family is locally flat with respect to $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ on

$$\Delta^\circ(X^\ell,\mathcal{P}')\cap\left(\cup_{\mathcal{P}<\mathcal{P}'}\tilde{\mathcal{U}}(X^\ell,\mathcal{P},g_{\mathcal{P}})\right). \tag{4.36}$$

Then, apply the flattening procedure of Lemma 4.10 to this interpolated family. Because this family is already flat with respect to $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ for $\mathbf{x}$ in the open space (4.36), the flattening procedure does not change the interpolated family there. Thus, the new, flattened family satisfies both properties of the conclusion of the lemma, completing the induction. □

4.4.2. *The tubular distance function.* Given the family of metrics constructed in the previous section, the tubular distance functions $t(\mathcal{P})$ are defined by the composition of a bundle norm on $\tilde{\nu}(X^\ell,\mathcal{P})$ with the inverse of the exponential map $e(X^\ell,g_{\mathcal{P}})^{-1}$. These do not satisfy the second Thom-Mather identity, (4.1), immediately.

We define a new tubular distance function $t(X^\ell,\mathcal{P})$ and a rescaling of it $t_r(X^\ell,\mathcal{P})$ on a possibly smaller tubular neighborhoods. These new functions will satisfy (4.1).

**Lemma 4.12.** *For $\mathcal{P}$ a partition of $N_\ell$, let $g_{\mathcal{P}}$ be the smooth family of metrics parameterized by $\Delta^\circ(X^\ell,\mathcal{P})$ and let $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ be the tubular neighborhood of $\Delta^\circ(X^\ell,\mathcal{P})$ constructed in Lemma 4.11. Then there are tubular neighborhoods $\tilde{\mathcal{U}}''(X^\ell,\mathcal{P},g_{\mathcal{P}})$ and $\mathfrak{S}(\mathcal{P})$-invariant smooth functions*

$$t(X^\ell,\mathcal{P}):\tilde{\mathcal{U}}''(X^\ell,\mathcal{P},g_{\mathcal{P}})\to[0,1],\quad\textit{and}\quad t_r(X^\ell,\mathcal{P}):\tilde{\mathcal{U}}''(X^\ell,\mathcal{P},g_{\mathcal{P}})\to[0,1], \tag{4.37}$$

*such that*

1. *$t_r(X^\ell,\mathcal{P})$ is a rescaling of $t(X^\ell,\mathcal{P})$,*
2. *$t(X^\ell,\mathcal{P})^{-1}(0)=t_r(X^\ell,\mathcal{P})^{-1}(0)=\Delta^\circ(X^\ell,\mathcal{P})$,*

(3) $t_r(X^\ell,\mathcal{P})^{-1}([0,\frac{1}{2})) \Subset \tilde{\mathcal{U}}''(X^\ell,\mathcal{P},g_{\mathcal{P}})$
(4) *For all partitions $\mathcal{P}<\mathcal{P}'$, the equalities*

$$
\begin{aligned}
t(X^\ell,\mathcal{P})\circ\pi(X^\ell,\mathcal{P}',g_{\mathcal{P}'}) &= t(X^\ell,\mathcal{P}),\\
t_r(X^\ell,\mathcal{P})\circ\pi(X^\ell,\mathcal{P}',g_{\mathcal{P}'}) &= t_r(X^\ell,\mathcal{P}),
\end{aligned}
\tag{4.38}
$$

*hold on $\mathcal{U}''(X^\ell,g_{\mathcal{P}})\cap\mathcal{U}''(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$.*

*Proof.* Because the functions $t(Z_P):Z_P\to[0,\infty)$ defined in Lemma 4.3 are invariant under the SO(4) action on $Z_P$, the function $\sum_{P\in\mathcal{P}}|P|t(Z_P)^2$ is defined on

$$\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}}\times_{G(T,\mathcal{P})}\prod_{P\in\mathcal{P}}Z_P,$$

and hence

$$t(X^\ell,\mathcal{P}) = \left(\sum_{P\in\mathcal{P}}|P|t(Z_P)^2\right)\circ e(X^\ell,g_{\mathcal{P}})^{-1} \tag{4.39}$$

is defined on $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ and satisfies item (2) of the lemma. By definition, $t(X^\ell,\mathcal{P})$ is invariant under the action of $\mathfrak{S}(\mathcal{P})$. By the construction of the tubular neighborhoods $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})$ in Lemma 4.11, for $\mathcal{P}<\mathcal{P}'$ the intersection $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})\cap\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ is contained in the image of the neighborhood $\tilde{\mathcal{O}}(X^\ell,\mathcal{P},\mathcal{P}',g_{\mathcal{P}})$ under the maps in the diagram (4.23). Under the identification of $\tilde{\mathcal{U}}'(X^\ell,\mathcal{P},g_{\mathcal{P}})\cap\tilde{\mathcal{U}}'(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ with a subspace of

$$\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}})\times_{G(T,\mathcal{P})}\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}Z_{P'}\right)$$

(see (4.22)), the projection map $\pi(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$ is identified with the projection

$$
\begin{array}{c}
\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}})\times_{G(T,\mathcal{P})}\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_P,\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}Z_{P'}\right)\\
\downarrow\\
\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}})\times_{G(T,\mathcal{P})}\prod_{P\in\mathcal{P}}\Delta^\circ(Z_P,\mathcal{P}'_P)
\end{array}
$$

By item (2) in Lemma 4.3, by the definition of the map $\rho^{X,d}_{\mathcal{P},\mathcal{P}'}$, and by the definition of $t(X^\ell,\mathcal{P})$ in (4.39) as a sum of the functions $t(Z_P)$, the function $t(X^\ell,\mathcal{P})$ satisfies (4.38).

We define $t_r(X^\ell,\mathcal{P})$ to be a rescaling of $t(X^\ell,\mathcal{P})$. The function $f:\Delta^\circ(X^\ell,\mathcal{P})$ used to rescale $t(X^\ell,\mathcal{P})$ is chosen to ensure item (3) holds. Then, the equality

$$t_r(X^\ell,\mathcal{P})(\cdot) = t(X^\ell,\mathcal{P})(\cdot)f(\pi(X^\ell,\mathcal{P},g_{\mathcal{P}})(\cdot)),$$

and the equality

$$f\circ\pi(X^\ell,\mathcal{P},g_{\mathcal{P}}) = f\circ\pi(X^\ell,\mathcal{P},g_{\mathcal{P}})\circ\pi(X^\ell,\mathcal{P}',g_{\mathcal{P}'})$$

(which follows from (4.30)), and the fact that (4.38) holds for $t(X^\ell,\mathcal{P})$ ensure that $t_r(X^\ell,\mathcal{P})$ also satisfies (4.38). □

**Remark 4.13.** All the constructions of this section can be made $\mathfrak{S}_\ell$ equivariant and thus descend to the symmetric product $\mathrm{Sym}^\ell(X)$.

## 5. The instanton moduli space with spliced ends

The first step in constructing the space of global splicing data is to define the moduli space with spliced ends. This moduli space will be identical to $\bar{M}^{s,\natural}_\kappa(\delta)$, the Uhlenbeck compactification of the mass-centered, charge $\kappa$, framed instantons on $\mathbb{R}^4$ with scale less than $\delta$, except on an Uhlenbeck neighborhood of the punctured, centered symmetric product,

$$[\Theta] \times \left(\mathrm{Sym}^{\kappa,\natural}_\delta(\mathbb{R}^4) \setminus c_\kappa\right) \tag{5.1}$$

where $\Theta$ is the trivial connection, $\mathrm{Sym}^{\kappa,\natural}_\delta(\mathbb{R}^4)$ is the subspace of mass-centered points in the $\kappa$-th symmetric product of $\mathbb{R}^4$ with a scale constraint (see (5.4)), and $c_\kappa \in \mathrm{Sym}^\kappa(\mathbb{R}^4)$ is the point totally concentrated at the origin.

A neighborhood of the strata (5.1) in $\bar{M}^{s,\natural}_\kappa(\delta)$ is parameterized by the union of the images of the gluing maps $\boldsymbol{\gamma}_{\Theta,\Sigma}$, as $\Sigma$ varies over the strata in (5.1). Roughly speaking, if the stratum $\Sigma$ of $\mathrm{Sym}^\kappa(\mathbb{R}^4)$ is given by the partition $\kappa = \kappa_1 + \cdots + \kappa_r$, and $\mathcal{P}$ is a partition of $N_\kappa$ corresponding to $\Sigma$ then the gluing map is an smoothly stratified embedding

$$\boldsymbol{\gamma}_{\Theta,\mathcal{P}} : \Sigma \times \prod_{i=1}^r \bar{M}^{s,\natural}_{\kappa_i}(\delta_i) \to \bar{M}^{s,\natural}_\kappa(\delta).$$

The scale parameters on the moduli spaces $\bar{M}^{s,\natural}_{\kappa_i}(\delta)$ make the image of $\boldsymbol{\gamma}_{\Theta,\Sigma}$ into a kind of tubular neighborhood of $\{[\Theta]\} \times \Sigma$ in $\bar{M}^{s,\natural}_\kappa(\delta)$. The moduli space with spliced ends will be defined, roughly speaking, by replacing the image of $\boldsymbol{\gamma}_{\Theta,\Sigma}$ with the image of the splicing map $\boldsymbol{\gamma}'_{\Theta,\Sigma}$ on the interior of this tubular neighborhood. We summarize the properties of the spliced-ends moduli space in the following theorem.

**Theorem 5.1.** *Let $\bar{\mathcal{B}}^s_\kappa(\delta)$ be the Uhlenbeck extension of the framed quotient space defined in* (5.17). *There exists a smoothly-stratified moduli space with spliced ends, $\bar{M}^{s,\natural}_{spl,\kappa}(\delta) \subset \bar{\mathcal{B}}^s_\kappa(2\delta)$, closed under the* $\mathrm{SO}(3)\times\mathrm{SO}(4)$ *action, satisfying the following properties*

(1) *There is a smoothly-stratified,* $\mathrm{SO}(3)\times\mathrm{SO}(4)$*-equivariant, homeomorphism, $\bar{M}^{s,\natural}_{spl,\kappa}(\delta) \cong \bar{M}^{s,\natural}_\kappa(\delta)$, which is the identity on the trivial strata,*

$$[\Theta] \times \mathrm{Sym}^{\kappa,\natural}_\delta(\mathbb{R}^4),$$

*where $\mathrm{Sym}^{\kappa,\natural}_\delta(\mathbb{R}^4) \subset \mathrm{Sym}^\kappa(\mathbb{R}^4)$ is defined in* (5.4).

(2) *There is an Uhlenbeck neighborhood $W_\kappa$ of the punctured centered symmetric product,* (5.1)*, in $\bar{\mathcal{B}}^s_\kappa(2\delta)$ such that*

$$\bar{M}^{s,\natural}_{spl,\kappa}(\delta) \setminus W_\kappa = \bar{M}^{s,\natural}_\kappa(\delta) \setminus W_\kappa.$$

(3) *To each stratum $\{[\Theta]\}\times\Sigma$ in the punctured, centered symmetric product,* (5.1)*, there is an Uhlenbeck neighborhood $W(\Sigma) \subset \bar{\mathcal{B}}^s_\kappa(\delta)$ of the trivial stratum*

$$[\Theta] \times \Sigma, \tag{5.2}$$

*and an open neighborhood $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},\delta) \subset \nu(\Theta,\mathcal{P},\delta)$ of the trivial stratum* (5.53) *in the domain of the splicing map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ defined in* (5.19)*, such that*

$$\bar{M}^{s,\natural}_{spl,\kappa}(\delta) \cap W(\Sigma) = \boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},\delta)\right). \tag{5.3}$$

(4) *For all $[A,\mathbf{x}] \in \bar{M}^{s,\natural}_{spl,\kappa}$, $\|F^+_A\|_{L^{\sharp,2}} \le M$ where $M$ is the constant appearing in* [7, Proposition 7.6].

We construct $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$ using induction on $\kappa$. We first define the *spliced end* of the moduli space. The spliced end will be an Uhlenbeck neighborhood of the punctured, centered symmetric product in $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$. It is defined as the union of the images of splicing maps $\gamma'_{\Theta,\Sigma}$ where $\Sigma$ is a stratum in $\mathrm{Sym}^{\kappa,\natural}_{\delta}(\mathbb{R}^4) - \{c_\kappa\}$ and where the domain of the splicing map is not defined by the lower charge moduli spaces $\bar{M}^{s,\natural}_{\kappa_i}(\delta_i)$ but rather by the lower charge *spliced ends moduli spaces* $\bar{M}^{s,\natural}_{\kappa_i}(\delta_i)(\delta_i)$. A technical result on the equality of two splicing maps, Proposition 5.12 controls the overlaps of the images of the different splicing maps, proving that this union is a smoothly stratified space.

In §5.1 we introduce some vocabulary to describe the trivial strata. In §5.2, we define the relevant splicing maps. In §5.3, we prove the technical result Proposition 5.12 controlling the overlap of two splicing maps. Then in §5.4, we finish the first stage of the proof of Theorem 5.1 by constructing the spliced end in Proposition 5.13.

The second stage of the proof of Theorem 5.1 appears in §5.6. There, we show how to isotope the complement of a neighborhood of the punctured, centered symmetric product in the spliced end to the actual moduli space $\bar{M}^{s,\natural}_{\kappa}(\delta)$. This isotopy is defined by the composition of the isotopy defining the gluing map and the isotopy defined by the centering map. That such an isotopy exists follows immediately from the estimates on $\|F^+_A\|_{L^{\sharp,2}}$ for $A$ in the spliced end; most of the work in §5.6 lies in constructing the parameter on the spliced end for the isotopy. The existence of this parameter is established in §5.5 by constructing a partial Thom-Mather structure on the spliced end.

5.1. **The trivial strata.** We begin with some vocabulary, mostly adapted from §3 to describe the trivial strata (5.1).

The symmetric product, $\mathrm{Sym}^{\kappa,\natural}(\mathbb{R}^4)$ is the quotient of the zero locus of the center of mass map $z_\kappa : \oplus_{i\in N_\kappa}\mathbb{R}^4 \to \mathbb{R}^4$ (defined following (4.3)) by the symmetric group:

$$\mathrm{Sym}^{\kappa,\natural}(\mathbb{R}^4) = z_\kappa^{-1}(0)/\mathfrak{S}_\kappa.$$

We shall abbreviate $z_\kappa^{-1}(0)$ by $Z_\kappa$. We define a scale for elements of $\mathrm{Sym}^{\kappa}(\mathbb{R}^4)$ by

$$\lambda([v_1,\dots,v_\kappa])^2 = \sum_i |v_i|^2, \tag{5.4}$$

Then, we define $Z_\kappa(\delta) = z_\kappa^{-1}(0)\cap\lambda^{-1}([0,\delta))$ and

$$\mathrm{Sym}^{\kappa,\natural}_{\delta}(\mathbb{R}^4) = Z_\kappa(\delta)/\mathfrak{S}_\kappa. \tag{5.5}$$

We describe the strata of $\mathrm{Sym}^{\kappa,\natural}_{\delta}(\mathbb{R}^4)$ by the quotients of the diagonals of $Z_\kappa(\delta)$. For any partition $\mathcal{P}$ of $N_\kappa$, define

$$\begin{aligned}\Delta^\circ(Z_\kappa(\delta),\mathcal{P}) &= \Delta^\circ(Z_\kappa,\mathcal{P})\cap Z_\kappa(\delta),\\ \Sigma(Z_\kappa(\delta),\mathcal{P}) &= \Delta^\circ(Z_\kappa(\delta),\mathcal{P})/\mathfrak{S}(\mathcal{P}) \subset \mathrm{Sym}^{\kappa,\natural}_{\delta}(\mathbb{R}^4)\end{aligned} \tag{5.6}$$

where $\Delta^\circ(Z_\kappa,\mathcal{P})$ is defined in (4.7). Let $\Sigma(Z_\kappa(\delta),\mathcal{P})$ be the image of $\Delta^\circ(Z_\kappa(\delta),\mathcal{P})$ under the projection $Z_\kappa(\delta)\to\mathrm{Sym}^{\kappa,\natural}_{\delta}(\mathbb{R}^4)$. We then have:

**Lemma 5.2.** *If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $N_\kappa$ and $\pi_\kappa : Z_\kappa \subset \oplus_{i\in N_\kappa}\mathbb{R}^4 \to \mathrm{Sym}^{\kappa,\natural}(\mathbb{R}^4)$ is the projection, then:*

1. $\Sigma(Z_\kappa(\delta),\mathcal{P}) = \Delta^\circ(Z_\kappa(\delta),\mathcal{P})/\mathfrak{S}(\mathcal{P})$,
2. $\pi_\kappa^{-1}(\Sigma(Z_\kappa(\delta),\mathcal{P})) = \sqcup_{\mathcal{P}'\in[\mathcal{P}]}\Delta^\circ(Z_\kappa(\delta),\mathcal{P}')$.
3. $\Sigma(Z_\kappa(\delta),\mathcal{P}) \subset \mathrm{cl}\,(\Sigma(Z_\kappa(\delta),\mathcal{P}'))$ *if and only if there is* $\mathcal{P}''\in[\mathcal{P}]$ *such that* $\mathcal{P}''<\mathcal{P}$.

If $(v_1,\dots,v_\kappa)\in\Delta^\circ(Z_\kappa,\mathcal{P})$ and $P\in\mathcal{P}$, then $v_i=v_j$ for $i,j\in P$. Thus, we will write $v_P:=v_i$ for any $i\in P$. With this notation, we will write $(y_1,\dots,y_\kappa)=(y_P)_{P\in\mathcal{P}}$ for an element of $\Delta^\circ(Z_\kappa,\mathcal{P})$.

5.1.1. *Tubular neighborhoods.* A tubular neighborhood of a diagonal in $Z_\kappa(\delta)$ can be obtained using the exponential map with respect to the product of the flat metrics on $\oplus_{i\in N_\kappa}\mathbb{R}^4$:

**Lemma 5.3.** *The normal bundle of the diagonal $\Delta^\circ(Z_\kappa(\delta),\mathcal{P})$ in $Z_\kappa$ is*

$$\begin{aligned}\nu(Z_\kappa(\delta),\mathcal{P})=\{((x_1,\dots,x_\kappa),(v_1,\dots,v_\kappa))\in\Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\oplus_{i\in N_\kappa}\mathbb{R}^4:\\ \text{for all } P\in\mathcal{P}\ \sum_{i\in P}v_i=0\}.\end{aligned}\tag{5.7}$$

*There is an open neighborhood, $\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P})$, of the zero section in $\nu(Z_\kappa(\delta),\mathcal{P})$ such that the restriction of the exponential map $e(Z_\kappa,\mathcal{P})$ defined by*

$$e(Z_\kappa,\mathcal{P})\left((x_1,\dots,x_\kappa),(v_1,\dots,v_\kappa)\right)=(x_1+v_1,\dots,x_\kappa+v_\kappa)\tag{5.8}$$

*to $\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P})$ is injective and $\mathfrak{S}(\mathcal{P})$ equivariant.*

Let $\tilde{\mathcal{U}}(Z_\kappa(\delta),\mathcal{P})$ be the image of $\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P})$ under the exponential map (5.8). By the $\mathfrak{S}(\mathcal{P})$ equivariance of the exponential map $e(Z_\kappa(\delta),\mathcal{P})$, there is a homeomorphism,

$$\mathcal{O}(Z_\kappa(\delta),\mathcal{P})\cong\mathcal{U}(Z_\kappa(\delta),\mathcal{P})\tag{5.9}$$

where $\mathcal{O}(Z_\kappa(\delta),\mathcal{P})=\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P})/\mathfrak{S}(\mathcal{P})$ and $\mathcal{U}(Z_\kappa(\delta),\mathcal{P})=\tilde{\mathcal{U}}(Z_\kappa(\delta),\mathcal{P})/\mathfrak{S}(\mathcal{P})$.

To describe the pre-image of $\Delta^\circ(Z_\kappa(\delta),\mathcal{P}')$ under $e(Z_\kappa,\mathcal{P})$ where $\mathcal{P}<\mathcal{P}'$, we introduce the following notation. First, for $P\subseteq N_\kappa$ let $Z_P\subset\oplus_{i\in P}\mathbb{R}^4$ be the mass-centered subspace defined in (4.3).

**Lemma 5.4.** *Let $\mathcal{P}<\mathcal{P}'$ be partitions of $N_\kappa$. For $P\in\mathcal{P}$, let $\mathcal{P}'_P$ be the partition of $P$ defined in* (4.6). *Then there is an inclusion*

$$\nu(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')=\Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\prod_{P\in\mathcal{P}}\Delta^\circ(Z_P,\mathcal{P}'_P)\to\nu(Z_\kappa(\delta),\mathcal{P})\tag{5.10}$$

*such that if we define*

$$\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')=\tilde{\mathcal{O}}(Z_\kappa(\delta)(\delta),\mathcal{P})\cap\nu(Z_\kappa(\delta),\mathcal{P},\mathcal{P}'),\tag{5.11}$$

*then $\tilde{\mathcal{O}}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')=e(Z_\kappa,\mathcal{P})^{-1}(\Delta^\circ(Z_\kappa(\delta),\mathcal{P}'))$.*

*Proof.* We observe that

$$\Delta^\circ(Z_P,\mathcal{P}'_P)=\{(v_i)_{i\in P}\in Z_P: v_i=v_j\quad\text{if and only if there is } P'\in\mathcal{P}'_P \text{ with } i,j\in P'\}.$$

Because $\mathcal{P}$ is a partition of $N_\kappa$, there is an isomorphism

$$\oplus_{P\in\mathcal{P}}\oplus_{i\in P}\mathbb{R}^4\simeq\oplus_{i\in N_\kappa}\mathbb{R}^4,$$

which induces an inclusion

$$\prod_{P\in\mathcal{P}}\Delta^\circ(Z_P,\mathcal{P}'_P)\to Z_\kappa\quad((v_i)_{i\in P,P\in\mathcal{P}})\mapsto(v_i)_{i\in N_\kappa}.\tag{5.12}$$

By the $\mathfrak{S}(\mathcal{P})$ equivariance of the exponential map $e(Z_\kappa,\mathcal{P})$, an element of the pre-image of $\Delta^\circ(Z_\kappa(\delta),\mathcal{P}')$ under $e(Z_\kappa,\mathcal{P})$ can be written as

$$((x_1,\dots,x_\kappa),(v_1,\dots,v_\kappa))\in\Delta^\circ(Z_\kappa,\mathcal{P})\times\oplus_{i\in N_\kappa}\mathbb{R}^4$$

where $(v_1,\dots,v_\kappa)\in\oplus_{i\in N_\kappa}\mathbb{R}^4$ satisfies

1. $\sum_i v_i=0$ for all $P\in\mathcal{P}$,
2. For all $i,j$, $v_i=v_j$ if and only if there is $P'\in\mathcal{P}'$ with $i,j\in P'$.

Then one observes that the elements $(v_1, \dots, v_\kappa) \in \oplus_{i \in N_\kappa} \mathbb{R}^4$ satisfying these two conditions are precisely the image of the inclusion (5.12). $\square$

By the second and third items of Lemma 5.2, to describe the end of $\Sigma(Z_\kappa(\delta), \mathcal{P}')$ near $\Sigma(Z_\kappa(\delta), \mathcal{P})$, we must describe not only the end of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$ near $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ as is done in Lemma 5.4, but also the ends of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}'')$ near $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ for all $\mathcal{P}'' \in [\mathcal{P}']$ with $\mathcal{P} < \mathcal{P}''$.

Thus, define

$$\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}']) = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \coprod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \left( \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}''_P) \right). \tag{5.13}$$

There is an action of $\mathfrak{S}(\mathcal{P})$ on $\nu(Z_\kappa, [\mathcal{P} < \mathcal{P}'])$ by the standard action on $\Delta^\circ(Z_\kappa, \mathcal{P})$ and the action of $\mathfrak{S}(\mathcal{P})$ on $[\mathcal{P} < \mathcal{P}']$. Although $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ is not a subset of $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$, we use the phrase *a neighborhood of* $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ *in* $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$ to refer to the obvious parallel to subspaces of $\nu(Z_\kappa(\delta), \mathcal{P})$.

**Lemma 5.5.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $N_\kappa$ with $\mathcal{P} < \mathcal{P}'$. Then a neighborhood of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma(Z_\kappa(\delta), \mathcal{P}')$ is homeomorphic to a neighborhood of the zero section in*

$$\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])/\mathfrak{S}(\mathcal{P}).$$

*Denote this neighborhood by $\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$.*

*Proof.* The bundle $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$ is the union of the bundles $\nu(Z_\kappa(\delta), \mathcal{P}, \mathcal{P}'')$ defined in (5.10) as $\mathcal{P}''$ varies in $[\mathcal{P} < \mathcal{P}']$ along the subspace $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ of each of these bundles. Thus a neighborhood of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ in

$$\cup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}'') \tag{5.14}$$

is homeomorphic to a neighborhood of the zero-section in $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$. By the characterization of the pre-image of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ under the projection $Z_\kappa(\delta) \to \mathrm{Sym}^{\natural,\kappa}_\delta(\mathbb{R}^4)$ in Item (2) of Lemma 5.2, a neighborhood of $\Sigma^\natural(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma^\natural(Z_\kappa(\delta), \mathcal{P}')$ is homeomorphic to the quotient of (5.14) by $\mathfrak{S}(\mathcal{P})$, proving the lemma. $\square$

5.2. **Splicing to the trivial on $\mathbb{R}^4$.** In this section, we define the basic operation of splicing as used in [9]. The main result is in Lemma 5.11 where we show that the composition of two splicing maps is equal to a single splicing map under suitably small open sets.

Let $\mathcal{B}_\kappa$ denote the quotient space of SU(2) connections on a bundle $E_\kappa \to S^4$ with $c_2(E_\kappa) = \kappa$. Let

$$\bar{\mathcal{B}}_{|P|} = \cup_{i=0}^{p} \left( \mathcal{B}_{|P|-i} \times \mathrm{Sym}^i(\mathbb{R}^4) \right),$$

denote the space of ideal connections on $S^4$, given the topology induced by Uhlenbeck convergence. We write elements of this space as $[A, \mathbf{x}]$ where $[A] \in \mathcal{B}_{|P|-\ell}$ and $\mathbf{x} \in \mathrm{Sym}^\ell(\mathbb{R}^4)$. We define the *center of mass* $z([A, \mathbf{x}])$, for $\mathbf{x} = (x_1, \dots, x_\ell)$ by:

$$z([A, \mathbf{x}]) = \frac{1}{8\pi^2 |P|} \int_{\mathbb{R}^4} y |F_A|^2 d^4 y + \sum_{i=1}^{\ell} x_i \tag{5.15}$$

and the *scale*, $\lambda([A, \mathbf{x}])$ by

$$\lambda([A, \mathbf{x}])^2 = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} |y - z([A, F^s])|^2 |F_A|^2 d^4 y + \sum_{i=1}^{\ell} |x_i|^2. \tag{5.16}$$

Define

$$\bar{\mathcal{B}}_{|P|}(\varepsilon) = \lambda^{-1}([0, \varepsilon]).$$

If $[A,\mathbf{x}] \in \bar{\mathcal{B}}_{|P|}(\varepsilon)$ and $\varepsilon << 1$, then the support of $\mathbf{x}$ is disjoint from the south pole. Thus, we can extend the framed quotient space $\mathcal{B}^s_{|P|}(\varepsilon) \to \mathcal{B}_{|P|}(\varepsilon)$ over the quotient space of ideal connections:

$$\bar{\mathcal{B}}^s_{|P|}(\varepsilon) = \lambda^{-1}([0,\varepsilon]) \cap \left( \sqcup_{i=0}^{p} \left( \mathcal{B}^s_{|P|-i} \times \mathrm{Sym}^i(\mathbb{R}^4) \right) \right). \tag{5.17}$$

We now define the operation of splicing connections to the trivial connection. Let $\mathcal{P}$ be a partition of $N_\kappa$. We define the splicing map on an open subspace of

$$\Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{\mathcal{B}}^{s,\natural}_{|P|}(\delta_P),$$

where the constants $\delta_P$ are invariant under the action of $\mathfrak{S}(\mathcal{P})$ and where $\mathfrak{S}(\mathcal{P})$ acts on the generalized connections by permuting them, sending and element of $\bar{\mathcal{B}}^{s,\natural}_{|P|}(\delta_P)$ to an element of $\bar{\mathcal{B}}^{s,\natural}_{|\sigma(P)|}(\delta_{|\sigma(P)|})$. Note that elements of $\Delta^\circ(Z_\kappa(\delta),\mathcal{P})$ can be written as $(y_P)_{P\in\mathcal{P}}$, where $y_P \in \mathbb{R}^4$ while elements of $\prod_{P\in\mathcal{P}} \bar{\mathcal{B}}^{s,\natural}_{|P|}(\delta_P)$ can be written as $([A_P, F^s_P, \mathbf{x}_P])_{P\in\mathcal{P}}$, where $[A_P, F^s_P, \mathbf{x}_P] \in \bar{\mathcal{B}}^{s,\natural}_{|P|}(\delta_P)$. The open subspace on which the splicing map will be defined is

$$\begin{aligned} \tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta) := \{ (y_P, [A_P, F^s_P, \mathbf{x}_P])_{P\in\mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta),\mathcal{P}) \times \prod_{P\in\mathcal{P}} \bar{\mathcal{B}}^s_{|P|}(\delta_P) : \\ \text{for all } P \neq P',\ 8\sqrt{\lambda([A_P,\mathbf{x}_P])} + 8\sqrt{\lambda([A_{P'},\mathbf{x}_{P'})} < \mathrm{dist}(x_P, x_{P'}) \}. \end{aligned} \tag{5.18}$$

Then, we define

$$\gamma'_{\Theta,\mathcal{P}} : \mathcal{O}(\mathbb{R}^4,\mathcal{P}) = \tilde{\mathcal{O}}(\mathbb{R}^4,\mathcal{P})/\mathfrak{S}(\mathcal{P}) \to \bar{\mathcal{B}}^s_\kappa(2\delta), \tag{5.19}$$

as follows.

For $P \in \mathcal{P}$, let $E_P \to S^4$ be the bundle supporting the connection $A_P$. The framed connection $[A_P, F^s_P]$ defines a section of $\mathrm{Fr}(E_P)$, $\phi(A_P, F^s_P)$, over the complement of the north pole by parallel translation along great circles from the south pole. Let $\Theta(A_P, F^s_P)$ be the flat connection defined by this section. Note that if $A_P$ is flat on the domain of $\phi(A_P, F^s_P)$, then $\Theta(A_P, F^s_P) = A_P$.

If $\Theta$ denotes the trivial connection on $\mathbb{R}^4 \times \mathrm{SO}(3)$, then the flat connection $\Theta(A_P, F^s_P)$ is canonically identified with the restriction of $\Theta$ to the domain of the $\phi(A_P, F^s_P)$ as follows. The section $\phi(A_P, F^s_P)$ gives a trivialization of the bundle $E$ and thus identifies it with the restriction of the trivial bundle $\mathbb{R}^4 \times \mathrm{SO}(3)$ to the domain of the section. Under this identification the connection $\Theta$ is identified with $\Theta(A_P, F^s_P)$.

We also note that given any two connections, $A_1$ and $A_2$, a convex linear combination of them, $tA_1 + (1-t)A_2$ also defines a connection. For example, if $A_1$ is a connection which is trivial in a particular trivialization and $A_2 = A_1 + a_2$, we would write the connection one form for $tA_1 + (1-t)A_2$ in this trivialization as $(1-t)a_2$.

Let $c_{y,\lambda} : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $c_{\lambda,y}(z) = (z-y)/\lambda$. Thus, $c_{y,\lambda}$ maps the ball $B(y,\lambda)$ to $B(0,1)$.

Let $\chi : \mathbb{R} \to [0,2)$ be a smooth function satisfying $\chi(x) = 0$ for $x \leq \frac{1}{2}$ and $\chi(x) = 1$ for $x \geq 1$. Define $\chi_{y,\lambda} : \mathbb{R}^4 \to [0,2)$ by $\chi_{y,\lambda}(x) = \chi(|x-y|/\lambda)$.

Then, for $\mathbf{y} = (y_P)_{P\in\mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta),\mathcal{P})$, $\lambda_P = \lambda([A_P, F^s_P, \mathbf{x}_P])$, we define:

$$\gamma'_{\Theta,\mathcal{P}}\left(\mathbf{x}, ([A_P, F^s_P, \mathbf{x}_P])_{P\in\mathcal{P}}\right) = [A', F^s_P, \mathbf{x}'] \tag{5.20}$$

where the framed connection $[A', F_P^s]$ is defined by

(5.21)
$$A' = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P\in\mathcal{P}} B(y_P, 4\sqrt{\lambda_P}) \\ (1-\chi_{y_P,4\sqrt{\lambda_P}})c_{y_P,1}^* A_P + \chi_{y_P,4\sqrt{\lambda_P}} c_{y_P,1}^* \Theta(A_P, F_P^s) & \text{on } \Omega(y_P, 2\sqrt{\lambda_P}, 4\sqrt{\lambda_P})) \\ c_{y_P,1}^* A_P & \text{on } B(y_P, 2\sqrt{\lambda_P}). \end{cases}$$

and the frame in $[A', F_P^s]$ is given by the canonical frame for the trivial connection $\Theta$.

The point $\mathbf{x}'$ in the expression (5.20) is defined by translating the singular support (and multiplicity) of the point $\mathbf{x}_P$ to $c_{y_P}(\mathbf{x}_P)$.

Note that we have not computed the behavior of the function $\lambda \circ \boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ precisely so we cannot assert that the image of $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ is contained in $\mathcal{B}_\kappa^s(2\delta)$. However, that containment will follow from the continuity of $\lambda$ and of $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ with respect to Uhlenbeck limits if we shrink the domain $\mathcal{O}(\Theta, \mathcal{P}, \delta)$ by requiring the scales $\lambda_P$ to be sufficiently small.

We note the following result on the behavior of this splicing map at the cone point of $\bar{\mathcal{B}}_{|P|}^s(\delta_P)$.

**Lemma 5.6.** *For any partition $\mathcal{P}$ of $N_\kappa$, the map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ is continuous with respect to Uhlenbeck limits. If $c_{|P|} \in \mathrm{Sym}_\delta^{|P|,\natural}(\mathbb{R}^4)$ denotes the cone point, then for any $(y_P)_{P\in\mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$,*

(5.22)
$$\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left((y_P)_{P\in\mathcal{P}}, ([\Theta, c_{|P|})_{P\in\mathcal{P}})\right) = [\Theta, (y_p)_{P\in\mathcal{P}}]$$

It is important that our constructions in §5.4 be equivariant with respect to the $\mathrm{SO}(3)\times\mathrm{SO}(4)$ actions on the space of connections so that those constructions can be used in the space of gluing data which is defined with this action. Thus, we prove the following result on the equivariance of the splicing map.

**Lemma 5.7.** *Let $\mathcal{P}$ be a partition of $N_\kappa$. Let $\mathrm{SO}(3)\times\mathrm{SO}(4)$ act on*

$$\mathcal{O}(\Theta, \mathcal{P}, \delta) \subset \Delta^\circ(Z_\kappa, \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{\mathcal{B}}_{|P|}^s(\delta_P),$$

*by, for $A \in \mathrm{SO}(3)$ and $R \in \mathrm{SO}(4)$,*

(5.23)
$$((x_P)_{P\in\mathcal{P}}, [A_P, F_P^s]_{P\in\mathcal{P}}, A, R) \to \left((Rx_P)_{P\in\mathcal{P}}, [(R^{-1})^* A_P, RF_P^s A^{-1}]_{P\in\mathcal{P}}\right),$$

*where $R \in \mathrm{SO}(4)$ acts on the frame by a lift to the principal bundle. Then, the splicing map*

$$\boldsymbol{\gamma}_{\Theta,\mathcal{P}} : \mathcal{O}(\Theta, \mathcal{P}, \delta) \to \bar{\mathcal{B}}_\kappa^s(2\delta),$$

*is $\mathrm{SO}(3)\times\mathrm{SO}(4)$ equivariant.*

*Proof.* The equivariance of the SO(4) action follows from noting that the inclusion

$$\coprod_{P\in\mathcal{P}} B(0, 4\sqrt{\lambda_P}) \to \mathbb{R}^4,$$

given by identifying $B(0, 4\sqrt{\lambda_P})$ with $B(y_P, 4\sqrt{\lambda_P})$ is SO(4) equivariant if $R \in \mathrm{SO}(4)$ acts on $(x_P)_{P\in\mathcal{P}}$ by

$$(x_P)_{P\in\mathcal{P}} \to (Rx_P)_{P\in\mathcal{P}}.$$

(This follows because the action of SO(4) on a connection is given by taking any bundle map lift of the action on $\mathbb{R}^4$.) The equivariance of the SO(3) action follows from observing that splicing in $A_P$ by identifying the section $\phi(A_P, F_P^s)$ with the identity section of $\mathbb{R}^4\times\mathrm{SO}(3)$ gives the same connection as splicing in $A_P$ by identifying the section $\phi(A_P, F_P^s A^{-1})$ with the section $x \to (x, A^{-1})$ of $\mathbb{R}^4\times\mathrm{SO}(3)$. □

We will also require the following observation about centering maps.

**Lemma 5.8.** *Let $c_{y,\lambda}:\mathbb{R}^4\to\mathbb{R}^4$ be defined by $c_{\lambda,y}(z)=(z-y)/\lambda$. For any $\lambda_1,\lambda_2>0$ and $x_1,x_2\in\mathbb{R}^4$, we have:*

$$\begin{aligned} c_{x_2,\lambda_2}\circ c_{x_1,\lambda_1} &= c_{x_1+\lambda_1 x_2,\lambda_1\lambda_2},\\ c^*_{x_2,\lambda_2}\chi_{x_1,\lambda_1} &= \chi_{x_2+\lambda_2 x_1,\lambda_1\lambda_2}. \end{aligned}$$

*Proof.* Elementary algebra. □

If one splices in a mass-centered connection of charge $\kappa$ (to the trivial background) and scale $\lambda$ at a point $y\in\mathbb{R}^4$, it is not immediately clear what the scale of the resulting connection will be. However, the limit of the scale of the spliced up connection (as $\lambda$ goes to zero) will be $\sqrt{\kappa|y|^2}$. The following exploitation of that fact will be used in Lemma 5.11 to prove that, on a suitable open set of splicing data, when composing two splicing maps, the cutting-off operation of the second splicing map does not interact with that of the first splicing map.

**Lemma 5.9.** *Define $T(\Theta,\mathcal{P},\delta)\subset\mathcal{O}(\Theta,\mathcal{P},\delta)$ by*

$$T(\Theta,\mathcal{P},\delta)=\{(x_P)_{P\in\mathcal{P}},[\Theta,c_{|P|}]_{P\in\mathcal{P}})\in\mathcal{O}(\Theta,\mathcal{P},\delta)\} \tag{5.24}$$

*Then,*

$$\gamma'_{\Theta,\mathcal{P}}(T(\Theta,\mathcal{P},\delta))=\{[\Theta]|\times\Sigma(Z_\kappa(\delta),\mathcal{P}) \tag{5.25}$$

*and there is an open neighborhood, $D(\Theta,\mathcal{P},\delta)$ of $T(\Theta,\mathcal{P},\delta)$ in $\mathcal{O}(\Theta,\mathcal{P},\delta)$ such that for every*

$$\mathbf{A}:=(x_P,[A_P,F^s_P,\mathbf{x}_P])_{P\in\mathcal{P}}\in D(\Theta,\mathcal{P},\delta),$$

*and for every $P\in\mathcal{P}$, if $\lambda_P=\lambda([A_P,F^s_P,\mathbf{x}_P])$ and $\lambda=\lambda(\gamma'_{\Theta,\mathcal{P}}(\mathbf{A}))$, then*

$$B(x_P,4\lambda_P^{1/2})\Subset B(0,2\lambda^{1/2}) \tag{5.26}$$

*and*

$$B(x_P,\tfrac14\lambda_P^{1/3})\Subset B(0,\tfrac18\lambda^{1/2}). \tag{5.27}$$

*Proof.* The equality (5.25) follows immediately from Lemma 5.6.

The functions $\lambda_P$ vanish on $T(\Theta,\mathcal{P},\delta)$ while $\lambda\circ\gamma'_{\Theta,\mathcal{P}}$ is equal to the square root of $\sum_{P\in\mathcal{P}}|P||x_P|^2$, and thus non-zero. The continuity of the functions involved then implies the existence of the desired open subspace. □

5.3. **Composing splicing maps.** To construct the moduli space with spliced ends we need to show the images of the splicing maps $\gamma'_{\Theta,\mathcal{P}}$ form a smoothly stratified space. Let $\mathcal{P},\mathcal{P}'$ be partitions of $N_\kappa$ with $\mathcal{P}<\mathcal{P}'$. For each $P\in\mathcal{P}$, let $\mathcal{P}'_P$ be the partition of $P$ defined in (4.6). We will need to understand the overlaps of the images of the maps $\gamma'_{\Theta,\mathcal{P}}$ and $\gamma'_{\Theta,\mathcal{P}'}$. To that end, we define a space $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ with maps to $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta)$ and $\tilde{\mathcal{O}}(\Theta,\mathcal{P}',\delta)$ so that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) & \xrightarrow{\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}} & \tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta) \\ {\scriptstyle \rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}}\downarrow & & {\scriptstyle\gamma'_{\Theta,\mathcal{P}}}\downarrow \\ \tilde{\mathcal{O}}(\Theta,\mathcal{P}',\delta) & \xrightarrow{\gamma'_{\Theta,\mathcal{P}'}} & \bar{\mathcal{B}}^s_\kappa(2\delta) \end{array} \tag{5.28}$$

We note that the $d$ and $u$ appearing in the maps $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ and $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ are supposed to imply "down" and "up" respectively as $\mathcal{P}$ and $\mathcal{P}'$ correspond to the lower and upper strata.

5.3.1. *Defining the overlap data.* We now define the objects appearing in the diagram (5.28). The open set $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ will be defined to be a subspace of

$$\nu(\Theta,\mathcal{P},\mathcal{P}',\delta) = \Delta^\circ(Z_\kappa(\delta),\mathcal{P}) \times \prod_{P\in\mathcal{P}} \left( \Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P) \times \prod_{P'\in\mathcal{P}'_P} \bar{\mathcal{B}}^s_{|P'|}(\delta_{P'}) \right), \tag{5.29}$$

where the constants $\delta_P$ and $\delta_{P'}$ will not need to be defined explicitly. We introduce the following projections from $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$:

$$\begin{aligned}
\pi_{\mathcal{P},\mathbf{x}} &: \nu(\Theta,\mathcal{P},\mathcal{P}',\delta) \to \Delta^\circ(Z_\kappa(\delta),\mathcal{P}),\\
\pi_{\mathcal{P}'_P,\mathbf{x}} &: \nu(\Theta,\mathcal{P},\mathcal{P}',\delta) \to \Delta^\natural(Z_{|P|}(\delta_P),\mathcal{P}'_P),\\
\pi_{\mathbf{x}} &: \nu(\Theta,\mathcal{P},\mathcal{P}',\delta) \to \Delta^\circ(Z_\kappa(\delta),\mathcal{P}) \times \prod_{P\in\mathcal{P}} \Delta^\natural(Z_{|P|}(\delta_P),\mathcal{P}'_P),\\
\pi_P &: \nu(\Theta,\mathcal{P},\mathcal{P}',\delta) \to \Delta^\natural(Z_{|P|}(\delta_P),\mathcal{P}'_P) \times \prod_{P'\in\mathcal{P}'_P} \bar{\mathcal{B}}^s_{|P'|},\\
\pi_{P'} &: \nu(\Theta,\mathcal{P},\mathcal{P}',\delta) \to \bar{\mathcal{B}}^s_{|P'|}(\delta_P).
\end{aligned} \tag{5.30}$$

Use the inclusion given in (5.10),

$$\nu(Z_\kappa(\delta),\mathcal{P},\mathcal{P}') = \Delta^\circ(Z_\kappa(\delta),\mathcal{P}) \times \prod_{P\in\mathcal{P}} \Delta^\circ(Z_P(\delta_P),\mathcal{P}'_P) \to \nu(Z_\kappa(\delta),\mathcal{P}),$$

to consider $\nu(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')$ as a subspace of $\nu(Z_\kappa(\delta),\mathcal{P})$. We may then apply the exponential map $e(Z_\kappa,\mathcal{P})$ of (5.8) to elements of $\nu(Z_\kappa,\mathcal{P},\mathcal{P}')$. We desire to define the map $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ appearing in the diagram (5.28) by

$$\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'} = (e(Z_\kappa,\mathcal{P}) \circ \pi_{\mathbf{x}}) \times \prod_{P'\in\mathcal{P}'} \pi_{P'}. \tag{5.31}$$

That is, the map $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ leaves the connection data in $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ alone and maps the points in $\nu(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')$ to the diagonal $\Delta^\circ(Z_\kappa(\delta),\mathcal{P}')$.

For the composition $e(Z_\kappa,\mathcal{P})\circ\pi_{\mathbf{x}}$ in (5.31) to be defined, we must restrict the domain of $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ to an open subspace $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ of $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ satisfying

$$\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) \subset \pi_{\mathbf{x}}^{-1}\left(\mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')\right), \tag{5.32}$$

where $\mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')$ is defined in (5.11).

To define the composition $\gamma'_{\Theta,\mathcal{P}'}\circ\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ appearing in the diagram (5.28), the open subspace $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ must satisfy

$$\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) \subset (\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'})^{-1}\left(\tilde{\mathcal{O}}(\Theta,\mathcal{P}',\delta)\right). \tag{5.33}$$

Before defining $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$, we note that if $P\in\mathcal{P}\cap\mathcal{P}'$ then the partition $\mathcal{P}'_P$ is just the set $P$ and $\Delta^\natural(Z_{|P|},\mathcal{P}'_P)$ is a single point (the zero vector). In this case, we define the splicing map

$$\gamma'_{\Theta,\mathcal{P}'_P} : \Delta^\natural(Z_{|P|}(\delta),\mathcal{P}'_P) \times \bar{\mathcal{B}}^s_{|P|}(\delta_P) \to \bar{\mathcal{B}}^s_{|P|}(2\delta),$$

to be the projection onto $\bar{\mathcal{B}}^s_{|P|}(\delta_P) \subset \bar{\mathcal{B}}^s_{|P|}(2\delta)$. Then, the map $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ of (5.28) will be defined by:

$$\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'} := \pi_{\mathcal{P},\mathbf{x}} \times \left( \prod_{P\in\mathcal{P}} \gamma'_{\Theta,\mathcal{P}'_P} \circ \pi_P \right) \tag{5.34}$$

That is, the map $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ leaves the point in $\Delta^\circ(Z_\kappa,\mathcal{P})$ alone and for each $P\in\mathcal{P}$ applies the splicing map $\gamma'_{\Theta,\mathcal{P}'_P}$ to the space

$$\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P) \times \prod_{P'\in\mathcal{P}'_P} \bar{\mathcal{B}}^s_{|P'|}(\delta_{P'}).$$

For this map to be defined on $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ we need the image of $\pi_P$ to be contained in $\mathcal{O}(\Theta,\mathcal{P}'_P,\delta_P)$, the domain of the splicing map $\gamma'_{\Theta,\mathcal{P}'_P}$. However, we will make a stronger requirement (to be used in Lemma 5.11):

$$\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) \subset \pi_P^{-1}\left(D(\Theta,\mathcal{P}'_P,\delta_P)\right) \quad \text{for all } P\in\mathcal{P}\setminus\mathcal{P}', \tag{5.35}$$

where $D(\Theta,\mathcal{P}'_P,\delta_P)$ is the open subspace of $\tilde{\mathcal{O}}(\Theta,\mathcal{P}'_P,\delta_P)$ defined in Lemma 5.9. Since $D(\Theta,\mathcal{P}'_P,\delta_P) \subset \mathcal{O}(\Theta,\mathcal{P}'_P,\delta_P)$, the condition (5.35) implies $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ is defined on $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$.

Finally, for the composition $\gamma'_{\Theta,\mathcal{P}}\circ\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ to be well defined, the open subspace $\tilde{\mathcal{O}}(\Theta,\mathcal{P}'_P,\delta_P)$ must satisfy

$$\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) \subset (\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'})^{-1}\left(\tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta)\right). \tag{5.36}$$

We now record a result showing that the image of the restriction of $\gamma'_{\Theta,\mathcal{P}'}\circ\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ to an appropriate open subspace $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ of the overlap data $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ contains

$$\{[\Theta,\mathbf{x}] : \mathbf{x}\in e(Z_\kappa,\mathcal{P})(\mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}'))\}. \tag{5.37}$$

Thus, the image of $\gamma'_{\Theta,\mathcal{P}'}\circ\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ will contain all of the trivial stratum $\{[\Theta]\}\times\Sigma(Z_\kappa(\delta),\mathcal{P}')$ in the image of $\gamma'_{\Theta,\mathcal{P}}$.

**Lemma 5.10.** *Let $\mathcal{P},\mathcal{P}'$ be partitions of $N_\kappa$ satisfying $\mathcal{P}<\mathcal{P}'$. Let $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ be the subspace of the space $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ given by*

$$T(\Theta,\mathcal{P},\mathcal{P}',\delta) := \pi_{\mathbf{x}}^{-1}\left(\mathcal{O}(Z_\kappa,\mathcal{P},\mathcal{P}')\right) \cap \left(\cap_{P'\in\mathcal{P}'}\pi_{P'}^{-1}([\Theta,c_{|P'|}])\right). \tag{5.38}$$

*Then, there is an open subspace, $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$, of $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ satisfying the conditions* (5.32),(5.33), (5.35), *and* (5.36) *and also satisfying*

1. $T(\Theta,\mathcal{P},\mathcal{P}',\delta)\subset\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$,
2. $\gamma'_{\Theta,\mathcal{P}'}\circ\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}(T(\Theta,\mathcal{P},\mathcal{P}',\delta)) = \{[\Theta,\mathbf{x}] : \mathbf{x}\in e(Z_\kappa,\mathcal{P})\left(\mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}')\right)\}$,
3. $(\gamma'_{\Theta,\mathcal{P}})^{-1}\left([\Theta]\times e(Z_\kappa,\mathcal{P})(\mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}'))\right) = \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}(T(\Theta,\mathcal{P},\mathcal{P}',\delta))$.

*Proof.* First,

$$\pi_{\mathbf{x}}(T(\Theta,\mathcal{P},\mathcal{P}',\delta)) = \mathcal{O}(Z_\kappa(\delta),\mathcal{P},\mathcal{P}'), \tag{5.39}$$

by definition so there is an open neighborhood of $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ in $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ satisfying (5.32). Second, for every $P'\in\mathcal{P}'$,

$$\pi_{P'}(T(\Theta,\mathcal{P},\mathcal{P}',\delta)) = [\Theta,c_{|P'|}]$$

so $\pi_P(T(\Theta,\mathcal{P},\mathcal{P}',\delta)) \subset T(\Theta,\mathcal{P}'_P,\delta_P) \subset D(\Theta,\mathcal{P},\delta)$ for all $P\in\mathcal{P}$ and there is an open neighborhood of $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ in $\nu(\mathbb{R}^4,\mathcal{P},\mathcal{P}')$ satisfying (5.35).

The second item in the conclusions of the lemma follows from the definitions of the spaces and maps and the value of the splicing map given in (5.22). The second item implies that $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}(T(\Theta,\mathcal{P},\mathcal{P}',\delta)) \subset \tilde{\mathcal{O}}(\Theta,\mathcal{P}',\delta)$, so there is an open neighborhood of $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ in $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ satisfying (5.33).

Next, equation (5.22) implies that

$$\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}|_{T(\Theta,\mathcal{P},\mathcal{P}',\delta)} = \pi_{\mathbf{x}}|_{T(\Theta,\mathcal{P},\mathcal{P}',\delta)}.$$

This equality and (5.39) imply the third item of the conclusions of the lemma. This third item implies that $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}(T(\Theta,\mathcal{P},\mathcal{P}',\delta))$ is contained in the domain of $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}$, $\mathcal{O}(\Theta,\mathcal{P},\delta)$, so there is an open neighborhood of $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ in $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ satisfying (5.36).

By taking the intersection of these open neighborhoods of $T(\Theta,\mathcal{P},\mathcal{P}',\delta)$ in $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$, we obtained the desired open neighborhood $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$. □

5.3.2. *Equality of splicing maps.* We now are in a position to prove that the diagram (5.28) commutes. The key point is the restriction (5.35) on the domain $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$. This ensures that in the iterated splicing construction defining the composition $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \circ \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ (first the splicings $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}'_P}$ defining the map $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ are performed, then the splicing $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ is performed), the cut-off function of the second splicing has no effect. This follows from Lemma 5.9 which ensures that the connections given by $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ are *already* trivial on the annuli on which the splicing map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ interpolates between them and the trivial connection. Thus, this iterated splicing procedure is equivalent to a single splicing, that given by the composition $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}'} \circ \rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$.

**Lemma 5.11.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $N_\kappa$ with $\mathcal{P} < \mathcal{P}'$. Let $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ be the open set defined in Lemma 5.10 of the space $\nu(\Theta,\mathcal{P},\mathcal{P}',\delta)$ defined in* (5.29)*. Then, the following diagram commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta) & \xrightarrow{\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}} & \tilde{\mathcal{O}}(\Theta,\mathcal{P}',\delta) \\ {\scriptstyle \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}}\downarrow & & {\scriptstyle \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}}\downarrow \\ \tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta) & \xrightarrow{\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}} & \bar{\mathcal{B}}^s_\kappa(2\delta) \end{array} \tag{5.40}$$

*Proof.* To simplify the notation, we write $Q$ for an element of the partition $\mathcal{P}'$. Let $\mathbf{A} \in \tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ be given by:

$$\mathbf{A} = \Big((y_P)_{P\in\mathcal{P}}, ((x_Q)_{P'\in\mathcal{P}'_P}, [A_Q, F^s_Q]_{Q\in\mathcal{P}'_P})\Big),$$

where $(y_P)_{P\in\mathcal{P}} \in \Delta^\circ(Z_\kappa,\mathcal{P})$ and $((x_Q)_{Q\in\mathcal{P}'_P}, [A_Q, F^s_Q]_{Q\in\mathcal{P}'_P}) \in D(\Theta,\mathcal{P}'_P,\delta_P)$. (For simplicity of exposition, we assume $[A_Q, F^s_Q] \in \bar{\mathcal{B}}^s_{|Q|}(\delta_Q)$ has no ideal points; the proof is no more difficult without that assumption, but the notation becomes more opaque.)

The image of $\mathbf{A}$ under $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}$ is given by the same connections, $[A_Q, F^s_Q]$ as given in $\mathbf{A}$, but by the point $(x'_Q)_{Q\in\mathcal{P}'} \in \Delta^\circ(Z_\kappa(\delta),\mathcal{P}')$, where

$$x'_Q = y_P + x_Q \quad \text{if } Q \in \mathcal{P}'_P. \tag{5.41}$$

(Note that $x'_Q = y_P$ if $Q \in \mathcal{P} \cap \mathcal{P}'$ as $x_Q$ is then the zero vector.) If we set $\lambda_Q = \lambda([A_Q])$, then

(5.42)

$$\begin{aligned}&\boldsymbol{\gamma}'_{\Theta,\mathcal{P}'} \circ \rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}(\mathbf{A})\\ &= \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q\in\mathcal{P}'_P, P\in\mathcal{P}} B(x'_Q, 4\sqrt{\lambda_Q}),\\ (1-\chi)c^*_{x'_Q,4\sqrt{\lambda_Q}} A_Q + \chi_{x'_Q,4\sqrt{\lambda_Q}} c^*_{x'_Q}\Theta(A_Q, f_{E_Q}) & \text{on } \Omega(x'_Q, 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}),\\ c^*_{x'_Q,1}A_Q & \text{on } B(x'_Q, 2\sqrt{\lambda_Q}).\end{cases}\end{aligned}$$

We now compare (5.42) with $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \circ \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}(\mathbf{A})$.

If, for $P \in \mathcal{P} \setminus \mathcal{P}'$, we define

$$[A_P, F^s_P] = \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'_P}(\pi_P(\mathbf{A})) = \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'_P}\Big((x_Q)_{Q\in\mathcal{P}'_P}, [A_Q, f_{E_Q}]_{Q\in\mathcal{P}'_P}\Big),$$

and $[A_P, F^s_P] = [A_Q, F^s_Q]$ for $P = Q \in \mathcal{P} \cap \mathcal{P}'$, then

$$\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}(\mathbf{A}) = ((y_P)_{P\in\mathcal{P}}, ([A_P, F^s_P])_{P\in\mathcal{P}}) \in \mathcal{O}(\Theta, \mathcal{P}, \delta).$$

The composition $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \circ \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ is then given by:

(5.43)

$$\begin{aligned}&\boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \circ \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}\\ &= \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P\in\mathcal{P}} B(y_P, 4\sqrt{\lambda_P}),\\ (1-\chi_{y_P,4\sqrt{\lambda_P}})c^*_{y_P,1}A_P + \chi_{y_P,4\sqrt{\lambda_P}} c^*_{y_P,1}\Theta(A_P, f_{E_P}) & \text{on } \Omega(y_P, 2\sqrt{\lambda_P}, 4\sqrt{\lambda_P}),\\ c^*_{y_P,1}A_P & \text{on } B(y_P, 2\sqrt{\lambda_P}),\end{cases}\end{aligned}$$

where $\lambda_P = \lambda([A_P])$. On the balls $B(x_P, 4\sqrt{\lambda_P})$ for $P \in \mathcal{P}\cap\mathcal{P}'$, and on $\mathbb{R}^4 \setminus \cup_{P\in\mathcal{P}} B(x_P, 4\sqrt{\lambda_P})$ the connections (5.42) and (5.43) are identical. We thus focus our attention on the balls $B(x_P, 4\sqrt{\lambda_P})$ for $P \in \mathcal{P} \setminus \mathcal{P}'$. For such $P$,

(5.44)

$$\begin{aligned}A_P &= \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'_P}(\pi_P(\mathbf{A}))\\ &= \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q\in\mathcal{P}'_P} B(x_Q, 4\sqrt{\lambda_Q}),\\ (1-\chi_{x_Q,\sqrt{4\lambda_Q}})c^*_{x_Q,1}A_Q + \chi_{x_Q,\sqrt{4\lambda_Q}} c^*_{x_Q,1}\Theta(A_Q, f_{E_Q}) & \text{on } \Omega(x_Q, 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}),\\ c^*_{x_Q,1}A_Q & \text{on } B(z_Q, 2\sqrt{\lambda_Q}).\end{cases}\end{aligned}$$

Because the points $(x_Q)_{Q\in\mathcal{P}'_P} \in \Delta^\natural(\mathbb{R}^4, \mathcal{P}'_P)$ and the connections $[A_Q, f_{E_Q}] \in \bar{\mathcal{B}}^s_{|Q|}$ form a point in $D(\Theta, \mathcal{P}'_P, \delta_P)$ by the construction of $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, Lemma 5.9 and equation (5.26) imply that for all $Q \in \mathcal{P}'_P$,

$$B(x_Q, 4\sqrt{\lambda_Q}) \subset B(0, 2\sqrt{\lambda_P}).$$

Thus, the construction in (5.44) implies $A_P$ is already equal to $\Theta$ on

$$\mathbb{R}^4 \setminus B(0, 2\sqrt{\lambda_P}),$$

so $c^*_{y_P,1}A_P$ is equal to $\Theta$ on
$$\mathbb{R}^4 \setminus B(y_P, 2\sqrt{\lambda_P}).$$
Thus, the convex combination in the second line of (5.43) is equal to $\Theta$. We examine the final line of (5.43) by applying Lemma 5.8:

(5.45)
$$\begin{aligned} &c^*_{y_P,1}A_P \\ &= \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q\in\mathcal{P}'_P} B(y_P + x_Q, 4\sqrt{\lambda_Q}), \\ (1-\chi)c^*_{y_P+x_Q,4\sqrt{\lambda_Q}}A_Q + \chi_{y_P+x_Q,4\sqrt{\lambda_Q}} c^*_{y_P+x_Q}\Theta(A_Q, f_{E_Q}) & \text{on } \Omega(y_P + x_Q, 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c^*_{y_P+x_Q,1}A_Q & \text{on } B(y_P + x_Q, 2\sqrt{\lambda_Q}). \end{cases} \end{aligned}$$

Thus, we see that $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \circ \rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}(\mathbf{A})$ is given by:

(5.46)
$$\begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q\in\mathcal{P}'_P, P\in\mathcal{P}} B(y_P + x_Q, 4\sqrt{\lambda_Q}), \\ (1-\chi)c^*_{y_P+x_Q,4\sqrt{\lambda_Q}}A_Q + \chi_{y_P+x_Q,4\sqrt{\lambda_Q}} c^*_{y_P+x_Q}\Theta(A_Q, f_{E_Q}) & \text{on } \Omega(y_P + x_Q, 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c^*_{y_P+x_Q,1}A_Q & \text{on } B(y_P + x_Q, 2\sqrt{\lambda_Q}). \end{cases}$$

Then comparing (5.46) with (5.42) and noting the definition of $x_Q$ in (5.41) completes the proof. $\square$

5.3.3. *Symmetric group quotients.* Finally, we make some observations on the action of the symmetric group necessary to define a quotient of the diagram (5.28) by the symmetric group. Recall from Lemma 5.5 that for $\mathcal{P} < \mathcal{P}'$, to describe a neighborhood of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma(Z_\kappa, \mathcal{P})$, we must describe all the diagonals $\Delta^\natural(Z_\kappa(\delta), \mathcal{P}'')$ where $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. We define

$$\nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) = \sqcup_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \nu(\Theta, \mathcal{P}, \mathcal{P}'', \delta) \tag{5.47}$$

The group $\mathfrak{S}(\mathcal{P})$ acts on $\nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ by the obvious permutation. Note that if $\sigma \in \mathfrak{S}(\mathcal{P})$ and $P' \in \mathcal{P}'$, then it is possible that $\sigma(P') \in \mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ with $\mathcal{P}'' \neq \mathcal{P}'$, thus explaining the need for the additional partitions $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. Let $\mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \subset \nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ be the analogue of the open subspace $\mathcal{O}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ defined in Lemma 5.10. We define (analogously to the definition of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in (5.38))

$$\begin{aligned} T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) =& \{\big((y_P)_{P\in\mathcal{P}}, (x_Q)_{Q\in\mathcal{P}''}, ([\Theta, c_{|Q|}])_{Q\in\mathcal{P}''}\big) \in \nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) : \\ & \big((y_P)_{P\in\mathcal{P}}, (x_Q)_{Q\in\mathcal{P}''}\big) \in \mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])\}, \end{aligned} \tag{5.48}$$

where $\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$ is defined in Lemma 5.5.

The maps $\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}''}$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ define a map

$$\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']} : \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \to \nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta) = \left( \coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \nu(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}). \tag{5.49}$$

Similarly, the coproduct of the maps $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}''}$ is invariant under the action of $\mathfrak{S}(\mathcal{P})$ and thus defines a map

$$\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']} : \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \to \mathcal{O}(\Theta, \mathcal{P}, \delta). \tag{5.50}$$

The coproduct of the splicing maps $\gamma'_{\Theta,\mathcal{P}''}$, over $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$, is invariant under the action of $\mathfrak{S}(\mathcal{P})$ and thus defines a map
(5.51)

$$\gamma'_{\Theta,\mathcal{P},[\mathcal{P}']}=\left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\gamma'_{\Theta,\mathcal{P}''}\right)/\mathfrak{S}(\mathcal{P}):\left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\mathcal{O}(\mathbb{R}^4,\mathcal{P}'',\delta)\right)/\mathfrak{S}(\mathcal{P})\to\bar{\mathcal{B}}^s_\kappa(2\delta),$$

which has the same image as $\gamma'_{\Theta,\mathcal{P}'}$. Lemma 5.11 then implies the following.

**Proposition 5.12.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $N_\kappa$ with $\mathcal{P}<\mathcal{P}'$. Let $\mathcal{O}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ be the open subspace defined following*(5.47). *Then, the following diagram commutes:*

$$\begin{array}{ccc}\mathcal{O}(\Theta,\mathcal{P},[\mathcal{P}'],\delta) & \xrightarrow{\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}} & \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\mathcal{O}(\mathbb{R}^4,\mathcal{P}'')\right)/\mathfrak{S}(\mathcal{P}) \\ {\scriptstyle\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}}\big\downarrow & & {\scriptstyle\gamma'_{\Theta,\mathcal{P},[\mathcal{P}']}}\big\downarrow \\ \mathcal{O}(\Theta,\mathcal{P},\delta) & \xrightarrow{\gamma'_{\Theta,\mathcal{P}}} & \bar{\mathcal{B}}^s_\kappa(2\delta)\end{array}\tag{5.52}$$

*where $\gamma'_{\Theta,[\mathcal{P}<\mathcal{P}']}$ is defined in* (5.51).

5.4. **The spliced end.** We now construct the spliced end, given by the union of the images of the splicing maps $\gamma'_{\Theta,\mathcal{P}}$ restricted to appropriate subspaces of $\mathcal{O}(\Theta,\mathcal{P},\delta)$.

This construction is inductive. We therefore assume that the spliced ends moduli space $\bar{M}^{s,\natural}_{spl,|P|}(\delta)$ with the properties stated in Theorem 5.1 has already been constructed for $|P|<\kappa$. Note that because we only need to describe Uhlenbeck neighborhoods of the *punctured* trivial strata, we only need to use splicing maps $\gamma'_{\Theta,\mathcal{P}}$ where $|\mathcal{P}|>1$. For a partition $\mathcal{P}$ of $N_\kappa$ with $|\mathcal{P}|>1$, $|P|<\kappa$ for all $P\in\mathcal{P}$. Therefore, the inductive hypothesis suffices for this construction.

The splicing maps will be restricted to the following spaces (where the precise choice of the constants $\delta_P>0$ will not be relevant to the rest of the discussion)

$$\begin{aligned}\tilde{\mathcal{O}}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)&=\tilde{\mathcal{O}}(\Theta,\mathcal{P},\delta)\cap\left(\Delta^\circ(Z_\kappa(2\delta),\mathcal{P})\times\prod_{P\in\mathcal{P}}M^{s,\natural}_{spl,|P|}(\delta_P)\right),\\ \mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)&=\tilde{\mathcal{O}}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)/\mathfrak{S}(\mathcal{P}).\end{aligned}\tag{5.53}$$

Observe that $T(\Theta,\mathcal{P},\delta)\subset\tilde{\mathcal{O}}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ and the dimension of $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ is equal to that of $\bar{M}^{s,\natural}_\kappa(\delta)$.

The spliced end is constructed by the following proposition.

**Proposition 5.13.** *Assume that the spliced end moduli space $\bar{M}^{s,\natural}_{spl,\kappa'}(\delta)$ with the properties enumerated in Theorem 5.1 has already been constructed. Then for every partition $\mathcal{P}$ of there is an $\mathrm{SO}(3)\times\mathrm{SO}(4)$-invariant, open neighborhood $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},\delta)$ of $T(\Theta,\mathcal{P},\delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ such that the space,*

$$W_\kappa=\cup_{\mathcal{P}}\,\gamma'_{\Theta,\mathcal{P}}\left(\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},\delta)\right),\tag{5.54}$$

*is a smoothly-stratified subspace of $\bar{\mathcal{B}}^s_\kappa(\delta)$ with every point $[A,F^s,\mathbf{x}]\in W_\kappa$ satisfying the estimate $\|F^+_A\|_{L^{\sharp,2}}\le M$.*

We prove Proposition 5.13 by using Proposition 5.12 to control the overlaps of the images of the splicing maps. The transition maps $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$ and $\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}$ will be restricted to the following subspace of $\mathcal{O}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$

$$\begin{aligned}
&\tilde{\mathcal{O}}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\\
&\quad = \tilde{\mathcal{O}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\\
&\quad \cap \left( \Delta^\circ(Z_\kappa(\delta),\mathcal{P}) \times \coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \prod_{P\in\mathcal{P}} \left( \Delta^\circ(Z_P(\delta_P),\mathcal{P}''_P) \times \prod_{Q\in\mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) \right)\right), \qquad (5.55)\\
&\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta) = \tilde{\mathcal{O}}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)/\mathfrak{S}(\mathcal{P}).
\end{aligned}$$

(Compare the definition (5.47)). Observe that $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\subset\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ where $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ is defined in (5.48) because $[\Theta,c_{|Q|}]\in\bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q)$ for all $|Q|<\kappa$.

We must first verify that the transition maps $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$ and $\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}$ map $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ to $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P}',\delta)$ and $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ respectively.

**Lemma 5.14.** *Assume $\bar{M}^{s,\natural}_{spl,\kappa'}(\delta)$ has been constructed satisfying the properties enumerated in Theorem 5.1 for all $\kappa'<\kappa$. Let $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ be the subspace of $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ defined in* (5.48)*. Then there is an open neighborhood $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ of $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ such that the restriction of $\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}$ to $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ is an open embedding of $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ into $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$.*

*Proof.* Induction and the property (5.3) in Theorem 5.1 imply that for each $P\in\mathcal{P}\setminus\mathcal{P}'$, there is an open neighborhood $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P}'_P,\delta_P)$ of $T(\Theta,\mathcal{P}'_P,\delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P}'_P,\delta)$ such that

$$\gamma'_{\Theta,\mathcal{P}'_P}\left(\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P}'_P,\delta_P)\right)\subset\bar{M}^{s,\natural}_{spl,|P|}(\delta_P).$$

Thus, if in the definition of $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$, (5.55), we replace the factor

$$\Delta^\circ(Z_P(\delta_P),\mathcal{P}''_P)\times\prod_{Q\in\mathcal{P}''_P}\bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q)$$

with the open subspace

$$\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P}'_P,\delta_P)\subset\Delta^\circ(Z_P(\delta_P),\mathcal{P}''_P)\times\prod_{Q\in\mathcal{P}''_P}\bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q)$$

the resulting open subspace $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ of $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ will satisfy

$$\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}\left(\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\right)\subset\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta). \qquad (5.56)$$

(Note that we are not concerned with the subsets $P\in\mathcal{P}\cap\mathcal{P}'$ because $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ is the identity on these factors and thus they do not affect the inclusion (5.56).)

That the restriction of $\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}$ to $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ is an open embedding into $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ follows from the assumption that $\gamma'_{\Theta,\mathcal{P}'_P}$ gives an open embedding of $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P}'_P,\delta_P)$ into $\bar{M}^{s,\natural}_{spl,|P|}(\delta_P)$. □

We now prove an analogy of Lemma 5.14 for the map $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$. First, we must define the appropriate range of $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$, to take into account the appearance of the conjugate partitions

$\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. Let

$$\mathcal{O}^{\mathrm{asd}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta) = \left( \coprod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}) \subset \nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta), \tag{5.57}$$

where $\nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$ is defined in (5.49). Then,

**Lemma 5.15.** *Continue the assumptions and hypotheses of Lemma 5.14. Then the restriction of the map $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$ to $\mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ is an open embedding into $\mathcal{O}^{\mathrm{asd}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$.*

*Proof.* The lemma follows immediately from the definition of $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$ because, being defined largely by the exponential map $e(Z_\kappa, \mathcal{P})$ (see (5.31)), the map $\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}$ does not change the connections in the domain $\mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$. □

The following lemma shows the existence of suitably small, $\mathrm{SO}(3) \times \mathrm{SO}(4)$ invariant neighborhoods of $T(\Theta, \mathcal{P}, \delta)$.

**Lemma 5.16.** *For any subspace $V$ of $\bar{\mathcal{B}}^s_\kappa$, let $\mathrm{cl}(V)$ denote the closure of $V$ in the Uhlenbeck topology. If $V$ is any set in $\bar{\mathcal{B}}^s_\kappa$ with satisfying*

$$\mathrm{cl}(V) \cap \boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \left( T(\Theta, \mathcal{P}, \delta) \right) = \varnothing,$$

*then there is an $\mathrm{SO}(3) \times \mathrm{SO}(4) \times \mathfrak{S}(\mathcal{P})$ invariant neighborhood, $\mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta)$, of $T(\Theta, \mathcal{P}, \delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}, \delta)$ such that the intersection*

$$\mathrm{cl}(V) \cap \boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \left( \mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta) \right)$$

*is empty and the inclusion,*

$$\mathrm{cl}(V) \cap \mathrm{cl} \left( \boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \left( \mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta) \right) \right) \subset \mathrm{cl}(V) \cap \mathrm{cl} \left( \boldsymbol{\gamma}'_{\Theta,\mathcal{P}} \left( T(\Theta, \mathcal{P}, \delta) \right) \right),$$

*is satisfied.*

*Proof.* For $\mathbf{A} \in \mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}, \delta)$ given by $\mathbf{A} = ((x_P)_{P\in\mathcal{P}}, ([A_P, F^s_P])_{P\in\mathcal{P}})$, let $\lambda_P(\mathbf{A}) = \lambda(A_P)$. For $\mathbf{x} = (x_P)_{P\in\mathcal{P}} \in \Delta^\circ(Z_\kappa, \mathcal{P})$, define

$$V(\mathbf{x}, \mathcal{P}) := \{(\mathbf{x}, (A_P)_{P\in\mathcal{P}}) \in (\boldsymbol{\gamma}'_{\Theta,\mathcal{P}})^{-1} (\mathrm{cl}(V))\}$$

and $\boldsymbol{\lambda}_V : \Delta^\circ(Z_\kappa, \mathcal{P}) \to (0, \infty)$ by

$$\boldsymbol{\lambda}_V(\mathbf{x}) = \min_{\mathbf{A} \in V(\mathbf{x},\mathcal{P})} \sum_{P\in\mathcal{P}} \lambda^2_P(\mathbf{A}).$$

By assumption $\boldsymbol{\lambda}_V$ is non-zero on $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. Let $f : \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \to (0, \varepsilon)$ be a continuous function with $f(\mathbf{x}) < \frac{1}{2}\boldsymbol{\lambda}_V(\mathbf{x})$, and

$$f\left((x_P)_{P\in\mathcal{P}}\right) < \frac{1}{2} \min_{P_1 \neq P_2 \in \mathcal{P}} |x_{P_1} - x_{P_2}|. \tag{5.58}$$

Then we define the neighborhood $\mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta)$ by

$$\mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta) := \{\mathbf{A} = (\mathbf{x}, ([A_P, F^s_P])_{P\in\mathcal{P}}) \in \mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}, \delta) : \sum_{P\in\mathcal{P}} \lambda^2_P(\mathbf{A}) < f(\mathbf{x})\}.$$

This set has the desired invariance and its image does not intersect $\mathrm{cl}(V)$.

If $\{\mathbf{A}(\alpha)\}_{\alpha=1}^\infty \subset \mathcal{O}^{\mathrm{asd}}_1(\Theta, \mathcal{P}, \delta)$, with

$$\mathbf{A}(\alpha) = [\mathbf{x}(\alpha), ([A_P(\alpha), F^s_P(\alpha)])_{P\in\mathcal{P}}],$$

where $\mathbf{x}(\alpha) \in \Delta^\circ(Z_\kappa, \mathcal{P})$ is the splicing points of $\mathbf{A}(\alpha)$ and if

$$\lim_{\alpha\to\infty} \boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\mathbf{A}(\alpha)\right) \in \mathrm{cl}(V),$$

then

$$\lim_{\alpha\to\infty} \mathbf{x}(\alpha) \notin \Delta^\circ(Z_\kappa, \mathcal{P}).$$

The inequality (5.58) then implies that

$$\lim_{\alpha\to\infty} f(\mathbf{x}(\alpha)) = 0,$$

so the condition

$$\sum_{P\in\mathcal{P}} \lambda_P^2(\mathbf{A}(\alpha)) < f(\mathbf{x}(\alpha))$$

in the definition of $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ implies that for all $P\in\mathcal{P}$

$$\lim_{\alpha\to\infty} \lambda(A_P(\alpha)) = 0,$$

so $\lim_{\alpha\to\infty} A_P(\alpha) = [\Theta, c_{|P|}]$, giving the final conclusion of the lemma. □

We now construct the subspaces $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ referred to in Proposition 5.13. We will construct these subspaces to be $\mathfrak{S}_\kappa$ invariant in the sense that if $\mathcal{P}''\in[\mathcal{P}]$, then the open subspaces $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ and $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P}'',\delta)$ are related by the are identified by the natural action of $\mathfrak{S}_\kappa$ on $\coprod_{\mathcal{P}_i\in[\mathcal{P}_1]}\nu(\Theta,\mathcal{P}_i)$. Thus, defining $\mathcal{O}_1^{\mathrm{asd}}(\mathbb{R}^4,\mathcal{P}')$ for one $\mathcal{P}'\in[\mathcal{P}']$ suffices to define the space

$$\mathcal{O}_1^{\mathrm{asd}}(\Theta,[\mathcal{P}<\mathcal{P}']) = \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P}'',\delta)\right)\mathfrak{S}(\mathcal{P}) \subset \mathcal{O}^{\mathrm{asd}}(\Theta,[\mathcal{P}<\mathcal{P}'],\delta), \tag{5.59}$$

where $\mathcal{O}^{\mathrm{asd}}(\Theta,[\mathcal{P}<\mathcal{P}'],\delta)$ is defined in (5.57).

**Lemma 5.17.** *Assume $\bar M^{s,\natural}_{spl,\kappa'}(\delta)$ has been constructed satisfying the properties enumerated in Theorem 5.1 for all $\kappa'<\kappa$. Then, to every partition $\mathcal{P}$ of $N_\kappa$ there is an open neighborhood $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ of $T(\Theta,\mathcal{P},\delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$, such that*

(1) *These neighborhoods are closed under the $\mathrm{SO}(3)\times\mathrm{SO}(4)$ action given in Lemma 5.23,*
(2) *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are conjugate under the action of $\mathfrak{S}_\kappa$, then the neighborhoods $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P}_1,\delta)$ and $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P}_2,\delta)$ are identified by the natural action of $\mathfrak{S}_\kappa$ on $\coprod_{\mathcal{P}_i\in[\mathcal{P}_1]}\nu(\Theta,\mathcal{P}_i)$,*
(3) *If $\mathcal{P}\not<\mathcal{P}'$ and $\mathcal{P}'\not<\mathcal{P}$, then*

$$\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}(\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)) \cap \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}(\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P}',\delta)) = \varnothing,$$

(4) *If $\mathcal{P}<\mathcal{P}'$ then there is an open neighborhood $\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ of the space $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ in $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ such that*

$$\begin{aligned}
&\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)\right) \cap \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}\left(\mathcal{O}_1^{\mathrm{asd}}(\mathbb{R}^4,\mathcal{P}')\right)\\
&\subseteq \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}\left(\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}(\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta))\right)\\
&= \boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']}(\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta))\right)
\end{aligned} \tag{5.60}$$

*Proof.* By induction, assume the open sets $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ have been defined for all $[\mathcal{P}] < \mathcal{P}'$. Note that the conclusion still holds if we shrink the sets $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$. We now construct $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}', \delta)$.

It suffices to construct $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}', \delta)$ for a single $\mathcal{P}' \in [\mathcal{P}']$ because for another element, $\mathcal{P}'' \in [\mathcal{P}']$, one can define $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}'', \delta)$ to be the image of $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}', \delta)$ under the action of $\mathfrak{S}_\kappa$.

If $\mathcal{P} \not< \mathcal{P}'$ and $\mathcal{P}' \not< \mathcal{P}$, then we assume $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ has already been constructed. If $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ has not already been constructed, then this step will appear later when the induction reaches $\mathcal{P}$. The relation $\mathcal{P} \not< \mathcal{P}'$ and $\mathcal{P}' \not< \mathcal{P}$ implies that

$$\operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right)\right) \cap \gamma'_{\Theta,\mathcal{P}'}\left(T(\Theta,\mathcal{P}',\delta)\right) = \varnothing, \tag{5.61}$$

$$\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}'}\left(T(\Theta,\mathcal{P}',\delta)\right)\right) = \varnothing. \tag{5.62}$$

Lemma 5.16 and (5.61) imply there is an $\mathrm{SO}(3)\times\mathrm{SO}(4)$ invariant neighborhood $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}', \delta)$ such that

$$\operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right)\right) \cap \gamma'_{\Theta,\mathcal{P}'}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P}',\delta)\right) = \varnothing.$$

Lemma 5.16 also implies that

$$\begin{aligned}
&\operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right)\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}'}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P}',\delta)\right)\right) \\
&\quad\subseteq \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right)\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}'}\left(T(\Theta,\mathcal{P}',\delta)\right)\right).
\end{aligned} \tag{5.63}$$

Equations (5.63) and (5.62) imply

$$\begin{aligned}
&\gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}'}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P}',\delta)\right)\right) \\
&\qquad\subset \gamma'_{\Theta,\mathcal{P}}\left(T(\Theta,\mathcal{P},\delta)\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}'}\left(T(\Theta,\mathcal{P}',\delta)\right)\right) = \varnothing.
\end{aligned}$$

Thus, Lemma 5.16 implies there is a (smaller) $\mathrm{SO}(3) \times \mathrm{SO}(4)$ invariant neighborhood $\mathcal{O}_2^{\text{asd}}(\mathbb{R}^4, \mathcal{P}) \subset \mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ of $T(\Theta, \mathcal{P}, \delta)$ with

$$\gamma'_{\Theta,\mathcal{P}}\left(\mathcal{O}_2^{\text{asd}}(\mathbb{R}^4,\mathcal{P})\right) \cap \operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right)\right) = \varnothing$$

as desired. Replace the open neighborhood $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ with $\mathcal{O}_2^{\text{asd}}(\mathbb{R}^4, \mathcal{P})$. For all other $\mathcal{P}'' \in [\mathcal{P}]$, replace $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}'', \delta)$ with the image of $\mathcal{O}_2^{\text{asd}}(\mathbb{R}^4, \mathcal{P})$ under the action of $\mathfrak{S}_\kappa$. This completes the case of $\mathcal{P} \not< \mathcal{P}'$ and $\mathcal{P}' \not< \mathcal{P}$ in the induction.

If $[\mathcal{P}] < \mathcal{P}'$, by conjugating $\mathcal{P}$ if necessary and using the invariance of the neighborhoods $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$ under the action of $\mathfrak{S}_\kappa$, we can assume $\mathcal{P} < \mathcal{P}'$. By Lemma 5.9, the closure of

$$[\Theta] \times \left(\Sigma(Z_\kappa(\delta), \mathcal{P}') - \operatorname{Im}(e(Z_\kappa, \mathcal{P}))\right),$$

(where $e(Z_\kappa, \mathcal{P})$ is the exponential map defined in (5.8)) does not intersect $\gamma'_{\Theta,\mathcal{P}}(T(\Theta, \mathcal{P}, \delta))$. Thus, applying Lemma 5.16 and shrinking $\mathcal{O}_1^{\text{asd}}(\Theta, \mathcal{P}, \delta)$, we can assume

$$\operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right)\right) \cap \left([\Theta] \times \left(\Sigma(Z_\kappa, \mathcal{P}'') - \operatorname{Im}(e(Z_\kappa,\mathcal{P}))\right)\right) = \varnothing$$

for all $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ and thus

$$\begin{aligned}
&\operatorname{cl}\left(\gamma'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right)\right) \cap \left(\left(\cup_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\gamma'_{\Theta,\mathcal{P}'}(T(\Theta,\mathcal{P}'',\delta))\right)/\mathfrak{S}(\mathcal{P})\right) \\
&\quad\Subset [\Theta] \times e(Z_\kappa(\delta), \mathcal{P})\left(\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])\right),
\end{aligned} \tag{5.64}$$

where $\mathcal{O}(Z_\kappa(\delta),[\mathcal{P}<\mathcal{P}'])$ is defined in Lemma 5.5. Let $T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ and $\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)$ be the sets defined in (5.48). The equality and inclusion

$$\begin{aligned}
[\Theta]\times e(Z_\kappa,\mathcal{P})\left(\mathcal{O}(Z_\kappa(\delta),[\mathcal{P}<\mathcal{P}'])\right) &= \boldsymbol{\gamma}'_{\Theta,[\mathcal{P}<\mathcal{P}']}\left(\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}\left(T(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\right)\right)\\
&\subset \boldsymbol{\gamma}'_{\Theta,[\mathcal{P}<\mathcal{P}']}\left(\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\right)\right),
\end{aligned}$$

together with (5.64) then imply that

$$\operatorname{cl}\left(\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right)\right)\setminus \boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}\left(\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\mathcal{P}',\delta)\right)\right) \tag{5.65}$$

is disjoint from $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}''}(T(\Theta,\mathcal{P}'',\delta))$ for all $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$. Lemma 5.16 gives an $\mathrm{SO}(3)\times\mathrm{SO}(4)\times\mathfrak{S}(\mathcal{P})$ invariant neighborhood $\mathcal{O}_1^{\text{asd}}(\Theta,[\mathcal{P}<\mathcal{P}'],\delta)$ of

$$\left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} T(\Theta,\mathcal{P}'',\delta)\right)/\mathfrak{S}(\mathcal{P}) \quad\text{in}\quad \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \mathcal{O}^{\text{asd}}(\Theta,\mathcal{P}'',\delta)\right)/\mathfrak{S}(\mathcal{P})$$

whose image under $\boldsymbol{\gamma}'_{\Theta,[\mathcal{P}<\mathcal{P}']}$ is disjoint from the last difference (5.65). This disjointess implies

$$\begin{aligned}
\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right)&\cap \boldsymbol{\gamma}'_{\Theta,\mathcal{P}''}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P}'',\delta)\right)\\
&\subset \boldsymbol{\gamma}'_{\Theta,[\mathcal{P}<\mathcal{P}']}\left(\rho^{\Theta,u}_{\mathcal{P},[\mathcal{P}']}\left(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},[\mathcal{P}'],\delta)\right)\right),
\end{aligned}$$

for all $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$ and so as required in the conclusion of the lemma. □

*Proof of Proposition 5.13.* For each partition $\mathcal{P}$ of $N_\kappa$, let $\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)$ be the $\mathrm{SO}(3)\times\mathrm{SO}(4)$ invariant subspace of $\mathcal{O}^{\text{asd}}(\Theta,\mathcal{P},\delta)$ constructed in Lemma 5.17. For any two partitions $\mathcal{P}$ and $\mathcal{P}'$ the images $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta))$ and $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}'}(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P}',\delta))$ are either disjoint or, by the last item in Lemma 5.17, are given by:

$$\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}\left((\rho^{\Theta,d}_{\mathcal{P},[\mathcal{P}']})^{-1}(\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)\right).$$

Thus, the overlaps are open subspaces and the union of the images $W_\kappa$ has the desired properties. The estimates on $F_A^+$ in the proposition follow from the arguments in [7], details will appear in [9, 10]. □

5.5. **Tubular neighborhoods of the spliced end moduli space.** We now introduce the tubular neighborhood structure of the spliced end $W_\kappa$ defined in the previous section. This structure will satisfy the conditions discussed in the beginning of §4. In this lemma, we introduce the projection map.

**Lemma 5.18.** *For each partition $\mathcal{P}$ of $N_\kappa$ with $|\mathcal{P}|>1$, let $\mathcal{U}(\Theta,\mathcal{P})\subset W_\kappa$ be the image of the open subspace $\mathcal{O}_1^{\text{asd}}(\Theta,\mathcal{P},\delta)$ defined in Lemma 5.17. There is an* $\mathrm{SO}(3)\times\mathrm{SO}(4)$ *equivariant map*

$$\pi(\Theta,\mathcal{P}):\mathcal{U}(\Theta,\mathcal{P})\to\{[\Theta]\}\times\Sigma(Z_\kappa(\delta),\mathcal{P}), \tag{5.66}$$

*satisfying*

1. *If $\mathcal{P}'\in[\mathcal{P}]$, then $\mathcal{U}(\Theta,\mathcal{P})=\mathcal{U}(\Theta,\mathcal{P}')$, $\Sigma(Z_\kappa(\delta),\mathcal{P})=\Sigma(Z_\kappa(\delta),\mathcal{P}')$, and $\pi(\Theta,\mathcal{P})=\pi(\Theta,\mathcal{P}')$.*
2. *On the overlap $\mathcal{U}(\Theta,\mathcal{P})\cap\mathcal{U}(\Theta,\mathcal{P}')$,*

$$\pi(\Theta,\mathcal{P})\circ\pi(\Theta,\mathcal{P}')=\pi(\Theta,\mathcal{P}) \tag{5.67}$$

*Proof.* The map $\pi(\Theta,\mathcal{P})$ is defined to be the composition of the projection

$$\mathcal{O}_1^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)\to\Sigma(Z_\kappa,\mathcal{P}),$$

with the inverse of the splicing map $\gamma'_{\Theta,\mathcal{P}}$. The equality (5.67) follows from (5.60). □

The obvious identification

$$\{[\Theta]\}\times Z_\kappa/\mathfrak{S}_\kappa\cong Z_\kappa/\mathfrak{S}_\kappa, \tag{5.68}$$

allows us to define the projection maps $\pi(Z_\kappa,\mathcal{P})$ and functions $t(Z_\kappa)$ defined in §4.1 as maps and functions on the trivial strata of $W_\kappa$. We identify the restriction of the projection $\pi(\Theta,\mathcal{P})$ with such a map in the following.

**Lemma 5.19.** *Under the identification* (5.68)*, the restriction of the projection map $\pi(\Theta,\mathcal{P})$ of Lemma 5.18 to the intersection*

$$(\{[\Theta]\}\times Z_\kappa/\mathfrak{S}_\kappa)\cap\mathcal{U}(\Theta,\mathcal{P})$$

*is equal to the map $\pi(Z_\kappa,\mathcal{P}):U(Z_\kappa,\mathcal{P})/\mathfrak{S}(\mathcal{P})\to\Delta^\circ(Z_\kappa,\mathcal{P})=\Sigma(Z_\kappa,\mathcal{P})/\mathfrak{S}(\mathcal{P})$ defined in* (5.67)*.*

*Proof.* The lemma follows from noting that the restriction of the splicing map $\gamma'_{\Theta,\mathcal{P}}$ to points where the connections being spliced in are trivial is exactly the map $e(Z_\kappa,\mathcal{P})$ of (4.9). □

Before defining the tubular distance functions, we define a perturbation of the scale function which is constant along the fibers of the projections $\pi(\Theta,\mathcal{P})$. This scale shall be used in the construction of the tubular distance function and in the definition of the link of the reducible PU(2) monopoles.

**Lemma 5.20.** *There are continuous maps*

$$\tilde\lambda_\kappa:\bar M^{s,\natural}_{spl,\kappa}(\delta)\to[0,2\delta), \tag{5.69}$$

*and*

$$t_r(\Theta,\mathcal{P}):\mathcal{U}(\Theta,\mathcal{P})\to[0,1],\quad \textit{and}\quad t(\Theta,\mathcal{P}):\mathcal{U}(\Theta,\mathcal{P})\to[0,1]. \tag{5.70}$$

*which are smooth on each stratum,* $\mathrm{SO}(3)\times\mathrm{SO}(4)$ *invariant, and satisfy for some neighborhoods $\mathcal{U}'(\Theta,\mathcal{P})\Subset\mathcal{U}(\Theta,\mathcal{P})$ of $\{[\Theta]\}\times\Sigma(Z_\kappa(\delta),\mathcal{P})$*

1. *The function $\tilde\lambda_\kappa$ is equal to the scale function $\lambda$ on the complement of $W_\kappa$,*
2. *Under the identification* (5.68)*, the restriction of $\tilde\lambda_\kappa$ to the trivial strata in* (5.68) *is equal to the function $t(Z_\kappa)$ defined in Lemma 4.3,*
3. *For all partitions $\mathcal{P}$ of $N_\kappa$ with $|\mathcal{P}|>1$, $\tilde\lambda_\kappa=\tilde\lambda_\kappa\circ\pi(\Theta,\mathcal{P})$ on $\mathcal{U}'(\Theta,\mathcal{P})$,*
4. *If $\mathcal{P}'\in[\mathcal{P}]$ then, $t_r(\Theta,\mathcal{P})=t(\Theta,\mathcal{P}')$ and $t(\Theta,\mathcal{P})=\tilde t(\Theta,\mathcal{P}')$,*
5. *On the overlap $\mathcal{U}'(\Theta,\mathcal{P})\cap\mathcal{U}'(\Theta,\mathcal{P}')$,*

$$t_r(\Theta,\mathcal{P})\circ\pi(\Theta,\mathcal{P}')=t_r(\Theta,\mathcal{P}),\quad \textit{and}\quad t(\Theta,\mathcal{P})\circ\pi(\Theta,\mathcal{P}')=t(\Theta,\mathcal{P}) \tag{5.71}$$

6. *$t_r(\Theta,\mathcal{P})^{-1}(0)=t(\Theta,\mathcal{P})^{-1}(0)=\{\Theta\}\times\Sigma(Z_\kappa(\delta),\mathcal{P})$,*
7. *There are open neighborhoods*

$$U_1(\Theta,\mathcal{P})\Subset\mathcal{U}_2(\Theta,\mathcal{P})\Subset\mathcal{U}(\Theta,\mathcal{P})$$

*of $\{\Theta\}\times\Sigma(Z_\kappa,\mathcal{P})$, such that*

$$\begin{aligned}\mathcal{U}_1(\Theta,\mathcal{P})&=t_r(\Theta,\mathcal{P})^{-1}([0,\tfrac12)),\\ \mathcal{U}(\Theta,\mathcal{P})-\mathcal{U}_2(\Theta,\mathcal{P})&=t_r(\Theta,\mathcal{P})^{-1}(\{1\})\end{aligned} \tag{5.72}$$

*Proof.* The function $\tilde\lambda_\kappa$ is constructed using induction on $\kappa$. Thus, we assume the functions $\tilde\lambda_{\kappa'}$, $t_r(\Theta,\mathcal{P})$, and $t(\Theta,\mathcal{P})$ have been constructed for all $\kappa'<\kappa$ and all partitions $\mathcal{P}$ of $N_{\kappa'}$.

Begin the induction with $\tilde\lambda_1=\lambda_1$. First, observe that the restriction of $\tilde\lambda_\kappa$ to the trivial strata in $W_\kappa$ appearing in (5.68) is determined by item (2). This is consistent with item (3) by virtue of the identification of $\pi(\Theta,\mathcal{P})$ in Lemma 5.19 and the equality in item (2) of Lemma 4.3. Then construct $\tilde\lambda_\kappa$ following the pattern of the construction of the function $t(Z_\kappa)$ in Lemma 4.3 using $t_r(\Theta,\mathcal{P})$ in place of $t_r(Z_P,\mathcal{P})$.

For each partition $\mathcal{P}$ of $N_\kappa$, the function $t(\Theta,\mathcal{P})$ is then defined by

$$t(\Theta,\mathcal{P})=\sum_{P\in\mathcal{P}}|P|\tilde\lambda_P^2, \tag{5.73}$$

where the functions $\tilde\lambda_P$ are defined on $\mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\delta)$ by the new scale function $\tilde\lambda_{|P|}$ on the $S^4$ connection being spliced in at $x_P$.

The equality (5.67) holds for $t(\Theta,\mathcal{P})$ by the same argument used to establish (4.38), using the observation that for $\mathcal{P}<\mathcal{P}'$, the projection $\pi(\Theta,\mathcal{P}')$ on $\mathcal{U}(\Theta,\mathcal{P})\cap\mathcal{U}(\Theta,\mathcal{P}')$ is given by the restriction of the projection

$$\begin{gathered}\Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\textstyle\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}\bar M^{s,\natural}_{spl,|P'|}(\delta_{P'})\right)\\ \downarrow\\ \Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\textstyle\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P)\times\prod_{P'\in\mathcal{P}'_P}\{[\Theta,c_{P'}]\}\right)\end{gathered}$$

to the appropriate subspaces (see (5.52) and (5.55)).

The function $t_r(\Theta,\mathcal{P})$ is then a rescaling of the function $t(\Theta,\mathcal{P})$ in the sense of Lemma 4.12. That is, $t_r(\Theta,\mathcal{P})$ is defined by multiplying $t(\Theta,\mathcal{P})$ by a non-vanishing, smooth function pulled back from $T(\Theta,\mathcal{P},\delta)$ by the map $\pi(\Theta,\mathcal{P})$. The property (5.67) of $\pi(\Theta,\mathcal{P})$ implies that the pullback of a function $f$ by $\pi(\Theta,\mathcal{P})$ satisfies

$$(f\circ\pi(\Theta,\mathcal{P}))\circ\pi(\Theta,\mathcal{P}')=f\circ\pi(\Theta,\mathcal{P})$$

so defining $t_r(\Theta,\mathcal{P})$ by such a rescaling of $t(\Theta,\mathcal{P})$ implies that if $t(\Theta,\mathcal{P})$ has the property (5.67) then $t_r(\Theta,\mathcal{P})$ does as well. $\square$

5.6. **The isotopy of the spliced end.** The final step in the proof of Theorem 5.1 is to use the isotopy provided by the gluing and centering maps to piece together the spliced end $W_\kappa$ with the actual moduli space $\bar M^{s,\natural}_\kappa(\delta)$.

We first construct the isotopy of $W_\kappa$ in $\bar{\mathcal{B}}^s_\kappa(\delta)$.

**Lemma 5.21.** *There is continuous, smoothly-stratified map*

$$R:(-\infty,1]\times W_\kappa\to\bar{\mathcal{B}}^s_\kappa(\delta), \tag{5.74}$$

*such that*

1. *For all $t\in(-\infty,\frac12]$ and all $[A,F^s,\mathbf{x}]\in W_\kappa$, $R(t,[A,F^s,\mathbf{x}])=[A,F^s,\mathbf{x}]$,*
2. *For all $t\in[\frac34,1]$ and all $[A,F^s,\mathbf{x}]\in W_\kappa$, $R(t,[A,F^s,\mathbf{x}])\in\bar M^s_\kappa(\delta)$,*
3. *For all $[A,F^s,\mathbf{x}]\in W_\kappa$, $R(1,[A,F^s,\mathbf{x}])\in\bar M^{s,\natural}_\kappa(\delta)$,*
4. *For all $t\in(-\infty,1]$, the map $R(t,\cdot):W_\kappa\to\bar{\mathcal{B}}^s_\kappa(\delta)$ is $\mathrm{SO}(3)\times\mathrm{SO}(4)$ equivariant.*

*Proof.* The bound on the $L^{\sharp,2}$-norm of the $F^+_A$ for all $[A,\mathbf{x}]\in W_\kappa$ implies (see [7, Proposition 7.6]) that there is a continuous, smoothly-stratified, $\mathrm{SO}(3)\times\mathrm{SO}(4)$ equivariant embedding, $G:W_\kappa\to\bar M^{s,\natural}(\delta)$

$$G([A,F^s,\mathbf{x}])=[A+a(A),F^s,\mathbf{x}]. \tag{5.75}$$

For $t \in [\frac{1}{2}, \frac{3}{4}]$, we define

$$R(t, [A, F^s, \mathbf{x}]) = [A + 4(t - \tfrac{1}{2})a(A), F^s, \mathbf{x}]. \tag{5.76}$$

The remainder of the isotopy, for $t \in [\frac{3}{4}, 1]$ is defined by the mass-centering map. That is, pulling any $[A, F^s, \mathbf{x}]$ back by an appropriate conformal diffeomorphism ensures it is mass-centered (see [6, §3.2.1]). This defines an natural isotopy, parameterized by $t \in [\frac{3}{4}, 1]$, of the image of $W_\kappa$ under $R(\frac{1}{2}, \cdot)$ into the space of mass-centered connections. □

*Proof of Theorem 5.1.* The function $1 - t_r(\Theta, \mathcal{P})$ is compactly supported on $\mathcal{U}(\Theta, \mathcal{P})$ and thus defines an $\mathrm{SO}(3) \times \mathrm{SO}(4)$-invariant function on $W_\kappa$. Observe that by (5.72), the map

$$t = 1 - \sum_{\mathcal{P}} (1 - t_r(\Theta, \mathcal{P})). \tag{5.77}$$

satisfies

$$t(\cup_{\mathcal{P}} \mathcal{U}_1(\Theta, \mathcal{P})) \subset (-\infty, \tfrac{1}{2}],$$
$$t\left(\cap_{\mathcal{P}} (W_\kappa - \mathcal{U}_2(\Theta, \mathcal{P}))\right) = 1.$$

That is, if $[A, F^s, \mathbf{x}] \in \mathcal{U}_1(\Theta, \mathcal{P})$ for some $\mathcal{P}$ then $t([A, F^s, \mathbf{x}]) < \frac{1}{2}$ while if $[A, F^s, \mathbf{x}] \notin \mathcal{U}_2(\Theta, \mathcal{P})$ for all $\mathcal{P}$, then $t([A, F^s, \mathbf{x}]) = 1$. Alternately, $t < \frac{1}{2}$ on the "interior" of $W_\kappa$, near the trivial strata while $t$ is equal to one the "exterior", away from the trivial strata. Define $R(W_\kappa)$ to be the image of $W_\kappa$ under the map $R(t(\cdot), \cdot)$:

$$R(W_\kappa) = \{R(t([A, F^s, \mathbf{x}]), [A, F^s, \mathbf{x}]) : [A, F^s, \mathbf{x}] \in W_\kappa\}.$$

Then if we define the spliced end moduli space by

$$\bar{M}^{s,\natural}_{spl,\kappa}(\delta) = \left(\bar{M}^{s,\natural}_{\kappa}(\delta) - R(1, W_\kappa)\right) \cup R(W_\kappa),$$

it will have the properties specified in Theorem 5.1. □

## 6. The space of global splicing data

In this section, we construct the space of global splicing data. This space will be the union of spaces of splicing data, $\mathcal{O}(\mathfrak{t}, \mathcal{P})$, associated to diagonals $\Sigma(X^\ell, \mathcal{P})$, as $\mathcal{P}$ varies over partitions of $N_\ell$. (Of course, we will do this equivariantly with respect to the symmetric group.) To each partition $\mathcal{P}$, we will define a *crude splicing map* which is identical to the splicing map defined in [9] except

1. The connections on $S^4$ being spliced in are elements of the spliced ends moduli space rather than the moduli space of anti-self-dual connections on $S^4$,
2. The background pair, $(A_0, \Phi_0)$, are "flattened" (i.e. the connection $A_0$ is replaced with one which is flat while the section $\Phi_0$ is multiplied by a vanishing cut-off function) on balls of fixed radius around the splicing point rather than balls whose radius vanishes as the connection on $S^4$ becomes more concentrated.

This last difference is necessary to mimic the result which holds when splicing connections to the trivial connection on $\mathbb{R}^4$ (Proposition 5.12), allowing us to control the overlap of the images of the crude splicing map as we did when constructing the spliced end $W_\kappa$ by defining a space of overlap data and overlap maps. With this control of the overlaps of the images of the crude splicing maps, we can define the space of global splicing data is then the union of the images of the crude splicing maps.

After constructing the domain of the splicing map in §6.1, we show how to flatten pairs as described above in §6.2 which defines the crude splicing map in §6.3. With the understanding of the overlaps of the images of the crude splicing maps developed in 6.4, we construct the space of global splicing data in §6.5. In §6.6, we construct the partial Thom-Mather structure which will be used in the construction of the link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. In §6.7, we produce a global splicing map whose image will contain the reducible monopoles. Finally, in §6.8, we construct a projection map from the space of global splicing data to $\mathrm{Sym}^\ell(X)$.

6.1. **The splicing data.** We begin by recalling the domains of the splicing maps from [9].

6.1.1. *The background pairs.* Let $\tilde{M}_{\mathfrak{s}} \subset \tilde{\mathcal{C}}_{\mathfrak{t}_0}$ be the image of the embedding of the solutions of the perturbation of the Seiberg-Witten equations considered in [15]. The gauge group quotient, $M_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$ embeds in $\mathcal{C}_{\mathfrak{t}_0}$. A virtual normal bundle $N_{\mathfrak{t}_0,\mathfrak{s}} \to M_{\mathfrak{s}}$ with an embedding of a $\delta$-disk subbundle $N_{\mathfrak{t}_0,\mathfrak{s}}(\delta) \subset N_{\mathfrak{t}_0,\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}_0}$, was defined in [15]. We will abbreviate $N_{\mathfrak{t}_0,\mathfrak{s}}(\delta)$ by $N(\delta)$. If $\tilde{N} \to \tilde{M}_{\mathfrak{s}}$ is the pullback of $N_{\mathfrak{t}_0,\mathfrak{s}}$ by the projection $\tilde{M}_{\mathfrak{s}} \to M_{\mathfrak{s}}$, we showed in [11] that there is an embedding $\tilde{N}(\delta) \to \tilde{\mathcal{C}}_{\mathfrak{t}_0}$ covering the embedding of $N(\delta) \to \mathcal{C}_{\mathfrak{t}_0}$.

6.1.2. *The metrics.* For each partition $\mathcal{P}$ of $N_\ell$, let $g_{\mathcal{P}}$ be the smooth family of metrics on $X$ parameterized by $\Delta^\circ(X^\ell,\mathcal{P})$ and let $\tilde{\mathcal{U}}(X^\ell,\mathcal{P}) \subset X^\ell$ be the tubular neighborhood of $\Delta^\circ(X^\ell,\mathcal{P})$ constructed in Lemma 4.11. Recall that $\pi(X^\ell,\mathcal{P}) : \tilde{\mathcal{U}}(X^\ell,\mathcal{P}) \to \Delta^\circ(X^\ell,\mathcal{P})$ is the projection map of this tubular neighborhoods. We list here the relevant properties:

(1) For all $\mathbf{y} \in \Delta^\circ(X^\ell,\mathcal{P})$, the metric $g_{\mathcal{P},\mathbf{y}}$ is flat on the support of $\tilde{\mathcal{U}}(X^\ell,\mathcal{P}) \cap \pi(X^\ell,\mathcal{P})^{-1}(\mathbf{y})$,
(2) For $\mathcal{P} < sP'$ partitions of $N_\ell$ and for $\mathbf{y}' \in \tilde{\mathcal{U}}(X^\ell,\mathcal{P}) \cap \Delta^\circ(X^\ell,\mathcal{P}')$ with $\pi(X^\ell,\mathcal{P})(\mathbf{y}') = \mathbf{y}$, the equality $g_{\mathcal{P},\mathbf{y}} = g_{\mathcal{P},\mathbf{y}'}$ holds,
(3) The metrics are $\mathfrak{S}_\ell$ invariant in the sense that $g_{\mathcal{P},\mathbf{y}} = g_{\sigma(\mathcal{P}),\sigma(\mathbf{y})}$ for all $\sigma \in \mathfrak{S}_\ell$.

6.1.3. *The frame bundles.* For $\mathcal{P}$ be a partition of $N_\ell$, let

$$\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}}) \to \Delta^\circ(X^\ell,\mathcal{P}),$$

be the fiber bundle defined before (4.18). Similarly, for $\mathfrak{g}_0 = \mathfrak{g}_{V_0}$ we define

$$\mathrm{Fr}(\mathfrak{g}_0,\mathcal{P}) \to \Delta^\circ(X^\ell,\mathcal{P}),$$

as

$$\begin{aligned} \mathrm{Fr}(\mathfrak{g}_0,\mathcal{P}) = \{&(F_1^{\mathfrak{g}},\dots,F_\ell^{\mathfrak{g}}) \in \prod_{i=1}^{\ell} \mathrm{Fr}(\mathfrak{g}_0) : \\ &F_i^{\mathfrak{g}} = F_j^{\mathfrak{g}} \text{ if and only if there is } P \in \mathcal{P} \text{ and } i,j \in P\}. \end{aligned} \tag{6.1}$$

Then, the gluing data bundle is:

$$\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) = \mathrm{Fr}(TX,\mathcal{P},g_{\mathcal{P}}) \times_{\Delta^\circ(X^\ell,\mathcal{P})} \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}_0},\mathcal{P}). \tag{6.2}$$

Recall that we denote elements of $\Delta^\circ(X^\ell,\mathcal{P})$ as $(y_P)_{P\in\mathcal{P}}$ where $y_P \in X$. We will similarly denote elements of the bundle $\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ lying over $(y_P)_{P\in\mathcal{P}}$ as $(F_P^T,F_P^{\mathfrak{g}})_{P\in\mathcal{P}}$ where $F_P^T \in \mathrm{Fr}(TX)|_{y_P}$ and $F_P^{\mathfrak{g}} \in \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}_0})|_{y_P}$. We also write $(F_P^{T,\mathfrak{g}})_{P\in\mathcal{P}}$ for such an element of $\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$.

The structure groups of $\mathrm{Fr}(X,\mathcal{P})$ over $\Delta^\circ(X^\ell,\mathcal{P})$ and over $\Sigma(X^\ell,\mathcal{P})$ are $\widetilde{G}(\mathcal{P})$ and $G(\mathcal{P})$ respectively where

$$\begin{aligned} \widetilde{G}(\mathcal{P}) &= \{((R_1,M_1),\dots,(R_\ell,M_\ell)) \in (\mathrm{SO}(4)\times\mathrm{SO}(3))^\ell : \\ &\qquad \text{for every } i,j \in P \in \mathcal{P}\ R_i = R_j \text{ and } M_i = M_j\}, \\ G(\mathcal{P}) &= \widetilde{G}(\mathcal{P}) \ltimes \mathfrak{S}(\mathcal{P}). \end{aligned} \tag{6.3}$$

Observe that for $\mathcal{P} < \mathcal{P}'$, there is an inclusion,

$$\tilde{G}(\mathcal{P}) \to \tilde{G}(\mathcal{P}'). \tag{6.4}$$

We can write elements of $\widetilde{G}(\mathcal{P})$ as $(R_P, M_P)_{P\in\mathcal{P}}$. With this notation the action of $G(\mathcal{P})$ on

$$\prod_{P\in\mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|}(\delta)$$

is given by the factor $(R_P, M_P)$ acting by the standard $\mathrm{SO}(3)\times\mathrm{SO}(4)$ action on $\bar{M}^{s,\natural}_{spl,|P|}(\delta)$ and the natural permutation action of $\mathfrak{S}(\mathcal{P})$.

6.1.4. *The space of splicing data.* The splicing map will be defined on a subspace

$$\tilde{N}(\delta)\times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

where

$$\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) = \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \times_{G(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P), \tag{6.5}$$

is an open neighborhood of

$$\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) = \{\Big((F^{T,\mathfrak{g}}_P)_{P\in\mathcal{P}}, ([\Theta, F^s_P, c_{|P|}])_{P\in\mathcal{P}}\Big) \in \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\} \tag{6.6}$$

where $c_{|P|}$ is the cone point of $\mathrm{Sym}^{|P|,\natural}(\mathbb{R}^4)$. The notation in (6.6) is motivated by the observation that because $\mathrm{SO}(3)\times\mathrm{SO}(4)$ acts trivially on $[\Theta, F^s_P, c_{|P|}]$, there is an identification,

$$\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \cong \Sigma(X^\ell,\mathcal{P}). \tag{6.7}$$

We also define a larger subspace of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ by:

$$T(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) = \{\Big((F^{T,\mathfrak{g}}_P)_{P\in\mathcal{P}}, ([\Theta, F^s_P, \mathbf{v}_P])_{P\in\mathcal{P}}\Big) \in \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\}. \tag{6.8}$$

One requirement in the definition of $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ will be that for any point,

$$\mathbf{A} = \big((A_0,\Phi_0)(F^T_P, F^{\mathfrak{g}}_P)_{P\in\mathcal{P}}, ([A_P, F^s_P, \mathbf{v}_P])_{P\in\mathcal{P}}\big) \in \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

where $F^T_P$ and $F^{\mathfrak{g}}_P$ lie over $y_P \in X$ that

$$8\lambda([A_P, F^s_P, \mathbf{v}_P]) < \min_{P'\in\mathcal{P}, P'\neq P} d(y_P, y'_P) \tag{6.9}$$

for all $P\in\mathcal{P}$. Finally, we define

$$\pi_\Sigma : \tilde{N}(\delta)\times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \to \Sigma(X^\ell,\mathcal{P}) \tag{6.10}$$

to be the obvious projection. The $S^1$ action on $\tilde{N}_{\mathfrak{t}_0,\mathfrak{s}}$ defined in 2 defines an $S^1$ action on

$$\tilde{N}(\delta)\times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$$

and we will see that all the constructions of this section are equivariant with respect to this $S^1$ action. See [11] for different ways of presenting this $S^1$ action.

6.2. **Flattening pairs.** The commutativity of the diagram (5.52) for the spliced ends moduli space depended on the flatness of the metric on $\mathbb{R}^4$ and on the flatness of the trivial connection $\Theta$. The locally flat metrics $g_{\mathcal{P}}$ described in §6.1.2 have this property. We now introduce a method for achieving the same kind of local flatness for the background connection $A_0$. This method also yields a way of cutting off the background section consistently from one stratum to another.

**Definition 6.1.** A collection of smooth functions $s_P : \Delta^\circ(X^\ell, \mathcal{P}) \to (0,1)$, indexed by $P \in \mathcal{P}$ are a *separating family* if

(1) The functions are $\mathfrak{S}(\mathcal{P})$ invariant in the sense that $s_P = s_{\sigma(P)}$ and $s_P = s_P \circ \sigma$, for all $\sigma \in \mathfrak{S}(\mathcal{P})$,
(2) For all $\mathbf{y} = (y_P)_{P\in\mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$ and for all $P \neq P' \in \mathcal{P}$, the balls $B(y_P, 4s_P(\mathbf{y})^{1/2})$ and $B(y_{P'}, 4s_{P'}(\mathbf{y})^{1/2})$ are disjoint.

**Definition 6.2.** Let $\{s_P\}$ be a separating family of smooth functions on $\Delta^\circ(X^\ell, \mathcal{P})$. A connection $\hat{A}_0$ is *flat with respect to* $\{s_P\}$ at $\mathbf{y} = (y_P)_{P\in\mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$ if the connection $\hat{A}_0$ is flat on

$$\cup_{P\in\mathcal{P}}\ B(y_P, 2s_P(\mathbf{y})^{1/2}). \tag{6.11}$$

A pair $(A, \Phi) \in \mathcal{C}_{\mathfrak{t}_0}$ is *flat with respect to* $\{s_P\}$ if $\hat{A}$ is flat with respect to $\{s_P\}$ and $\Phi$ vanishes on

$$\cup_{P\in\mathcal{P}}\ B(y_P, 4s_P(\mathbf{y})^{1/3}). \tag{6.12}$$

The following lemma is the analogue of the flattening construction for metrics given in Lemma 4.10.

**Lemma 6.3.** *Let $\mathcal{P}$ be a partition of $N_\ell$. Let $\{s_P\}$ be a separating family of smooth functions on $\Delta^\circ(X^\ell, \mathcal{P})$. Then, there is an $S^1$-equivariant map,*

$$\mathbf{\Theta}'_{\mathcal{P}} : \tilde{N}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \to \tilde{\mathcal{C}}_{\mathfrak{t}_0} \tag{6.13}$$

*which is $\mathcal{G}_{\mathfrak{s}}$-equivariant and descends to the quotient*

$$N(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \to \mathcal{C}_{\mathfrak{t}_0}$$

*In addition, for all $\mathbf{y} = (y_P)_{P\in\mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, if $(A', \Phi') = \mathbf{\Theta}'_{\mathcal{P}}((A_0, \Phi_0), \mathbf{y})$ then*

(1) *The connection $\hat{A}'$ is flat on*

$$\cup_{P\in\mathcal{P}} B(y_P, 2s_P(\mathbf{y})^{1/2}),$$

*and equal to $A_0$ on the complement of*

$$\cup_{P\in\mathcal{P}} B(y_P, 4s_P(\mathbf{y})^{1/2}) \tag{6.14}$$

(2) *The section $\Phi'$ vanishes on*

$$\cup_{P\in\mathcal{P}} B(y_P, 4s_P(\mathbf{y})^{1/3})$$

*and is equal to $\Phi$ on the complement of*

$$\cup_{P\in\mathcal{P}} B(y_P, 8s_P(\mathbf{y})^{1/3}). \tag{6.15}$$

*If the connection $\hat{A}_0$ is already flat on the space* (6.14)*, then $A' = A_0$. If the section $\Phi$ already vanishes on the space* (6.15)*, then $\Phi' = \Phi_0$.*

*Proof.* The map $\mathbf{\Theta}_{\mathcal{P}'}$ is initially defined as a map,

$$\mathbf{\Theta}'_{\mathcal{P}} : \tilde{N}(\delta) \times \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \to \tilde{\mathcal{C}}_{\mathfrak{t}_0}$$

as described in [9, §3.2]. Specifically, the frames and radial parallel translation define a trivial connection on the space (6.14). If $\hat{A}_0$ is flat on the space (6.14), this trivial connection is equal to $\hat{A}_0$. Then, use a cut-off function to interpolate between the trivial connection and $\hat{A}_0$ on the annuli around the points $y_P$, defining a new connection $\hat{A}'$ which is flat on the inner balls and equal to $\hat{A}_0$ outside the space (6.14). Observe that the resulting connection $\hat{A}'$ is independent of the choice of the frames used as it is equal to the connection obtained by splicing in the trivial connection which has stabilizer $\mathrm{SO}(3) \times \mathrm{SO}(4)$. Thus, the map $\mathbf{\Theta}'_{\mathcal{P}}$ descends to the domain stated in the lemma.

The section $\Phi'$ is defined by multiplying $\Phi_0$ by a cut-off function equal to on the space (6.15) and vanishing on the balls of radius $4s_P(\mathbf{y})^{1/3}$. □

To define the crude splicing maps in such a way that an analogue of Proposition 5.12 controls the overlaps of their images, we must redefine the flattening maps in such a way that for $\mathbf{y}' \in \Delta^\circ(X^\ell, \mathcal{P}')$ near $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$ the flattened background pairs $\mathbf{\Theta}_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}') = \mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y})$. (Compare the second property of the metrics $g_{\mathcal{P}}$ appearing in §6.1.2.) To this end, we introduce a refinement of the flattening map of Lemma 6.3.

**Lemma 6.4.** *Continue the assumptions of Lemma 6.3. Then, for every partition $\mathcal{P}$ of $N_\ell$ there are separating functions $\{\tilde{s}_P\}$ and is an $S^1$-equivariant map*

$$\mathbf{\Theta}_{\mathcal{P}} : \tilde{N}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \to \tilde{\mathcal{C}}_{\mathfrak{t}_0}, \tag{6.16}$$

*which is $\mathfrak{S}(\mathcal{P})$-invariant, $\mathcal{G}_{\mathfrak{s}}$-equivariant and descends to the quotient*

$$N(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \to \mathcal{C}_{\mathfrak{t}_0}$$

*such that if $(A', \Phi') = \mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y})$ where $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$ then*

(1) *The connection $\hat{A}'$ is flat on*

$$\cup_{P \in \mathcal{P}} B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2}),$$

(2) *The section $\Phi'$ vanishes on*

$$\cup_{P \in \mathcal{P}} B(y_P, 4\tilde{s}_P(\mathbf{y})^{1/3}).$$

*Furthermore, if $\mathcal{P} < \mathcal{P}'$ and the tubular neighborhood $\mathcal{U}(X^\ell, \mathcal{P})$ is suitably small, if $\mathbf{y}' \in \mathcal{U}(X^\ell, \mathcal{P}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ and $\pi(X^\ell, \mathcal{P})(\mathbf{y}') = \mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$, then*

$$\mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y}) = \mathbf{\Theta}_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}'). \tag{6.17}$$

*Proof.* The proof is identical to the construction of the consistent flat families of metrics in Lemma 4.11. One constructs $\mathbf{\Theta}_{\mathcal{P}}$ by induction on $\mathcal{P}$.

For $\mathcal{P}_0 = \{N_\ell\}$ the crudest partition of $N_\ell$, define $\mathbf{\Theta}_{\mathcal{P}_0} = \mathbf{\Theta}'_{\mathcal{P}_0}$ (as constructed in Lemma 6.3).

Assume $\mathbf{\Theta}_{\mathcal{P}}$ has been constructed for all $\mathcal{P} < \mathcal{P}'$. To construct $\mathbf{\Theta}_{\mathcal{P}'}$, we first define

$$\mathbf{\Theta}_{\mathcal{P},\mathcal{P}'} : \tilde{N}(\delta) \times \Big(\mathcal{U}(X^\ell, \mathcal{P} \cap \Delta^\circ(X^\ell, \mathcal{P}')\Big) \to \tilde{\mathcal{C}}_{\mathfrak{t}_0},$$

by

$$\mathbf{\Theta}_{\mathcal{P}',\mathcal{P}}(A_0, \Phi_0, \mathbf{y}') = \mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \pi(X^\ell, \mathcal{P})(\mathbf{y}')).$$

Observe that if $\mathcal{P}_1 < \mathcal{P}$ and $\mathcal{P}_2 < \mathcal{P}$ and $\mathcal{U}(X^\ell, \mathcal{P}_1) \cap \mathcal{U}(X^\ell, \mathcal{P}_2)$ is non-empty, then we can assume $\mathcal{P}_1 < \mathcal{P}_2 < \mathcal{P}$. The Thom-Mather condition on the projections $\pi(X^\ell, \mathcal{P})$, (4.30):

$$\pi(X^\ell, \mathcal{P}_1) \circ \pi(X^\ell, \mathcal{P}_2) = \pi(X^\ell, \mathcal{P}_1),$$

and the inductive assumption imply that for $\mathbf{y}' \in \mathcal{U}(X^\ell, \mathcal{P}_1) \cap \mathcal{U}(X^\ell, \mathcal{P}_2) \cap \Delta^\circ(X^\ell, \mathcal{P}')$,

$$\begin{aligned}
\mathbf{\Theta}_{\mathcal{P}',\mathcal{P}_2}(A_0, \Phi_0, \mathbf{y}') &= \mathbf{\Theta}_{\mathcal{P}_2}(A_0, \Phi_0, \pi(X^\ell, \mathcal{P}_2)(\mathbf{y}')) \\
&= \mathbf{\Theta}_{\mathcal{P}_1}(A_0, \Phi_0, \pi(X^\ell, \mathcal{P}_1)\pi(X^\ell, \mathcal{P}_2)(\mathbf{y}')) \quad \text{By inductive hypothesis} \\
&= \mathbf{\Theta}_{\mathcal{P}_1}(A_0, \Phi_0, \pi(X^\ell, \mathcal{P}_1)(\mathbf{y}')) \quad \text{By Thom-Mather property (4.30)} \\
&= \mathbf{\Theta}_{\mathcal{P}',\mathcal{P}_1}(A_0, \Phi_0, \mathbf{y}').
\end{aligned}$$

By the preceding equality, we can define

$$\mathbf{\Theta}_{\mathcal{P}',\mathcal{U}}(A_0, \Phi_0, \mathbf{y}') = \mathbf{\Theta}_{\mathcal{P}',\mathcal{P}}(A_0, \Phi_0, \mathbf{y}') \quad \text{for } \mathbf{y}' \in \mathcal{U}(X^\ell, \mathcal{P})$$

on $\tilde{N}(\delta) \times \mathcal{U}(\mathcal{P}')$ where

$$\mathcal{U}(\mathcal{P}') = \cup_{\mathcal{P}<\mathcal{P}'} \mathcal{U}(X^\ell, \mathcal{P}) \cap \Delta^\circ(X^\ell, \mathcal{P}'). \tag{6.18}$$

Then, we define

$$\mathbf{\Theta}_{\mathcal{P}',1} : \tilde{N}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}') \to \tilde{\mathcal{C}}_{\mathfrak{t}_0},$$

by

$$\mathbf{\Theta}_{\mathcal{P}',1}(A_0, \Phi_0, \mathbf{y}') = (1 - \boldsymbol{\chi}(\mathbf{y}'))(A_0, \Phi_0) + \boldsymbol{\chi}(\mathbf{y}')\mathbf{\Theta}_{\mathcal{P}',\mathcal{U}}(A_0, \Phi_0, \mathbf{y}')$$

where $\boldsymbol{\chi} : \Delta^\circ(X^\ell, \mathcal{P}) \to [0,1]$ is an $\mathfrak{S}(\mathcal{P})$ invariant function supported in (6.18) and equal to one on a neighborhood $\mathcal{U}_2$ of the end of $\Delta^\circ(X^\ell, \mathcal{P})$ (hence $\mathcal{U}_2$ is a proper subspace of $\mathcal{U}(\mathcal{P}')$). By shrinking the tubular neighborhoods $\mathcal{U}(X^\ell, \mathcal{P})$, we can find a family of separating functions $\{\tilde{s}_{P'}\}_{P' \in \mathcal{P}'}$, such that for $\mathbf{y}' = (y'_{P'})_{P' \in \mathcal{P}'} \in \mathcal{U}(X^\ell, \mathcal{P}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ with $\pi(X^\ell, \mathcal{P})(\mathbf{y}') = \mathbf{y} = (y_P)_{P \in \mathcal{P}}$

$$\begin{aligned}
&B(y'_{P'}, 8\tilde{s}_{P'}(\mathbf{y}')^{1/3}) \Subset B(y_P, 4\tilde{s}_P(\mathbf{y})^{1/3}) \\
&\quad \text{and} \quad B(y'_{P'}, 4\tilde{s}_{P'}(\mathbf{y}')^{1/2}) \Subset B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2})
\end{aligned} \tag{6.19}$$

for all $P' \in \mathcal{P}'_P$. By these inclusions, by the inductive assumption, and by the final statement of Lemma 6.3, we have the equality, for $\mathbf{y}' \in \mathcal{U}_2$,

$$\mathbf{\Theta}'_{\mathcal{P}'}\left(\mathbf{\Theta}_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}'), \mathbf{y}'\right) = \mathbf{\Theta}_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}').$$

That is, for $\mathbf{y}' \in \mathcal{U}_2$, the pair $\mathbf{\Theta}_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}')$ is already flat on the relevant balls and thus the flattening construction of Lemma 6.3 does not change the pair. Hence, if we define

$$\mathbf{\Theta}_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}') = \mathbf{\Theta}'_{\mathcal{P}'}(\mathbf{\Theta}_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}'), \mathbf{y}'),$$

this flattening map will satisfy (6.17), completing the construction of the map $\mathbf{\Theta}_{\mathcal{P}'}$ and thus completing the induction. □

6.3. **The crude splicing map.** The crude splicing map:

$$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}} : \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \to \bar{\mathcal{C}}_{\mathfrak{t}} \tag{6.20}$$

is $S^1$-equivariant and defined on the domain

$$\begin{aligned}
&\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \\
&\quad = \{(F_P^{T,\mathfrak{g}}, [A_P, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}} \in \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) : 8\lambda([A_P, F_P^s, \mathbf{v}_P])^{1/3} \le s_P(F_P^{T,\mathfrak{g}})\},
\end{aligned} \tag{6.21}$$

where $\{s_P\}$ is the pullback of the family of separating functions $\{\tilde{s}_P\}$ constructed in Lemma 6.4 by the projection $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \to \Sigma(X^\ell, \mathcal{P})$.

The crude splicing map is defined exactly as is the splicing map in [9, §3.2] with the following exceptions.

First, the connections on $S^4$ being spliced in are elements of the spliced ends moduli space $\bar{M}^{s,\natural}_{spl,|P|}$ constructed in §5 rather than elements of the anti-self-dual moduli space $\bar{M}^{s,\natural}_{|P|}$.

Second, the metric on $X$ used to identify balls in $X$ with balls around the north pole of $S^4$ is fixed for the standard splicing map. In the crude splicing map, the metric varies with the splicing points $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$, varying in a smooth family of metrics $g_{\mathcal{P}}$.

Third, in the definition of the splicing map in [9, §3.2], the background connection $A_0$ is flattened on the balls

$$B(y_P, 2\lambda_P^{1/2})$$

while the background section is multiplied by a cut-off function vanishing on

$$B(y_P, 4\lambda_P^{1/3}),$$

where $\lambda_P$ is the scale of the connection on $S^4$ being spliced in at $x_P \in X$. The crude splicing map is defined instead by replacing the background pair $(A_0, \Phi_0)$ with the background pair $\mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y})$ constructed in Lemma 6.4. Thus, the background pair is flattened on balls whose radius does not depend on the scale of the connections on $S^4$.

The definition of the crude splicing map immediately implies the following observation about the crude splicing map.

**Lemma 6.5.** *Let* $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}})$ *be as defined in* (6.6). *Let* $\mathrm{id}_\Sigma : \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \cong \Sigma(X^\ell, \mathcal{P})$ *be the identification in* (6.7). *Then the restriction of the crude splicing map* $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ *to*

$$\tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) = N(\delta) \times \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}})$$

*takes values in*

$$\mathcal{C}_{\mathfrak{t}_0} \times \Sigma(X^\ell, \mathcal{P}) \subset \bar{\mathcal{C}}_{\mathfrak{t}}$$

*and for any* $\mathbf{F} \in \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}})$ *and* $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}_0,\mathfrak{s}}$,

$$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}((A_0, \Phi_0), \mathbf{F}) = (\mathbf{\Theta}_{\mathcal{P}}(A_0, \Phi_0, \mathrm{id}_\Sigma(\mathbf{F})), \mathrm{id}_\Sigma(\mathbf{F}))\,.$$

**Remark 6.6.** In contrast to the result of Lemma 6.5, the standard splicing map would satisfy

$$\boldsymbol{\gamma}'_{X,\mathcal{P}}((A_0, \Phi_0), \mathbf{F}) = ([A_0, \Phi_0], \mathrm{id}_\Sigma(\mathbf{F}))\,,$$

that is, the background pair are not flattened.

6.4. **The overlap spaces.** We now introduce spaces of overlap data $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, attempting to mimic the construction of the spliced end in §5.

6.4.1. *The overlap space.* For $\mathcal{P} < \mathcal{P}'$ partitions of $N_\ell$ and $g_{\mathcal{P}}$ a family of smooth metrics on $X$ parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$, we define the space of overlap data by:

$$\begin{aligned} &\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \\ &\quad = \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \times_{G(\mathcal{P})} \coprod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \left( \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) \right). \end{aligned} \tag{6.22}$$

Recall that we denote elements of $\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P)$ as $(v_Q)_{Q \in \mathcal{P}''_P}$ where $v_Q \in \mathbb{R}^4$. We then denote elements of $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$ by

$$\mathbf{A} = \left( (F_P^{T,\mathfrak{g}})_{P \in \mathcal{P}}, \left( (v_Q)_{Q \in \mathcal{P}''_P}, ([A_Q, F_Q^s, \mathbf{v}_Q])_{Q \in \mathcal{P}''_P} \right)_{P \in \mathcal{P}} \right) \in \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}), \tag{6.23}$$

where $(F_P^{T,\mathfrak{g}})_{P \in \mathcal{P}} \in \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}})$, $(v_Q)_{Q \in \mathcal{P}''_P} \in \Delta^\circ(Z_{|P|}, \mathcal{P}''_P)$, and $[A_Q, F_Q^s, \mathbf{v}_Q] \in \bar{M}^{s,\natural}_{spl,|Q|}(\delta)$.

The projection $\mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \to \Sigma(X^\ell, \mathcal{P})$ defines a projection,

$$\pi_\Sigma : \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \to \Sigma(X^\ell, \mathcal{P}). \tag{6.24}$$

Because $[\Theta, F^s, c_{|Q|}] \in \bar{M}^{s,\natural}_{spl,|Q|}$ is a fixed point of the action of $\mathrm{SO}(3)\times\mathrm{SO}(4)$, the subspace

$$\begin{aligned}&\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\\ &\quad= \{\left((F^{T,\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left((v_Q)_{Q\in\mathcal{P}''_P}, ([\Theta, F^s_Q, c_{|Q|}])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right) \in \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\}\end{aligned} \tag{6.25}$$

is identified with the normal bundle $\nu(X^\ell,\mathcal{P},[\mathcal{P}'])$

$$k_{\Sigma,\mathcal{P},[\mathcal{P}']} : \Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \cong \nu(X^\ell,\mathcal{P},[\mathcal{P}']) \tag{6.26}$$

as, the reader may recall, $\nu(X^\ell,\mathcal{P},[\mathcal{P}'])$ is defined by

$$\mathrm{Fr}(TX^\ell,\mathcal{P},g_{\mathcal{P}}) \times_{G_T(\mathcal{P})} \coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}''_P).$$

There is thus a projection,

$$\pi_{\Sigma,\mathcal{P},[\mathcal{P}']} : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \to \Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}), \tag{6.27}$$

given by

$$\begin{aligned}&\pi_T\left((F^T_P,F^{\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left((v_Q)_{Q\in\mathcal{P}''_P}, ([A_Q, F^s_Q, \mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right)\\ &\quad= \left((F^T_P,F^{\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left((v_Q)_{Q\in\mathcal{P}''_P}, ([\Theta, F^s_Q, c_{|Q|}])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right).\end{aligned}$$

We will then write

$$\pi_{\nu(\Sigma)} : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \to \nu(X^\ell,\mathcal{P},[\mathcal{P}']), \quad \pi_{\nu(\Sigma)} = k_{\Sigma,\mathcal{P},[\mathcal{P}']}\circ\pi_{\Sigma,\mathcal{P},[\mathcal{P}']}, \tag{6.28}$$

for the projection to the normal bundle.

6.4.2. *The upwards overlap map.* To work simultaneously with all the diagonals $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$, we introduce the gluing space,

$$\mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}) = \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}''}} \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''}) \times_{\tilde{G}(\mathcal{P}'')} \prod_{Q\in\mathcal{P}''} \bar{M}^{s,\natural}_{spl,|Q|}\right)/\mathfrak{S}(\mathcal{P}).$$

Note that by the $\mathfrak{S}(\mathcal{P})$ invariance of the families of metrics $g_{\mathcal{P}''}$ (see §6.1.2), the space $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'})$ depends only on the choice of one family of metrics $g_{\mathcal{P}'}$. By comparing the orbits under the symmetric group actions, one can see there is an identification

$$\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''}) \simeq \mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}),$$

for any $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$. Then, for each background connection $\hat{A}_0$ on $\mathfrak{g}_0$, we will define an upwards overlap map,

$$\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']} : \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}). \tag{6.29}$$

In words, the map $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ will be defined by keeping the $S^4$ connections, and parallel translating the frames $(F^T_P,F^{\mathfrak{g}}_P)$ from $y_P\in X$ to each $x_Q\in X$, where $(x_Q)_{Q\in\mathcal{P}''}$ is the image of $\mathbf{A}$ under the composition $e(X^\ell,\mathcal{P})\circ\pi_{\nu(\Sigma)}$.

We assume that the domain $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ of $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ satisfies

$$\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subset \pi^{-1}_{\nu(\Sigma)}\left(\mathcal{O}(X^\ell,\mathcal{P},[\mathcal{P}'])\right), \tag{6.30}$$

where $\mathcal{O}(X^\ell,\mathcal{P},[\mathcal{P}'])\subset\nu(X^\ell,\mathcal{P},[\mathcal{P}'])$ is the subspace mapped to the tubular neighborhood $\mathcal{U}(X^\ell,\mathcal{P})$ in Lemma 6.4. Then, the map $e(X^\ell,\mathcal{P})\circ\pi_{\nu(\Sigma)}$ is defined on $\mathcal{O}(X^\ell,\mathcal{P},[\mathcal{P}'])$ and takes values in $\mathcal{U}(X^\ell,[\mathcal{P}<\mathcal{P}'])$, the intersection of $\mathcal{U}(X^\ell,\mathcal{P})$ with the diagonals $\Sigma(X^\ell,\mathcal{P}'')$ for $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$.

Assume $x$ is a point in a geodesic ball around $y\in X$. Then for any connection $A$ on a bundle over $X$ and $F$ a frame of the bundle lying over $y$, let $T^A_{x,y}(F)$ denote parallel translation (parallel with respect to the connection $A$) of the frame $F$ from $x$ to $y$ along the radial geodesic. We will use $T^{\mathrm{LC}}_{x,y}$ to denote parallel translation with respect to the Levi-Civita connection.

We now define the upwards overlap map. Assume $\mathbf{A}\in\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])$ satisfies

$$\pi_\Sigma(\mathbf{A})=\mathbf{y}=(y_P)_{P\in\mathcal{P}}\quad\text{and}\quad e(X^\ell,\mathcal{P})\circ\pi_{\nu(\Sigma)}(\mathbf{A})=(x_Q)_{Q\in\mathcal{P}''}.$$

Then,

$$\begin{aligned}\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}(\mathbf{A})&=\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}\left((F^T_P,F^{\mathfrak{g}}_P)_{P\in\mathcal{P}},\left((v_Q)_{Q\in\mathcal{P}''_P},([A_Q,F^{\mathfrak{s}}_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right)\\&=\left((T^{\mathrm{LC}}_{x_Q,y_P}(F^T_P),T^{\hat A_0}_{x_Q,y_P}(F^{\mathfrak{g}}_P))_{Q\in\mathcal{P}''},([A_Q,F^{\mathfrak{s}}_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right),\end{aligned}\tag{6.31}$$

where the indices $P$ and $Q$ appearing in the parallel translations $T^{\mathrm{LC}}_{x_Q,y_P}$ and $T^A_{x_Q,y_P}$ satisfy $Q\subseteq P$. The Levi-Civita connection used to define $T^{\mathrm{LC}}_{x_Q,y_P}$ is the metric $g_{\mathcal{P},\mathbf{y}}$. The map $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ thus depends on the background connection $\hat A_0$ used to define $T^{\hat A_0}_{x_Q,y_P}$.

We define the subspace,

$$\begin{aligned}&T(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])\\&\quad=\{\left((F^T_P,F^{\mathfrak{g}}_P)_{P\in\mathcal{P}},\left((v_Q)_{Q\in\mathcal{P}''_P},[\Theta,F^{\mathfrak{s}}_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right)\in\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])\}.\end{aligned}\tag{6.32}$$

That is, $T(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])$ is the subspace of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ where all the $S^4$ connections are trivial. The proof of the following is trivial.

**Lemma 6.7.** *Let $\mathcal{P}<\mathcal{P}'$ be partitions of $N_\ell$. Assume the open subspace $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ satisfies the condition* (6.30) *Then, for each connection $\hat A_0$ on $\mathfrak{g}_0$, the map*

$$\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}:\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\to\mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}),$$

*defines an open embedding of $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ into $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'})$.*

The upwards overlap map $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ depends on the choice of a connection $\hat A_0$ on $\mathfrak{g}_0$. For a background pair $(A_0,\Phi_0)\in\tilde{\mathcal{C}}_{\mathfrak{t}_0}$, we will write $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}(A_0,\Phi_0)$ for the upwards overlap map defined by the connection $\hat A_0$. Then, we define

$$\rho^{N,\mathfrak{t},\mathfrak{s},u}_{[\mathcal{P}<\mathcal{P}']}:\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\to\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'})\tag{6.33}$$

by, for $(A_0,\Phi_0)\in\tilde N_{\mathfrak{t}_0,\mathfrak{s}}$ and $\mathbf{F}\in\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$,

$$\rho^{N,\mathfrak{t},\mathfrak{s},u}_{[\mathcal{P}<\mathcal{P}']}((A_0,\Phi_0),\mathbf{F})=\left((A_0,\Phi_0),\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}(\mathbf{\Theta}_{\mathcal{P}}(A_0,\Phi_0,\pi_\Sigma(\mathbf{F}))(\mathbf{F}))\right).$$

In words, $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{[\mathcal{P}<\mathcal{P}']}$ leaves the background pair alone and the restriction of $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{[\mathcal{P}<\mathcal{P}']}$ to

$$\mathcal{G}_{\mathfrak{s}}(A_0,\Phi_0)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})|_{\pi_\Sigma^{-1}(\mathbf{y})}$$

is equal to the upwards overlap map $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ given by the background connection $\hat A'$ where $(A',\Phi')=\mathbf{\Theta}_{\mathcal{P}}(A_0,\Phi_0,\mathbf{y})$.

6.4.3. *The downwards overlap map.* The downwards overlap map is a map,

$$\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},\mathcal{P}'} : \mathcal{O}_d(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}),$$

defined analogously to the map $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$. We wish to define, for $\mathbf{A} \in \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$,

$$\begin{aligned} \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}(\mathbf{A}) &= \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}\left((F^{T,\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left((v_Q)_{Q\in\mathcal{P}''_P}, ([A_Q,F^s_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right) \\ &= \left((F^{T,\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left(\boldsymbol{\gamma}'_{\mathbb{R}^4,\mathcal{P}''_P}\left((v_Q)_{Q\in\mathcal{P}''_P}, ([A_Q,F^s_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)\right)\right). \end{aligned} \tag{6.34}$$

In words, the map $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ is defined by applying the splicing map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}''_P}$ of (5.19) to the data in

$$\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}''_P) \times \prod_{Q\in\mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q). \tag{6.35}$$

Thus, for the map $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ to be defined, we must restrict $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ to the points where the data in (6.35) lie in the domain of the splicing map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}''_P}$. More formally, we must assume that domain of $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ satisfies

$$\begin{aligned} &\mathcal{O}_d(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \\ &\subseteq \{\left((F^T_P,F^{\mathfrak{g}}_P)_{P\in\mathcal{P}}, \left((v_Q)_{Q\in\mathcal{P}''_P}, ([A_Q,F^s_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right)_{P\in\mathcal{P}}\right) \in \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) : \\ &\quad \left((v_Q)_{Q\in\mathcal{P}''_P}, ([A_Q,F^s_Q,\mathbf{x}_Q])_{Q\in\mathcal{P}''_P}\right) \in \mathcal{O}^{\mathrm{asd}}_1(\mathbb{R}^4,\mathcal{P}''_P) \quad \text{for all } P\in\mathcal{P}\}, \end{aligned} \tag{6.36}$$

where $\mathcal{O}^{\mathrm{asd}}_1(\Theta,\mathcal{P}''_P,\delta_P)$ is the open set defined in Lemma 5.17.

**Lemma 6.8.** *Let* $\mathcal{O}_d(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subseteq \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ *be any open neighborhood of* $\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ *satisfying* (6.36)*. Then the restriction of* $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ *to* $\mathcal{O}_d(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ *is an open embedding whose image is an open neighborhood of* $\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ *in* $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$.

6.4.4. *The splicing equality.* We now show how the constructions of the overlap maps and crude splicing maps lead to a diagram similar to the diagram (5.52) used to construct the spliced end $W_\kappa$.

For $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, let $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''})$ be a $\mathfrak{S}(\mathcal{P})$-invariant collection of open subspaces on which the crude splicing maps $\boldsymbol{\gamma}''_{X,\mathcal{P}''}$ are defined. Define

$$\mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}) = \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''})\right)/\mathfrak{S}(\mathcal{P}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}). \tag{6.37}$$

Then, by the $\mathfrak{S}(\mathcal{P})$-invariance of the crude splicing maps $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}''}$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, the maps$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}''}$ define a crude splicing map

$$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']} : \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'}) \to \bar{\mathcal{C}}_{\mathfrak{t}}. \tag{6.38}$$

Given subspaces, $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ and an $\mathfrak{S}(\mathcal{P})$-invariant collection of subspaces $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''})$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, we define an open subspace,

$$\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \tag{6.39}$$

to be the points satisfying the conditions (6.30) and (6.36) and, in addition, satisfying

$$\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \subseteq (\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']})^{-1}\left(\mathcal{O}(\mathfrak{t},[\mathcal{P}<\mathcal{P}'])\right) \cap (\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})^{-1}\left(\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\right).$$

Then we have,

**Proposition 6.9.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Assume the families of metrics $g_{\mathcal{P}''}$ satisfy the conditions in §6.1.2. Let $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \subset \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ be an open subspace on which the crude splicing map is defined. Let $\{\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}'',g_{\mathcal{P}''})\}_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}$ be an $\mathfrak{S}(\mathcal{P})$-invariant collection of open subspaces on which the crude splicing maps, $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}''}$ are defined and let $\mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'})$ be the resulting space defined by (6.37). Write $\mathrm{id}_N$ for the identity map on $\tilde N = \tilde N_{\mathfrak{t}_0,\mathfrak{s}}$. Then, if $\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ is the subspace defined in (6.39), the maps in the diagram*

$$\begin{CD}
\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) @>{\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}}>> \tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'],g_{\mathcal{P}'})\\
@V{\mathrm{id}_N\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}}VV @VV{\gamma''_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}}V\\
\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) @>{\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}}>> \bar{\mathcal{C}}_{\mathfrak{t}}
\end{CD} \tag{6.40}$$

*are $S^1$-equivariant and the diagram commutes.*

*Proof.* The conditions in the definition of the subspace $\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ imply that the compositions

$$\gamma''_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}\circ\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']} \quad\text{and}\quad \gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}\circ(\mathrm{id}_N\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})$$

are defined on $\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$.

The $S^1$-equivariance follows from the $S^1$-equivariance of the crude splicing maps and the observation that the overlap maps in (6.40) are $S^1$-equivariant because the $S^1$ action is simply scalar multiplication on the section in $\tilde N_{\mathfrak{t}_0,\mathfrak{s}}$ and the overlap maps do not change the background section.

To see that the diagram (6.40) commutes, begin by restricting the diagram to

$$\mathcal{G}_{\mathfrak{s}}(A_0,\Phi_0)\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})|_{\pi_\Sigma^{-1}(\mathbf{y})}\subset\tilde N\times_{\mathcal{G}_{\mathfrak{s}}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$$

for $(A_0,\Phi_0)\in\tilde N$ and $\mathbf{y}\in\Delta^\circ(X^\ell,\mathcal{P})$. The definition of the overlap maps imply that the crude splicing maps are defined by splicing in $S^4$ connections at $\mathbf{y}$ for $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and at some $\mathbf{y}'$ for $\gamma''_{X,[\mathcal{P}<\mathcal{P}'}$ where $\pi(X^\ell,\mathcal{P})(\mathbf{y}')=\mathbf{y}$. Then, the consistency of the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ imply $g_{\mathcal{P},\mathbf{y}}=g_{\mathcal{P}',\mathbf{y}'}$. Similarly, the consistency condition of the flattening maps in (6.17) imply that $(A_0',\Phi_0')=\mathbf{\Theta}_{\mathcal{P}}(A_0,\Phi_0,\mathbf{y})=\mathbf{\Theta}_{\mathcal{P}'}(A_0,\Phi_0,\mathbf{y}')$. This proves that the sections obtained by the two compositions in the diagram (6.40) are equal to $\Phi_0'$. The connections obtained by the two compositions in the diagram (6.40) are defined by splicing connections on $S^4$ to the background connection $A_0'$.

The property (6.30) of $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ implies that $\mathbf{y}'$ lies in the suitably small tubular neighborhood $\mathcal{U}(X^\ell,\mathcal{P})$ defined in Lemma 6.4. Thus, from (6.19) the inclusions

$$B(y'_{P'},4\tilde s_{P'}(\mathbf{y}')^{1/2})\Subset B(y_P,2\tilde s_P(\mathbf{y})^{1/2})$$

hold for the balls where the splicing takes place. Then because the connection $\hat A_0'$ and metric $g_{\mathcal{P},\mathbf{y}}$ are flat on each ball $B(y_P,2\tilde s_P(\mathbf{y})^{1/2})$, the proof of the proposition reduces to Lemma 5.11. □

6.5. **Constructing the space of global splicing data.** The construction of the space of global splicing data follows from Proposition 6.9 and the arguments on shrinking the neighborhoods $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ given in the construction of the spliced end $W_\kappa$ in §5.4.

**Theorem 6.10.** *For every partition $\mathcal{P}$ of $N_\ell$, there is a neighborhood $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ of $\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ in $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ such that the intersection of the images*

$$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}(\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}))\cap\boldsymbol{\gamma}''_{\mathcal{P}'}(\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}',g_{\mathcal{P}'}))$$

*is empty if $\mathcal{P}\not<\mathcal{P}'$ and $\mathcal{P}'\not<\mathcal{P}$ and is contained in*

$$\begin{aligned}\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}&\left(\mathrm{id}_N\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}\left(\tilde N\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\right)\right)\\&=\boldsymbol{\gamma}''_{[\mathcal{P}<\mathcal{P}']}\left(\rho^{N,d}_{\mathcal{P},[\mathcal{P}']}\left(\tilde N\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})\right)\right)\end{aligned}$$

*if $\mathcal{P}<\mathcal{P}'$, where $\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ is the subspace appearing in Proposition 6.9.*

*Proof.* The proof is identical to the arguments appearing in Lemma 5.17 for the construction of $W_\kappa$. □

We then define the *space of global splicing data* to be

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}=\sqcup_{\mathcal{P}}\left(\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\right)/\sim \tag{6.41}$$

where $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ is the space appearing in Theorem 6.10 and points in

$$\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\quad\text{and}\quad\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}',g_{\mathcal{P}'})$$

are identified by the relation $\sim$ if their images under the crude splicing maps are equal. Thus, the space $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ is the homotopy pushout of the diagram (6.40) in the sense of the following lemma.

**Lemma 6.11.** *If $\mathcal{P}<\mathcal{P}'$ are partitions of $N_\ell$, $(A_0,\Phi_0,\mathbf{F})\in\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ and $(A_0',\Phi_0',\mathbf{F}')\in\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}',g_{\mathcal{P}'})$, then*

$$(A_0,\Phi_0,\mathbf{F})\sim(A_0',\Phi_0',\mathbf{F}')$$

*if and only if $(A_0,\Phi_0)=(A_0',\Phi_0')$ and there is $(A_0,\Phi_0,\mathbf{A})\in\tilde N\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ satisfying*

$$\begin{aligned}(A_0,\Phi_0,\mathbf{F})&=(\mathrm{id}_N\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})(A_0,\Phi_0,\mathbf{F}_0)\\(A_0',\Phi_0',\mathbf{F}')&=\rho^{N,d}_{\mathcal{P},[\mathcal{P}']}(A_0,\Phi_0,\mathbf{F}_0).\end{aligned}\tag{6.42}$$

Because the overlaps of the images of $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ are controlled by the diagram (6.40) and because the maps $\mathrm{id}_N\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ and $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ are smoothly-stratified embeddings, we have the following.

**Corollary 6.12.** *The space $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ is a smoothly-stratified space admitting an $S^1$-equivariant fibration,*

$$\pi_N:\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}\to N_{\mathfrak{t}_0,\mathfrak{s}}(\delta),\tag{6.43}$$

*and an $S^1$-equivariant embedding*

$$\boldsymbol{\gamma}''_{\mathcal{M}}:\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}\to\bar{\mathcal{C}}_{\mathfrak{t}},\tag{6.44}$$

*such that the restriction of $\boldsymbol{\gamma}''_{\mathcal{M}}$ to $\tilde N(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ is equal to $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$.*

6.6. **Thom-Mather structures on the space of global splicing data.** On each open subspace

$$\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) = \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \subset \bar{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}},$$

there is a projection map

$$\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}) : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to N(\delta) \times \Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \tag{6.45}$$

defined by $\mathrm{id}_N \times \pi_\Sigma$, where $\pi_\Sigma$ is the restriction of the projection defined in (6.10) to $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ and $\mathrm{id}_N$ is the identity on $\tilde{N}(\delta)$.

To prove the relations (4.1) holds for the projection maps $\pi(\mathfrak{t},\mathfrak{s},\mathcal{P})$ we make the following observations on the overlap maps. First, note that the projection maps,

$$\pi(\Theta,\mathcal{P}'_P) : \mathcal{U}(\Theta,\mathcal{P}'_P) \subset \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) \to \{[\Theta]\} \times \Sigma(Z_P(\delta_P),\mathcal{P}'_P)$$

defined in (5.66) are $\mathrm{SO}(3)\times\mathrm{SO}(4)$ equivariant. Together with the identify map $\mathrm{id}_N$ on $\tilde{N}(\delta)$, the maps $\pi(\Theta,\mathcal{P}''_P)$ (for $\mathcal{P}'' \in [\mathcal{P}<\mathcal{P}']$) define a map, $\mathrm{id}_N \times p_{\mathcal{P},[\mathcal{P}']}$,

$$\begin{array}{c} \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \times_{\tilde{G}(\mathcal{P})} \prod_{P\in\mathcal{P}} \coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \mathcal{U}(\Theta,\mathcal{P}''_P) \\ \Big\downarrow{\scriptstyle \mathrm{id}_N \times p_{\mathcal{P},[\mathcal{P}']}} \\ \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \times_{\tilde{G}(\mathcal{P})} \prod_{P\in\mathcal{P}} \coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \{[\Theta]\} \times \Sigma(Z_P(\delta_P),\mathcal{P}''_P) \end{array} \tag{6.46}$$

Note that because

$$\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \mathcal{U}(\Theta,\mathcal{P}''_P) \subset \bar{M}^{s,\natural}_{spl,|P|}(\delta_P), \quad \text{and}$$

$$\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']} \{[\Theta]\} \times \Sigma(Z_P(\delta_P),\mathcal{P}''_P) \subset \bar{M}^{s,\natural}_{spl,|P|}(\delta_P)$$

the map $\mathrm{id}_N \times p_{\mathcal{P},\mathcal{P}'}$ is defined on a subspace of $\tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$. Observe that the relation

$$(\mathrm{id}_N \times \pi_\Sigma) \circ (\mathrm{id}_N \times p_{\mathcal{P},\mathcal{P}'}) = (\mathrm{id}_N \times \pi_\Sigma) \tag{6.47}$$

holds where $\pi_\Sigma$ is the projection given in (6.10) because $p_{\mathcal{P},\mathcal{P}'}$ respects the fibers of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$.

Then for $\mathcal{P} < \mathcal{P}'$, the following lemma describes the projection map $\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ on the image of the overlap map $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$.

**Lemma 6.13.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. For $\pi_{\Sigma,\mathcal{P},[\mathcal{P}']}$ the projection defined in (6.27), let*

$$\mathrm{id}_N \times \pi_{\Sigma,\mathcal{P},[\mathcal{P}']} : \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}}) \to N(\delta) \times \Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$$

*be the product of $\pi_{\Sigma,\mathcal{P},[\mathcal{P}']}$ with the identity map on $\tilde{N}(\delta)$. Then, the relations*

$$\begin{gathered} \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}') \circ \rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']} = \rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']} \circ (\mathrm{id}_N \times \pi_{\Sigma,\mathcal{P},[\mathcal{P}']}) \\ (\mathrm{id}_N \times p_{\mathcal{P},[\mathcal{P}']}) \circ (\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}) = (\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}) \circ (\mathrm{id}_N \times \pi_{\Sigma,\mathcal{P},[\mathcal{P}']}) \end{gathered} \tag{6.48}$$

*hold on $\tilde{N} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$.*

We now prove the first Thom-Mather relation between the projections.

**Lemma 6.14.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Then, restricted to the intersection*

$$\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \cap \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}'),$$

*the equality*

$$\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}) \circ \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}') = \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}) \tag{6.49}$$

*holds.*

*Proof.* By Theorem 6.10, the intersection $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ is contained in the image of the overlap maps in the diagram (6.40). Therefore every point $\mathbf{A}'$ in the intersection is given by

$$\mathbf{A}' = (\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})(\mathbf{A}) = \rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}(\mathbf{A}).$$

for some $\mathbf{A}\in \tilde N \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$. Then

$$\begin{aligned}
\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}')(\mathbf{A}') &= \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}')\circ \rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}(\mathbf{A})\\
&= \rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}\circ(\mathrm{id}_N\times\pi_{\Sigma,\mathcal{P},[\mathcal{P}']})(\mathbf{A}) \quad \text{By equation (6.48)}\\
&= (\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})\circ(\mathrm{id}_N\times\pi_{\Sigma,\mathcal{P},[\mathcal{P}']})(\mathbf{A}) \quad \text{By the definition of } \sim\\
&= (\mathrm{id}_N\times p_{\mathcal{P},[\mathcal{P}']})\circ(\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']})(\mathbf{A}) \quad \text{By equation (6.48)}
\end{aligned}$$

The preceding equality implies

$$\begin{aligned}
&(\pi(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}'))(\mathbf{A}')\\
&\quad= (\pi(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ(\mathrm{id}_N\times p_{\mathcal{P},[\mathcal{P}']})\circ(\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}))(\mathbf{A})\\
&\quad= ((\mathrm{id}_N\times\pi_\Sigma)\circ(\mathrm{id}_N\times p_{\mathcal{P},[\mathcal{P}']})\circ(\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}))(\mathbf{A}) \quad \text{By the definition following (6.45)}\\
&\quad= (\mathrm{id}_N\times\pi_\Sigma)\circ(\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}))(\mathbf{A}) \quad \text{By equation (6.47)}\\
&\quad= (\pi(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ(\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}))(\mathbf{A}) \quad \text{By the definition following (6.45)}\\
&\quad= (\pi(\mathfrak{t},\mathfrak{s},\mathcal{P})(\mathbf{A}'),
\end{aligned}$$

thus proving (6.49). □

We now define a tubular distance function,

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P}) : \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to [0,1)$$

which is invariant under the actions of $\mathcal{G}_{\mathfrak{s}}$ and $S^1$ on $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ and which satisfies:

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P})^{-1}(0) = \Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}).$$

The natural choice is

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P}) = \sum_{P\in\mathcal{P}} |P|\tilde\lambda_P^2, \tag{6.50}$$

where $\tilde\lambda_P$ is the modified scale parameter defined in (5.69). We shall also use a rescaling of this map.

We note that all maps defined on subspaces of $\mathrm{Sym}^\ell(X)$ define maps on the subspace $N(\delta)\times\mathrm{Sym}^\ell(X)$ by pullback.

**Lemma 6.15.** *There are tubular distance functions,*

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P}) : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to [0,1), \quad \textit{and} \quad t_r(\mathfrak{t},\mathfrak{s},\mathcal{P}) : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to [0,1) \tag{6.51}$$

*satisfying*

(1) $t(\mathfrak{t},\mathfrak{s},\mathcal{P})^{-1}(0) = t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})^{-1}(0) = N(\delta)\times\Sigma(X^\ell,\mathcal{P})$,

(2) *The restriction of* $t(\mathfrak{t},\mathfrak{s},\mathcal{P})$ *to the subspace*

$$\Big(N(\delta)\times\mathcal{U}''(X^\ell,\mathcal{P},g_{\mathcal{P}})\Big)\cap U(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

*is equal to the pullback of the map* $t(X^\ell,\mathcal{P})$ *defined in* (4.37) *under the obvious projection,*

(3) *On the intersection* $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ *the relations,*

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}')=t(\mathfrak{t},\mathfrak{s},\mathcal{P})\quad\text{and}\quad t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}')=t_r(\mathfrak{t},\mathfrak{s},\mathcal{P}) \tag{6.52}$$

*hold.*

(4) *There are open neighborhoods*

$$U_1(\mathfrak{t},\mathfrak{s},\mathcal{P})\Subset\mathcal{U}_2(\mathfrak{t},\mathfrak{s},\mathcal{P})\Subset\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

*of* $N(\delta)\times\Sigma(X^\ell,\mathcal{P})$, *such that*

$$\begin{aligned}\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})&=t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})^{-1}([0,\tfrac{1}{2})),\\ \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})-\mathcal{U}_2(\mathfrak{t},\mathfrak{s},\mathcal{P})&=t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})^{-1}(\{1\})\end{aligned} \tag{6.53}$$

*Proof.* The first and second items follow immediately from the definition. To prove the equality (6.52) for the function $t(\mathfrak{t},\mathfrak{s},\mathcal{P})$ defined in (6.50), we begin by recalling that the modified scale function $\tilde{\lambda}_P$ satisfied

$$\tilde{\lambda}_P\circ\pi(\Theta,\mathcal{P}''_P)=\tilde{\lambda}_P.$$

Then the definition of the projection $\mathrm{id}_N\times p_{\mathcal{P},[\mathcal{P}']}$ in (6.46) implies

$$t(\mathfrak{t},\mathfrak{s},\mathcal{P})\circ(\mathrm{id}_N\times p_{\mathcal{P},[\mathcal{P}']})=t(\mathfrak{t},\mathfrak{s},\mathcal{P}).$$

The equality (6.52) then follows by the argument in Lemma 6.14.

The function $t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})$ is defined by a rescaling of $t(\mathfrak{t},\mathfrak{s},\mathcal{P})$ (i.e. multiplying by a function pulled back from $N(\delta)\times\Sigma(X^\ell,\mathcal{P})$) which does not affect the property (6.52). The third item then follows by the argument given for item (6) in Lemma 5.20. □

6.7. **The global splicing map.** The global crude splicing map $\gamma''_{\mathcal{M}}$ of (6.44) is unsatisfactory because its image does not include $M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X)$ (see Lemma 6.5 and Remark 6.6). To remedy this, we define an isotopy of the embedding $\gamma''_{\mathcal{M}}$.

First, observe that if $\beta:[0,1]\to[0,1]$ is a smooth function with $\beta([0,\frac{1}{4}])=1$ and $\beta([\frac{1}{2},1])=0$, then $\beta\circ t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})$ is supported in $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})$ and vanishes on $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})-\mathcal{U}_2(\mathfrak{t},\mathfrak{s},\mathcal{P})$ where the spaces $\mathcal{U}_i(\mathfrak{t},\mathfrak{s},\mathcal{P})$ are given in appearing in Lemma 6.15. Hence, the function

$$\beta_{\mathcal{P}}=\beta\circ t_r(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

can be considered a global function on $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ satisfying:

$$\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}-\mathcal{U}_2(\mathfrak{t},\mathfrak{s},\mathcal{P})\subset\beta_{\mathcal{P}}^{-1}(0)\quad\text{and}\quad\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})\subset\beta_{\mathcal{P}}^{-1}(1). \tag{6.54}$$

Pick one representative $\mathcal{P}$ of each conjugacy class $[\mathcal{P}]$ and enumerate these representatives in such a way that

$$\Sigma(X^\ell,\mathcal{P}_i)\subset\mathrm{cl}_{\mathrm{Sym}^\ell(X)}\Sigma(X^\ell,\mathcal{P}_j)$$

implies $i<j$. Assume $\mathcal{P}_1=\{N_\ell\}$ is the crudest partition. We define a finite sequence of embeddings,

$$\gamma''_{\mathcal{M},i}:\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}\to\bar{\mathcal{C}}_{\mathfrak{t}},$$

by, if $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ is the standard splicing map defined on $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$,

$$\begin{aligned}\gamma''_{\mathcal{M},0} &= \gamma''_{\mathcal{M}},\\ \gamma''_{\mathcal{M},i} &= \beta_{\mathcal{P}}(\cdot)\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}(\cdot) + (1-\beta_{\mathcal{P}})(\cdot)\gamma''_{\mathcal{M},i-1}(\cdot).\end{aligned} \tag{6.55}$$

Observe that although $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ is only defined on the subspace $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \subset \bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$, the convex combination of the embeddings $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\gamma''_{\mathcal{M},i-1}$ is defined on all of $\bar{\mathcal{M}}^{ir}_{\mathfrak{t},\mathfrak{s}}$ because $\beta_{\mathcal{P}}$ vanishes on the complement of $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$. By (6.54), $\gamma''_{\mathcal{M},i}$ is equal to the standard splicing map $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ on $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})$ and equal to $\gamma''_{\mathcal{M},i-1}$ on $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}} - \mathcal{U}_2(\mathfrak{t},\mathfrak{s},\mathcal{P})$. Assume that $\mathcal{P}_r = N_\ell$ is the finest partition. We then define

$$\gamma'_{\mathcal{M}} = \gamma''_{\mathcal{M},r}, \tag{6.56}$$

It is straightforward to verify that

(1) The map $\gamma'_{\mathcal{M}}$ is a continuous $S^1$-equivariant embedding and is smooth on each stratum,
(2) The restriction of $\gamma'_{\mathcal{M}}$ to $N(\delta)\times \operatorname{Sym}^\ell(X)$ is equal to the product of the embedding $N(\delta)\to \mathcal{C}_{\mathfrak{t}_0}$ with the identity on $\operatorname{Sym}^\ell(X)$.

6.8. **Projection to** $\operatorname{Sym}^\ell(X)$**.** We now define a projection map,

$$\pi_X : \bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}} \to \operatorname{Sym}^\ell(X), \tag{6.57}$$

which will be used in describing cohomology classes on $\bar{\mathcal{M}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$.

We will denote the projection $\pi_\Sigma$ of (6.10) by

$$\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}) : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \Sigma(X^\ell,\mathcal{P}),$$

to avoid confusion between the projection maps of the different strata. These local projections maps do not agree on the intersections $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ but can be related by the following.

**Lemma 6.16.** *If $\mathcal{P} < \mathcal{P}'$ are partitions of $N_\ell$, then*

$$\pi(X^\ell,\mathcal{P})\circ\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}') = \pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}) \tag{6.58}$$

*on $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$.*

*Proof.* The lemma follows immediately from Lemma 6.11 and the definitions of the projection maps. □

For $\mathcal{P}$ a partition of $N_\ell$, let

$$r_{\mathcal{P}} : \mathcal{U}(X^\ell,\mathcal{P})\times[0,1] \to \mathcal{U}(X^\ell,\mathcal{P}),$$

be the deformation retraction of $\mathcal{U}(X^\ell,\mathcal{P})$ to $\Sigma(X^\ell,\mathcal{P})$ defined by identifying $\mathcal{U}(X^\ell,\mathcal{P})$ with the normal bundle $\nu(X^\ell,\mathcal{P})$ by the exponential map $e(X^\ell,\mathcal{P},g_{\mathcal{P}})$ and fiber multiplication by the parameter $t\in[0,1]$. Thus, $r_{\mathcal{P}}(\cdot,1)$ is the identity map and $r_{\mathcal{P}}(\cdot,0) = \pi(X^\ell,\mathcal{P},g_{\mathcal{P}})$. Because $r_{\mathcal{P}'}(\cdot,1)$ is the identity, we can define

$$r_{\mathcal{P}}\left(\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}')(\cdot), 1-\beta_{\mathcal{P}}(\cdot)\right) : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}') \to \Sigma(X^\ell,\mathcal{P}')\cup\Sigma(X^\ell,\mathcal{P}).$$

Observe that on $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ the preceding map is equal to $\pi(X^\ell,\mathcal{P})\circ\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ which by Lemma 6.16 is equal to $\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$. We now prove the existence of the global projection $\pi_X$ on an open subspace of $\bar{M}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$.

**Lemma 6.17.** *There is an open neighborhood* $\mathcal{U}_1 \subset \bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ *of* $N(\delta) \times \mathrm{Sym}^\ell(X)$ *and a projection map,*

$$\pi_X : \mathcal{U}_1 \to \mathrm{Sym}^\ell(X),$$

*such that the restriction of* $\pi_X$ *to* $\mathcal{U}_1 \cap \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ *is homotopic to* $\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$.

*Proof.* We construct the map

$$\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}') : \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}') \to \mathrm{cl}\Sigma(X^\ell,\mathcal{P}')$$

inductively so that

(1) The map $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ is homotopic to $\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$,
(2) For all $\mathcal{P} < \mathcal{P}'$, on $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P}) \cap \mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})$, we have $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}') = \tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$.

The open neighborhood $\mathcal{U}_1$ refered to in the statement of the lemma will be the union of $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ and the map $\pi_X$ will be defined by $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$ on each subspace $\mathcal{U}_1(\mathfrak{t},\mathfrak{s},\mathcal{P})$.

Let $\mathcal{P}_0 = \{N_\ell\}$ be the crudest partition of $N_\ell$. Let $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}_0) = \pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}_0)$.

Assume $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$ has been constructed for all $\mathcal{P} < \mathcal{P}'$. The projection $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$ is constructed by induction on $\mathcal{P}'$ with $\mathcal{P} < \mathcal{P}'$ using the deformation $r_{\mathcal{P}'}$ and the parameter given by the function $t_r(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ to homotope between $\pi_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$ and a new map $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ equal to $\tilde{\pi}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$. The inductive assumption ensures there is no conflict for different $\mathcal{P}'$. Let $\tilde{\pi}_\Sigma(\mathfrak{t},\mathfrak{s},\mathcal{P})$ be the end of this string of homotopies. □

## 7. The spliced-end obstruction bundle

The gluing map, described in Theorem 8.2, does not map all points in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ to the moduli space $\bar{\mathcal{M}}_{\mathfrak{t}}$. Instead, there is a section (the *obstruction section*) of a pseudo-vector bundle (the *obstruction bundle*) over $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ such that the intersection of $\mathcal{M}_{\mathfrak{t}}$ with the image of the gluing map is given by the image of the zero-locus of the obstruction section.

The obstruction bundle has two components: the background obstruction and the instanton obstruction. In this section, we define a pseudo-bundle, the *spliced-end obstruction* bundle, $\bar{\Upsilon}^i_{spl,\kappa} \to \bar{M}^{s,\natural}_{spl,\kappa}(\delta)$ which will be used to define the instanton obstruction over $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$.

The restriction of $\bar{\Upsilon}^i_{spl,\kappa}$ to the top stratum of $\bar{M}^{s,\natural}_{spl,\kappa}$, $\Upsilon^i_{spl,\kappa} \to M^{s,\natural}_{spl,\kappa}$ is a finite-dimensional subbundle of the infinite-dimensional vector bundle,

$$\mathfrak{V}_\kappa = \mathcal{A}^\natural_\kappa(2\delta) \times \mathrm{Fr}(E_\kappa)|_s \times_{\mathcal{G}_\kappa} L^2_{k-1}(V^-_\kappa) \to \mathcal{B}^s_\kappa(2\delta), \tag{7.1}$$

restricted to $M^{s,\natural}_{spl,\kappa}(\delta)$, where $V^\pm_\kappa = \mathbf{S}^\pm \otimes E_\kappa$ for the standard spin structure $(\mathfrak{S}^\pm,\rho)$ on $S^4$. The pseudo-bundle $\bar{\Upsilon}^i_{spl,\kappa}$ will be a sub-bundle of the space

$$\bar{\mathfrak{V}}_\kappa = \sqcup^\kappa_{\ell=0} \mathfrak{V}_{\kappa-\ell} \times \mathrm{Sym}^\ell(X), \tag{7.2}$$

but this extension will only be a pseudo-bundle rather than as a vector bundle as the rank of the extension will decrease on the lower strata.

The bundle $\Upsilon^i_{spl,\kappa}$ is an approximation to the index bundle $\mathrm{Index}(\mathbf{D}_\kappa)$ associated to the Dirac operator $D_A : \Gamma(V^-_\kappa) \to \Gamma(V^+_\kappa)$ (as defined in §7.1) but an approximation that works more conveniently with the splicing construction of the space of global splicing data. To use the $\bar{\Upsilon}^i_{spl,\kappa}$ to define an obstruction bundle over the space of splicing data, the $\bar{\Upsilon}^i_{spl,\kappa}$ must be equivariant with respect to a $\mathrm{Spin}^u(4)$ action described in §7.2, just as the spliced ends moduli space is closed under the $\mathrm{SO}(3) \times \mathrm{SO}(4)$ action. The construction of the $\bar{\Upsilon}^i_{spl,\kappa}$ is analogous to the that of the spliced-ends moduli space. Using induction, we require the restriction of $\bar{\Upsilon}^i_{spl,\kappa}$ to the image of the splicing map $\gamma'_{\Theta,\mathcal{P}}$ to be given by the image of a splicing map for sections of (7.2). An inductive argument, analogous to that used in constructing the subspace $W_\kappa$ in §5.4. Then, following the argument in §5.6, the resulting bundle over $W_\kappa$ is isotoped to one which is equal to $\mathrm{Index}(\mathbf{D}^*_\kappa)$ away from the trivial strata. These last arguments, because they are so similar to those in §5 are carried out in a compressed fashion in §7.3; the properties of $\bar{\Upsilon}^i_{spl,\kappa}$ are enumerated in Proposition 7.1.

7.1. **The equivariant Dirac index bundle.** We begin by defining the bundle $\mathrm{Index}(\mathbf{D}^*_\kappa)$. Let $\mathfrak{s}_{S^4} = (\rho, \mathbf{S}^+, \mathbf{S}^-)$ be the standard spin structure on $S^4$ with respect to the round metric. Let $E_\kappa \to S^4$ be the rank-two, complex Hermitian bundle with $c_2(E_\kappa) = \kappa$. For $[A] \in M^{s,\natural}_\kappa(\delta)$ we define the Dirac operator

$$D_A : L^2_k(\mathbf{S}^+ \otimes E_\kappa) \to L^2_{k-1}(\mathbf{S}^- \otimes E_\kappa), \tag{7.3}$$

to be the Dirac operator defined by the Levi-Civita connection on $\mathfrak{s}_{S^4}$ and $A$ on $E_\kappa$. The spliced-end obstruction bundle is an approximation to the bundle

$$\mathrm{Index}(\mathbf{D}^*_\kappa) \to M^{s,\natural}_\kappa(\delta),$$

defined by the cokernels of the Dirac operators:

$$\mathrm{Index}(\mathbf{D}^*_\kappa) = \{(A, F^s, \Psi) \in \tilde{M}^{s,\natural}_\kappa \times \mathrm{Fr}(E_\kappa)|_s \times L^2_{k-1}(\mathbf{S}^- \otimes E_\kappa) : D^*_A \Psi = 0\}/\mathcal{G}_\kappa. \tag{7.4}$$

Because of the positive scalar curvature on $S^4$, the Weitzenbock estimates in [14] imply that the cokernel of $D^*_A$ vanishes for $[A] \in M^{s,\natural}_\kappa(\delta)$ and thus $\mathrm{Index}(\mathbf{D}^*_\kappa)$ actually forms a vector bundle. An index theorem computation shows that $\mathrm{Index}(\mathbf{D}^*_\kappa)$ is a complex, rank $\kappa$ vector bundle. We will also use the notation $\mathrm{Index}(\mathbf{D}^*_\kappa)$ to denote the bundle over $\bar{M}^{s,\natural}_\kappa(\delta)$ given by,

$$\{[A, F^s, \mathbf{x}, \Psi] : [A, F^s, \mathbf{x}] \in \bar{M}^{s,\natural}_\kappa(\delta) \quad \text{and} \quad D_A\Psi = 0\}.$$

Because the index of the Dirac operator depends on $\kappa$, this extension of $\mathrm{Index}(\mathbf{D}^*_\kappa)$ is only a pseudo-vector bundle, that is, the rank of the fiber depends on the stratum.

7.2. **The action of Spin$^u$(4).** The group

$$\mathrm{Spin}^u(4) = (\mathrm{U}(2) \times \mathrm{Spin}^c(4))/S^1$$

is defined in [11, §3.2] as are a pair of surjective homomorphisms

$$\mathrm{Ad}^u_{\mathrm{SO}(3)} : \mathrm{Spin}^u(4) \to \mathrm{SO}(3), \quad \mathrm{Ad}^u_{\mathrm{SO}(4)} : \mathrm{Spin}^u(4) \to \mathrm{SO}(4)$$

(see [11, Equation (3.13)]). The bundle $\mathfrak{V}_\kappa \to \mathcal{B}^s_\kappa$ is an equivariant $\mathrm{Spin}^u(4)$ bundle with respect to the action of $\mathrm{Spin}^u(4)$ on $\mathcal{B}^s_\kappa$ defined by the homomorphism

$$\mathrm{Spin}^u(4) \to \mathrm{SO}(3) \times \mathrm{SO}(4), \quad A \mapsto (\mathrm{Ad}(A), \mathrm{Ad}(A)).$$

The bundle $\mathrm{Index}(\mathbf{D}^*_\kappa)$ is invariant under this action and thus is also an $\mathrm{Spin}^u(4)$ equivariant bundle.

7.3. **The splicing map.** The spliced-ends obstruction bundle will be identical to the index bundle $\mathrm{Index}(\mathbf{D}^*_\kappa)$ on the complement of the spliced ends of $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$. On the spliced end, $W_\kappa$, the spliced-ends obstruction bundle will be given by the image of a splicing map which we now introduce.

Recall that the end of $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$ near the trivial stratum,

$$\{[\Theta]\} \times \Sigma(Z_\kappa(\delta), \mathcal{P})$$

where $\mathcal{P}$ is a partition of $N_\kappa$ with $|\mathcal{P}| > 1$, was given by the image of a splicing map

$$\gamma'_{\Theta,\mathcal{P}} : \mathcal{O}^{\mathrm{asd}}(\Theta, \mathcal{P}) \subset \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) \to \bar{M}^{s,\natural}_{spl,\kappa}(\delta).$$

We will define the spliced-ends obstruction bundle inductively.

An element of the space $\bar{\mathfrak{V}}_\kappa$ from (7.2) can be written as $[A, F^s, \mathbf{x}, \Psi]$ where $[A, F^s, \mathbf{x}] \in \bar{\mathcal{B}}^s_\kappa(2\delta)$, $\mathbf{x} \in \operatorname{Sym}^\ell(\mathbb{R}^4)$, and $\Psi \in L^2_{k-1}(\mathbf{S}^- \otimes E_{\kappa-\ell})$. Then, for each partition $\mathcal{P}$ of $N_\kappa$ we define an embedding, covering the splicing map $\boldsymbol{\gamma}_{\Theta,\mathcal{P}}$ by,
(7.5)

$$\varphi'_{\Theta,\mathcal{P}} : \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{\mathfrak{V}}_{|P|} \to \bar{\mathfrak{V}}_\kappa,$$

$$\varphi'_{\Theta,\mathcal{P}}\left([x_P, [A_P, F^s_P, v_P, \Psi_P]]_{P\in\mathcal{P}}\right) = \left(\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}([x_P, [A_P, F^s_P, v_P]_{P\in\mathcal{P}}), \sum_{P\in\mathcal{P}} \chi_{x_P,\lambda_P} c^*_{x_P,1}\Psi_P\right).$$

Here $\lambda_P = \lambda([A_P, F^s_P, \mathbf{x}_P])$ and $\chi_{x,\lambda}$ is a cut-off function supported on $B(x, \frac{1}{4}\lambda^{1/3})$ and equal to one on $B(x, \frac{1}{8}\lambda^{1/2})$. Observe that the map (7.5) is equivariant with respect to the $\operatorname{Spin}^u(4)$ action if $\operatorname{Spin}^u(4)$ acts diagonally on the domain, by the projection $\operatorname{Ad}^u_{\mathrm{SO}(4)}$ to $\mathrm{SO}(4)$ on the factor $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. With this vocabulary established, we can now state the proposition detailing the properties of the spliced-end obstruction bundle.

**Proposition 7.1.** *There is a pseudo-bundle*

$$\bar{\Upsilon}^i_{spl,\kappa} \to \bar{M}^{s,\natural}_{spl,\kappa}(\delta) \tag{7.6}$$

*which is a $\mathit{Spin}^u(4)$-invariant subbundle of the space $\bar{\mathfrak{V}}_\kappa \to \bar{\mathcal{B}}^{s,\natural}_\kappa(2\delta)$ and satisfies*

1. *The restriction of $\bar{\Upsilon}^i_{spl,\kappa}$ to the top stratum, $M^{s,\natural}_{spl,\kappa}(\delta)$, is a complex, rank $\kappa$ vector bundle,*
2. *For any partition $\mathcal{P}$ of $\kappa$ with $|\mathcal{P}| > 1$, the restriction of $\bar{\Upsilon}^i_{spl,\kappa}$ to the neighborhood of $\{[\Theta]\} \times \Sigma(Z_\kappa(\delta), \mathcal{P})$ given in* (5.3) *is given by the image of*
$$\Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{\Upsilon}^i_{spl,|P|}, \tag{7.7}$$
*under the splicing map $\varphi'_{\Theta,\mathcal{P}}$,*
3. *The restriction of $\bar{\Upsilon}^i_{spl,\kappa}$ to the complement of $W_\kappa$ is given by the restriction of the bundle* $\operatorname{Index}(\mathbf{D}^*_\kappa)$ *defined in* (7.4) *to the complement of $W_\kappa$,*
4. *For any $[A', F^s, \mathbf{x}] \in \bar{M}^{s,\natural}_{spl,\kappa}(\delta)$, $L^2$-orthogonal projection defines an isomorphism,*
$$\Upsilon^i_{spl,\kappa}|_{[A,F^s,\mathbf{x}]} \cong \operatorname{Coker} D_{A_t}$$
*for all $t \in [0,1]$ where $A_t = A' + ta$ and $A = A' + a$ is the value of the gluing map at $A'$.*

The construction of the bundle $\bar{\Upsilon}^i_{spl,\kappa}$ uses induction on $\kappa$ and is parallel to the construction of the spliced-ends moduli space. Thus, the first step (done in §7.3.1) is show that the images of the splicing maps $\varphi'_{\Theta,\mathcal{P}}$ form a vector bundle over $W_\kappa$. We then use the final property in the proposition property that $L^2$-orthogonal projection gives an isomorphism to define the isotopy between the image of the splicing maps over $W_\kappa$ and the bundle $\operatorname{Index}(\mathbf{D}^*_\kappa)$.

7.3.1. *The overlap bundles.* If the restriction of the spliced end index bundle to the image of the splicing map $\boldsymbol{\gamma}'_{\Theta,\mathcal{P}}$ is to be given by the image of the splicing map $\varphi'_{\Theta,\mathcal{P}}$ restricted to the subspace (7.7), then we need to describe the overlaps of the images of the splicing maps $\varphi'_{\Theta,\mathcal{P}}$ and $\varphi'_{\Theta,\mathcal{P}'}$ where $\mathcal{P} < \mathcal{P}'$. This is done by constructing a space of overlap data analogous to that appearing in Proposition 5.12.

Begin by initially considering the vector pseudo-bundle $\tilde\Upsilon(\Theta,\mathcal{P},\mathcal{P}')$ defined by restricting the vector bundle

$$\begin{array}{c} \Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P)\times\prod_{Q\in\mathcal{P}'_P}\bar\Upsilon^i_{spl,|Q|}\right) \\ \downarrow \\ \Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P)\times\prod_{Q\in\mathcal{P}'_P}\bar M^{s,\natural}_{spl,|Q|}(\delta_Q)\right) \end{array} \tag{7.8}$$

to the intersection of its base with the space $\tilde{\mathcal{O}}(\Theta,\mathcal{P},\mathcal{P}',\delta)$ defined in Lemma 5.10. By induction on $\kappa$, the splicing maps $\varphi_{\Theta,\mathcal{P}'_P}$ define maps

$$\varphi_{\Theta,\mathcal{P}'_P}:\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}'_P)\times\prod_{Q\in\mathcal{P}'_P}\Upsilon^i_{spl,|Q|}\to\Upsilon^i_{spl,|P|},$$

covering the splicing map $\gamma'_{\Theta,\mathcal{P}'_P}$. Hence, there is a map

$$\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}:\tilde\Upsilon(\Theta,\mathcal{P},\mathcal{P}')\to\Upsilon(\Theta,\mathcal{P})=\Delta^\circ(Z_\kappa(\delta),\mathcal{P})\times\prod_{P\in\mathcal{P}}\Upsilon^i_{spl,|P|}$$

defined by

$$\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}=\mathrm{id}_\Delta\times\prod_{P\in\mathcal{P}}\varphi'_{\Theta,\mathcal{P}'_P},$$

where $\mathrm{id}_\Delta$ is the identity map on $\Delta^\circ(Z_\kappa(\delta),\mathcal{P})$. By the definition of the overlap map $\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}$ in (5.34), the map $\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}$ fits into the vector bundle diagram

$$\begin{array}{ccc} \tilde\Upsilon(\Theta,\mathcal{P},\mathcal{P}') & \xrightarrow{\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}} & \Upsilon(\Theta,\mathcal{P}) \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\mathcal{P}',\delta) & \xrightarrow{\rho^{\Theta,d}_{\mathcal{P},\mathcal{P}'}} & \mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P}) \end{array} \tag{7.9}$$

The map $\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}$ is not injective because of the action of the symmetric group, but this will not affect our construction.

We also define a map

$$\rho^{\Psi,u}_{\mathcal{P},\mathcal{P}'}:\tilde\Upsilon(\Theta,\mathcal{P},\mathcal{P}')\to\Upsilon(\Theta,\mathcal{P}') \tag{7.10}$$

which fits into the vector bundle diagram

$$\begin{array}{ccc} \tilde\Upsilon(\Theta,\mathcal{P},\mathcal{P}') & \xrightarrow{\rho^{\Psi,u}_{\mathcal{P},\mathcal{P}'}} & \Upsilon(\Theta,\mathcal{P}') \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P},\mathcal{P}',\delta) & \xrightarrow{\rho^{\Theta,u}_{\mathcal{P},\mathcal{P}'}} & \mathcal{O}^{\mathrm{asd}}(\Theta,\mathcal{P}') \end{array} \tag{7.11}$$

Then, the following lemma, analogous to Lemma 5.11, is the key to the construction of the bundle $\Upsilon^i_{spl,\kappa}$.

**Lemma 7.2.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\kappa$ with $|\mathcal{P}| > 1$. Let $\tilde{\Upsilon}(\Theta, \mathcal{P}, [\mathcal{P}'])$ be the space defined in* (7.8)*. Then, the following diagram of vector bundles,*

$$\begin{array}{ccc}
\tilde{\Upsilon}(\Theta,\mathcal{P},\mathcal{P}') & \xrightarrow{\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}} & \Upsilon(\Theta,\mathcal{P}) \\
{\scriptstyle \rho^{\Psi,u}_{\mathcal{P},\mathcal{P}'}}\big\downarrow & & {\scriptstyle \varphi'_{\Theta,\mathcal{P}}}\big\downarrow \\
\Upsilon(\Theta,\mathcal{P}') & \xrightarrow{\varphi'_{\Theta,\mathcal{P}'}} & \bar{\mathfrak{V}}_\kappa
\end{array} \tag{7.12}$$

*covers the diagram* (5.40)*, is* Spin$^u(4)$ *equivariant, and commutes.*

*Proof.* That the diagram (7.12) covers the diagram (5.40) follows from the definition of the maps $\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}$ and $\rho^{\Psi,u}_{\mathcal{P},\mathcal{P}'}$ (see the diagrams (7.9) and (7.11)) and the fact that the of the spicing maps $\varphi'_{\Theta,\mathcal{P}}$ cover the splicing maps $\gamma'_{\Theta,\mathcal{P}}$.

The Spin$^u(4)$ equivariance of the diagram follows from that of the splicing maps $\varphi'_{\Theta,\mathcal{P}}$.

The commutativity of the diagram follows from the definition of the splicing maps $\varphi'_{\Theta,\mathcal{P}}$ and the property (5.27) which appears, in (5.35), in the definition of the overlap space $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$.

In detail, assume that the point in $\tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}')$ is given by data consisting of points $x_P \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ (for $P \in \mathcal{P}$), points $y_Q \in \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P)$ (for $Q \in \mathcal{P}'_P$), and $[A_Q, F^s_Q, \mathbf{v}_Q, \Psi_Q] \in \bar{\Upsilon}_{spl,|Q|}$. Then, the first map in the composition $\varphi'_{\Theta,\mathcal{P}} \circ \rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}$ entails defining sections $\Psi_P$ (for $P \in \mathcal{P}$) through the splicing maps $\varphi_{\Theta,\mathcal{P}'}$ in the definition of the map $\rho^{\Psi,d}_{\mathcal{P},\mathcal{P}'}$ by

$$\Psi_P = \sum_{Q \in \mathcal{P}'_P} \chi_{y_Q,\lambda_Q} c^*_{y_Q,1} \Psi_Q,$$

where $\lambda_Q = \lambda([A_Q, F^s_Q, \mathbf{v}_Q])$. Applying the splicing map $\varphi'_{\Theta,\mathcal{P}}$ to the sections $\Psi_P$ will give, for

$$\lambda_P = \lambda\left(\gamma'_{\Theta,\mathcal{P}'_P}((y_Q, [A_Q, F^s_Q, \mathbf{v}_Q])_{Q \in \mathcal{P}'_P})\right),$$

the section

$$\begin{aligned}
\sum_{P \in \mathcal{P}} \chi_{x_P,\lambda_P} c^*_{x_P,1} \Psi_P &= \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{P}'_P} \chi_{x_P,\lambda_P} c^*_{x_P,1} \chi_{y_Q,\lambda_Q} c^*_{y_Q,1} \Psi_Q \\
&= \sum_{Q \in \mathcal{P}'} \chi_{x_P,\lambda_P} \chi_{x_P+y_Q,\lambda_Q} c^*_{x_Q+y_Q,1} \Psi_Q
\end{aligned} \tag{7.13}$$

where the last equality follows from Lemma 5.8. Recall that by the condition (5.35), the points $x_P$ and $y_Q$ in the definition of the subspace $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}')$ will satisfy (see condition (5.27)),

$$B(x_P + y_Q, \tfrac{1}{4}\lambda_Q^{1/3}) \Subset B(x_P, \tfrac{1}{8}\lambda_P^{1/2}).$$

Because $\chi_{x,\lambda}$ is supported on $B(x, \frac{1}{4}\lambda^{1/3})$ and equal to one on $B(x, \frac{1}{8}\lambda^{1/2})$. the preceding inclusion implies that $\chi_{x_P,\lambda_P}$ is equal to one on the support of $\chi_{x_P+y_Q,\lambda_Q}$, so the section in (7.13) is equal to

$$\sum_{Q \in \mathcal{P}'} \chi_{x_P+y_Q,\lambda_Q} c^*_{x_Q+y_Q,1} \Psi_Q.$$

This is exactly the section resulting from the applying the map $\varphi'_{\Theta,\mathcal{P}'} \circ \rho^{\Psi,u}_{\mathcal{P},\mathcal{P}'}$ to the same data, proving the diagram (7.12) commutes. □

*Proof of Proposition 7.1.* The proof of this proposition is largely identical to the proof of Theorem 5.1 so we omit many of the details.

The induction begins with setting $\Upsilon^i_{spl,1}$ equal to the bundle $\operatorname{Index}(\mathbf{D}^*_1)$ on the top stratum of $\bar{M}^{s,\natural}_{spl,1}(\delta) = (0,\delta)\times \mathrm{SO}(3)$ (which is given by)

$$(0,\delta)\times \mathrm{SU}(2)\times_{\pm 1}\mathbb{C} \to (0,\delta)\times \mathrm{SO}(3).$$

The bundle extends as a pseudo-vector bundle over the Uhlenbeck compactification, $c(\mathrm{SO}(3))$ as

$$c\left(\mathrm{SU}(2)\times_{\pm 1}\mathbb{C}\right) \to c(\mathrm{SO}(3)).$$

Assuming that $\bar{\Upsilon}^i_{spl,\kappa'}$ has been constructed for all $\kappa' < \kappa$, we define a pseudo-vector bundle $\bar{\Upsilon}^i(W_\kappa)\to W_\kappa$ over the spliced end by the image of the splicing maps $\varphi'_{\Theta,\mathcal{P}}$ on the domains in (7.7). This space is a pseudo-vector bundle because the overlap of the images of the maps $\varphi'_{\Theta,\mathcal{P}}$ is controlled by Lemma 7.2.

The final property in Proposition 7.1 on the surjectivity of $L^2$-orthogonal projection follows from the results of [9, §8] (see also [3, Proposition 7.1.32]). We can then define an isotopy $R_D$ of $\bar{\Upsilon}^i(W_\kappa)$ by, for $t\in[\frac{1}{2},\frac{3}{4}]$,

$$R_D\left([A,F^s,\mathbf{x},\Psi],t\right) = \big(R([A,F^s,\mathbf{x},\Psi],t),(1-4(t-\tfrac{1}{2}))\Psi + 4(t-\tfrac{1}{2})\Pi_{A+a(A)}\Psi\big)$$

where $R$ is the isotopy defined in (5.76) and for $G([A,F^s,\mathbf{x}] = [A+a(A),F^s,\mathbf{x}]$ is the gluing map from (5.75), $\Pi_{A+a(A)}$ denotes $L^2$-orthogonal onto $\operatorname{Coker} D_{A+a(A)}$. For for $t\in(-\infty,\frac{1}{2}]$, the isotopy $R_D(\cdot,t)$ is equal to the identity and for $t\in[\frac{3}{4},1]$, the isotopy $R_D(\cdot,t)$ is defined by pulling back the section $\Psi$ by the centering map.

The bundle $\bar{\Upsilon}^i_{spl,\kappa}$ is then defined by the union of the image $\bar{\Upsilon}^i(W_\kappa)$ under the map $R_D(\cdot,t(\cdot))$ where $t$ is the function defined in (5.77) with $\operatorname{Index}(\mathbf{D}^*_\kappa)$ restricted to the complement of

$$\left(\bar{M}^{s,\natural}_{\kappa}(\delta) - R(1,W_\kappa)\right)$$

as in the definition of $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$ in the proof of Theorem 5.1. □

## 8. The obstruction bundle

The gluing map is an embedding of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ into $\bar{\mathcal{C}}_{\mathfrak{t}}$ defined by a perturbation of the image of the splicing map. The intersection of the image of the gluing map with $\bar{\mathcal{M}}_{\mathfrak{t}}$ is defined by the zero-locus of a finite-dimensional pseudo-bundle, the *obstruction bundle*, which is a smooth vector bundle on each stratum of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. In this section, we define this obstruction bundle.

The obstruction bundle is the direct sum of the Seiberg-Witten obstruction bundle, defined in §8.2, and the instanton obstruction pseudo-bundle, defined in §8.3. The gluing map, defined in Theorem 8.2, is homotopic to the global splicing map and the image of the gluing map contains a neighborhood of $M_{\mathfrak{s}}\times\mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$. The intersection of the image of the gluing map with $\bar{\mathcal{M}}_{\mathfrak{t}}$ is given by the zero-locus of a section of the obstruction bundles as discussed in §8.4.

### 8.1. The infinite-dimensional obstruction bundle.

The infinite-dimensional obstruction bundle in which the monopole map $\mathfrak{S}$ takes values is given by

$$\mathfrak{V}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}\times_{\mathcal{G}_{\mathfrak{t}}} L^2_{k-1}(\Lambda^+\otimes\mathfrak{g}_{\mathfrak{t}})\oplus L^2_{k-1}(V^-)\to\mathcal{C}_{\mathfrak{t}}. \tag{8.1}$$

The $S^1$ action on $\mathcal{C}_{\mathfrak{t}}$ lifts to an $S^1$ action on the bundle (8.1) given by scalar multiplication on the element of $L^2_{k-1}(V^-)$.

We extend the above bundle to a pseudo-bundle over $\bar{\mathcal{C}}_{\mathfrak{t}}$ by

$$\bar{\mathfrak{V}}_{\mathfrak{t}} = \sqcup_\ell \left(\mathfrak{V}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)\right) \to \bar{\mathcal{C}}_{\mathfrak{t}} = \sqcup_\ell \left(\mathcal{C}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)\right). \tag{8.2}$$

The monopole map is a continuous map $\mathfrak{S} : \bar{\mathcal{C}}_{\mathfrak{t}} \to \bar{\mathfrak{V}}_{\mathfrak{t}}$.

8.2. **The Seiberg-Witten obstruction.** Recall from [9] or [11, §3.6.1] that the background or Seiberg-Witten component of the obstruction is constructed by splicing in sections of the obstruction bundle

$$\pi_N^* \Xi_{\mathfrak{t},\mathfrak{s}} \to N_{\mathfrak{t},\mathfrak{s}}(\delta),$$

where $\pi_N : N_{\mathfrak{t},\mathfrak{s}}(\delta) \to M_{\mathfrak{s}}$ and $\Xi_{\mathfrak{t},\mathfrak{s}} \cong M_{\mathfrak{s}} \times \mathbb{C}^{r_\Xi}$ is a trivial bundle. The embedding $N_{\mathfrak{t},\mathfrak{s}}(\delta) \to \mathcal{C}_{\mathfrak{t}(\ell)}$ is covered by an vector bundle embedding:

$$\begin{array}{ccc} \pi_N^* \Xi_{\mathfrak{t},\mathfrak{s}} & \longrightarrow & \mathfrak{V}_{\mathfrak{t}(\ell)} \\ \Big\downarrow & & \Big\downarrow \\ N_{\mathfrak{t},\mathfrak{s}}(\delta) & \longrightarrow & \mathcal{C}_{\mathfrak{t}(\ell)} \end{array}$$

The above vector bundle map is $S^1$ equivariant with respect to the actions described in [11, §3.6.1] and is defined by the gauge group quotient of a vector bundle embedding

$$\pi_N^* \tilde{\Xi}_{\mathfrak{t},\mathfrak{s}} \to \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)} \times L^2_{k-1}(\Lambda^+ \otimes \mathfrak{g}_{\mathfrak{t}(\ell)} \oplus V^-_{\mathfrak{t}(\ell)}),$$

where $\tilde{\Xi}_{\mathfrak{t},\mathfrak{s}} \to \tilde{M}_{\mathfrak{s}}$ satisfies $\tilde{\Xi}_{\mathfrak{t},\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}} = \Xi_{\mathfrak{t},\mathfrak{s}}$.

Define

$$\Xi_{\mathfrak{s}}(\mathfrak{t},\mathcal{P}) = \pi_N^* \tilde{\Xi}_{\mathfrak{t},\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) \subset \pi_N^* \tilde{\Xi}_{\mathfrak{t},\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}).$$

Then, the crude splicing construction from §6.3 (multiplying the obstruction sections by the cut-off functions used to define the flattened section $\Phi'$ in the definition of the flattening map $\boldsymbol{\Theta}_{\mathcal{P}}$) defines a vector bundle embedding,

$$\varphi''_{\mathfrak{s}}(\mathcal{P}) : \Xi_{\mathfrak{s}}(\mathfrak{t},\mathcal{P}) \to \bar{\mathfrak{V}}_{\mathfrak{t}},$$

covering the crude splicing map:

$$\begin{array}{ccc} \pi_N^* \tilde{\Xi}_{\mathfrak{t},\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) & \xrightarrow{\varphi''_{\mathfrak{s}}(\mathcal{P})} & \bar{\mathfrak{V}}_{\mathfrak{t}} \\ \Big\downarrow & & \Big\downarrow \\ \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}) & \xrightarrow{\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}} & \bar{\mathcal{C}}_{\mathfrak{t}} \end{array}$$

The crude obstruction splicing maps $\varphi''_{\mathfrak{s}}(\mathcal{P}')$ are invariant under the action of the symmetric group and thus, for $\mathcal{P} < \mathcal{P}'$ define a crude obstruction splicing map

$$\varphi''_{\mathfrak{s}}([\mathcal{P} < \mathcal{P}']) : \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']) \to \bar{\mathfrak{V}}_{\mathfrak{t}}$$

covering the crude splicing map $\gamma''_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}$ defined in (6.38).

We can control the overlaps of the images of the embeddings $\varphi''_{\mathfrak{s}}(\mathcal{P})$ exactly as was done in the construction of the space of global splicing data. The overlap maps, $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ and $\mathrm{id}_N \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$, appearing in the diagram (6.40) are covered by obvious bundle maps,

$$\begin{aligned} &\rho^{\Xi,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']} : \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}']) \to \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']) \\ &\mathrm{id}_\Xi \times \rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']} : \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}']) \to \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}). \end{aligned} \tag{8.3}$$

Then, exactly as in the proof of Proposition 6.9 (specifically the equality of the section $\Phi_0'$), the diagram

$$\begin{array}{ccc} \pi_N^*\tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}']) & \xrightarrow{\rho^{\Xi,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}} & \pi_N^*\tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']) \\ {\scriptstyle \mathrm{id}_{\Xi}\times\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}} \big\downarrow & & \big\downarrow {\scriptstyle \varphi''_{\mathfrak{s}}([\mathcal{P}<\mathcal{P}'])} \\ \pi_N^*\tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}) & \xrightarrow{\varphi''_{\mathfrak{s}}(\mathcal{P})} & \bar{\mathfrak{V}}_{\mathfrak{t}} \end{array} \tag{8.4}$$

commutes and covers the diagram (6.40).

We can therefore define a bundle

$$\bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \tag{8.5}$$

as the union of the bundles $\Xi_s(\mathfrak{t},\mathcal{P})$ as $\mathcal{P}$ varies over the partitions $\mathcal{P}$ of $N_\ell$, patched together on the overlaps by the diagram (8.4). The embeddings $\varphi_s''(\mathcal{P})$ fit together to define a vector bundle embedding

$$\varphi_s'' : \bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$$

covering the embedding given by the crude splicing maps.

The embedding $\varphi_s''$ suffers from the same failings as the crude splicing map: it is not equal to the identity. We therefore define a new vector bundle embedding following the isotopies of (6.55). That is, let $\varphi_s'(\mathcal{P})$ be the standard slicing map

$$\varphi_s'(\mathcal{P}) : \Xi_s(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \bar{\mathfrak{V}}_{\mathfrak{t}}$$

defined by using the cut-off functions vanishing on balls $B(x_P, 4\lambda_P^{1/3})$. Define a sequence of vector bundle embeddings

$$\varphi_{s,j}'' : \bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathfrak{V}}_{\mathfrak{t}}$$

exactly the embeddings $\gamma''_{\mathcal{M},j}$ were defined in (6.55), with the map $\varphi_s'(\mathcal{P})$ in place of $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$. Define

$$\varphi_s'(\mathfrak{t},\mathfrak{s}) : \bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathfrak{V}}_{\mathfrak{t}} \tag{8.6}$$

to be the end of the sequence of vector bundle embeddings $\varphi_{s,j}''$. It is a vector bundle embedding, covering the global splicing map $\gamma'_{\mathcal{M}}$ of (6.56).

8.3. **The instanton obstruction.** The other component of the obstruction is given by splicing in the spliced-ends obstruction bundle over $\bar{M}^{s,\natural}_{spl,\kappa}(\delta)$.

8.3.1. *The frame bundles.* In [11, §3.2], we defined a frame bundle

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \to \mathrm{Fr}(TX) \to X$$

which is a U(2) bundle over $\mathrm{Fr}(TX)$ and a $\mathrm{Spin}^u(4)$ bundle over $X$. The fiber over $F \in \mathrm{Fr}(TX)|_x$ is given by the Clifford module isomorphisms from the standard Clifford module $\Delta \otimes \mathbb{C}^2$ to $V|_x$ with respect to the isomorphism $\mathbb{R}^4 \cong T^*X|_x$ defined by $F$. The homomorphisms from $\mathrm{Spin}^u(4) \to \mathrm{SO}(3)\times\mathrm{SO}(4)$ define a bundle map

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \to \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}}) \times_X \mathrm{Fr}(TX).$$

Analogously to the definition of the gluing data bundle $\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})$ in (6.2), we define

$$\mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}}) \to \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}}), \tag{8.7}$$

by

$$\begin{aligned}\mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}}) = \{(F_1^V,\dots,F_\ell^V) \in \prod_{i=1}^{\ell} \mathrm{Fr}_{\mathbb{C}\ell(T^*)}(V_{\mathfrak{t}(\ell)})|_{\Delta^\circ(X^\ell,\mathcal{P})} :\\ F_i^V = F_j^V \quad \text{if and only there is } P\in\mathcal{P} \text{ with } i,j\in P\}.\end{aligned}$$

The structure group of $\mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}}) \to \Delta^\circ(X^\ell,\mathcal{P})$ is

$$\begin{aligned}\tilde{G}^V(\mathcal{P}) = \{(G_1,\dots,G_\ell) \in \prod_{i=1}^{\ell} \mathrm{Spin}^u(4) :\\ G_i = G_j \quad \text{if there is } P\in\mathcal{P} \text{ with } i,j\in P\}.\end{aligned} \tag{8.8}$$

The structure group of the bundle $\mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}}) \to \Sigma(X^\ell,\mathcal{P})$ is

$$G^V(\mathcal{P}) = \tilde{G}^V(\mathcal{P}) \ltimes \mathfrak{S}(\mathcal{P}). \tag{8.9}$$

We then define the instanton obstruction bundle for the partition $\mathcal{P}$ by

$$\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P}) = \mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}}) \times_{G^V(\mathcal{P})} \prod_{P\in\mathcal{P}} \bar{\mathbf{D}}^{s,\natural}_{spl,|P|}. \tag{8.10}$$

8.3.2. *The splicing map.* For each partition, $\mathcal{P}$, of $N_\ell$, we define a splicing map for the instanton obstruction,

$$\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}} : \tilde{N}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \bar{\mathfrak{V}}_{\mathfrak{t}}, \tag{8.11}$$

which will cover the crude splicing map $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$. Frames $(F_P^V)_{P\in\mathcal{P}} \in \mathrm{Fr}(V,\mathcal{P})|_{\mathbf{y}}$, where $\mathbf{y} = (y_P)_{P\in\mathcal{P}}$ and a spin$^u$ connection $A$ determine embeddings

$$\phi(A,F_P^V) : V_{\kappa_i}|_{B(0,8\lambda^{1/3})} \cong V_{\mathfrak{t}}|_{B(y_P,8\lambda^{1/3})} \tag{8.12}$$

as described in [11, §3.4.1]. If $\chi$ is a cut-off function supported on $B(0,8\lambda^{1/3})$ and $\Psi$ is a section of $V_{\kappa_i}$, then we write $\phi(A,F^V)(\chi\Psi)$ for the section of $V_{\mathfrak{t}}|_{B(x,8\lambda^{1/3})}$ given by the isomorphism (8.12).

To ensure the images of the different splicing maps fit together coherently, we introduce a flattening map on spin$^u$ connections as was done for metrics and connections in §6.2. That is, given a spin$^u$ connection $A_0$ on $V_{\mathfrak{t}(\ell)}$ we follow the construction of Lemma 6.4 and for each $\mathbf{y}\in\Sigma(X^\ell,\mathcal{P})$ define a locally flattened spin$^u$ connection $\mathbf{\Theta}^u(A,\mathbf{y})$ on $V_{\mathfrak{t}(\ell)}$ satisfying

(1) For $\mathbf{y} = (y_P)_{P\in\mathcal{P}}$, $\mathbf{\Theta}^u(A_0,\mathbf{y})$ is flat on

$$\cup_{P\in\mathcal{P}} B(y_p, 4\tilde{s}_P(\mathbf{y})^{1/3})$$

(2) If $\mathcal{P} < \mathcal{P}'$ and $\mathbf{y}\in\Sigma(X^\ell,\mathcal{P})$ and $\mathbf{y}'\in\Sigma(X^\ell,\mathcal{P}')\cap\mathcal{U}(X^\ell,\mathcal{P})$ satisfy $\pi(X^\ell,\mathcal{P})(\mathbf{y}') = \mathbf{y}$, then

$$\mathbf{\Theta}^u(A_0,\mathbf{y}) = \mathbf{\Theta}^u(A_0,\mathbf{y}').$$

Henceforth, $\phi(A_0,F^V)$ will denote the isomorphism obtained by the locally flattened connection $\mathbf{\Theta}^u(A_0,\mathbf{y})$ where $F^V$ lies over the support of $\mathbf{y}$.

For $(A_0,\Phi_0) \in N(\delta) \subset \mathcal{C}_{\mathfrak{t}(\ell)}$ and a point in $\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ given by the data $(F_P^V)_{P\in\mathcal{P}} \in \mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}})$, $[A_P,F_P^s,\mathbf{v}_P] \in \bar{M}^{s,\natural}_{spl,|P|}(\delta_P)$ $[A_P,F_P^s,\mathbf{v}_P,\Psi] \in \bar{\Upsilon}_{spl,|P|}$, and

$$\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}((A_0,\Phi_0),F_P^V,[A_P,F_P^s,\mathbf{v}_P])_{P\in\mathcal{P}} = (A'',\Phi'',\mathbf{x}),$$

the instanton obstruction splicing map is then defined by

(8.13)
$$\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}\left((A_0,\Phi_0),(F_P^V,[A_P,F_p^s,\mathbf{v}_P,\Psi_P])_{P\in\mathcal{P}}\right) = \left(A'',\Phi'',\mathbf{x},\sum_{P\in\mathcal{P}}\phi(A_0,F_P^V)(\chi_{0,\lambda_P}\Psi_P)\right)$$

where the cut-off function $\chi_{0,\lambda_P}$ vanishes on $B(0,\frac{1}{8}\lambda_P^{1/3})$ and is equal to one on $B(0,\frac{1}{4}\lambda_P^{1/2})$.

8.3.3. *The overlap space and maps.* As is now standard, to control the overlap of the images of the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}'}$, we introduce a space of overlap data and overlap maps. Analogously to the definition of the overlap space $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])$ in (6.22), we define

$$\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}']) \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])$$

by

$$(8.14)\quad \begin{aligned}&\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])\\ &= \mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}})\times_{G^V(\mathcal{P})}\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}''_P)\times\prod_{Q\in\mathcal{P}''_P}\bar{\Upsilon}_{spl,|Q|}\right).\end{aligned}$$

We define overlap maps,

$$\rho^{V,d}_{\mathcal{P},\mathcal{P}'}:\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])\to\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

covering the overlap maps $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ by applying the obstruction splicing map $\varphi_{\Theta,\mathcal{P}''_P}$ defined in (7.5) to the data given by the factor

$$\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}''_P)\times\prod_{Q\in\mathcal{P}''_P}\bar{\Upsilon}_{spl,|Q|}$$

in the definition of $\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])$. More formally, the downwards overlap map is defined by

$$(8.15)\quad \rho^{V,d}_{\mathcal{P},\mathcal{P}'} = \mathrm{id}_{\mathrm{Fr}(V)}\times\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\prod_{P\in\mathcal{P}}\varphi_{\Theta,\mathcal{P}''_P}$$

where $\mathrm{id}_{\mathrm{Fr}(V)}$ is the identity map on $\mathrm{Fr}(V,\mathcal{P},g_{\mathcal{P}})$.

We define an upwards overlap map to the space,

$$\mathrm{Ob}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']) = \left(\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P}'')\right)/\mathfrak{S}(\mathcal{P})$$

exactly as was done in (6.31) and (6.33). Recall that the map $\rho^{\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ was defined by a parallel translation of the frames in the domain $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'],g_{\mathcal{P}})$ and leaving the $S^4$ connections unchanged. Note that this parallel translation is done with respect to the locally flattened connection $\mathbf{\Theta}^u(A_0,\mathbf{y})$. Thus, the map

$$(8.16)\quad \rho^{N,V,u}_{\mathcal{P},[\mathcal{P}']}:\tilde{N}(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}'])\to\tilde{N}(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathrm{Ob}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'])$$

can be defined identically, using the flattened spin$^u$ connection $A_0$ to parallel translate the frames of $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\mathfrak{t}(\ell)})$ and leaving the elements of $\bar{\Upsilon}_{spl,|Q|}$ unchanged.

By the invariance of the obstruction splicing maps under the action of the symmetric group, the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}''}$ for $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$ define a splicing map,

$$(8.17)\quad \varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}:\tilde{N}(\delta)\times_{\mathcal{G}_\mathfrak{s}}\mathrm{Ob}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}'])\to\bar{\mathfrak{V}}_\mathfrak{t}$$

which covers the crude splicing map $\boldsymbol{\gamma}''_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}$ of (6.38). We then have the following relation between the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}$.

**Lemma 8.1.** *Let $\mathcal{P} < \mathcal{P}'$ be partitions of $N_\ell$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
\tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P},[\mathcal{P}']) & \xrightarrow{\rho^{N,V,u}_{\mathcal{P},[\mathcal{P}']}} & \tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Ob}(\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']) \\
{\scriptstyle \mathrm{id}_N \times \rho^{V,d}_{\mathcal{P},[\mathcal{P}']}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P}<\mathcal{P}']}} \\
\tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P}) & \xrightarrow{\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}} & \bar{\mathfrak{V}}_\mathfrak{t}
\end{array}
\tag{8.18}
$$

*where* $\mathrm{id}_N$ *is the identity map on $\tilde{N}(\delta)$, covering the diagram* (6.40).

*Proof.* That the diagram (8.18) covers the diagram (6.40) follows immediately from the definition of the overlap and obstruction splicing maps. Then, as in the proof of Proposition 6.9, assume that the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}'}$ are defined at points $\mathbf{y} \in \Sigma(X^\ell,\mathcal{P})$ and $\mathbf{y}' \in \Sigma(X^\ell,\mathcal{P}')$ respectively where $\pi(X^\ell,\mathcal{P})(\mathbf{y}') = \mathbf{y}$. Thus, for $P' \in \mathcal{P}'$ and $P' \subseteq P \in \mathcal{P}$, the inclusion

$$B(y_{P'}, 8\tilde{s}(\mathbf{y}')^{1/3}) \Subset B(y_P, 4\tilde{s}(\mathbf{y})^{1/2})$$

holds by (6.19). Because the metric and spin$^u$ connection used in the splicing argument are flat on the above balls, the lemma then follows from the argument giving Lemma 7.2. □

The commutativity of the diagram (8.18) and the definition of the space $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ as the union of the spaces

$$\tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

implies that we can define a vector pseudo-bundle

$$\bar{\Upsilon}^i_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \tag{8.19}$$

as the union of the pseudo-bundles

$$\tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathrm{Ob}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P}),$$

with the overlaps identified by the diagram (8.18).

The splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ define a vector bundle embedding of $\bar{\Upsilon}^i_{\mathfrak{t},\mathfrak{s}}$ into $\bar{\mathfrak{V}}_\mathfrak{t}$ which covers the embedding of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ into $\bar{\mathcal{C}}_\mathfrak{t}$ given by the crude splicing maps. Finally, we define an embedding

$$\varphi'_i : \bar{\Upsilon}^i_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathfrak{V}}_\mathfrak{t} \tag{8.20}$$

covering the global splicing map $\boldsymbol{\gamma}'_{\mathcal{M}}$ of (6.56) by replacing the triple $(A'',\Phi'',\mathbf{x})$ in the definition (8.13) with the triple $(A',\Phi',\mathbf{x})$ given by the global splicing map $\boldsymbol{\gamma}'_{\mathcal{M}}$.

8.4. **The gluing theorem.** We are now in a position to state the relevant gluing theorem.

**Theorem 8.2.** *There is a continuous, $S^1$-equivariant embedding*

$$\boldsymbol{\gamma}_{\mathcal{M}} : \bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{C}}_\mathfrak{t} \tag{8.21}$$

*which is homotopic through $S^1$-equivariant embeddings to the global splicing map $\boldsymbol{\gamma}'_{\mathcal{M}}$, smooth on each stratum of $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}$, and equal to the identity on $N(\delta) \times \mathrm{Sym}^\ell(X)$. In addition, there are continuous, $S^1$-equivariant sections $\mathfrak{o}_s$ and $\mathfrak{o}_i$ of the bundles $\bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}}$ and $\bar{\Upsilon}^i_{\mathfrak{t},\mathfrak{s}}$ such that*

1. *The section $\mathfrak{o} = \mathfrak{o}_s \oplus \mathfrak{o}_i$ of $\bar{\Upsilon}^s_{\mathfrak{t},\mathfrak{s}} \oplus \bar{\Upsilon}^i_{\mathfrak{t},\mathfrak{s}}$ vanishes transversely on the top stratum of $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}$,*

(2) *The restriction of* $\boldsymbol{\gamma}_{\mathcal{M}}$ *to the zero-locus* $\mathfrak{o}^{-1}(0)$ *parameterizes a neighborhood of* $M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X)$ *in* $\bar{\mathcal{M}}_{\mathfrak{t}}$.

## 9. Cohomology

In this section, we compute the pullback of the cohomology classes by the global splicing map. We define the cohomology classes and maps in terms of which these pullbacks will be expressed in §9.1. We compute the pullbacks of the cohomology classes $\mu_p(\beta)$ and $\mu_c$ in §9.2 the Euler class of the Seiberg-Witten component of the obstruction in §9.3. The computation of the Euler class of the instanton component of the obstruction occupies §9.4, where we compute the restriction of this Euler class to the top stratum of the subspaces $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i) \subset \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ and §9.5 where discuss how these local computations yield the global cohomology class. Finally, in §9.6, we define some extensions of these cohomology classes from the top stratum of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ the complement of the reducible monopoles.

### 9.1. **Definitions.**

9.1.1. *Subspace and maps.* We begin by defining subspaces of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. First,

$$\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \tag{9.1}$$

will denote the top stratum which is mapped to $\mathcal{C}_{\mathfrak{t}}$ by the splicing and gluing maps $\boldsymbol{\gamma}'_{\mathcal{M}}$ and $\boldsymbol{\gamma}_{\mathcal{M}}$. Let

$$\iota : \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \tag{9.2}$$

be the inclusion. Next, we define the complement of the reducible points,

$$\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}} = \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} - M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X). \tag{9.3}$$

The $S^1$ action is free on the subspace $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}$.

The map

$$\pi_N : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to N_{\mathfrak{t}_0,\mathfrak{s}}(\delta),$$

was defined in (6.43) and we define

$$\pi_{\mathfrak{s}} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}} \tag{9.4}$$

as the composition of $\pi_N$ with the projection $N_{\mathfrak{t}_0,\mathfrak{s}}(\delta) \to M_{\mathfrak{s}}$. Recall that the projection

$$\pi_X : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to \operatorname{Sym}^{\ell}(X)$$

was defined in (6.57). We will also write

$$\pi_{\mathfrak{s},X} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}} \times \operatorname{Sym}^{\ell}(X) \tag{9.5}$$

and

$$\pi_{N,X} : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to N_{\mathfrak{t}_0,\mathfrak{s}}(\delta) \times \operatorname{Sym}^{\ell}(X) \tag{9.6}$$

for the projections.

We use the same notation to denote the projections from $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ (although note that $\pi_N$ maps this quotient to $N_{\mathfrak{t}_0,\mathfrak{s}}(\delta)/S^1$).

9.1.2. *Cohomology classes.* We define some cohomology classes on $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1$.

**Definition 9.1.** Let $\nu \in H^2(\bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{Z})$ be the first Chern class of the $S^1$ bundle,

$$\bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}/S^1. \tag{9.7}$$

If

$$\mathbb{L}_\nu = \bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathbb{C}, \tag{9.8}$$

to be the complex line bundle associated to the $S^1$ action on $\bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}$ so $c_1(\mathbb{L}_\nu) = \nu$.

**Definition 9.2.** For $\beta \in H_\bullet(X;\mathbb{R})$, let $\mu_{\mathfrak{s}}(\beta) \in H^2(M_{\mathfrak{s}};\mathbb{Z})$ be the $\mu$-class defined in §2. We will use the same notation for the pullback, by the projection $\pi_{\mathfrak{s}}$, of these classes to $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$. Define

$$\mathbb{L}_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times \mathbb{C} \to M_{\mathfrak{s}} \times X. \tag{9.9}$$

**Definition 9.3.** If $\beta \in H^\bullet(X;\mathbb{R})$ and $\pi_i : X^\ell \to X$ is projection onto the $i$-th factor, let $S^\ell(\beta) \in H^\bullet(\mathrm{Sym}^\ell(X);\mathbb{R})$ be the cohomology class which pulls back to $\sum_i \pi_i^*\beta$ under the projection $X^\ell \to \mathrm{Sym}^\ell(X)$.

The incidence locus,

$$\mathcal{V}_\ell(\Delta) \subset \mathrm{Sym}^\ell(X) \times X \tag{9.10}$$

is defined to be the points $(\mathbf{x}, y) \in \mathrm{Sym}^\ell(X) \times X$ such that $y$ is in the support of $\mathbf{x}$. Alternately, one can describe it

$$\mathcal{V}_\ell(\Delta) = \left(\cup_i (\pi_i \times \mathrm{id}_X)^{-1}(\Delta)\right)/\mathfrak{S}_\ell,$$

where $\pi_i : X^\ell \to X$ is projection onto the $i$-th factor and $\Delta \subset X \times X$ is the diagonal. Similarly, we define

$$\mathcal{V}_\ell(\nu(\Delta)) = \left(\cup_i (\pi_i \times \mathrm{id}_X)^{-1}(\nu(\Delta))\right)/\mathfrak{S}_\ell, \tag{9.11}$$

where $\nu(\Delta) \subset X \times X$ is the normal bundle of $\Delta$. Let

$$\mathrm{PD}[\mathcal{V}] \in H^4(\mathrm{Sym}^\ell(X) \times X;\mathbb{Z}), \tag{9.12}$$

be the image of the Thom class of the disk bundle $\mathcal{V}_\ell(\nu(\Delta)) \to \mathcal{V}_\ell(\Delta)$ under the excision isomorphism and the map

$$H^4(\mathrm{Sym}^\ell(X) \times X, \mathrm{Sym}^\ell(X) \times X - \mathcal{V}_\ell(\Delta);\mathbb{Z}) \to H^4(\mathrm{Sym}^\ell(X) \times X;\mathbb{Z}).$$

From the equality on Poincare duality given in [18, Theorem 30.6], we have

**Lemma 9.4.** *For any $\beta \in H_\bullet(X;\mathbb{R})$,*

$$\mathrm{PD}[\mathcal{V}]/\beta = S^\ell(\mathrm{PD}[\beta]).$$

9.2. **The $\mu$-classes.** We describe the pullbacks of the cohomology classes $\mu_c$ and $\mu_p(\beta)$ by the gluing map in terms of the cohomology classes introduced in §9.1.

The computation of the pullback of $\mu_c$ is identical to that appearing in [11, Lemma 4.2].

**Lemma 9.5.** *If $\mu_c \in H^2(\mathcal{C}^{*,0}_{\mathfrak{t}}/S^1;\mathbb{Z})$ is the cohomology class defined in* (2.44), *$\gamma_{\mathcal{M}}$ is the gluing map of* (8.21), *and $\iota$ is the inclusion* (9.2) *then*

$$\gamma^*_{\mathcal{M}}\mu_c = -\iota^*\nu. \tag{9.13}$$

To compute the pullback of the $\mu_p$ classes, we must study the incidence locus $\mathcal{V}_\ell(\Delta)$ more carefully. Define

$$\mathcal{O} = \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1 \times X - (\pi_X \times \mathrm{id}_X)^{-1}\mathcal{V}_\ell(\nu(\Delta)) \tag{9.14}$$

We will consider compare the pullback of the universal bundle $(\boldsymbol{\gamma} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}}$, restricted to $\mathcal{O}$, with the same restriction of the bundle

$$\mathbb{F}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} = \cup_{\mathcal{P}} \left(\tilde{N}(\delta) \times \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathfrak{g}_{\mathfrak{t}}\right) \tag{9.15}$$

which is defined analogously to the bundle in [11, Equation (4.7)]. The proof of [11, Lemma 4.7] then yields the following expression for $p_1(\mathbb{F}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}})$.

**Lemma 9.6.** *Assume the spin$^u$ structure $\mathfrak{t}_\ell$ admits a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. Let $\mathbb{F}^{vir}_{\mathfrak{t},\mathfrak{s}} \to \mathcal{M}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1 \times X$ be the bundle defined in* (9.15)*. Let $\pi_{X,2} : \mathcal{M}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1 \times X \to X$ be the projection onto the second factor. Then,*

$$p_1(\mathbb{F}^{vir}_{\mathfrak{t},\mathfrak{s}}) = \iota^* \left((2\pi_{\mathfrak{s}} \times \mathrm{id}_X)^* c_1(\mathbb{L}_{\mathfrak{s}}) - \nu + \pi^*_{X,2} c_1(L)\right)^2. \tag{9.16}$$

The following comparison of $\mathbb{F}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ with $(\boldsymbol{\gamma}_{\mathcal{M}} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}}$ is proven in [11, Lemma 4.3].

**Lemma 9.7.** *Assume the spin$^u$ structure $\mathfrak{t}_\ell$ admits a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. Let $\mathcal{O} \subset \mathcal{M}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1 \times X$ be the subspace defined in* (9.14)*. Then,*

$$(\boldsymbol{\gamma}_{\mathcal{M}} \times \mathrm{id}_X)^*\mathbb{F}_{\mathfrak{t}}|_{\mathcal{O}} \cong \mathbb{F}^{vir}_{\mathfrak{t},\mathfrak{s}}|_{\mathcal{O}}. \tag{9.17}$$

We specify the difference between the Pontrjagin classes of the bundles in Lemma 9.7 in the following.

**Lemma 9.8.** *Assume the spin$^u$ structure $\mathfrak{t}_\ell$ admits a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. Let $\mathrm{PD}[\mathcal{V}]$ be the cohomology class defined in* (9.12)*.*

$$(\boldsymbol{\gamma}_{\mathcal{M}} \times \mathrm{id}_X)^* p_1(\mathbb{F}_{\mathfrak{t}}) = p_1(\mathbb{F}^{vir}_{\mathfrak{t},\mathfrak{s}}) - 4(\iota \times \mathrm{id}_X)^*(\pi_X \times \mathrm{id}_X)^* \mathrm{PD}[\mathcal{V}]. \tag{9.18}$$

Let $\gamma_1, \dots, \gamma_b \in H_1(X;\mathbb{R})$ be a basis with $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_b \in H^1(X;\mathbb{R})$ the basis satisfying $\langle \boldsymbol{\gamma}^*_i, \boldsymbol{\gamma}_j \rangle = \delta_{ij}$. Let $\boldsymbol{\gamma}^{J,*}_1, \dots, \boldsymbol{\gamma}^{J,*}_b \in H^1(M_{\mathfrak{s}};\mathbb{R})$ be the related basis as defined in [15, Definition 2.2] (there written as $\mathbf{r}^*\boldsymbol{\gamma}^{J,*}_i$). Standard computations (compare [11, Lemma 4.10] and [16, Corollary 4.7]), Lemma 9.4, and the equality

$$c_1(\mathbb{L}_{\mathfrak{s}}) = \mu_{\mathfrak{s}} \times 1 + \sum_{i=1}^{b_1} \boldsymbol{\gamma}^{J,*}_i \times \boldsymbol{\gamma}^*_i$$

from [15, Lemma 2.24] then yield the following.

**Corollary 9.9.** *Assume the spin$^u$ structure $\mathfrak{t}_\ell$ admits a splitting $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, so $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. Let $\boldsymbol{\gamma}_{\mathcal{M}}$ be the gluing map defined in* (8.21) *and $\iota$ the inclusion defined in* (9.2)*. Then for $x \in H_0(X;\mathbb{Z})$ a generator, $\boldsymbol{\gamma}_i \in H_1(X;\mathbb{Z})$ a generator as above, $\beta \in H_2(X;\mathbb{R})$,* *and* $[Y] \in H_3(X;\mathbb{R})$,

(9.19)

$$\begin{aligned}
\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_c(x) &= -\iota^*\nu,\\
\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_p(x) &= \iota^*\left(-\tfrac{1}{4}(2\mu_{\mathfrak{s}}-\nu)^2 + S^\ell(x)\right),\\
\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_p(\gamma_i) &= \iota^*\left(-\tfrac{1}{2}(2\mu_{\mathfrak{s}}-\nu)\gamma_i^{J,*} + S^\ell(\gamma_i)\right),\\
\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_p(\beta) &= \iota^*\left(-\tfrac{1}{2}2\mu_{\mathfrak{s}}-\nu)\langle c_1(\mathfrak{t})-c_1(\mathfrak{s}),\beta\rangle - 2\sum_{i<j}\mu_{\mathfrak{s}}(\gamma_i\gamma_j)\langle\gamma_i^*\smile\gamma_j^*,\beta\rangle + S^\ell(\beta)\right),\\
\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_p(\beta_3) &= \iota^*\left(-\sum_i(c_1(\mathfrak{t})-c_1(\mathfrak{s})\smile\gamma_i^*,[Y]\rangle\mu_{\mathfrak{s}}(\gamma_i) + S^\ell(\beta)\right),
\end{aligned}$$

9.3. **The Seiberg-Witten obstruction Euler class.** The computation of the Euler class of the Seiberg-Witten, or background, component of the obstruction is identical to that given in [11, Lemma 4.11].

**Lemma 9.10.** *Let $r_\Xi$ be the complex rank of the background obstruction bundle $\Upsilon^s_{\mathfrak{t},\mathfrak{s}} \to \bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}$. Then,*

$$e(\Upsilon^s_{\mathfrak{t},\mathfrak{s}}/S^1) = \iota^*(-\nu)^{r_\Xi}. \tag{9.20}$$

9.4. **The local instanton obstruction Euler class.** We first give a description of the Euler class of the instanton component of the obstruction bundle, $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1 \to \mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ restricted to $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$.

We first introduce an additional cohomology class. Let $\pi_{\mathfrak{s},i} : M_{\mathfrak{s}} \times X^\ell \to M_{\mathfrak{s}} \times X$ be defined by the identity on $M_{\mathfrak{s}}$ and projection from $X^\ell$ onto the $i$-th factor of $X$. Then, the Chern classes of the bundle $\oplus_{i=1}^\ell \pi^*_{\mathfrak{s},i}\mathbb{L}_{\mathfrak{s}}$ are invariant under the action of the symmetric group and thus define cohomology classes

$$c_{\mathfrak{s},\ell,i} \in H^{2i}(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X);\mathbb{R}). \tag{9.21}$$

In addition, we will write

$$\iota_{\mathcal{P}} : \Sigma(X^\ell,\mathcal{P}) \to \mathrm{Sym}^\ell(X), \quad \tilde{\iota}_{\mathcal{P}} : \Delta^\circ(X^\ell,\mathcal{P}) \to X^\ell,$$

for the inclusions. Finally, we use the generic term

$$c(\mathfrak{t}) \in H^\bullet(X;\mathbb{R}), \tag{9.22}$$

for rational characteristic classes of the bundle $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \to X$. These characteristic classes include $c_1(\mathfrak{t})$, $p_1(\mathfrak{t})$, $p_1(X)$, and $e(X)$.

**Proposition 9.11.** *The Euler class of the restriction of the obstruction bundle $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1$ to $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ is given by a polynomial in the cohomology classes*

$$\nu, \quad \pi_X^*\iota_{\mathcal{P}}^*S^\ell(c(\mathfrak{t})), \quad \textit{and} \quad \pi^*_{X,\mathfrak{s}}\iota^*_{\mathcal{P}}c_{\mathfrak{s},\ell,i}, \tag{9.23}$$

*which is otherwise independent of $X$.*

To prove Proposition 9.11, we begin by observing that the space $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ retracts to the intersection of $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ with the subspace

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}). \tag{9.24}$$

We compute the pullback of the restriction of the Euler class of the instanton obstruction to the subspace (9.24) by the map,

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \tilde{\mathrm{Gl}}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \tag{9.25}$$

defined by the projection

$$\tilde{\mathrm{Gl}}(\mathfrak{t},\mathfrak{s},\mathcal{P}) = \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \times_{\widetilde{G}(\mathcal{P})} \prod_{P\in\mathcal{P}} M^{s,\natural}_{spl,|P|}(\delta_P) \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}),$$

where the group $\widetilde{G}(\mathcal{P})$ is defined in (6.3) (essentially by omitting the symmetric group component of $G(\mathcal{P})$). Observe that the preceding map appears in the diagram,

$$\begin{array}{ccc} \tilde{\mathrm{Gl}}(\mathfrak{t},\mathfrak{s},\mathcal{P}) & \longrightarrow & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \\ \downarrow & & \downarrow \\ \Delta^\circ(X^\ell,\mathcal{P}) & \longrightarrow & \Sigma(X^\ell,\mathcal{P}) \end{array}$$

For each $P\in\mathcal{P}$, the projection $\tilde{\mathrm{Gl}}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \to \Delta^\circ(X^\ell,\mathcal{P})$ fits into a diagram

$$\begin{array}{ccc} \tilde{\mathrm{Gl}}(\mathfrak{t},\mathfrak{s},\mathcal{P}) & \xrightarrow{\pi_{\tilde{\mathrm{Gl}},P}} & \mathrm{Fr}(\mathfrak{g}_0)\times_X \mathrm{Fr}(TX) \times_{\mathrm{SO}(3)\times\mathrm{SO}(4)} M^{s,\natural}_{spl,|P|}(\delta_P) \\ \downarrow & & \downarrow \\ \Delta^\circ(X^\ell,\mathcal{P}) & \xrightarrow{\pi_P} & X \end{array} \tag{9.26}$$

Then, we have the following:

**Lemma 9.12.** *The pullback of the restriction of* $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1$ *to* $\mathcal{O}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ *by the covering map* (9.25) *splits into a direct sum,* $\oplus_{P\in\mathcal{P}}\pi^*_{\tilde{\mathrm{Gl}},P}\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1$*, where the bundle*

$$\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1 \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Fr}(\mathfrak{g}_\ell)\times_X \mathrm{Fr}(TX)\times_{\mathrm{SO}(3)\times\mathrm{SO}(4)} M^{s,\natural}_{spl,|P|}(\delta_P) \tag{9.27}$$

*is defined by*

$$\begin{array}{c} \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{Spin^u(4)} \Upsilon^i_{spl,|P|} \\ \downarrow \\ \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Fr}(\mathfrak{g}_\ell)\times_X \mathrm{Fr}(TX)\times_{\mathrm{SO}(3)\times\mathrm{SO}(4)} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) \end{array} \tag{9.28}$$

*and the map* $\pi_{\tilde{\mathrm{Gl}},P}$ *is defined in the diagram* (9.26)*.*

The $S^1$ action in (9.28) reflects the action of $S^1$ on the infinite-dimensional obstruction space. Thus, following [11, Lemma 3.7], the $S^1$ action on

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V) \times_{\mathrm{Spin}^u(4)} \Upsilon^i_{spl,|P|} \tag{9.29}$$

is given by the diagonal action of the action by $\varrho_{\mathfrak{s}}(e^{-i\theta})$ on $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ and by scalar multiplication with weight one on the fibers of $\Upsilon^i_{spl,|P|}$.

Lemma 9.12 implies that to compute the restriction of the instanton obstruction bundle to the subspace (9.24), it suffices to characterize the Euler class of the bundle (9.28). We use the notation

$$\mathrm{Gl}(\mathfrak{t},\mathfrak{s},P) = \mathrm{Fr}(\mathfrak{g}_\ell)\times_X \mathrm{Fr}(TX)\times_{\mathrm{SO}(3)\times\mathrm{SO}(4)} M^{s,\natural}_{spl,|P|}(\delta_P)$$

Then, the base of the bundle $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1$ admits a product decomposition:

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},P) \cong M_{\mathfrak{s}} \times S^1\backslash \mathrm{Gl}(\mathfrak{t},\mathfrak{s},P). \tag{9.30}$$

Define

$$\Upsilon^i_{X,P}/S^1 = M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\operatorname{Spin}^u(4)} \Upsilon^i_{spl,|P|} \to M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P). \tag{9.31}$$

As in the discussion of the $S^1$ action on (9.29), the $S^1$ action in (9.31) is given by the diagonal action of the action by $\varrho_{\mathfrak{s}}(e^{-i\theta})$ on $\operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ and by scalar multiplication on the fibers of $\Upsilon^i_{spl,|P|}$.

**Lemma 9.13.** *Let $\mathbb{L}_{\mathfrak{s}} \to M_{\mathfrak{s}} \times X$ be the universal Seiberg-Witten line bundle defined in ??. Let $\pi_{X,\mathfrak{s}} : M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P) \to M_{\mathfrak{s}} \times X$ be the projection. Then,*

$$\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1 \cong \pi^*_{X,\mathfrak{s}}\mathbb{L}^*_{\mathfrak{s}} \otimes \Upsilon^i_{X,P}/S^1. \tag{9.32}$$

*Proof.* By the technical result [15, Lemma 3.27], the tensor product in (9.32) can be described as:

$$\left( \left( \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times S^1 \right)_{M_{\mathfrak{s}} \times X} \Upsilon^i_{X,P}/S^1 \right) /S^1, \tag{9.33}$$

where $S^1$ acts diagonally on the factor of $S^1$ and on the fiber of $\Upsilon^i_{X,P}/S^1$. Because the $S^1$ quotient in the definition of $\Upsilon^i_{X,P}/S^1$ acts anti-diagonally (see the remarks following (9.31)), we can then say that the final $S^1$ quotient in (9.33) acts diagonally on the factor of $S^1$ and by the action $\varrho_{\mathfrak{s}}(e^{i\theta})$ on the factor of $\operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V)$ in the definition of $\Upsilon^i_{X,P}/S^1$. One then defines a bundle map from the fibered product (9.33) to the bundle $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1$ by

$$\left( (A_0,\Phi_0,x,e^{i\theta},\tilde{F}_u,[A,F_s,\Psi] \right) \to \left( (A_0,\Phi_0,e^{-i\theta}\tilde{F}_u,[A,F_s,\Psi] \right),$$

where $(A_0,\Phi_0) \in \tilde{M}_{\mathfrak{s}}$, $x \in X$, $e^{i\theta} \in S^1$, $\tilde{F}_u \in \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V)|_x$, and $[A,F_s,\Psi] \in \Upsilon^i_{spl,|P|}$. Observe that the map is well-defined because the diagonal $S^1$ action described earlier in the proof vanishes in the product $e^{-i\theta}\tilde{F}_u$. The equivariance of this map with respect to the $\mathcal{G}_{\mathfrak{s}}$ action is correct because the action of $\mathcal{G}_{\mathfrak{s}}$ defining $\mathbb{L}_{\mathfrak{s}}$ is given by:

$$(u,(A_0,\Phi_0,x,z)) \to \left( u_*(A_0,\Phi_0), x, u(x)^{-1}z \right),$$

and thus using $e^{-i\theta}\tilde{F}_u$ takes the preceding action of $\mathcal{G}_{\mathfrak{s}}$ to the desired diagonal action on $\tilde{M}_{\mathfrak{s}}$ and $\operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V)$. □

Next, we characterize the (rational) Euler class of $\Upsilon^i_{X,P}/S^1$. To this end, we introduce the cohomology class,

$$\nu_{\mathrm{Gl}} \in H^2(M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P)), \tag{9.34}$$

(denoted $\nu_{\mathfrak{t}}$ in [11, Definition 5.22]) to be the first Chern class of the $S^1$ bundle,

$$M_{\mathfrak{s}} \times \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P) \to M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P). \tag{9.35}$$

In [11, Lemma 5.23], the equation

$$\nu = \nu_{\mathrm{Gl}} + 2\pi^*_{X,\mathfrak{s}} c_1(\mathbb{L}_{\mathfrak{s}}) \tag{9.36}$$

is proven. The advantage of the bundle (9.34) over $\pi^*_{X,\mathfrak{s}}\mathbb{L}_{\mathfrak{s}}$ is that the former pulls back by the projection $M_{\mathfrak{s}} \times \S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P) \to S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P)$.

**Lemma 9.14.** *The rational Euler class,*

$$e(\Upsilon^i_{X,P}/S^1) \in H^{2|P|}(M_{\mathfrak{s}} \times S^1 \backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P);\mathbb{R}),$$

*is given by a polynomial in $\pi^*_X c(\mathfrak{t})$ and $\nu_{\mathrm{Gl}}$ which is otherwise independent of $X$.*

*Proof.* Observe that the bundle $\Upsilon^i_{X,P}/S^1$ is the pullback by the projection

$$M_{\mathfrak{s}} \times S^1\backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P) \to S^1\backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},P)$$

of the bundle

$$S^1\backslash \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\operatorname{Spin}^u(4)} \Upsilon^i_{spl,|P|} \to S^1\backslash \operatorname{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}). \tag{9.37}$$

The lemma will then follow if we can that the Euler class of the bundle (9.37) satisfies the conclusion of the lemma.

The kernel of the projection

$$(\operatorname{Ad}^u_{\mathrm{SO}(3)}, \operatorname{Ad}^u_{\mathrm{SO}(4)}) : \operatorname{Spin}^u(4) \to \mathrm{SO}(3) \times \mathrm{SO}(4) \tag{9.38}$$

is the central $S^1$ in $\operatorname{Spin}^u(4)$. By the identity (see [11, Equation (3.14)])

$$\operatorname{Fr}_{\mathbb{C}\ell(T^*X)}/S^1 = \operatorname{Fr}(\mathfrak{g}_\ell) \times_X \operatorname{Fr}(TX).$$

(where the $S^1$ is the kernel of the homomorphism (9.38)), we can rewrite the space $\operatorname{Gl}(\mathfrak{t},\mathfrak{s},P)$ as

$$\operatorname{Gl}(\mathfrak{t},\mathfrak{s},P) = \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\operatorname{Spin}^u(4)} M^{s,\natural}_{spl,|P|}(\delta_P)$$

where $\operatorname{Spin}^u(4)$ acts on $M^{s,\natural}_{spl,|P|}(\delta_P)$ via the projection (9.38) and the action of $\mathrm{SO}(3)\times\mathrm{SO}(4)$ on $M^{s,\natural}_{spl,|P|}(\delta_P)$. Let $\tilde\iota_u : \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \to \operatorname{ESpin}^u(4)$ be the classifying map, appearing in the diagram:

$$\begin{array}{ccc}
\operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) & \xrightarrow{\tilde\iota_u} & \operatorname{ESpin}^u(4) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\iota_u} & \operatorname{BSpin}^u(4)
\end{array}$$

Recall from [1, §4], that the map $[A] \to [D_A]$ where $D_A$ is the Dirac operator defined by the connection $A$ defines a continuous map,

$$f_\kappa : \mathcal{B}^s_\kappa \to \mathcal{F}_\kappa,$$

where $\mathcal{F}_\kappa$ is the space of Fredholm operators of index $\kappa$. The space $\mathcal{F}_\kappa$ is homotopic to $\mathrm{BU} = \lim_n \mathrm{BU}(n)$. Although there is no index bundle defined for the Dirac operators parameterized by $\mathcal{B}^s_\kappa$ because it is not compact, the restriction of the pullback of the universal Chern classes in BU by $f_\kappa$ to the subspace $\mathcal{K}^s_\kappa \subset \mathcal{B}^s_\kappa$ where the cokernel of the Dirac operator vanishes are equal to the Chern class of the vector bundle defined by $\operatorname{Index}(\mathbf{D})$ on this subspace. By property (iv) in Proposition 7.1, the Chern classes of the bundle

$$\Upsilon^i_{spl,|P|} \to M^{s,\natural}_{spl,|P|}$$

are given by the pullback of the universal Chern classes in $H^\bullet(\mathrm{BU})$ by the map $f_{|P|}$. By the $\operatorname{Spin}^u(4)$-equivariance of the Dirac operator, the map $f_\kappa$ extends to a map

$$F_\kappa : \operatorname{ESpin}^u(4) \times_{\operatorname{Spin}^u(4)} \mathcal{B}^s_\kappa \to \mathcal{F}_\kappa,$$

such that the restriction of $F_\kappa$ to

$$\operatorname{ESpin}^u(4) \times_{\operatorname{Spin}^u(4)} \mathcal{K}^s_\kappa$$

is the composition of a classifying map for the index bundle with the inclusion $\mathrm{BU}(\kappa) \to \mathrm{BU}$. We conclude that the classifying map for the bundle

$$\Upsilon^i_{X,P} = \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\operatorname{Spin}^u(4)} \Upsilon^i_{spl,|P|} \to \operatorname{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\operatorname{Spin}^u(4)} M^{s,\natural}_{spl,|P|}(\delta_P),$$

factors through the composition

$$\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell)\times_{\mathrm{Spin}^u(4)} M^{s,\natural}_{spl,|P|}(\delta_P) \xrightarrow{\tilde\iota_u\times\iota_M} \mathrm{ESpin}^u(4)\times_{\mathrm{Spin}^u(4)}\mathcal{B}^s_{|P|} \xrightarrow{F_{|P|}} \mathcal{F}_{|P|} \tag{9.39}$$

where $\iota_M : M^{s,\natural}_{spl,|P|}(\delta_P)\to\mathcal{B}^s_{|P|}$ is the inclusion. The bundle map $\tilde\iota_u\times\iota_M$ descends to the $S^1$ quotients:

$$\tilde\iota_u\times\iota_M : S^1\backslash\left(\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell)\times_{\mathrm{Spin}^u(4)} M^{s,\natural}_{spl,|P|}(\delta_P)\right)\to S^1\backslash\left(\mathrm{ESpin}^u(4)\times_{\mathrm{Spin}^u(4)}\mathcal{B}^s_{|P|}\right)$$

Then, because the rational cohomology of $\mathcal{B}^s_{|P|}$ is trivial (see [3, Lemma 5.1.14]), the rational cohomology of the space

$$S^1\backslash\left(\mathrm{ESpin}^u(4)\times_{\mathrm{Spin}^u(4)}\mathcal{B}^s_{|P|}\right)$$

is generated by the first Chern class of the $S^1$ action and cohomology classes pulled back by the projection

$$S^1\backslash\left(\mathrm{ESpin}^u(4)\times_{\mathrm{Spin}^u(4)}\mathcal{B}^s_{|P|}\right)\to\mathrm{BSpin}^u(4).$$

Under the map $\tilde\iota_u\times\iota_M$, the first Chern class of the $S^1$ action pulls back to $\nu_{\mathrm{Gl}}$ while the cohomology classes pulled back from $\mathrm{BSpin}^u(4)$ pullback to characteristic classes of $\mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V)$, which we are denoting by $c(\mathfrak{t})$. The conclusion of the lemma then follows from the factoring of the classifying map for the bundle (9.37) given in (9.39). □

**Remark 9.15.** Unlike the case $\ell=1$, the action of $\mathcal{G}_{\mathfrak{s}}$ on $S^1\backslash\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ is not trivial because there can be more than one frame in the definition of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P})$. That is, if $F_1$ and $F_2$ are frames over separate points $x_1$ and $x_2$ in $X$, the action of $S^1$ identifies $(F_1,F_2)$ with $(e^{i\theta}F_1,e^{i\theta}F_2)$. There can be $u\in\mathcal{G}_{\mathfrak{s}}$ with $u(x_1)\neq u(x_2)$. For such $u\in\mathcal{G}_{\mathfrak{s}}$, $(uF_1,uF_2)$ would not be identified with $(F_1,F_2)$ by the $S^1$ action.

*Proof of Proposition 9.12.* By Lemmas 9.12, 9.13, and 9.14, we see that the Euler class of the pullback, by the map (9.25), of the restriction of $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1$ to the space (9.24) is given by

$$\prod_{P\in\mathcal{P}}\pi^*_{\tilde{\mathrm{G}}\mathrm{l},P}e(\Upsilon^i_{\mathfrak{t},\mathfrak{s}}(P)/S^1)=\prod_{P\in\mathcal{P}}\pi^*_{\tilde{\mathrm{G}}\mathrm{l},P}e(\pi^*_{X,\mathfrak{s}}\mathbb{L}^*_{\mathfrak{s}}\otimes\Upsilon^i_{X,P}/S^1). \tag{9.40}$$

Now, by the splitting principal, we can assume that $\Upsilon^i_{X,P}/S^1=\oplus_{i\in P}L_i$, where $L_i$ is a complex line bundle and all the symmetric polynomials in $c_1(L_i)$ are given by polynomials in $c(\mathfrak{t})$ and, by (9.36), in $\nu-2\pi^*_{X,\mathfrak{s}}c_1(\mathbb{L}_{\mathfrak{s}})$. Observe that $\pi_{X,\mathfrak{s}}\circ\pi_{\tilde{\mathrm{G}}\mathrm{l},P}=\pi_{\mathfrak{s},i}\circ\pi_X$, so the symmetric polynomials in $\pi^*_{\tilde{\mathrm{G}}\mathrm{l},P}\pi^*_{X,\mathfrak{s}}\mathbb{L}_{\mathfrak{s}}$ are given by the Chern classes $\tilde\iota^*_{\mathcal{P}}\pi^*_X c_{\mathfrak{s},\ell,i}$ defined in (9.21). Hence, we can rewrite (9.40) as

$$=\prod_{P\in\mathcal{P}}\prod_{i\in P}\left(\tilde\iota^*_{\mathcal{P}}\pi^*_{\mathfrak{s},i}c_1(\mathbb{L}_{\mathfrak{s}})\right)\pi^*_{\tilde{\mathrm{G}}\mathrm{l},P}c_1(L_i). \tag{9.41}$$

One then verifies the preceding product can be expressed in terms of symmetric polynomials in $c_1(L_i)$ and $\pi^*_{\mathfrak{s},i}c_1(\mathbb{L}_{\mathfrak{s}})$. □

9.5. **The global instanton obstruction Euler class.** We now piece together the local computations of the preceding section to give a global characterization of the instanton obstruction $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1\to\mathcal{M}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1$.

**Proposition 9.16.** *The Euler class of the instanton obstruction bundle $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1\to\mathcal{M}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1$ is given by a polynomial in $\iota^*\nu$, $\iota^*\pi^*_X S^\ell(c(\mathfrak{t}))$, and $\iota^*\pi^*_{X,\mathfrak{s}}c_{\mathfrak{s},\ell,i}$ which is otherwise independent of $X$.*

*Proof.* The construction of the bundle $\Upsilon^i_{spl,\kappa} \to M^{s,\natural}_{spl,\kappa}(\delta)$ and the $\mathrm{Spin}^u(4)$ equivariance of the diagram (7.12) implies that the particular classifying maps of the local instanton obstruction bundles which admit the factorization (9.39) can be chosen to be equal on the overlap given in Lemma 8.1. Hence, the local equality of cohomology classes given in Proposition 9.11 is an equality of cocycles and thus an equality of global cohomology classes. □

**Definition 9.17.** We write $e(\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1)$ for the real cohomology class given by the Euler class of $\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1$ as described in Proposition 9.16.

9.6. **Extensions.** We define extensions of the cohomology classes $\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_p(\beta)$, $\boldsymbol{\gamma}^*_{\mathcal{M}}\mu_c$, and $e(\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1)$ from the top stratum $\mathcal{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ to $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1$ as follows.

**Definition 9.18.** For $\beta \in H_\bullet(X;\mathbb{R})$, let

$$\bar{\mu}_p(\beta) \in H^{4-\bullet}(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R}), \tag{9.42}$$

be the cohomology class obtained by omitting the pullback $\iota^*$ on the right-hand-side of (9.19). Similarly, let

$$\bar{\mu}_c = -\nu \in H^2(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R}), \tag{9.43}$$

be the cohomology class obtained by omitting the pullback $\iota^*$ on the right-hand-side of (9.19). Let

$$\bar{e}_i \in H^{2\ell}(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R}) \tag{9.44}$$

be the extensions of $e(\Upsilon^i_{\mathfrak{t},\mathfrak{s}}/S^1)$ obtained by omitting the pullback $\iota^*$ from the expression in Proposition 9.16.

**Remark 9.19.** Note that there is no need to extend the Euler class of the Seiberg-Witten obstruction bundle, $e(\Upsilon^s_{\mathfrak{t},\mathfrak{s}}/S^1)$, as it is already defined on $\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1$ by Lemma 9.10.

## 10. The link

We now use the Thom-Mather structure defined in §6.6 to construct an ambient link, $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, given by the boundary of a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ and then the actual link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, given by the boundary of a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

We construct the link in §10.1 using the tubular distance functions $t(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)$ constructed in Lemma 6.15. We use our understanding of the overlap maps to show that the ambient link $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ can be decomposed into pieces (enumerated by the strata of $\mathrm{Sym}^\ell(X)$) each of which admits a fiber bundle structure. Each piece is a smoothly-stratified space, with strata given by smooth manifolds with corners. The intersections of the different pieces are given by these corners. In §10.2, we show that the intersection pairing appearing in (2.51) can be computed by a pairing of the appropriate cohomology classes with the fundamental class of the ambient link. We reduce the pairing with the fundamental class of the ambient link to one with a simpler subspace, $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ in §10.3. This subspace is also a union of pieces each of which is a smoothly-stratified space with the strata given by manifolds with corners. In addition, each piece of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ is a fiber bundle over a stratum of $\mathrm{Sym}^\ell(X)$. In §10.4, we describe the intersections of the pieces of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ which are given by the corners as described above.

For use in this section and the following, pick a representative from each conjugacy class of partitions of $N_\ell$ and enumerate them in such a way that

$$\Sigma(X^\ell,\mathcal{P}_i) \subset \mathrm{cl}\Sigma(X^\ell,\mathcal{P}_j) \quad \text{only if} \quad i<j.$$

Write these partitions as $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$ so that $\Sigma(X^\ell, \mathcal{P}_1)$ is the lowest stratum and $\Sigma(X^\ell, \mathcal{P}_r)$ is the highest.

10.1. **Defining the link.** We define the link by first constructing an ambient link in §10.1.1. We discuss the orientations of the link in §10.1.2 which leads to a formal statement of the cobordism formula (2.51) in Theorem 10.7. Then in §10.1.3, we give a more detailed description of the pieces of the link, showing how they admit a fiber bundle structure.

10.1.1. *The ambient link.* The ambient link is defined as the boundary of the neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$. The subspace $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ is the intersection of the zero-locus of the function defined by the length of the fiber vector of the vector bundle $N(\delta) \to M_{\mathfrak{s}}$,

$$t_N : \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1 \to [0, \delta], \tag{10.1}$$

with the union of the zero-loci of the tubular distance function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ defined on $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ in Lemma 6.15. We will define the ambient link as the union of two pieces:

$$\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} = \bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}} \cup \bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}. \tag{10.2}$$

The first piece, $\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}}$, is defined to be the codimension-one subspace

$$\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}} = t_N^{-1}(\delta) \subset \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1. \tag{10.3}$$

The definition of the instanton component of the link, $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}$, is more involved as it will be the union of the hypersurfaces of the local functions $\tilde{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$. Pick a small constant $\varepsilon_i$ for each stratum $\Sigma(X^\ell, \mathcal{P}_i)$ (these will be chosen to be generic with respect to requirements obtained later). Then, we define the local link by

$$\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) = \mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \cap t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\varepsilon_j) - \cup_{i \neq j} t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)^{-1}([0, \varepsilon_i)). \tag{10.4}$$

Taking the 'hypersurface', $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\varepsilon_j)$ makes the subspace $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ a link for $M_{\mathfrak{s}} \times \Sigma(X^\ell, \mathcal{P}_j)$. Removing the subspaces $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)^{-1}([0, \varepsilon_i))$ from $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ ensures that $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ is disjoint from the other strata of reducibles $M_{\mathfrak{s}} \times \Sigma(X^\ell, \mathcal{P}_i)$ (for $i \neq j$). We then define the instanton component of the link:

$$\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}} = \cup_i \, \bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i). \tag{10.5}$$

The next lemma follows immediately from the construction.

**Lemma 10.1.** *The ambient link $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ defined in* (10.2), (10.3), *and* (10.5) *is the boundary of a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1$. For generic choices of the constants $\varepsilon_i$, the intersection of $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ with each stratum of $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1$ is a smooth manifold with corners.*

We can now define the link:

**Definition 10.2.** The link, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is defined to be the intersection of the ambient link $\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ defined in (10.2) with the zero-locus of the obstruction section.

We refer to the intersection of any two or more of the pieces $\bar{\mathbf{L}}^{\mathrm{vir},s}_{\mathfrak{t},\mathfrak{s}}$ or $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ as an edge. For generic choices of the parameters $\delta, \varepsilon_i$, the intersection of each of these edges with the strata of $\bar{\mathcal{M}}^{vir}_{\mathfrak{t},\mathfrak{s}}/S^1$ will also be a smooth manifold with corners.

**Lemma 10.3.** *For generic values of the parameters $\delta, \varepsilon_i$ used to define the ambient link, the link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is the boundary of a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. The link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is smoothly-stratified subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$; the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with each stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$*

*and each edge of* $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ *is a smooth, oriented manifold with corners. The intersections of* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ *with the lower strata of* $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ *have codimension at least two in* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$.

The following result then translates immediately from [11].

**Lemma 10.4.** *Assume* $w \in H^2(X;\mathbb{Z})$ *is such that* $w \pmod 2$ *is good in the sense of Definition 2.3. Given a spin*$^u$ *structure* $\mathfrak{t}$ *on* $X$ *and a spin*$^c$ *structure* $\mathfrak{s}$ *on* $X$ *satisfying* $\ell(\mathfrak{t},\mathfrak{s}) \geq 0$ *and* $w_2(\mathfrak{t}) \equiv w \pmod 2$, *there are positive constants* $\varepsilon_0$ *and* $\delta_0$ *such that the following hold for all generic choices of* $\varepsilon_i \leq \varepsilon_0$ *and* $\delta \leq \delta_0$ *defining* $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$.

- *The link* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ *is disjoint from* $\bar{M}^w_\kappa$ *and* $\bar{\mathcal{M}}^{\text{red}}_{\mathfrak{t}}$ *in the stratification* (2.13) *of* $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.
- *For all* $z \in \mathbb{A}(X)$ *and positive integers* $\eta$ *satisfying* (10.10), *and for* $\bar{\mathcal{V}}(z)$ *and* $\bar{\mathcal{W}}$ *the geometric representatives discussed in §2.4, the intersection*

$$\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \tag{10.6}$$

  *is a finite collection of points contained in the top stratum* $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ *of* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, *and is disjoint from the edges of* $\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$.

10.1.2. *Orientations of the link.* To define an intersection number from the intersection (10.6), it is necessary to discuss orientations of the link. An orientation for $\mathcal{M}_{\mathfrak{t}}$ determines one for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ through the convention introduced in [16, Equations (2.16), (2.16) & (2.25)] by considering $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ as a boundary of $(\mathcal{M}_{\mathfrak{t}} - \boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}))/S^1$. Specifically, at a point $[A,\Phi] \in \mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, if

- $\vec{r} \in T\mathcal{M}^{*,0}_{\mathfrak{t}}$ is an outward-pointing radial vector with respect to the open neighborhood $\mathcal{M}_{\mathfrak{t}} \cap \boldsymbol{\gamma}(\mathcal{M}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}})$ and complementary to the tangent space of $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$,
- $v_{S^1} \in T\mathcal{M}^{*,0}_{\mathfrak{t}}$ is tangent to the orbit of $[A,\Phi]$ under the (free) circle action (where $S^1 \subset \mathbb{C}$ has its usual orientation), and
- $\lambda_{\mathcal{M}} \in \det(T\mathcal{M}^{*,0}_{\mathfrak{t}})$ is an orientation for $T\mathcal{M}_{\mathfrak{t}}$ at $[A,\Phi]$,

then we define an orientation $\lambda_L$ for $T\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ at $[A,\Phi]$ by

$$\lambda_{\mathcal{M}} = -v_{S^1} \wedge \vec{r} \wedge \tilde{\lambda}_L, \tag{10.7}$$

where the lift $\tilde{\lambda}_L \in \Lambda^{\max -2}(T\mathcal{M}^{*,0}_{\mathfrak{t}})$ at $[A,\Phi]$ of $\lambda_L \in \det(T\mathbf{L}_{\mathfrak{t},\mathfrak{s}}) \subset \Lambda^{\max -1}(T(\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1))$, obeys $\pi_*\tilde{\lambda}_L = \lambda_L$, if $\pi : \mathcal{M}_{\mathfrak{t}} \to \mathcal{M}_{\mathfrak{t}}/S^1$ is the quotient map.

**Definition 10.5.** If $O$ is an orientation for $\mathcal{M}_{\mathfrak{t}}$, we call the orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ related to $O$ by equation (10.7) the *boundary orientation defined by* $O$.

The *standard orientation* for $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is defined in [11, Definition 3.12]. The standard orientation arises from the local topology of $\bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$ and thus is more natural for computations on this space. The standard and boundary orientations are related in the following.

**Lemma 10.6.** [11, Lemmas 3.13 & 3.14] *If* $\mathfrak{t}$ *and* $\mathfrak{t}_\ell$ *are spin*$^u$ *structures on* $X$ *satisfying*

$$p_1(\mathfrak{t}) = p_1(\mathfrak{t}_\ell) - 4\ell, \quad c_1(\mathfrak{t}) = c_1(\mathfrak{t}_\ell), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{t}_\ell).$$

*Let* $\Omega$ *be a homology orientation and let* $w$ *be an integral lift of* $w_2(\mathfrak{t})$. *If* $\mathfrak{t}_\ell$ *admits a splitting* $\mathfrak{t}_\ell = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, *then the standard orientation for* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ *and the boundary orientation for* $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ *defined through the orientation* $O^{\text{asd}}(\Omega,w)$ *for* $\mathcal{M}_{\mathfrak{t}}$ *differ by a factor of*

$$(-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})}, \quad \textit{where} \quad o_{\mathfrak{t}}(w,\mathfrak{s}) = \tfrac{1}{4}(w - c_1(L))^2. \tag{10.8}$$

For all $z \in \mathbb{A}(X)$ and positive integers $\eta$ satisfying (10.10), and for $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ the geometric representatives discussed in §2.4, we can then define the intersection number

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) \tag{10.9}$$

to be the oriented count of points in the intersection (10.6), using the standard orientation of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. This yields the following formal expression of the cobordism formula.

**Theorem 10.7.** *Let $\mathfrak{t}$ be a spin$^u$ structure on a smooth, oriented four-manifold $X$. Let $z \in \mathbb{A}(X)$ and $\eta$ a non-negative integer satisfy*

$$\deg(z) + 2\eta = \dim \mathcal{M}_{\mathfrak{t}}^{*,0} - 2. \tag{10.10}$$

*Assume there is $w \in H^2(X;\mathbb{Z})$ satisfying $w_2(\mathfrak{t}) \equiv w \pmod 2$ and which is good in the sense of Definition 2.3. Then the intersection numbers* (10.9) *satisfy*

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}\right) = -\sum_{\mathfrak{s}\in Spin^c(X)} (-1)^{o_{\mathfrak{t}}(w,\mathfrak{s})}\, \#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right), \tag{10.11}$$

*where $\bar{\mathbf{L}}^w_{\mathfrak{t},\kappa}$ is the link of the anti-self-dual connections in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ from* [15, Definition 3.7].

10.1.3. *The fiber bundle structure.* We now describe how the fiber bundle structure of the pieces $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ of the link. Observe that the functions $t(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$, while nominally defined on the subspaces $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ of $\bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1$ can also be understood as functions on subsets of $\mathrm{Sym}^\ell(X)$ by virtue of the inclusion,

$$N(\delta) \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}/S^1,$$

and by virtue of the equality in item (2) of Lemma 6.15, the restriction of $t(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ to

$$\Big(N(\delta) \times \mathrm{Sym}^\ell(X)\Big) \cap \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})$$

is equal to the pullback by the projection

$$N(\delta) \times \mathrm{Sym}^\ell(X) \to \mathrm{Sym}^\ell(X).$$

of the function $t(X^\ell,\mathcal{P})$ defined in Lemma 4.37.

With this understanding, define

$$K_j = \Sigma(X^\ell,\mathcal{P}_i) - \cup_{i<j}\, t(X^\ell,\mathcal{P}_i)^{-1}([0,\varepsilon_i)). \tag{10.12}$$

Recall that the open subspace $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ is a neighborhood of $N_{\mathfrak{t},\mathfrak{s}}(\delta)/S^1 \times \Sigma(X^\ell,\mathcal{P}_j)$ in the fiber bundle

$$\begin{array}{ccc} \prod_{P\in\mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) & \longrightarrow & \tilde{N}(\delta_P) \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\mathcal{P}) \\ & & \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)} \\ & & N(\delta)/S^1 \times \Sigma(X^\ell,\mathcal{P}_j) \end{array} \tag{10.13}$$

We will write

$$\bar{M}(\mathcal{P}) = \prod_{P\in\mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) \tag{10.14}$$

for the fiber of (10.13).

The first step in describing the fiber bundle structure of $\mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ in $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ is the following description of the base of the bundle.

**Lemma 10.8.** *The space $K_j$ defined in* (10.12) *is a compact, codimension-zero submanifold with corners of $\Sigma(X^\ell,\mathcal{P}_j)$. Then, the subspace*

$$\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j) - \cup_{i<j}\, t(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)^{-1}([0,\varepsilon_i)) \tag{10.15}$$

*is given by the intersection of $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ with the restriction of the fiber bundle* (10.13) *to $N(\delta)/S^1 \times K_j$. In addition, the subspace* (10.15) *is disjoint from the subspace $N(\delta)/S^1 \times K_i$ for all $i < j$.*

*Proof.* By (6.52), for $i < j$ the function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ is constant on the fibers of the projection map $\pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$. Hence, restricting $\pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ to $\pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(K_j)$ does not change the fibers of this projection. The equality between $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ and $t(X^\ell, \mathcal{P}_j)$ in item (2) of Lemma 6.15 then completes the proof. □

The function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ on $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ is defined by a function,

$$t_f(\mathcal{P}_j) = \sum_{P \in \mathcal{P}_j} |P| \tilde{\lambda}^2_{|P|} : \bar{M}(\mathcal{P}) = \prod_{P \in \mathcal{P}_j} \bar{M}^{s,\natural}_{spl,|P|}(\delta_P) \to [0, \infty) \tag{10.16}$$

(see (6.50)) on the fibers of (10.13). Because $t_f(\mathcal{P}_j)$ is $G(\mathcal{P}_j)$-invariant it extends to a function on (10.13); this function is $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$. Because the open subspace $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ is defined by a subspace of the fiber bundle $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ in (10.13), the following lemma is necessary to ensure that working with the subspace $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ does not restrict the fibers of the local link $\bar{\mathbf{L}}^{vir,i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$.

**Lemma 10.9.** *For any compact $K \Subset \Sigma(X^\ell, \mathcal{P}_j)$, there is $\varepsilon_K$ such that the intersection*

$$\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i) \cap \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1} \left( N(\delta)/S^1 \times K \right),$$

*contains $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}([0, \varepsilon_K)) \cap \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1} \left( N(\delta)/S^1 \times K \right)$.*

*Proof.* The lemma follows immediately from the compactness of $K$. □

Lemma 10.9 allows us to define a link of $N(\delta)/S^1 \times K_j$ using a subspace of (10.13) defined by replacing the fiber $\bar{M}(\mathcal{P}_j)$ with the subspace

$$\bar{M}(\mathcal{P}_j, \varepsilon_j) = t_f(\mathcal{P}_j)^{-1}(\varepsilon_j) \subset \bar{M}(\mathcal{P}_j) \tag{10.17}$$

where $t_f(\mathcal{P}_j)$ is defined in (10.16). The resulting subspace will be $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\varepsilon_j)$:

**Lemma 10.10.** *Let $\varepsilon_j > 0$ be a small constant given by Lemma 10.9 for the space $K_j$ defined in* (10.12). *The subspace*

$$t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\varepsilon_j) - \cup_{i<j}\ t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)^{-1}([0, \varepsilon_i)) \tag{10.18}$$

*is the codimension-one subspace of $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$, disjoint from $N(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}_i)$ for all $i \leq j$ and is given by*

$$\begin{array}{ccc} \bar{M}(\mathcal{P}_j, \varepsilon_j) & \longrightarrow & \tilde{N}(\delta) \times_{\mathcal{G}_\mathfrak{s} \times S^1} \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j, g_{\mathcal{P}_j})|_{K_j} \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P}, \varepsilon_j) \\ & & \Big\downarrow {\scriptstyle \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)} \\ & & N(\delta)/S^1 \times K_j \end{array} \tag{10.19}$$

*where $\bar{M}(\mathcal{P}_j, \varepsilon_j)$ is defined in* (10.17).

*Proof.* The subspace (10.18) is disjoint from the subspaces $N(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}_i)$ for $i < j$ by Lemma 10.8 and from $N(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}_j)$ because

$$N(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}_j) = t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(0).$$

The subspace (10.18) is a sub-fiber bundle of (10.13) because, as noted above, the function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ is defined by extending a function on the fibers over the bundle. □

The subspace (10.18) still intersects $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$, but only along higher strata $M_\mathfrak{s} \times \Sigma(X^\ell, \mathcal{P}_i)$ where $i > j$. We remove this intersection by removing neighborhoods of the subspace $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)^{-1}(0) \cap \mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$, giving the definition of $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ found in (10.4). To understand the subspace $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ of (10.18), we must examine the function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ on $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \cap \mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$.

Recall from Proposition 6.9 that the intersection $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ (where $\mathcal{P}<\mathcal{P}'$) is given by an open subspace of the fiber bundle

$$\begin{array}{c} \tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}\times S^1}\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\times_{G(\mathcal{P})}\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\bar M(\mathcal{P},\mathcal{P}'') \\ \big\downarrow \\ N(\delta)/S^1\times\Sigma(X^\ell,\mathcal{P}) \end{array} \tag{10.20}$$

where the fiber has components (up to a quotient by the symmetric subgroup $\mathfrak{S}(\mathcal{P})$)

$$\bar M(\mathcal{P},\mathcal{P}'')=\prod_{P\in\mathcal{P}}\left(\Delta^\circ(Z_{|P|}(\delta_P),\mathcal{P}''_P)\times\prod_{Q\in\mathcal{P}''_P}\bar M^{s,\natural}_{spl,|Q|}(\delta_Q)\right). \tag{10.21}$$

There is an $\mathfrak{S}(\mathcal{P})$-equivariant projection map,

$$\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\bar M(\mathcal{P},\mathcal{P}'')\to\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\prod_{Q\in\mathcal{P}''}\bar M^{s,\natural}_{spl,|Q|}(\delta_Q). \tag{10.22}$$

If $\tilde\lambda_Q$ is the modified scale parameter on $\bar M^{s,\natural}_{spl,|Q|}(\delta_Q)$ defined in (5.69)), then the function,

$$\sum_{Q\in\mathcal{P}'}|Q|\tilde\lambda_Q^2:\prod_{Q\in\mathcal{P}''}\bar M^{s,\natural}_{spl,|Q|}(\delta_Q)\to[0,\infty) \tag{10.23}$$

extends to a unique $\mathfrak{S}(\mathcal{P})$-invariant function on the image of the projection (10.22). The pullback of this $\mathfrak{S}(\mathcal{P})$-invariant function by the projection map (10.22) defines a function,

$$t_{o,f}(\mathcal{P},[\mathcal{P}']):\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\bar M(\mathcal{P},\mathcal{P}'')\to[0,\infty). \tag{10.24}$$

The function $t_{o,f}(\mathcal{P},[\mathcal{P}'])$ is invariant under the action of the group $G(\mathcal{P})$ and thus defines a function on the bundle (10.20):

$$\tilde t_o(\mathcal{P},[\mathcal{P}']):\tilde N(\delta)\times_{\mathcal{G}_{\mathfrak{s}}\times S^1}\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P},g_{\mathcal{P}})\times_{G(\mathcal{P})}\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\bar M(\mathcal{P},\mathcal{P}'')\to[0,\infty). \tag{10.25}$$

**Lemma 10.11.** *Let $\mathcal{P}<\mathcal{P}'$ be partitions of $N_\ell$. Let $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ be the upwards overlap map appearing in* (6.40). *The pullback $t(\mathfrak{t},\mathfrak{s},\mathcal{P}')\circ\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ is defined on the subspace $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ of the fiber bundle* (10.20) *and is equal to the function $\tilde t_o(\mathcal{P},[\mathcal{P}'])$ from* (10.25).

*Proof.* The lemma follows immediately from the definition of the function $t(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ and the observation that the overlap map $\rho^{N,\mathfrak{t},\mathfrak{s},u}_{\mathcal{P},[\mathcal{P}']}$ is the identity on the spaces $\bar M^{s,\natural}_{spl,|Q|}(\delta_Q)$. □

To understand the space (10.4), we now discuss the behaviour of the function $\tilde t_o(\mathcal{P},[\mathcal{P}'])$ under the downwards transition map $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ from (6.40). On the component of the intersection $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P})\cap\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}')$ corresponding to the partition $\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']$, the map $\rho^{\mathfrak{t},\mathfrak{s},d}_{\mathcal{P},[\mathcal{P}']}$ is defined by the inclusion of fibers (see the discussion around (6.35)) defined the product (over $P\in\mathcal{P}$) of splicing maps (of connections to the trivial connection on $\mathbb{R}^4$):

$$\coprod_{\mathcal{P}''\in[\mathcal{P}<\mathcal{P}']}\bar M(\mathcal{P},\mathcal{P}'')\to\bar M(\mathcal{P})=\prod_{P\in\mathcal{P}}\bar{\mathcal{M}}^{s,\natural}_{spl,|P|}(\delta_P). \tag{10.26}$$

Let

$$U_f(\mathcal{P},[\mathcal{P}'])\subset\bar M(\mathcal{P}) \tag{10.27}$$

be the image of the inclusion (10.26). (Note that the map (10.26) is actually injective only after taking the quotient of the domain and image by $\mathfrak{S}(\mathcal{P})$; we omit that quotient from the notation since it is not relevant to this discussion.) Let

$$t_f(\mathcal{P},[\mathcal{P}']) : U_f(\mathcal{P},[\mathcal{P}']) \to [0,\infty) \tag{10.28}$$

be the function whose pullback by the inclusion (10.26) is equal to the function $t_{o,f}(\mathcal{P},[\mathcal{P}'])$ from (10.24). We can characterize the function $t_f(\mathcal{P},[\mathcal{P}'])$ as follows. Let

$$t(\Theta,\mathcal{P}'_P) : \mathcal{U}(\Theta,\mathcal{P}'_P) \subset \bar{M}^{s,\natural}_{spl,|P|} \to [0,\infty),$$

be the tubular distance function defined in (5.73). These function (pulled back by the appropriate projections) define a function

$$\sum_{P\in\mathcal{P}} |P| t(\Theta,\mathcal{P}'_P)^2 : \bar{M}(\mathcal{P}) = \prod_{P\in\mathcal{P}} \bar{M}^{s,\natural}_{spl,|P|} \to [0,\infty), \tag{10.29}$$

which is independent of $\mathcal{P}' \in [\mathcal{P} < \mathcal{P}']$ and we have the following:

**Lemma 10.12.** *The function $t_f(\mathcal{P},[\mathcal{P}'])$ defined in* (10.28) *is equal to the function defined in* (10.29).

*Proof.* The lemma follows form the definition of $t_f(\mathcal{P},[\mathcal{P}'])$ and the definition (5.73). □

We then have:

**Lemma 10.13.** *For $i > j$, the function $t(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)$ on the subspace $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i) \cap \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ is defined by the extension of the $G(\mathcal{P})$-invariant function $t_f(\mathcal{P},[\mathcal{P}'])$ from the fiber $\mathcal{U}_f(\mathcal{P},[\mathcal{P}'])$ to the subspace $\mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i) \cap \mathcal{U}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)$ of the fiber bundle* (10.13).

For all $k > j$ we define

$$\bar{M}(\mathcal{P}_j,[\mathcal{P}_k],\varepsilon) = \bar{M}(\mathcal{P}_j) - t_f(\mathcal{P}_j,[\mathcal{P}_k])^{-1}([0,\varepsilon)). \tag{10.30}$$

Let $\boldsymbol{\varepsilon}_j = (\varepsilon_j,\varepsilon_{j+1},\dots,\varepsilon_r)$. Then define

$$\bar{M}(\mathcal{P}_j,\boldsymbol{\varepsilon}_j) = \bar{M}(\mathcal{P}_j,\varepsilon_j) \cap \left(\cap_{k>j}\bar{M}(\mathcal{P}_j,[\mathcal{P}_k],\varepsilon_k)\right) \subset \bar{M}(\mathcal{P}). \tag{10.31}$$

We can then conclude with the following description of the fiber bundle structure of $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$.

**Lemma 10.14.** *The local instanton link $\bar{\mathbf{L}}^{vir,i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ defined in* (10.4) *is the subbundle of the restriction of the fiber bundle* (10.13) *to $N(\delta)/S^1 \times K_j$ defined by the subspace* (10.31) *of the fiber $\bar{M}(\mathcal{P}_j)$:*

$$\begin{tikzcd} \bar{M}(\mathcal{P}_j,\boldsymbol{\varepsilon}_j) \arrow[r] & \tilde{N}(\delta_P) \times_{\mathcal{G}_\mathfrak{s}\times S^1} \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j,g_{\mathcal{P}_j}) \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P},\boldsymbol{\varepsilon}) \arrow[d, "\pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)"'] \\ & N(\delta)/S^1 \times K_j \end{tikzcd} \tag{10.32}$$

*This local instanton link is disjoint from $N(\delta)/S^1 \times \mathrm{Sym}^\ell(X)$.*

10.2. **Duality and the link.** We now discuss how the intersection number,

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) \tag{10.33}$$

can be expressed in terms of pairing a cohomology class with a homology class.

The proof of the following is analogous to that [11, Proposition 5.2]

**Theorem 10.15.** *For $\beta \in H_\bullet(X;\mathbb{R})$, let*

$$\bar{\mu}_p(\beta), \bar{\mu}_c, \quad \text{and} \quad \bar{e}_i \in H^\bullet(\bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R})$$

*be the cohomology classes from Definition 9.18. Let $\bar{e}_s = e(\Upsilon^s_{\mathfrak{t},\mathfrak{s}}/S^1)$ be the Euler class of the Seiberg-Witten obstruction bundle. If $d(\mathfrak{t}) = \dim \mathcal{M}^{*,0}_{\mathfrak{t},\mathfrak{s}}$, then let $[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}] \in H_{d(\mathfrak{t})-2}(\bar{\mathcal{M}}^{vir,*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R})$ be the homology class defined in* (10.35)*. Then*

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\delta_c} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) = \langle \bar{\mu}_p(z) \smile \bar{\mu}_c^{\delta_c} \smile \bar{e}_i \smile \bar{e}_s, [\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle. \tag{10.34}$$

10.2.1. *The homology class of the ambient link.* We now define the homology class of the ambient link,

$$[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}] \in H_{d(\mathfrak{t})-2}(\bar{\mathcal{M}}^{\mathrm{vir},*}_{\mathfrak{t},\mathfrak{s}}/S^1;\mathbb{R}).$$

The union of the lower strata in $\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}$ will be denoted $\mathbf{L}^{sing}_{\mathfrak{t},\mathfrak{s}} \subset \mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}$. By [24, Propositions 3.2], $\mathbf{L}^{sing}_{\mathfrak{t},\mathfrak{s}} \subset \mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}$ is a neighborhood deformation retract, so there is an open neighborhood $U \subset \mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}$ such that

(1) $\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U$ is a topological manifold with boundary,
(2) The neighborhood $U$ retracts onto $\mathbf{L}^{sing}_{\mathfrak{t},\mathfrak{s}}$.

The neighborhood $U$ and the retraction of $U$ onto $\mathbf{L}^{sing}_{\mathfrak{t},\mathfrak{s}}$ can also be constructed by noting that (i) the property of being a neighborhood deformation retraction is a local property and that (ii) the pre-image of the lower strata of $\bar{M}^{s,\natural}_\kappa(\delta)$ in the bubbletree compactification of $M^{s,\natural}_\kappa(\delta)$ as defined in [4] form the boundary of a topological manifold and thus are a neighborhood deformation retraction. Let

$$\jmath : \left(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}, \varnothing\right) \to \left(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}, U\right)$$

be the map of pairs. Because $U$ retracts to the codimension-four subspace $\mathbf{L}^{sing}_{\mathfrak{t},\mathfrak{s}}$,

$$\jmath_* : H_{d(\mathfrak{t})-2}\left(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}, \varnothing\right) \to H_{d(\mathfrak{t})-2}\left(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}, U\right)$$

is an isomorphism. If

$$[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U, \partial(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U)] \in H_{d(\mathfrak{t})-2}\left(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}, U\right)$$

is the relative fundamental class of the(topological) manifold with boundary $\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U$, then we define

$$[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}] = \jmath_*^{-1}[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U, \partial(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}} - U)], \tag{10.35}$$

to be the fundamental class of the ambient link.

*Proof of Theorem 10.15.* The proof is analogous to that of [11, Proposition 5.2] using relative Euler classes of the obstruction sections and representatives of the cohomology with compact support along the geometric representatives. □

10.3. **Reducing to a submanifold.** We now show how to reduce pairings with the homology class $[\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}]$ defined in (10.35) to pairings with the homology class of a subspace which eliminates the topology of the normal bundle $N(\delta) \to M_{\mathfrak{s}}$.

The deformation retraction $N(\delta) \to M_{\mathfrak{s}}$. defines a deformation retraction of $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}$ to the subspace

$$\bar{\mathbf{B}}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} = \cup_i \bar{\mathbf{B}}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) = \cup_i \left(t_N^{-1}(0) \cap \mathbf{L}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)\right) \tag{10.36}$$

where the function $t_N$ is defined in (10.1). Essentially, $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ is defined by replacing $\tilde{N}(\delta)$ with $\tilde{M}_{\mathfrak{s}}$ in the definition (10.4). Hence, we have the following fiber bundle description of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$.

**Lemma 10.16.** *The space*

$$\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) = t_N^{-1}(0) \cap \bar{\mathbf{L}}^{vir,i}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) \tag{10.37}$$

*is the restriction of the fiber bundle* (10.32) *to the subspace*

$$M_{\mathfrak{s}} \times K_j \subset N(\delta)/S^1 \times K_j$$

*of the base and is thus given by:*

$$\begin{array}{ccc} \bar{M}(\mathcal{P}_j,\boldsymbol{\varepsilon}_j) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_j,g_{\mathcal{P}_j}) \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P},\boldsymbol{\varepsilon}) \\ & & \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_j)} \\ & & M_{\mathfrak{s}} \times K_j \end{array} \tag{10.38}$$

Let $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ be the subspace defined in (10.36). We observe that $\bar{\mathbf{L}}^{\mathrm{vir},i}_{\mathfrak{t},\mathfrak{s}}$ is a disk bundle over $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. A fundamental class,

$$[\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}] \in H_{d(\mathfrak{t})-2r_N-2}(\bar{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}};\mathbb{R}), \tag{10.39}$$

(where $r_N$ is the complex rank of the bundle $N_{\mathfrak{t}_0,\mathfrak{s}} \to M_{\mathfrak{s}}$) is defined exactly as is done in (10.35). We then have:

**Proposition 10.17.** *Let $d_s = \dim M_{\mathfrak{s}}$. Let $s_j(N) \in H^{2j}(M_{\mathfrak{s}})$ and $r_N$ be the Segre classes and complex rank (respectively) of the bundle $N_{\mathfrak{t}_0,\mathfrak{s}} \to M_{\mathfrak{s}}$. Let $k$ and $m$ be non-negative integers satisfying $k+2m = d(\mathfrak{t})-2$. For any $\alpha \in H^k(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X);\mathbb{R})$, we have the equality*

$$\begin{aligned} &\langle \nu^m \smile \pi^*_{X,\mathfrak{s}}\alpha, [\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle \\ &= \sum_{j=0}^{d_2/2} (-1)^{r_N+j} \langle \nu^{m-r_N-j} \smile \pi^*_{\mathfrak{s}} s_j(N) \smile \pi^*_{X,\mathfrak{s}}\alpha, [\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle \end{aligned} \tag{10.40}$$

*Proof.* See [11, §5.2]. □

**Remark 10.18.** The Segre classes $s_i(N)$ have been computed under some assumptions on $H^1(X;\mathbb{R})$ in [15, Lemma 4.11]. From [15, Theorem 3.29] one can see that in general, these Segre classes will be given by a universal polynomial in $\mu_{\mathfrak{s}}(x)$ and $\mu_{\mathfrak{s}}(\gamma_i)$ with coefficients depending only on the indices $n'_s$ and $n''_s$ which depend only on $p_1(\mathfrak{t}_0)$ and $c_1(\mathfrak{t}_0)$.

10.4. **Boundary of link components.** Each of the pieces $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ as $\mathcal{P}_i$ is a stratified space where each stratum is a smooth manifold with corners. We describe this corner structure by denoting

$$\partial_{k_1}\dots\partial_{k_r}\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) = \left(\cap_{u=1}^r t(\mathfrak{t},\mathfrak{s},\mathcal{P}_{k_u})^{-1}(\varepsilon_{k_u})\right) \cap \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j). \tag{10.41}$$

Observe that the intersection of the pieces of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ is described by the equality:

$$\partial_k\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) = \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) \cap \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_k) = \partial_j\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_k) \tag{10.42}$$

which follows immediately from the definition. If

$$\mathrm{cl}\Sigma(X^\ell,\mathcal{P}_j) \cap \Sigma(X^\ell,\mathcal{P}_k) = \varnothing \quad \text{and} \quad \Sigma(X^\ell,\mathcal{P}_j) \cap \mathrm{cl}\Sigma(X^\ell,\mathcal{P}_k) = \varnothing,$$

then the boundaries in (10.42) are empty. By the symmetry of the definition (10.41) with respect to $k_u$ and the equality (10.42), it suffices to describe the boundary in (10.41) under the assumption that $k_1 < k_2 < \cdots < k_r < j$.

**Lemma 10.19.** *Let $K_j$ be the subspace defined in* (10.12). *For $k_1 < k_2 < \cdots < k_r < j$, define $\partial_{k_1} \dots \partial_{k_r} K_j$ by*

$$\partial_{k_1} \dots \partial_{k_r} K_j = \left( \cap_{u=1}^{r} t(X^\ell, \mathcal{P}_{k_u})^{-1}(\varepsilon_{k_u}) \right) \cap K_j \subset \Sigma(X^\ell, \mathcal{P}_j). \tag{10.43}$$

*Then, the corner $\partial_{k_1} \dots \partial_{k_r} \bar{\mathbf{B}\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ is the restriction of the fiber bundle*

$$\bar{\mathbf{B}\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) \to K_j$$

*to the subspace $\partial_{k_1} \dots \partial_{k_r} K_j \subset K_j$.*

*Proof.* The lemma follows immediately from the definitions (10.41) and (10.43) and the relation between the functions $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ and $t(X^\ell, \mathcal{P}_i)$ given in the proof of Lemma 10.8. □

We further extend the description of the corner (10.41) in Lemma 10.19 by describing the corner $\partial_{k_1} \dots \partial_{k_r} K_j \subset K_j$. We begin by describing the corner $\partial_k K_j$ where $k < j$. First, observe that for $\varepsilon_k$ sufficiently small $\partial_k K_j$ is in the image of the exponential map with domain an open subspace of

$$\mathrm{Fr}(TX^\ell, \mathcal{P}_k, g_{\mathcal{P}_k})|_{K_k} \times_{G(T,\mathcal{P})} Z(\mathcal{P}_k)/\mathfrak{S}(\mathcal{P}_k), \tag{10.44}$$

where the fiber $Z(\mathcal{P}_k)$ is given by

$$Z(\mathcal{P}_k) = \prod_{P \in \mathcal{P}_k} Z_P(\delta_P).$$

The function $t(X^\ell, \mathcal{P}_k)$ is defined by the $G(T, \mathcal{P})$-invariant functions on the fiber,

$$m_{\mathcal{P},[\mathcal{P}]} = \sum_{P \in \mathcal{P}} |P| t(Z_P)^2 : Z(\mathcal{P}_k) \to [0, \infty) \tag{10.45}$$

where $t(Z_P)$ is defined in Lemma 4.3. (The motivation for the appearance of two copies of $\mathcal{P}$ in $m_{\mathcal{P},[\mathcal{P}]}$ will be apparent shortly.) For other partitions $\mathcal{P}'$ with $\mathcal{P}_k < \mathcal{P}'$, a neighborhood of $K_k$ in $\Sigma(X^\ell, \mathcal{P}')$ is given by restricting the exponential map $e(X^\ell, g_{\mathcal{P}_k})$ to the subspace of (10.44) given by the subspace of the fiber,

$$\coprod_{\mathcal{P}'' \in [\mathcal{P}_k < \mathcal{P}']} \prod_{P \in \mathcal{P}_k} \Delta^\circ(Z_P, \mathcal{P}''_P) \subset Z(\mathcal{P}_k). \tag{10.46}$$

A neighborhood of this subspace of the fiber $Z(\mathcal{P}_k)$ is given by the union, over $\mathcal{P}'' \in [\mathcal{P}_k < \mathcal{P}']$, of the images of the exponential map

$$\prod_{P \in \mathcal{P}_k} e(Z_P, \mathcal{P}''_P) : \prod_{P \in \mathcal{P}_k} \left( \Delta^\circ(Z_P, \mathcal{P}''_P) \times \prod_{P'' \in \mathcal{P}''} Z_{P''} \right) \to Z(\mathcal{P}_k) \tag{10.47}$$

where $e(Z_P, \mathcal{P}''_P)$ is defined in (4.9). On the domain of the map (10.47), we can define a function

$$m_{\mathcal{P},[\mathcal{P}']} = \sum_{P'' \in \mathcal{P}''} |P''| (t(Z_{P''}) \circ \pi_{P''})^2 \tag{10.48}$$

where $\pi_{P''}$ is the projection from the domain of the map (10.47) to $Z_{P''}$. We also write $m_{\mathcal{P},[\mathcal{P}']}$ for the pushforward of this function to the image of the map (10.47). One can see that the function $m_{\mathcal{P},[\mathcal{P}']}$ is invariant under the choice of $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$.

**Lemma 10.20.** *For $\mathcal{P} \leq \mathcal{P}'$, the pullback of the function $t(X^\ell, \mathcal{P}')$ by the exponential map $e(X^\ell, \mathcal{P})$ to the domain* (10.44) *is equal to the function defined by the $G(\mathcal{P})$-invariant function $m_{\mathcal{P},[\mathcal{P}']}$ on the fiber $Z(\mathcal{P})$.*

*Proof.* The lemma follows from an examination of the diagram (4.22) and the definition of $t(X^\ell, \mathcal{P}')$. Compare the discussion prior to Lemma 10.11. □

Hence, for $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r)$ the choice of parameters used to define the link, we define a subspace of the fiber $Z(\mathcal{P}_k)$ by

$$Z(\mathcal{P}_k, \boldsymbol{\varepsilon}) = Z(\mathcal{P}_k) - \cup_{i \geq k} m^{-1}_{\mathcal{P}_k,[\mathcal{P}_i]}([0, \varepsilon_i)). \tag{10.49}$$

For $k \leq k_1 < \cdots < k_r$, we define

$$\partial_{k_1} \dots \partial_{k_r} Z(\mathcal{P}_k, \boldsymbol{\varepsilon}) = Z(\mathcal{P}_k) \cap \left( \cap_{i=1}^r m^{-1}_{\mathcal{P}_k,[\mathcal{P}_{k_i}]}(\varepsilon_{k_i}) \right). \tag{10.50}$$

Finally, recall from Lemma 4.5 that a neighborhood of $K_k$ in $\Sigma(X^\ell, \mathcal{P}_j)$ is parameterized by the subspace of (10.44) given (up to symmetric group quotients) by the subspace of the fiber:

$$Z(\mathcal{P}_k, [\mathcal{P}_j]) = \coprod_{\mathcal{P}'' \in [\mathcal{P}_k, \mathcal{P}_j]} \prod_{P \in \mathcal{P}_k} \Delta^\circ(Z_P(\delta_P), \mathcal{P}''_P). \tag{10.51}$$

Hence, for $k \leq k_1 < \cdots < k_r < j$ we define

$$\begin{aligned} Z(\mathcal{P}_k, [\mathcal{P}_j], \boldsymbol{\varepsilon}) &= Z(\mathcal{P}_k, \boldsymbol{\varepsilon}) \cap Z(\mathcal{P}_k, [\mathcal{P}_j]), \\ \partial_{k_1} \dots \partial_{k_r} Z(\mathcal{P}_k, [\mathcal{P}_j], \boldsymbol{\varepsilon}) &= \partial_{k_1} \dots \partial_{k_r} Z(\mathcal{P}_k, \boldsymbol{\varepsilon}) \cap Z(\mathcal{P}_k, [\mathcal{P}_j]). \end{aligned} \tag{10.52}$$

The above discussion yields the following result.

**Lemma 10.21.** *For $k_1 < k_1 < \cdots < k_r < j$, the subspace $\partial_{k_1} \dots \partial_{k_r} K_j$ is given by the image under the exponential map $e(X^\ell, \mathcal{P}_{k_1})$ of the subspace*

$$\operatorname{Fr}(TX^\ell, \mathcal{P}_{k_1}, g_{\mathcal{P}_{k_1}}) \times_{G(T,\mathcal{P}_{k_1})} \partial_{k_1} \dots \partial_{k_r} Z(\mathcal{P}_{k_1}, [\mathcal{P}_j], \boldsymbol{\varepsilon}) / \mathfrak{S}(\mathcal{P}_{k_1}) \tag{10.53}$$

*of the domain* (10.44).

The following relation between the boundaries $\partial_j K_i$ and $\partial_j \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ when $i < j$ will also be useful.

**Lemma 10.22.** *For $i < j$, let $\bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ be the space defined in* (10.31)*. Let $t_f(\mathcal{P}_i, [\mathcal{P}_j])$ be the function defined in* (10.28)*. Define*

$$\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i) = t_f(\mathcal{P}_i, [\mathcal{P}_j])^{-1}(\varepsilon_j) \cap \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i) \subset \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i). \tag{10.54}$$

*Then, the boundary $\partial_j \bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$, defined in* (10.41)*, is the fiber subbundle,*

$$\begin{array}{ccc} \partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \operatorname{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i, g_{\mathcal{P}_i}) \times_{G(\mathcal{P}_i)} \partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i) \\ & & \Big\downarrow {\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)} \\ & & M_{\mathfrak{s}} \times K_i \end{array} \tag{10.55}$$

*of the fiber bundle $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ given in* (10.38).

*Proof.* The lemma follows from the description in Lemma 10.13 of the relation between the functions $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ and $t_f(\mathcal{P}_i, [\mathcal{P}_j])$. □

We now give another description of the fiber $\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ appearing in (10.54). First, because the function $t_f(\mathcal{P}_i, [\mathcal{P}_j])$ is defined on $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ in (10.28), we have the inclusion

$$\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i) \subset U_f(\mathcal{P}_i, [\mathcal{P}_j]),$$

where the subspace $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ is defined in (10.27). Because $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ is defined as the image of the injective (up to symmetric group action) splicing map in (10.26), there is a homeomorphism (up to symmetric group action) between $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ and

$$\coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_i} \left( \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) \right).$$

This homeomorphism defines a projection map:

$$U_f(\mathcal{P}_i, [\mathcal{P}_j]) \to Z(\mathcal{P}_i, [\mathcal{P}_j])/\mathfrak{S}(\mathcal{P}_i) \tag{10.56}$$

where $Z(\mathcal{P}_i, [\mathcal{P}_j])$ is defined in (10.51). In addition, because

$$\begin{aligned}\coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}} \prod_{Q \in \mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) &= \coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{Q \in \mathcal{P}''} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) \\ &= \coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}''),\end{aligned}$$

where $\bar{M}(\mathcal{P}'')$ is defined in (10.14), there is a projection map

$$U_f(\mathcal{P}_i, [\mathcal{P}_j]) \to \coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'')/\mathfrak{S}(\mathcal{P}_i). \tag{10.57}$$

We then have:

**Lemma 10.23.** *For $i < j$, if*

$$U_f(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}_i) = U_f(\mathcal{P}_i, [\mathcal{P}_j]) \cap \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i).$$

*then the restriction of the projection map* (10.56) *to $U_f(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}_i)$ defines a projection map*

$$U_f(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}_i) \to Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon})/\mathfrak{S}(\mathcal{P}_i), \tag{10.58}$$

*where the subspace $Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon})$ is defined in* (10.52).

*Proof.* The subspace $\bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ is defined by the functions $t_f(\mathcal{P}_i, [\mathcal{P}_k])$ in (10.30) and (10.31). The subspace $Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}_i)$ is defined in (10.49) by the functions $m_{\mathcal{P}_i,[\mathcal{P}_k]}$ in (10.48). The projection map (10.56), is given by a product of the projection maps $\pi(\Theta, \mathcal{P}''_P)$ defined in Lemma 5.66. The tubular distance functions $t(\Theta, \mathcal{P}''_P)$ are constant on the fibers of these projections by (5.67). The lemma then follows from the relation between the functions $t_f(\mathcal{P}_i, [\mathcal{P}_k])$ and $t(\Theta, \mathcal{P}''_P)$ in Lemma 10.12 and the relation between $t(\Theta, \mathcal{P}'_P)$ and the function $m_{\mathcal{P},[\mathcal{P}'']}$ given by the definitions of these function. □

A similar argument yields the following characterization of the fiber $\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$.

**Lemma 10.24.** *For $i < j$, the restriction of the projection map* (10.56) *to the fiber $\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ appearing in* (10.54) *defines a projection map*

$$\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}) \subset \partial_j Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}),$$

*to the subspace $\partial_j Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}_i)$ defined in* (10.52).

We now characterize the fiber of the projection map (10.56) appearing in Lemmas 10.23 and 10.24.

**Lemma 10.25.** *For $i < j$, the fiber $\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ appearing in* (10.54) *is homeomorphic to the subspace of*

$$\coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_i} \left( \partial_j Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon}) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}^{s,\natural}_{spl,|Q|}(\delta_Q) \right) \tag{10.59}$$

*given by the pre-image of the subspace*

$$\coprod_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'', \boldsymbol{\varepsilon}_j)/\mathfrak{S}(\mathcal{P}_i)$$

*under the projection map* (10.57).

*Proof.* The fiber $\partial_j \bar{M}(\mathcal{P}_i, \boldsymbol{\varepsilon}_i)$ is defined by the functions $t_f(\mathcal{P}_i, [\mathcal{P}_k])$ where $k > i$. For $j > k > i$, the constraint $t_f(\mathcal{P}_i, [\mathcal{P}_k])^{-1}((\varepsilon_k, \infty))$, is given, as described in Lemma 10.23, by the restriction of the projection map (10.56) to the subspace $\partial_j Z(\mathcal{P}_i, [\mathcal{P}_j], \boldsymbol{\varepsilon})$. For $k \geq j > i$, the function $t_f(\mathcal{P}_i, [\mathcal{P}_k])$ on $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ and the function $t_f(\mathcal{P}'', [\mathcal{P}_k])$ (for $\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]$) on $U_f(\mathcal{P}'', [\mathcal{P}_k])$ are both defined by pullback of the function $t(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ (see Lemma 10.11). Hence, the constraints $t_f(\mathcal{P}_i, [\mathcal{P}_k])^{-1}((\varepsilon_k, \infty))$ for $k \geq j > i$ are as described in the statement of the Lemma. □

## 11. The computation

In this section, we perform the computation leading to a proof of the following theorems which form the technical heart of this paper.

**Theorem 11.1.** *Let $\mathfrak{t}$ be a spin$^u$ structure on a smooth four-manifold $X$ with $b^1(X) = 0$. Assume that $\mathfrak{s}$ is a spin$^c$ structure with $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ a subset of the space of ideal monopoles $I\mathcal{M}_{\mathfrak{t}}$ defined in* (2.12). *Let $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ be the link of the reducible* SO(3) *monopoles given by $\mathfrak{s} \in Spin^c(X)$ as defined in Definition 10.2. For $z = h^{\delta - 2m} x^m \in \mathbb{A}(X)$ and $\eta$ a non-negative integer satisfying*

$$\deg(z) + 2\eta = \dim \mathcal{M}_{\mathfrak{t}} - 2,$$

*let $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}^\eta$ be the geometric representatives defined in §2.4. Then*

$$\begin{aligned} &\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) \\ &\quad = SW_X(\mathfrak{s}) \sum_{i=0}^{\min(\ell, [\frac{\delta-2m}{2}])} \left( q_{\delta,\ell,m,i}(c_1(\mathfrak{s}) - c_1(\mathfrak{t}), c_1(\mathfrak{t})) Q_X^i \right)(h). \end{aligned} \tag{11.1}$$

*where $q_{\delta,\ell,m,i}$ are degree $\delta - 2m - 2i$ homogeneous polynomials which are universal functions of the constants given in Theorem 1.1.*

We note that a similar result can also be achieved by the methods of this paper without the assumption that $b^1(X) = 0$ and $z = h^{\delta-2m} x^m$, but the resulting expression becomes considerably more complicated. While the precise expression for the intersection number in (11.1) is still unknown, we note the following important result, referred to as the 'multiplicity conjecture' in [16, 6, 11], which holds even without the simplifying assumptions.

**Theorem 11.2.** *Let $X$ be a smooth, oriented four-manifold. Assume that $\mathfrak{t}$ is a spin$^u$ structure on $X$ and $\mathfrak{s}$ is a spin$^c$ structure on $X$ such that $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ is contained in the space of ideal monopoles $I\mathcal{M}_{\mathfrak{t}}$ defined in* (2.12). *Let $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ be the link of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ given in Definition 10.2. Then, if $SW_{X,\mathfrak{s}}(\omega)$ vanishes for all $\omega \in \mathbb{A}_2(X)$, then*

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\delta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right) = 0.$$

To prove Theorems 11.1 and 11.2, we begin by observing that Theorem 10.15 gives the equality

$$\#\left(\bar{\mathcal{V}}(z)\cap\bar{\mathcal{W}}^{\delta_c}\cap\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}\right)=\langle\bar{\mu}_p(z)\smile\bar{\mu}_c^{\delta_c}\smile\bar{e}_i\smile\bar{e}_s,[\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle. \tag{11.2}$$

The expression for the $\mu$-classes in Corollary 9.9, the expression for $\bar{e}_s$ in Lemma 9.10, the expression for $\bar{e}_i$ in Proposition 9.16, and the equality between the homology classes $[\bar{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}]$ and $[\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}]$ in Proposition 10.17 implies that the intersection number in (11.2) can be reduced to pairings of the form

$$\langle\nu^a\smile\pi_X^*S^\ell(\beta)\smile\pi_{\mathfrak{s}}^*\mu_s(z_2),[\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle \tag{11.3}$$

where $\beta\in H^\bullet(X;\mathbb{R})$ and $z_2\in\mathbb{A}_2(X)$.

We wish to compute the pairings (11.3) by writing it as a sum over pairings with the pieces $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ and then use the fiber bundle structure in Lemma 10.16 of each of these pieces. To do this, we must replace the cohomology classes in (11.3) by cohomology classes with compact support away from the boundaries $\partial_j\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$. Such a replacement requires a choice and this choice must be done consistently on each piece. We encode these choices in the following geometric data. We define a quotient, $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$, of $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ by replacing the boundaries $\partial_j\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ with spaces of codimension greater than or equal to two. The cohomology classes in (11.3) pull back from cohomology classes on this quotient. Because the image of the boundaries in the quotient has codimension greater than or equal to two, we can define cohomology classes with compact support by pulling back cocycles from the quotient. Using these cocycles is equivalent to computing (11.3) using the fundamental class of $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ which, because the image of the boundaries has codimension two, can be written as a sum of fundamental classes of pieces.

In §11.1, we construct the quotient space $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$. This construction shows that the images of each piece $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ in the quotient has a fiber bundle structure identical to that of $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ but with a compact fiber and base. In §11.2, we show how to use the quotient to perform the desired computations. Finally in §11.3 we give the proofs of the main theorems of the paper.

## 11.1. **The quotient space.**

We wish to take advantage of the fiber bundle structure

$$\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)\to K_j \tag{11.4}$$

and our control over the structure group of this fibration to compute cohomological pairings with $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. To do this, we must write the pairing in (11.3) with the homology class $[\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]$ as a sum over pairings with some kind of homology class representing the space $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$. However, the pieces $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ have boundaries as described in §10.4 and so would only define relative homology classes. To overcome this, we describe a quotient of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ obtained by leaving the interiors of the pieces $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ and replacing the boundaries $\partial_k\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ with quotients obtained by deleting part of the gluing data.

To be more precise, for $A\subset X$ a topological space and $f:A\to B$ a continuous map, we define the *compression of $X$ along $f$* to be the quotient space obtained from $X$ by identifying points in $A$ which have the same image under $f$. The following is an elementary exercise.

**Lemma 11.3.** *If $A\subset X$ has a neighborhood $N_A$ homeomorphic to a mapping cone of some map $p:Y\to A$, then the compression of $X$ along $f$ is homeomorphic to the space obtained by replacing $N_A$ with the mapping cone of the map $f\circ p$.*

We will define the quotient of $\bar{\mathbf{BL}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$ to be the compressions of $\bar{\mathbf{BL}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$ along a projection map from $\partial_k\bar{\mathbf{BL}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ given by deleting the data in the fiber of $\partial_k K_j \to K_k$ (given by (10.53)). Three observations then complete the computation. First, the quotient is the union of the quotients of the pieces, $\check{\mathbf{BL}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$, and the intersection of any two such pieces has codimension-four. Second, the cohomology classes in (11.2) pull back from this quotient and hence the computation can be done on the quotient. Third, the quotient pieces, $\check{\mathbf{BL}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$, all admit the desired fiber bundle structure, (11.4), albeit over a larger space, with the same structure group.

11.1.1. *The quotients of $K_i$.* We begin by defining a quotient $\check{K}_i$ of the compact subspaces $K_j$ of $\operatorname{Sym}^\ell(X)$ defined in (10.12). We define the quotients inductively. As before, we have enumerated the partitions respecting the ordering of $\Sigma(X^\ell,\mathcal{P}_i)$. For $j<i$, write

$$\pi_{j,i} : \partial_j K_i \to K_j \tag{11.5}$$

for the projection defined by the restriction of $\pi(X^\ell,\mathcal{P}_j)$ to the subspace in Lemma 10.21. For $k<j<i$, the equality

$$\pi_{k,i} = \pi_{k,j}\circ\pi_{j,i} \tag{11.6}$$

follows from the Thom-Mather equality (4.30) which is shown to hold in Lemmas 4.9-4.11.

If $\mathcal{P}_1$ is the lowest partition, $\check{K}_1 = K_1$. For all $j$ with $1<j$, let $\check{K}_j(1)$ be the compression of $K_j$ along the map $\pi_{1,j} : \partial_1 K_j \to K_1$. Let $\partial_i\check{K}_j(1)$ be the image of $\partial_i K_j$ under the projection $K_j \to \check{K}_j(1)$. The equalities

$$\pi_{1,j}\circ\pi_{j,i} = \pi_{1,i} \quad \text{for } 1<j<i \tag{11.7}$$

imply that the projections $\pi_{j,i}$ descend to maps on the quotients, $\check{\pi}_{j,i}(1) : \partial_i\check{K}_i(1) \to \check{K}_j(1)$. Define $\check{K}_2 = \check{K}_2(1)$.

Continue inductively. Assume that

(1) For all $i=1,\dots,r$, a quotient $\check{K}_i(1,2,\dots,p)$ of $K_i$ has been defined,
(2) If $i>j$ and $\partial_j\check{K}_i(1,2,\dots,p)$ is the image of $\partial_j K_i$ under the quotient map, then the map $\pi_{j,i} : \partial_j K_i \to K_j$ induces a map
$$\check{\pi}_{j,i}(1,2,\dots,p) : \partial_j\check{K}_i(1,2,\dots,p) \to \check{K}_j(1,2,\dots,p),$$
(3) The quotient $\check{K}_i(1,2,\dots,p)$ is the compression of $\check{K}_i(1,2,\dots,p-1)$ along the map $\check{\pi}_{k-1,i}(1,\dots,p-1) : \partial_{k-1}K_i(1,2,\dots,p-1)$

Because the maps $\check{\pi}_{j,i}(1,2,\dots,p)$ are defined by quotients of the maps $\pi_{j,i}$, the analogue of the equality (11.7) holds for $i>j>k$:

$$\check{\pi}_{k,j}(1,\dots,p)\circ\check{\pi}_{j,i}(1,\dots,p) = \check{\pi}_{k,i}(1,\dots,p). \tag{11.8}$$

For $i>p+1$, we then define $\check{K}_i(1,\dots,p+1)$ to be the compression of $\check{K}_i(1,\dots,p)$ along the map

$$\check{\pi}_{p+1,i}(1,\dots,p) : \partial_{p+1}\check{K}_i(1,\dots,p) \to K_{p+1}(1,\dots,p).$$

For $i\le p+1$, we define $\check{K}_i(1,\dots,p+1) = \check{K}_i(1,\dots,p)$. The equality (11.8) implies that the maps $\check{\pi}_{j,i}(1,\dots,p)$ descend to define maps $\check{\pi}_{j,i}(1,\dots,p+1)$ on the quotients, completing the induction.

Define $\check{K}_i = \check{K}_i(1,\dots,i-1)$. We have the following characterization of $\check{K}_i$.

**Lemma 11.4.** *The quotient $\check{K}_i$ is homeomorphic to* $\operatorname{cl}\Sigma(X^\ell,\mathcal{P}_i)$.

*Proof.* The lemma follows from the construction of $\check{K}_i$ and from Lemma 11.3. □

11.1.2. *The quotients of* $\mathbf{BL}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$. The construction of the quotient $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ is done in an identical manner to that of $\check{K}_i$. To control the resulting topology, we begin with the following lemma.

**Lemma 11.5.** *The fiber bundle* $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to K_i$ *is the pullback of an orbifold fiber bundle* $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to \check{K}_i$ *by the projection map* $K_i \to \check{K}_i$.

*Proof.* The fiber bundle

$$\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \to \Delta^\circ(X^\ell,\mathcal{P}_i)$$

defined in (6.2) extends from the open diagonal $\Delta^\circ(X^\ell,\mathcal{P}_i)$ to its closure $\Delta(X^\ell,\mathcal{P}_i)$. The action of the symmetric group, $\mathfrak{S}(\mathcal{P}_i)/\Gamma(\mathcal{P}_i)$, is no longer free, but only pseudo-free on this extension and hence an orbifold fiber bundle. The quotient space $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ is defined by using this extension of the above fiber bundle, the identification of $\check{K}_i$ with $\Sigma(X^\ell,\mathcal{P}_i)$, and leaving the fiber unchanged. □

By the characterization of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ as a pullback, there is a map

$$\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to \check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \tag{11.9}$$

which we refer to as the quotient map. The following observation about the pseudo-fiber bundle described above is an immediate consequence of its construction.

**Lemma 11.6.** *There is a free* $S^1$ *action on the bundle* $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to \check{K}_i$ *defined in Lemma 11.5 such that the quotient map* (11.9) *is equivariant with respect to the* $S^1$ *action defining the cohomology class* $\nu$ *on* $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ *and this* $S^1$ *action on the quotient.*

The quotient $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ is then defined, roughly speaking, by taking the compressions of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ along the top map in the diagram for $i<j$

$$\begin{array}{ccc} \partial_i\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) & \xrightarrow{\pi^{\mathbf{B}}_{i,j}} & \check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)|_{\check{K}_i} \\ {\scriptstyle p}\downarrow & & \downarrow \\ \partial_i K_j & \xrightarrow{\pi_{i,j}} & \partial_i\check{K}_j \end{array} \tag{11.10}$$

where we understand $\partial_i\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)$ to be a subspace of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$.

To carry out the construction of $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, we use a downwards induction, first taking the compression along the maps $\pi^{\mathbf{B}}_{i,r}$ (where $\mathcal{P}_r$ is the top stratum). That this may be done for more than one boundary $\partial_j\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_r)$ (i.e. the map $\pi^{\mathbf{B}}_{j,r}$ descends to the quotient obtained from a compression along $\pi^{\mathbf{B}}_{i,r}$) follows from the equality (11.6)) on the maps $\pi_{k,i}$ on $\partial_k K_i$ and the compatibility of the trivializations of the bundle $\mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_r)$ used to define the maps $\pi^{\mathbf{B}}_{i,r}$.

To continue the induction, taking the quotient of $\mathbf{BL}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$, for $r>i$ along maps $\pi^{\mathbf{B}}_{j,i}$, we must ensure that the maps $\pi^{\mathbf{B}}_{j,i}$ descend to the quotient of $\mathbf{BL}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ obtained by compressing along the map $\pi^{\mathbf{B}}_{i,j}$ where $i<j$. We use the following result describing the compression along the map $\pi^{\mathbf{B}}_{i,j}$, for $i<j$, as a compression of a map on a subspace of $\partial_j\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$.

**Lemma 11.7.** *If* $i<j$ *and* $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ *is compressed along the map*

$$\pi^{\mathbf{B}}_{i,j} : \partial_i\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) \to \check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j)|_{\check{K}_i}$$

*then the resulting quotient of* $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ *can be described as a fiber bundle,*

$$\begin{array}{ccc} \check{M}_j(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \operatorname{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \times_{G(\mathcal{P}_i)} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) \\ & & \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)} \\ & & M_{\mathfrak{s}} \times \check{K}_i \end{array} \tag{11.11}$$

*where the fiber* $\check{M}_j(\mathcal{P}_i,\boldsymbol{\varepsilon}_i)$ *is a compression of* $\bar{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i)$ *along the map*

$$\pi^{\mathbf{B},f}_{i,j} : \partial_j \bar{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) \to \coprod_{\mathcal{P}''\in[\mathcal{P}_i<\mathcal{P}_j]} \bar{M}(\mathcal{P}'',\boldsymbol{\varepsilon}_j)/\mathfrak{S}(\mathcal{P}_i) \tag{11.12}$$

*defined by the map* (10.57) *as described in Lemma 10.25.*

*Proof.* The compression along the map $\pi^{\mathbf{B}}_{i,j}$ is defined by deleting the data in the fiber of the map $\pi_{i,j}$ in the diagram (11.10). Assume the fiber lies over a point $\mathbf{x} \in \partial_j\check{K}_i$ with pre-image in $\partial_{k_1}\partial_{k_2}\dots\partial_{k_r}\partial_j K_i$ and whose pre-image is disjoint from all other corners $\partial_{k_0}\partial_{k_1}\partial_{k_2}\dots\partial_{k_r}\partial_j K_i$. Then, the fiber of the map $\pi_{j,i}$ in the diagram (11.10) given by the fiber of the projection

$$\partial_{k_1}\partial_{k_2}\dots\partial_{k_r}\partial_j K_i \to K_{k_1}$$

given by the space $\partial_{k_2}\dots\partial_{k_r}\partial_j Z(\mathcal{P}_{k_1},[\mathcal{P}_i],\boldsymbol{\varepsilon})$ defined in (10.52) (see Lemma 10.53). Using the equality of boundaries,

$$\partial_i \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_j) = \partial_j \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$$

and the description of the fiber of $\partial_j \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ over $\mathbf{x}$ in Lemma 10.25, we see the compression is as described in the lemma. ☐

Proceeding inductively, one observes that the quotient described in Lemma 11.7 is still the pullback from a bundle (with the same fiber) over $\check{K}_i$ because only the fiber has been changed and not the underlying bundle. Thus, one may assume (inductively) that a quotient, $\tilde{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$, of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ has been obtained by all the compressions along $\partial_j \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ for $i<j$ as described in Lemma 11.7. Because these compressions only alter the fiber of the bundle $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to K_i$, the quotient $\tilde{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ still admits a fiber bundle structure $\tilde{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i) \to K_i$ which is the pullback of a bundle $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ as in Lemma 11.5. The bundle $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ is a quotient of $\tilde{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ compressing along the maps in induced by $\pi^{\mathbf{B}}_{k,i}$ induced on the quotient, allowing us to proceed inductively.

We summarize:

**Lemma 11.8.** *Let*

$$\check{\operatorname{Fr}}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \to \Delta(X^\ell,\mathcal{P}_i)$$

*be the extension of the bundle*

$$\operatorname{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \to \Delta^\circ(X^\ell,\mathcal{P}).$$

*Then, the quotient* $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ *of* $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ *admits a fiber orbifold bundle structure,*

$$\begin{array}{ccc} \check{M}(\mathcal{P}_i) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \check{\operatorname{Fr}}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \times_{G(\mathcal{P}_i)} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) \\ & & \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)} \\ & & M_{\mathfrak{s}} \times \operatorname{cl}\Sigma(X^\ell,\mathcal{P}_i) \end{array} \tag{11.13}$$

*where the fiber* $\check{M}(\mathcal{P}_i)$ *is obtained by compressions of the fiber* $\bar{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i)$ *of* (10.38) *along the maps* $\pi^{\mathbf{B},f}_{i,j}$ *defined in* (11.12).

The orbifold bundle structure is due to the non-trivial stabilizer of $\mathfrak{S}(\mathcal{P})/\Gamma(\mathcal{P})$ acting on $\bar{\mathrm{Fr}}(\mathfrak{t},\mathfrak{s},\mathcal{P})$ on the points in $\Delta(X^\ell,\mathcal{P}) - \Delta^\circ(X^\ell,\mathcal{P})$. We then define

$$\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} = \cup_i \check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}, \tag{11.14}$$

to be the quotient of $\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ defined by the taking union of the quotients of the pieces as described above. Note that the same quotient is taken on each piece.

**Proposition 11.9.** *The quotient $\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ of $\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ defined above has the following properties:*

(1) *The maps $\pi_X : \bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}} \to \mathrm{Sym}^\ell(X)$ and $\pi_{\mathfrak{s}} : \bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}} \to M_{\mathfrak{s}}$ descend to the quotient,*
(2) *There is a cohomology class*

$$\check{\nu} \in H^2(\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}) \tag{11.15}$$

*which pulls back to the cohomology class $\nu \in H^2(\bar{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}})$ under the quotient map.*

*Proof.* The first property is immediate from the construction. The second property follows from Lemma 11.6. □

11.1.3. *The fundamental classes.* We can define a fundamental class

$$[\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}] \in H_{\max}(\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}})$$

as was done in (10.35). However, because the intersections of the different pieces $\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ have codimension greater than or equal to two, we can also define fundamental classes,

$$[\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)] \in H_{\max}(\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}).$$

If

$$\pi_{\mathbf{B}} : \bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \to \check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$$

is the quotient map, then the fundamental classes defined above satisfy:

$$(\pi_{\mathbf{B}})_*[\bar{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}] = [\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}], \tag{11.16}$$

and

$$[\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}] = \sum_i [\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)]. \tag{11.17}$$

The preceding discussion and the equality $\nu = \pi^*_{\mathbf{B}}\check{\nu}$ in Proposition 11.9 then imply:

**Lemma 11.10.** *For any $\mu_{\mathfrak{s}} \in H^\bullet(M_{\mathfrak{s}})$, $S^\ell(\beta) \in H^\bullet(\mathrm{Sym}^\ell(X))$, and $z_2 \in \mathbb{A}_2(X)$, the equality*

$$\begin{aligned} &\langle \nu^\delta \smile \pi^*_{\mathfrak{s}}\mu_{\mathfrak{s}}(z_2) \smile \pi^*_X S^\ell(\beta), [\mathbf{BL}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle \\ &\quad = \sum_i \langle \check{\nu}^\delta \smile \pi^*_{\mathfrak{s}}\mu_{\mathfrak{s}}(z_2) \smile \pi^*_X S^\ell(\beta), [\check{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle. \end{aligned} \tag{11.18}$$

11.2. **Pullback arguments.** The pairings on the right-hand-side of (11.18) can be computed using a pushforward-pullback argument as follows. The pseudo-fiber bundle in (11.11) is branch-covered by the fiber bundle,

$$\begin{array}{ccc} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}\times S^1} \mathrm{Fr}(\mathfrak{t},\mathfrak{s},\mathcal{P}_i,g_{\mathcal{P}_i}) \times_{\widetilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) \\ & & \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)} \\ & & M_{\mathfrak{s}} \times \Delta(X^\ell,\mathcal{P}_i) \end{array} \tag{11.19}$$

Let $\hat{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\mathcal{P}_i)$ denote the fiber bundle in (11.19). (The actual degree of the covering, $|\mathfrak{S}(\mathcal{P}_i)/\Gamma(\mathcal{P}_i)|$, is not relevant to this discussion.) Hence, we have

$$\langle \check{\nu}^\delta \smile \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}} \smile \pi_X^* S^\ell(\beta), [\check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]\rangle = q_i \langle \check{\nu}^\delta \smile \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}} \smile \pi_X^* S^\ell(\beta), [\hat{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}]\rangle \tag{11.20}$$

for some rational number $q$. Let $G_i = \widetilde{G}(\mathcal{P}_i)$. Let $f_{\mathfrak{t},\mathfrak{s},i} : \Delta(X^\ell,\mathcal{P}_i) \to \mathrm{BG}_i$ be the classifying map. We then have a diagram,

$$\begin{array}{ccc} \check{\mathbf{BL}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} & \xrightarrow{\tilde{f}_{\mathfrak{t},\mathfrak{s},i}} & S^1\backslash\, \mathrm{EG}_i \times_{\widetilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i) \\ \Big\downarrow{\scriptstyle \pi(\mathfrak{t},\mathfrak{s},\mathcal{P}_i)} & & \Big\downarrow \\ \Delta(X^\ell,\mathcal{P}_i) & \xrightarrow{f_{\mathfrak{t},\mathfrak{s},i}} & \mathrm{BG}_i \end{array} \tag{11.21}$$

Observe that $\check{\nu}$ is the pullback of first Chern class of the $S^1$ action on $\mathrm{EG}_i \times_{\widetilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i,\boldsymbol{\varepsilon}_i)$. Then, a pushforward pullback argument applied to the diagram (11.21) yields

**Lemma 11.11.** *There are universal polynomials, $m_{\ell,\delta,\mathfrak{t},\mathfrak{s},i} \in H^\bullet(M_{\mathfrak{s}} \times \Delta(X^\ell,\mathcal{P}_i))$, in the characteristic classes of the spin$^u$ structure $V_\ell$ and the cohomology classes $\mu_{\mathfrak{s}}(\cdot)$ such that*

$$\begin{aligned} &\langle \check{\nu}^\delta \smile \pi_{\mathfrak{s}}^*\mu_{\mathfrak{s}}(z_2) \smile \pi_X^* S^\ell(\beta), [\hat{\mathbf{BL}}^{vir}_{\mathfrak{t},\mathfrak{s}}]\rangle \\ &\quad = \langle m_{\ell,\delta,\mathfrak{t},\mathfrak{s},i} \smile \mu_{\mathfrak{s}}(z_2) \smile \pi_X^* S^\ell(\beta), [M_{\mathfrak{s}} \times \Delta(X^\ell,\mathcal{P}_i)]\rangle. \end{aligned} \tag{11.22}$$

11.3. **Proof of the main theorem.** We now prove Theorems 11.1, 1.1, and 11.2.

*Proof of Theorem 11.1.* The discussion prior to (11.3), Lemma 11.10, and Lemma 11.11 imply that the pairing on the left-hand-side of (11.1) can be reduced to an algebraic expression in terms of the form appearing on the right-hand-side of (11.22). The result then follows from a discussion on the type of terms which will appear. First, observe that factors of $Q_X(h)$ will arise only from products of $S^\ell(h) \smile S^\ell(h)$. There are at most $(\delta-2m)/2$ of these factors. In addition, the dimension of the highest stratum in $\mathrm{Sym}^\ell(X)$ is $4\ell$, so we can have no power of $S^\ell(h) \smile S^\ell(h)$ greater than $\ell$. Thus, the power of $Q_X(h)$ appearing in the formula must be less than $\min[(\delta-2m)/2,\ell]$.

The factors of $\langle c_1(\mathfrak{s}) - c_1(\mathfrak{t}), h\rangle$ and $\langle c_1(\mathfrak{t}), h\rangle$ appearing in (11.1) can be understood by the algebraic work in [11, §6.1]. □

*Proof of Theorem 11.2.* The condition that $\langle \mu_{\mathfrak{s}}(z_2), [M_{\mathfrak{s}}]\rangle = 0$ for all $z_2 \in \mathbb{A}_2(X)$ and the expression on the right-hand-side of (11.22), (to which the intersection number can be reduced) implies that all pairings will be multiples of $SW_{X,\mathfrak{s}}(\cdot)$. □

*Proof of Theorem 1.1.* Theorem 1.1 follows from Theorem 11.1 and the cobordism formula (10.11). The polynomials $p_{\delta,\ell,m,i}$ appearing in Theorem 1.1 differ from the polynomials $q_{\delta,\ell,m,i}$ appearing in Theorem 11.1 by the factors of two appearing in (2.52) and factors of negative one from the change of orientation expression in (10.8) as derived in the analogous formula in [11] (see [11, Theorem 1.4]). □

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