ORDERABLE GROUPS

FILIPPO CALDERONI

Contents

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1. Orderable groups: definitions and basic properties

Definition 1.1. A group G is **left-orderable** if there exists a strict total order on G such that

$$g < h \implies fg < fh \qquad \text{for all } f, g, h \in G.$$

Similarly, we say that G is **bi-orderable** if there exists a strict total order on G such that

$$g < h \implies (fg < fh \text{ and } gf < hf) \qquad \text{for all } f, g, h \in G.$$

As we do not assume G to be abelian, we stick to the multiplicative notation in most of the cases. Also note that the terminology left-order over right-order is totally arbitrary, since every left-invariant order $<$ on G induces a right-invariant order defined as

$$x \prec y \iff y^{-1} < x^{-1}.$$

Any abelian left-orderable group is obviously bi-orderable since every left-invariant order is also right-invariant.

Proposition 1.2. *The following are equivalent:*

(1) *G is left-orderable;*
(2) *There is $P \subseteq G$ such that:*
   (i) *P is a subsemigroup of G;*
   (ii) *$P \cap P^{-1} = \emptyset$;*
   (iii) *$G = P \cup \{1_G\} \cup P^{-1}$.*
(3) *G embeds into the automorphism group* $\mathrm{Aut}(A, <)$ *for some totally ordered set* $(A, <)$.

Proof. (1) $\iff$ (2) If $<$ is a left-order on G, define $P = \{g \in G \mid id < g\}$. Conversely, if P satisfies conditions (2)(i)–(iii), then declare $x <_P y$ if and only if $x^{-1}y \in P$.

(1) $\iff$ (3) If $<$ is a left-order on G then let $A = G$ and

$$\varphi\colon G \to \mathrm{Aut}(A, <),$$
$$g \mapsto \varphi(x) = gx.$$

Clearly each $\varphi(g)$ is a bijection and preserves the order relation $<$, which is left-invariant. On the other hand, note that for any set A and total order $<$, the set $\mathrm{Aut}(A, <)$ is left-orderable. Choose any well-order of X (completely unrelated to the ordering of A). Then, for all $f, g \in \mathrm{Aut}(A, <)$, let $x_0 = \min\{x \in A \mid f(x) \neq$

$g(x)\}$ declare $f \prec g \iff f(x_0) < g(x_0)$. It is immediate to check that $\prec$ is a left-order. $\square$

Definition 1.3. Whenever $<$ is a left-order on G, we call the set $P_< = \{g \in G \mid 1_G < g\}$ the **positive cone** of $<$.

Exercise 1. Let $<$ be a left order on G with associated positive cone $P = P_<$. Prove that $<$ is bi-orderable if and only if for all $g \in G$, we have $g^{-1}Pg = P$.

Exercise 2. Prove that bi-orderable groups have unique roots, that is, if $g^n = h^n$ for some $n > 0$, then $g = h$.

Exercise 3 (Neumann)**.** Prove that in a bi-orderable group G, g^n commutes with h if and only if g commutes with h. Show more generally that if g^n and h^m commute for some nonzero integers m and n, then g and h must commute.
[Hint: For the non-trivial direction, assume g and h do not commute, say $g < h^{-1}gh$, and multiply this inequality by itself several times to conclude g^n cannot commute with h.]

The direct product of two left-orderable groups is left-orderable. In fact, we can use the usual lexicographic construction. More generally, we can create plenty of left-orderable groups using extensions and the next trick, called the "sandwich" trick.

For groups K, G, H, the sequence $1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$ is called a **short exact sequence** if and only if i is injective, q is surjective, and $\mathrm{im}(i) = \ker(q)$.

Proposition 1.4. *Suppose that $P_K \subseteq K$ and $P_H \subseteq H$ are positive cones, and that*

$$1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$$

is a short exact sequence. Then $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone, in particular, G is left-orderable.

Proof. Exercise. $\square$

The sandwich trick has several applications.

Example 1.5. We can prove that the fundamental group of the Klein bottle $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ is left-orderable, since there is a short exact sequence

$$1 \to \langle b \rangle \xrightarrow{i} K \xrightarrow{q} K/\langle b \rangle \to 1$$

where $\langle b \rangle \cong K/\langle b \rangle \cong \mathbb{Z}$.

Example 1.6. The (discrete Heisenberg group) $H_3(\mathbb{Z})$ is the multiplication group of upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with } a, b, c \in \mathbb{Z}.$$

For simplicity let $x = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$, and $z = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. The center $Z(H_3(\mathbb{Z})) \cong \mathbb{Z}$ is the cyclic group generated by z. Also note that $\{x, y\}$ is a generating set for $H_3(\mathbb{Z})$ as $[x, y] = z$. One also notices that the map $\theta\colon H_3(\mathbb{Z}) \to \mathbb{Z} \times \mathbb{Z}$ defined as

$$\theta \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = (a, b)$$

is a group homomorphism and $\ker(\theta) = \langle z \rangle \cong \mathbb{Z}$. Therefore, bearing in mind that there is a short exact sequence

$$1 \to \langle z \rangle \xrightarrow{i} H_3(\mathbb{Z}) \xrightarrow{\theta} \mathbb{Z} \times \mathbb{Z} \to 1,$$

we conclude that $H_3(\mathbb{Z})$ is left-orderable by the sandwich trick.

Example 1.7. For an integer ℓ, consider the Baumslag-Solitar groups $\mathrm{BS}(1, \ell) = \langle a, b | aba^{-1} = b^{\ell} \rangle$ are all left-orderable.

For $\ell \geq 2$ note that the group $\mathrm{BS}(1, \ell)$ is isomorphic to the semidirect product $\mathbb{Z}\left[\frac{1}{\ell}\right] \rtimes \mathbb{Z}$ where $\alpha(n) \in \mathrm{Aut}(\mathbb{Z}\left[\frac{1}{\ell}\right])$ is the "multiplication by ℓ^n" automorphism of the additive group structure on $\mathbb{Z}[\frac{1}{\ell}]$. Therefore, $\mathrm{BS}(1, \ell)$ sits in a short exact sequence

$$1 \to \mathbb{Z}\left[\tfrac{1}{\ell}\right] \to \mathrm{BS}(1, \ell) \to \mathbb{Z} \to 1.$$

Exercise 4. Prove that a left-order defined from two bi-orders using the sandwich method needs not be a bi-order.

[Hint: Consider the fundamental group of the Klein bottle.]

Exercise 5 (Sandwich trick and bi-orderability)**.** If $P_K \leq K$ and $P_H \subseteq H$ are positive cones of bi-orderings, and

$$1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$$

is a short exact sequence, show that $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone of a bi-order if and only if $gi(P_K)g^{-1} \subseteq i(P_K)$ for all $g \in G$.

Exercise 6. Is $\mathrm{BS}(1, \ell)$ bi-orderable? Motivate your answer.

1.1. **Group rings and Kaplansky's conjectures.** Let R be a ring and G be a group. The **group ring** $R[G]$ consists of all finite formal sums $\sum_{g \in G} a_g\, g$ with $a_g \in R$. The finiteness of the sum means that only finitely many of the coefficients can be non-zero. If $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$ the addition and multiplication are defined by

$$\alpha+\beta = \Big(\sum_{g\in G} a_g g\Big) + \Big(\sum_{g\in G} b_g g\Big) = $$
$$= \sum_{g\in G}(a_g+b_g)g$$

$$\alpha\beta = \Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{g\in G} b_g g\Big) =$$
$$\sum_{g,h\in G}(a_g b_h)gh =$$
$$\sum_{k\in G}\Big(\sum_{gh=k} a_g b_h\Big)k$$

Note that we can identify G with $1_R g$, therefore we write $G \subseteq R[G]$. With this identification $1 \in G$ is the multiplicative identity element in $R[G]$.

Exercise 7. For $n \geq 2$, let $C_n = \langle x \rangle$ be the cyclic group of order n. Then $1 + \cdots + x^{n-1}$ is a zero-divisor of $R[C_n]$.

Conjecture 1.8 (Kaplansky's unit conjecture)**.** Suppose that G is torsion-free, then $R[G]$ has no non-trivial units. That is, if $\alpha, \beta \in R[G]$ and $\alpha\beta = \beta\alpha = 1$, then $\alpha = kg$ for some unit $k \in R$ and $g \in G$.

Conjecture 1.9 (Kaplansky's zero divisors conjecture)**.** Suppose that G is trosion-free and R has no non-trivial zero-divisors, then $R[G]$ has no nontrivial zero-divisors. That is, $\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$.

Conjecture 1.10 (Kaplansky's idempotent conjecture)**.** Suppose that G is trosion-free, then $R[G]$ has no non-trivial idempotents. That is, if $\alpha \in R[G]$ and $\alpha^2 = \alpha$, then $\alpha = 0$ or $\alpha = 1$.

Exercise 8. Prove that

$$\text{zero-divisors conjecture} \implies \text{idempotent conjecture.}$$

As of today it is only known that Kaplansky's unit conjecture is false, as Giles Gardam [10] built a counterexample.

Definition 1.11. A group G has the unique product property (or "has unique products", or "has UP") if for all non-empty finite subsets $A, B \subseteq G$ there exists $g \in G$ such that $g = ab$ for a unique pair $(a, b) \in A \times B$.

Example 1.12. In $(Z, +)$ the "product" $\max(A) + \max(B)$ is unique.

Proposition 1.13. *A group G with* UP *satisfies the zero divisor conjecture.*

Proof. Assume that R has no non-trivial zero divisors. Let $\alpha, \beta \in R[G]$ with $\alpha, \beta \neq 0$ and set $A = supp(\alpha), B = supp(\beta)$ so that

$$\alpha = \sum_{a \in A} r_a a \qquad \text{and} \qquad \beta = \sum_{b \in B} s_b b.$$

Then let $g_0 = a_0 b_0$ be a unique product for (A, B). Then the coefficient of g_0 in $\alpha\beta$ is

$$\sum_{gh = g_0} r_g s_h = r_{a_0} s_{b_0}.$$

Since r_{a_0}, s_{b_0} are in the supports of α and β, respectively, we know that they are non-zero. It follows that $r_{a_0} s_{b_0} \neq 0$ because of the assumption on R. Thus $\alpha\beta \neq 0$ □

Exercise 9. Prove that for every left-orderable group G, the group ring $R[G]$ satisfies Kaplansky's zero-divisor's conjecture.

[Hint: Prove that every left-orderable group satisfies UP.]

Solution. Suppose that G is a left-orderable group, so fix a left-order $<$. Let $A, B \neq \emptyset$ be finite subset of G. We are going to show that $\max AB$ is a unique product[1]. Let $b_0 = \max B$. Then for every $a \in A$ and $b \in B \setminus \{b_0\}$, we have

$$b < b_0 \implies ab < ab_0.$$

Thus $\max AB = ab_0$ for some $a \in A$. Note that this product must be unique in any group $a_i b_0 \neq a_j b_0$, whenever $a_i \neq a_j$. □

Exercise 10. Suppose that $N \lhd G$ such that both N and G/N have UP. Then G has UP.

1.2. **Direct and free products.** If $<_G$ and $<_H$ are left-orders on the groups G, H, respectively, we can define a left-order on $G \times H$ lexicographically by

$$(g_1, h_1) <_{lex} (g_2, h_2) \quad \iff \quad g_1 < g_2 \text{ or } (g_1 = g_2 \text{ and } h_1 < h_2).$$

Therefore, we have the following:

Proposition 1.14. *$G \times H$ is left-orderable (resp. bi-orderable) if and only if both G and H are left-orderable (resp. bi-orderable).*

A trick of Dicks and Sunic [8] can be leveraged to show that the free group $\mathbb{F}_2$ is left-orderable.

Proposition 1.15. *The non-abelian free group on two generators $\mathbb{F}_2$ is left-orderable.*

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[1] It is not necessarily true that $\max AB = \max A \max(B)$.

Proof. Let $\mathbb{F}_2 = \langle a, b\rangle$ so that every nontrivial element $w \in \mathbb{F}_2$ admits a unique presentation as a word
$$w = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$$
where x_i's are alternately a or b and ϵ_i are nonzero integers. Then for all $w \in \mathbb{F}_2 \setminus \{1\}$, define
$$\eta(w) = \begin{cases} 0 & \text{if } x_1 = x_n, \\ 1 & \text{if } x_1 = a \text{ and } x_n = b, \\ -1 & \text{if } x_1 = b \text{ and } x_n = a. \end{cases}$$
Next define $\tau\colon \mathbb{F}_2 \setminus \{1\} \to \mathbb{Z}$ by
$$\tau(w) = |\{i \mid \epsilon_i > 0\}| - |\{i \mid \epsilon_i < 0\}| + \eta(w).$$
Note that the value of τ is always odd, and in particular $\tau(w) \neq 0$. Then set $P = \{w \in \mathbb{F}_2 \setminus \{1\} \mid \tau(w) > 0\}$. It is routine to check that P is a positive cone. $\square$

While Proposition 1.15 is stated for $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$, the argument can be adapted to the general case of free products.

Proposition 1.16. *The free product is left-orderable if and only if both G and H are left-orderable. Moreover, for any two left-orders $<_G$ and $<_H$ on G and H, respectively, there is a left-ordering on $G * H$ extending both.*

With a more involved construction the proposition above can be upgraded to the more restrictive context of bi-orderability.

Theorem 1.17 (Vinogradov [24])**.** *The free product is bi-orderable if and only if both G and H are bi-orderable. Moreover, for any two bi-orders $<_G$ and $<_H$ on G and H, respectively, there is a bi-order on $G * H$ extending both.*

1.3. **Dynamical realization of left-orders.**

Theorem 1.18. *Suppose that G is a countable group. Then G is left-orderable if and only if there is an embedding $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$.*

Proof. Note that $\mathrm{Homeo}_+(\mathbb{R})$ is left-orderable (cf. Proposition 1.2), so is is left-orderable whenever $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$.

For the other direction fix a left-order $<$ on G. Enumerate $G = \{g_0, g_1, \dots\}$ with $g_0 = 1_G$. First define an order preserving function $t\colon (G, <) \to \mathbb{R}$ as follows. Set $t(g_0) = 0$. If $t(g_0), \dots t(g_n)$ are already defined, set

$$t(g_{n+1}) = \begin{cases} \max\{t(g_0), \dots, t(g_n)\} + 1 & \text{if } g_{n+1} > t(g_0), \dots, t(g_n); \\ \min\{t(g_0), \dots, t(g_n)\} - 1 & \text{if } g_{n+1} < t(g_0), \dots, t(g_n); \\ \frac{t(g_j) + t(g_k)}{2} & \begin{array}{l}\text{if } g_j \text{ and } g_k \text{ are consecutive in} \\ (\{g_0, \dots, g_n\}, <) \text{ and } g_j < g_{n+1} < g_k. \end{array} \end{cases}$$
Next we can build $\rho\colon G \to \mathrm{Homeo}_+(\mathbb{R}), g \mapsto \rho_g$ as follows:

(1) If $x \in t(G)$ then $x = t(h)$ for some $h \in G$ and set $\rho_g(x) = t(gh)$;
(2) If $x \in \overline{t(G)}$, then define $\rho_g(x)$ so that ρ_g is continuous on $\overline{t(G)}$;
(3) Extend ρ_g to $\mathbb{R} \setminus \overline{t(G)}$ to ensure continuity and order-preservation. E.g., if $x \in \mathbb{R} \setminus \overline{t(G)}$, then $x \in (a, b)$ for some $a, b \in \overline{t(G)}$. So we can write

$$x = (1 - s)a + sb$$

for some $s \in (0, 1)$, therefore define $\rho_g(x) = (1 - s)\rho_g(a) + s\rho_g(b)$. $\square$

This means that the collection of countable, left-orderable groups is precisely the collection of groups that embed into $\mathrm{Homeo}_+(\mathbb{R})$.

Remark 1.19. The hypothesis of countability in Theorem 1.18 is essential. Mann [15] showed that the group of germs at infinity of orientation-preserving homeomorphisms of $\mathbb{R}$ does not embed into $\mathrm{Homeo}_+(\mathbb{R})$, yet it has the same cardinality as $\mathrm{Homeo}_+(\mathbb{R})$.

An embedding $\rho\colon\ \mathrm{Homeo}_+(\mathbb{R})$ as in the proof of Theorem 1.18 is called dynamical realization of G. Such embedding is not unique in that it depends on the enumeration of G, but it is canonical up to conjugation.

Proposition 1.20. *Any two dynamic realizations of $(G, <)$ are conjugate. In other words, given two enumerations of G and corresponding dynamic realizations $\rho, \rho'\colon G \to$* $\mathrm{Homeo}_+(\mathbb{R})$*, there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that $\rho_g = h \circ \rho_g \circ h^{-1}$ for all $g \in G$.*

Theorem 1.18 is very useful to prove that certain group are left-orderable using a purely dynamical argument, avoiding group theoretic tricks. For example the Baumslag-Solitar groups $\mathrm{BS}(1, \ell) = \langle a, b \mid aba^{-1} = b^{\ell} \rangle$ embeds into $\mathrm{Homeo}_+(\mathbb{R})$ through the function

$$\begin{aligned} \rho\colon\ \mathrm{BS}(1, \ell) &\to \mathrm{Homeo}_+(\mathbb{R}) \\ a &\mapsto \rho_a(x) = \ell x \\ b &\mapsto \rho_b(x) = x + 1, \end{aligned}$$

therefore is left-orderable.

Recall that

$$\mathrm{SL}_3(\mathbb{Z}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A_{ij} \in \mathbb{Z} \text{ and } \det A = 1\}.$$

Definition 1.21. Suppose $<$ is a left-invariant total order on the group G. For $a, b \in G$, we write $a \ll b$ if a is **infinitely smaller than** b. I.e.,

$$a \ll b \iff a^n < |b|, \text{for all } n \in \mathbb{Z}, \qquad \text{where } |b| = \begin{cases} b & \text{if } b \geq 1, \\ b^{-1} & \text{otherwise.} \end{cases}$$

Following key fact about the Heisenberg group will be crucial later in this section.

Lemma 1.22 (Ault [3], Rhemtulla [19])**.** *If $<$ is any left-invariant total order on $H_3(\mathbb{Z})$, then either $z \ll x$ or $z \ll y$.*

Proof. Fix any left-invariant total order on $H_3(\mathbb{Z})$. Without losing generality[2] we assume that $x, y, z > 1$. It is routine to verify that for all $s, t \in \mathbb{Z}$,

$$x^s y^t = y^t x^s z^{st}.$$

Suppose that $z \ll x$ and $z \ll y$ towards contradiction. Then there are $k, m \in \mathbb{Z}$, such that $z^k > x$ and $z^m > y$. Therefore

$$1 < x, y, x^{-1}z^k, y^{-1}z^m.$$

Then for all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}
1 &< y^n x^n (y^{-1}z^m)^n (x^{-1}z^k)^n \\
&= y^n x^n y^{-n} x^{-n} z^{mn+kn} \\
&= z^{-n^2} z^{(m+k)n} \\
&= z^{\text{linear}(n) - \text{ quadratic}(n)}.
\end{aligned}$$

Therefore for n large enough the expression on the right hand side of the above inequality would be negative. And this is a contradiction. $\square$

Proposition 1.23. $\mathrm{SL}_3(\mathbb{Z})$ *is not left-orderable.*

Short proof. Note that $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_3(\mathbb{Z})$ and $A^3 = 1$. $\square$

More informative proof. Suppose there is a left-invariant total order $<$ on $\mathrm{SL}_3(\mathbb{Z})$. Let

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The group $\langle A_1, A_2, A_3 \rangle$ is precisely the Heisenberg group with $A_1 = x$, $A_2 = z$, $A_3 = y$ in the notation of Example 1.6.

Therefore, we have $A_2 \ll A_1$ or $A_2 \ll A_3$. Assume without loss of generality that

$$A_2 \ll A_3.$$

Actually, one can check that all groups

[2]Note that there is no harm in replacing some or all of x, y, and z by their inverses. This is because we can retain the relation $[x, y] = z$ by interchanging x and y if necessary, since $[y, x] = z^{-1}$.

$$\begin{array}{ccc} \langle A_1, A_2, A_3\rangle & \langle A_2, A_3, A_4\rangle & \langle A_3, A_4, A_5\rangle \\ \langle A_4, A_5, A_6\rangle & \langle A_5, A_6, A_1\rangle & \langle A_6, A_1, A_2\rangle \end{array}$$

are isomorphic to the Heisenberg group with the second generator generating the center. So, since $\langle A_2, A_3, A_4\rangle$ is also (isomorphic to) the Heisenberg group, we must have $A_3 \ll A_2$ or $A_2 \ll A_4$. However, $A_2 \ll A_3$ implies $A_3 \not\ll A_2$, so we conclude that $A_3 \ll A_4$. We can repeat this argument and obtain

$$A_2 \ll A_3 \ll A_4 \ll A_5 \ll A_6 \ll A_1 \ll A_2,$$

a contradiction. □

It is not too hard to tweak the more informative proof of Proposition 1.23 to prove the following:

Theorem 1.24 (Witte Morris [25]). *For all $n \geq 3$, all finite-index subgroups $G \leq \mathrm{SL}_3(\mathbb{Z})$ are not left-orderable.*

Theorem 1.25 (Deroin-Hurtado [7]). *If Γ is an irreducible lattice in a connected, semisimple, real Lie group G with finite centre, and $\operatorname{rank}_{\mathbb{R}} G \geq 2$, then Γ is not left-orderable.*

The methods of [7] also generalizes to the p-adic case. (See [17].)

1.4. **Generalized torsion.** A non-trivial element g of a group G is a generalized torsion element if $g^{a_1} g_2^{a} \cdots g^{a_n} = 1$ for some $n > 1$ and $a_1, \ldots, a_n \in G$.

Exercise 11. If G is bi-orderable, then G has no generalized torsion elements.

However it is clear that left-orderable groups may have generalized torsion elements.

Question 1.26 (Kopytov-Medvedev [14, Question 16.48]). *Does generalized torsion-free imply left-orderable?*

1.5. **Local characterizations.** Denote by $2^G = \{f \colon G \to \{0,1\}\}$. It is convenient to consider 2^G as the set of subsets of G, observing that any $A \subseteq G$ can be identified with the corresponding indicator function $\chi_A(x) = 1 \iff x \in A$.

Then we consider 2^G as a topological space by giving $\{0,1\}$ the discrete topology and then giving $2^G = \prod_{g \in G}\{0,1\}$ the product topology. The resulting product topology on 2^G is compact, Hausdorff, and totally disconnected.

The sets

$$V_g = \{A \subseteq G \mid g \in A\} \quad (V_g)^c = \{A \subseteq G \mid g \notin A\}$$

for $g \in G$, form a subbasis for the topology of 2^G. (Every open set of 2^G can be written as the union of finite intersections of V_g's and their complements.)

Exercise 12. For any $B \subseteq G$, the sets

$$\{A \in 2^G \mid B \nsubseteq A\} \quad \text{and} \quad \{A \in 2^G \mid A \cap B \neq \emptyset\}\}$$

are open subsets of 2^G.

Proposition 1.27. *The group G is left-orderable if and only if, for any finite sequence of nontrivial elements $g_1, \ldots, g_n \in G$, there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ such that 1_G does not belong to the semigroup generated by $\{g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n}\}$.*

Proof. ($\Rightarrow$) Let $<$ be a left-order on G and let $g_1, \ldots, g_n$ be any finite sequence of elements in G. For each $i \leq n$, choose ϵ_i so that $1 < a_i^{\epsilon_i}$. Then every element of the semigroup generated by $\{g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n}\}$ is positive, therefore $1 \notin sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$.

($\Leftarrow$) For any finite sequence $\vec{g} = g_1, \ldots, g_n \in G$ and $\vec{\epsilon} = \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ define the set

$$C(\vec{g}, \vec{\epsilon}) = \{A \subseteq G \setminus \{1\} \mid sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n}) \subseteq A \text{ and } sg(g_1^{-\epsilon_1}, \ldots, g_n^{-\epsilon_n}) \cap A = \emptyset\}$$

Then let $C_{\vec{g}} = \bigcup_{\epsilon \in \{\pm 1\}^n} C(\vec{g}, \vec{\epsilon})$. Note that because of the assumption $C_{\vec{g}} \neq \emptyset$. Moreover $C_{\vec{g}}$ does not depend on the enumeration, therefore, for any finite set $F \subseteq G \setminus \{1\}$, we simply write C_F.

Claim 1. For any finite $F \subseteq G \setminus \{1\}$, the set C_F is closed in 2^G.

Proof. Exercise. □

Claim 2. The set $\{C_F \mid F \subseteq_{fin} G \setminus \{1\}\}$ has the finite intersection property (FIP).

Proof. Exercise. □

Since 2^G is compact, we must have

$$\bigcap \{C_F \mid F \subseteq_{fin} G \setminus \{1\}\} \neq \emptyset.$$

So let P be an element of the above intersection.

Claim 3. P is a positive cone of G.

Proof. First note that 1 does not belong to any $A \in C_F$, for every $F \subseteq_{fin} G \setminus \{1\}$. So, $1 \notin P$.

Then we show that $P \subseteq G$ is a semigroup. Let $g, h \in P$. Since $P \in C_F$ for $F = \{g, h\}$, there are $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ such that

$$sg(g^{\epsilon_1}, h^{\epsilon_2}) \subseteq P \qquad \text{and} \qquad sg(g^{-\epsilon_1}, h^{-\epsilon_2}) \cap P = \emptyset.$$

Since $g, h \in P$, we must have $\epsilon_1 = \epsilon_2 = 1$, which yields $g^{\epsilon_1} h^{\epsilon_2} = gh \in P$.

Similarly we show that $P \cap P^{-1} = \emptyset$. Let $g \in P$. Then, since $P \in C_{\{g\}}$, there is $\epsilon \in \{\pm 1\}$ such that $sg(g^{\epsilon}) \subseteq P$ and $sg(g^{-\epsilon}) \cap P = \emptyset$. So, since $g \in P$ we conclude that $\epsilon = 1$ and $g^{-1} \notin P$.

Finally we need to show that $P \cup P^{-1} = G \setminus \{1\}$. Take $g \in G$ such that $g \neq 1$. As $P \in C_{\{g\}}$, there is $\epsilon = \pm 1$ such that $sg(g^{\epsilon}) \subseteq P$ and $sg(g^{-\epsilon}) \cap P \neq \emptyset$, so $g^{\epsilon} \in P$. □

□

An immediate consequence of Proposition 1.27 is that a group is left-orderable if and only if is locally left-orderable.

Corollary 1.28. *The group G is left-orderable if and only if every finitely generated subgroup of G is left-orderable.*

Recall that G is said to be **residually left-orderable** if and only if for every nontrivial $g \in G$, there exists a left-orderable group H, and a homomorphism $\varphi\colon G \to H$ such that $\varphi(g) \neq 1$.

Corollary 1.29. *Every residually left-orderable group is left-orderable.*

Proof. Let $g_1, \ldots, g_n$ be nontrivial elements of G. For each i, there is a left-orderable group H_i, and a homomorphism $\varphi_i\colon G \to H_i$, such that $\varphi_i(g_i) \neq 1$. Then consider the resulting homomorphism into

$$\begin{aligned} \varphi\colon G &\to H_1 \times \cdots \times H_n, \\ g &\mapsto (\varphi_1(g_1), \ldots, \varphi(g_n))). \end{aligned}$$

We have $\varphi(g_i) \neq 1$ for all i. Now fix some positive cone $P \subseteq H_1 \times \cdots \times H_n$, and let ϵ_i be such that $\varphi(g_i)^{\epsilon_i} \in P$. Let $S = sg(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$. It is clear that $\varphi(S)$ is contained in the positive cone P, therefore $1 \notin \varphi(S)$, which implies that $1_G \notin S$. Hence, the statement follows from Proposition 1.27. □

Another celebrated corollary of Proposition 1.27 is the following classical theorem stating that a sufficient (and necessary) condition for being left-orderable is being locally projectable into the class of left-orderable groups.

Theorem 1.30 (Burns-Hale [4])**.** *A group G is left-orderable if and only if for every nontrivial finitely generated $H \leq G$, there exists a surjection $H \to L$ onto some nontrivial left-orderable group L.*

Proof. $(\Rightarrow)$ Obvious.

$(\Leftarrow)$ We are going to find suitable exponents ϵ_i's for any finite sequence of nontrivial elements $g_1, \ldots, g_n \in G$ by induction.

For $n = 1$ we only have a nontrivial element g, then there is an onto homomorphism $\langle g \rangle \to \mathbb{Z}$ and any choice of exponent ϵ would show that $1_G \notin sg(g)$ because g has no torsion.

Next let $n \geq 1$ be any natural number and suppose that for all $k \leq n$ and all $g_1, \ldots, g_k$, there exist $\epsilon_i = \pm 1$ such that $1_G \notin sg(g_1^{\epsilon_1}, \ldots, g_k^{\epsilon_k})$. Consider a sequence of nontrivial elements $h_1, \dots, h_{n+1} \in G$. Set $H = \langle h_1, \dots, h_{n+1} \rangle$ and choose $\phi \colon H \to L$ where L is a nontrivial left-orderable group. Possibly some of the generators $h_1, \dots, h_{n+1}$ are in the kernel of ϕ. To simplify the exposition, assume that the h_i 's are indexed so that $\phi(h_i) = 1_L$ for $i = 1, \dots, r$ and $\phi(h_i) \neq 1$ for $i = r+1, \dots, n+1$. Note that there is at least one h_i such that $\phi(h_i) \neq 1$, since ϕ is a surjection, so $r \leq n$.

Now, since L is left-orderable fix a positive cone $P_L \subseteq L$. Then choose exponents $\epsilon_{r+1}, \dots, \epsilon_{n+1}$ so that $\phi(h_i)^{\epsilon_i} \in P_L$ for $i = r+1, \dots, n+1$. Next, by the inductive assumption there are $\epsilon_1, \dots, \epsilon_r$ such that $1_G \notin sg(h_1^{\epsilon_1}, \dots, h_r^{\epsilon_r})$. Given $w \in sg(h_1^{\epsilon_1}, \dots, h_{k+1}^{\epsilon_{k+1}})$, if w contains any occurrences of $h_{r+1}, ..., h_{k+1}$ then $\phi(w) \in P_L$ and so $w \neq 1_G$. Otherwise, if w contains no such occurrences, then $w \in sg(h_1^{\epsilon_1}, \dots, h_r^{\epsilon_r})$ and $w \neq 1$. Therefore, the statement follows from Proposition 1.27. $\square$

Exercises

Exercise 1. Let $<$ be a left order on G with associated positive cone $P = P_<$. Prove that $<$ is bi-orderable if and only if for all $g \in G$, we have $g^{-1}Pg = P$.

Exercise 2. Prove that bi-orderable groups have unique roots, that is, if $g^n = h^n$ for some $n > 0$, then $g = h$.

Exercise 3 (Neumann)**.** Prove that in a bi-orderable group G, g^n commutes with h if and only if g commutes with h. Show more generally that if g^n and h^m commute for some nonzero integers m and n, then g and h must commute.
[Hint: For the non-trivial direction, assume g and h do not commute, say $g < h^{-1}gh$, and multiply this inequality by itself several times to conclude g^n cannot commute with h.]

Exercise 4. Prove that a left-order defined from two bi-orders using the sandwich method needs not be a bi-order.

[Hint: Consider the fundamental group of the Klein bottle.]

Exercise 5 (Sandwich trick and bi-orderability)**.** If $P_K \leq K$ and $P_H \subseteq H$ are positive cones of bi-orderings, and

$$1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$$

is a short exact sequence, show that $P_G = i(P_K) \cup q^{-1}(P_H)$ is a positive cone of a bi-order if and only if $gi(P_K)g^{-1} \subseteq i(P_K)$ for all $g \in G$.

Exercise 6. Is $\mathrm{BS}(1, \ell)$ bi-orderable? Motivate your answer.

Exercise 7. For $n \geq 2$, let $C_n = \langle x \rangle$ be the cyclic group of order n. Then $1 + \cdots + x^{n-1}$ is a zero-divisor of $R[C_n]$.

Exercise 8. Prove that

$$\text{zero-divisors conjecture} \implies \text{idempotent conjecture.}$$

Exercise 9. Prove that for every left-orderable group G, the group ring $R[G]$ satisfies the Kaplansky's zero-divisor's conjecture.

[Hint: Prove that every left-orderable group satisfies UP.]

Exercise 10. Suppose that $N \lhd G$ such that both N and G/N have UP. Then G has UP.

Exercise 11. Prove Theorem 1.24

Exercise 12. If G is bi-orderable, then G has no generalized torsion elements.

Exercise 13. For any $B \subseteq G$, the sets

$$\{A \in 2^G \mid B \nsubseteq A\} \quad \text{and} \quad \{A \in 2^G \mid A \cap B \neq \emptyset\}\}$$

are open subsets of 2^G.

2. Conradian orders

2.1. **Archimedean orders.**

Definition 2.1. A left-order $<$ of a group G is **Archimedean** if, whenever $g, h > 1_G$ there exists $n > 0$ such that $g^n > h$.

Proposition 2.2 (Conrad [6, Proposition 3.8])**.** *Any Archimedean order is a bi-order.*

Proof. Let $<$ be any Archimedean order with positve cone $P_< = P$. It suffices to show that for all $g \in G$ we have $g^{-1}Pg = P$. Indeed, it suffices to show that $g^{-1}Pg \subseteq P$.

Let $h \in P$ and $g \in G$.

Case 1: Suppose that g is positive By the Archimedean type property, there is $n > 0$ suth that $g < h^n$, therefore $1 < g^{-1}h^n$. Since g is also positive we get $1 < g^{-1}h^n g$, and by taking roots we have $1 < g^{-1}hg$ which is equivalent to $g^{-1}hg \in P$.

Case 2: Let g be negative. Suppose that $g^{-1}hg \notin P$ towards contradiction. It follows that $1 < g^{-1}h^{-1}g$, and since g^{-1} is also positive, it follows that

$$1 < (g^{-1})^{-1}g^{-1}h^{-1}gg^{-1} = h^{-1}$$

by Case 1, a contradiction. □

Lemma 2.3. *If G is bi-ordered and does not have a least positive element, then for every $p > 1$ there is some positive $r \in G$ such that $1 < r^2 < p$.*

Proof. Let $1 < s < q < p$, and note that $1 < qs^{-1}$. If $(qs^{-1})^2 < q$, let $r = qs^{-1}$. Otherwise $q \leq (qs^{-1})^2$, then let $r = s$. This works because the assumption that $q \leq (qs^{-1})^2$ implies $1 \leq s^{-1}qs^{-1}$, therefore $s^2 \leq q$. □

Proposition 2.4. *If G admits an Archimedean order, then G is abelian.*

Proof. Let $<$ be an Archimedean order on G with positive cone $P = P_<$. Note that by Lemma 2.2 the ordering is bi-invariant. We consider two cases.

Case 1: The positive cone P has a least element a. Then we claim that G is precisely the infinite cyclic subgroup $<a>$. For if $g \in G \setminus \langle a \rangle$, there is n such that $a^n < g < a^{n+1}$ and therefore we must have $1 < a^{-n}g < a$, contradicting the minimality of a.

Case 2: P does not have a least element. Suppose there are two elements $g, h \in G$ that do not commute. Without loss of generality, we may assume g and h and their commutator $ghg^{-1}h^{-1}$ are all positive. By Lemma 2.3 there is some positive $r \in G$ such that $1 < r^2 < ghg^{-1}h^{-1}$. Therefore, by the Archimedean property, there exists integers m, n such that

$$r^m \leq g < r^{m+1} \quad \text{and} \quad r^n \leq h < r^{n+1}.$$

Then $g^{-1} \leq r^{-m}$ and $h^{-1} \leq r^{-n}$. Multiplying the appropriate inequalities we get

$$ghg^{-1}h^{-1} < r^{m+1}r^{n+1}r^{-m}r^{-n} = r^2,$$

a contradiction.

□

Theorem 2.5. *If G admits an Archimedean left-order, then G is isomorphic (as an ordered group) to a subgroup of $(\mathbb{R}, +)$.*

Proof. Let $\prec$ be an Archimedean order on G. Fix a positive element $k \in G$ and set $\varphi(k) = 1$.

For each $x \in G$ and $n \in \mathbb{N}$, there is a corresponding integer $a_n \in \mathbb{Z}$ such that $k^{a_n} \preceq x^n \prec k^{a_{n+1}}$.

Since any order-preserving homomorphism $\varphi \colon (G, \prec) \to (\mathbb{R}, <)$ with $\varphi(k) = 1$ must have

$$\varphi(k^{a_n}) < \varphi(x^n) < \varphi(k^{a_n+1})$$
$$a_n < n\varphi(x) < a_n + 1$$

for all $n \in \mathbb{N}$. Therefore, we set

$$\varphi(x) = \lim_{n\to\infty} \frac{a_n}{n}.$$

First we claim that the map $\varphi \colon g \to (R, +)$ is a group homomorphism. To see that let g and h with $\varphi(g) = \lim_{n\to\infty} \frac{a_n}{n}$ and $\varphi(h) = \lim_{n\to\infty} \frac{b_n}{n}$. Since the order $\prec$

is bi-invariant, the inequalities

$$k^{a_n} \preceq g^n \prec k^{a_n+1}$$
$$k^{b_n} \preceq h^n \prec k^{b_n+1}$$

implies that

$$k^{a_n+b_n} \preceq g^n h^n = (gh)^n \prec k^{a_n+b_n+2}.$$

Then

$$\varphi(g) + \varphi(h) = \lim_{n\to\infty} \frac{a_n + b_n}{n} \leq \varphi(gh) \leq \lim_{n\to\infty} \frac{a_n + b_n + 2}{n} = \varphi(g) + \varphi(h).$$

We conclude that $\varphi(gh) = \varphi(g) + \varphi(h)$ as desired.

The fact that φ is order-preserving (in the sense that if $g_1 \prec g_2$ then $\varphi(g_1) < \varphi(g_2)$ follows from the definition.

To show that φ is one-to-one. Let $g \in G$ with $\varphi(g) = 0$. Assume that $g \neq 1_G$. Then there exists $n \in \mathbb{Z}$ such that $k \preceq g^n$. Consequently, $1 = \varphi(k) \leq \varphi(g^n) = n\varphi(g) = 0$. □

2.2. **Conradian orders.**

Definition 2.6. A left-order $<$ is **Conradian** if and only if for all $g, h > 1$ there is $n \in \mathbb{N}$ such that $g < hg^n$.

For a countable left-orderable group G, define the space of Conradian order $\mathrm{CLO}(G) = \{P \in \mathrm{LO}(G) \mid P \text{ is Conradian}\}$. It turns out that $\mathrm{CLO}(G)$ is compact. This is not immediate from the definition as

$$P \in \mathrm{CLO}(G) \iff \forall g, h(g, h \in P \implies \exists n \in \mathbb{N}(g^{-1}hg^n \in P)).$$

It follows that

$$P \in \mathrm{CLO}(G) \iff \bigcap_{g,h\in G} (V_g^c \cup V_h^c \cup \bigcup_{n\in\mathbb{N}} V_{g^{-1}hg^n}),$$

showing that $\mathrm{CLO}(G)$ is G_δ in $\mathrm{LO}(G)$, i.e., countable intersection of open sets. However, the following proposition shows that the Conradian property is equivalent to a simpler definition.

Proposition 2.7 ([18, Proposition 3.7])**.** *Suppose that $<$ is a Conradian left-order of a group G. If $g, h > 1$, then $g < hg^2$.*

Proof. Suppose that $hg^2 < g$ towards contradiction[3]. Then $g^{-1}hg^2 = (g^{-1}hg)g < 1$. This implies that $g^{-1}hg$ must be negative since g is positive. Thus we also have

[3]We can exclude that $hg^2 = g$, because then $g^2 = gh^{-1}$, and so $g = h^{-1}$, a contradiction.

$g^{-1}hg < 1$, which is equivalent to $hg < g$. Now consider the positive elements h and $x = hg$. For all $n \geq 0$ we calculate

$$\begin{aligned} hx^n =& h(hg)^{n-2}(hg)(hg) \\ <& h(hg)^{n-2}(hg)g \\ =& h(hg)^{n-2}(hg^2) \\ <& h(hg)^{n-2}g \\ =& h(hg)^{n-3}hg^2 \\ <& h(hg)^{n-3}g \\ & \vdots \\ <& hg = x. \end{aligned}$$

Thus the positive elements h, x do not satisfy the Conradian condition, a contradiction. ☐

Corollary 2.8. *The set* $\mathrm{CLO}(G)$ *is closed in* $\mathrm{LO}(G)$*, therefore it is a compact Polish space with the subspace topology.*

2.3. **Conradian orders via convex subgroups.**

Definition 2.9. Let $<$ be a left-order on G. A subgroup H of G is **convex with respect to** $<$ of G, or $<$**-convex** for short, if for all $g, h \in H$ and $f \in G$, $g < f < h$ implies $f \in H$. Moreover, we say that H of G is **relatively convex** if it is $<$-convex for some left-order $<$ of G.

Proposition 2.10. *Suppose that G is left-orderable and N is a normal subgroup. The quotient G/N is left-orderable if and only if N is relatively convex in G.*

Proof. Suppose that $<$ is a left-order on G and that N is $<$-convex. Define a total order $\prec$ on G/N by declaring $N \prec hN$ if and only if $1 < h$. To see that this is well defined, suppose that $hN = h'N$ with $h, h' \notin N$ and $h' < 1 < h$. Then note that

$$1 < (h')^{-1} < (h')^{-1}h.$$

Since $hN = h'N$ we have $(h')^{-1}hN = N$, thus $(h')^{-1}h \in N$. It follows that $(h')^{-1} \in N$ by convexity of N, a contradiction. Checking that $\prec$ is a left-order is routine. ☐

Lemma 2.11. *Suppose that C, D are $<$-convex subgroups in the ordered group $(G, <)$. Then either $C \subseteq D$ or $D \subseteq C$.*

Proof. Exercise. ☐

Lemma 2.12. *If $\{C_i\}_{i \in I}$ are $<$-convex subgroups of G, then $\bigcup_{i \in I} C_i$ and $\bigcap_{i \in I} C_i$ are also $<$-convex subgroups.*

Proof. Exercise. ☐

Exercise 14. Prove Lemma 2.11 and Lemma 2.11

Exercise 15. An order $<$ of an Abelian group A is Archimedean if and only if there are no nontrivial $<$-convex subgroups in A.

Definition 2.13. Let $<$ be a left-order on G. A **$<$-convex jump** in an ordered group $(G, <)$ is a pair of $<$-convex subgroups (C, D) with $C \subsetneq D$ such that if there is a third $<$-convex subgroup $C \subseteq C' \subseteq D$ then either $C = C'$ or $C' = D$.

Whenver the order is clear from the context we simply write *convex jump*. The following particular case of convex jumps are crucial in any discussion of Conradian orders.

Definition 2.14. The convex jump (C, D) is **Conradian** if and only if

(i) $C \lhd D$;
(ii) The quotient left-order of D/C is Archimedean.

Moreover, we call any embedding $\rho(C, D) \colon D/C \to \mathbb{R}$ arising from Hölder's theorem the **Conrad homomorphism** associated to the convex jump (C, D).

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Example 2.15. Consider the fundamental group of the Klein bottle $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ with any left-order arising from the short exact sequence

$$1 \to \langle b \rangle \xrightarrow{i} K \xrightarrow{q} K/\langle b \rangle \to 1$$

as in example 1.5. Then the Conradian jumps associated to the given left-order are $(1, \langle b \rangle)$ and $(\langle b \rangle, K)$.

Example 2.16. Let $G = \mathbb{Z} \times \mathbb{Z}$ with an Archimedean left-order on it. For any element $g \in G$, the pair $(1, \langle g \rangle)$ is not a convex jump by Exercise 15.

Proposition 2.17. *For a left-order $<$ on G, the following are equivalent:*

(i) *$<$ is Conradian;*
(ii) *For every $g \in G$ there exists a Conradian convex jump (C, D) such that $g \in D \setminus C$.*

Proof. We omit the lengthy proof of $(\Rightarrow)$.

$(\Leftarrow)$ Suppose that $<$ is a left-order on G and that (ii) holds. For every $g \in G$, let (C_g, D_g) be the Conradian jump associated to $g \in G$. Now, let $g, h \in G$ be positive. It suffices to show that $1 < g^{-1}hg^2$. To do this, consider the Conradian jumps (C_g, D_g) and (C_h, D_h) associated to g and h, respectively. Since (C_g, D_g) and (C_h, D_h) have no intermediate convex subgroups, it follows that either

(I) $C_g \subset D_g \subseteq C_h \subset D_h$; or
(II) $C_g = C_h$ and $D_g = D_h$; or
(III) $C_h \subset D_h \subseteq C_g \subseteq D_h$.

Suppose that (I) holds, case (III) being similar. Let $\pi\colon D_h \to D_h/C_h$ be the canonical quotient map. Note that $g \in C_h$, therefore $\pi(g^{-1}hg^2) = \pi(h) = hC_h$. Since $1 < h$, we know that hC_h is positive in the order induced on the quotient. We conclude that $1 < g^{-1}hg^2$ as desired.

Next suppose that (II) holds. Let $\pi\colon D_h \to G/C_h$ as before and let $\rho\colon D_h/C_h \to \mathbb{R}$ be the Conrad homomorphism associated to the jump (C_h, D_h). Observe that D_h/C_h is Abelian since it embeds into $\mathbb{R}$ via ρ, therefore $\pi(g^{-1}hg^2)) = hgN$, which is a positive element in the order on D_h/C_h induced by the quotient. It follows that $1 < g^{-1}hg^2$ as desired. $\square$

Corollary 2.18. *If G admits a Conradian left-order, then it is locally indicable.*

Proof. Suppose that G admits a Conradian left-ordering $<$. Given a nontrivial $H \leq G$ finitely generated with $H = \langle h_1, \dots, h_n\rangle$, suppose the generators are $1 < h_1 < h_2 < \cdots < h_k$. Then consider the Conradian jump (C, D) with $h_k \in D \setminus C$. Since D is convex, we see that D must contain all generators of H, and therefore $H \subseteq D$. On the other hand, C does not contain H. So there is a Conrad homomorphism $\rho\colon D/C \to (\mathbb{R}, +)$ which maps $H/(H \cap C)$ onto a non-trivial torsion-free abelian group. This gives a homomorphism from H onto a torsion-free finitely generated abelian group, so there is a homomorphism from H onto $\mathbb{Z}$. $\square$

Corollary 2.19. *Any bi-orderable group is locally indicable.*

Proof. Any bi-order group is Conradian since for any positive $g, h > 1$, we have $g < hg$ since $<$ is right-invariant. $\square$

Example 2.20 (Thurston-Kropholler)**.** It is rather tricky to construct examples of left-orderable groups that are not locally indicable.

Consider the group $G = \langle x, y, z \mid x^2 = y^3 = z^7 = xyz\rangle$. We claim that G is not locally indicable. Since G is finitely generated it suffices to show that $G^{\mathrm{ab}} = 1$. Working in the abelianization we get $y^2 = xz$ from the relation $y^3 = xyz$. Then $x^2 = xyz$ yields $x = yz$, which combines with the previous relation to give $y^2 = (yz)z$ or $y = z^2$. Now $y^3 = z^7$ gives $z = 1$, so that $y = 1$ and $x = 1$ as well. In the end, since G^{ab} is trivial, there is no surjection $G \to \mathbb{Z}$.

To prove that G is left-orderable, one notes that G embeds into $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ as in [1].

Theorem 2.21 (Burns-Hale for Conradian left-orderable groups,[18])**.** *A group G admits a Conradian left-ordering if and only if for every nontrivial finitely-generated subgroup H of G, there exists a Conradian left-orderable group K and a nontrivial homomorphism $H \to K$.*

Corollary 2.22. *A group is locally indicable if and only if it admits a Conradian order.*

2.4. **A digression on group growth.** Let G be a group generated by a finite subset X. The **length function** $|\ |_X\colon G \to \mathbb{N}$ is defined as follows:

$$|g|_X = \text{length of shortest word in } X \cup X^{-1} \text{ representing } g.$$

Fact 2.23. *The length function satisfies the following norm-like properties:*

(i) $|g| = 0 \iff g = e$;
(ii) $|g| = |g|^{-1}$;
(iii) $|gh| \leq |g| + |h|$.

The norm-like properties of $| \ |$ give rise to a metric $d_X \colon G \times G \to \mathbb{N}$, defined by $d(g,h) = |g^{-1}h|$. Note that d is invariant under left multiplication: $d(ag, ah) = d(g,h)$.

Definition 2.24. We define the **growth function** of (G, X) as the function $\gamma_X \colon \mathbb{N} \to \mathbb{N}$, $\gamma_X(n) = \#B_e(n)$, i.e., the number of elements representable as words in $X \cup X^{-1}$ of length $\leq n$.

More geometrically, $B_c(r)$ denotes the closed ball of raldius r and center c in the Cayley graph of G with the set of generators , $X \cup X^{-1}$.

Example 2.25. (a) Let $G = \mathbb{Z} \times \mathbb{Z}$ and $X = \{(1,0),(0,1)\}$. Then $\gamma_X(n) = 2n^2 + 2n + 1$.
(b) Let $G = \mathbb{F}_2$ and $X = \{a, b\}$. Then $\gamma_X(n) = 2 \cdot 3^n - 1$.

Definition 2.26. Let $f, g \colon \mathbb{N} \to \mathbb{R}_+$. We say f **dominates** g, denoted by $g \lesssim f$, if there exists a constant $c > 0$ such that $g(n) \leq cf(n)$ for all $n \geq 1$. Furthermore, we say that f and g are **equivalent** if $f \lesssim g$ and $g \lesssim f$.

Given X and X' generating sets of G, their respective growth functions γ_X and $\gamma_{X'}$ are equivalent.

Definition 2.27. Let G be a finitely generated group with corresponding growth function γ (with respect to any set of generators). We say that G is of **polynomial growth** if there is some $d \in \mathbb{N}$ and $c > 0$ such that $\gamma(n) \leq cn^d$ for all n. Moreover, G is of **exponential growth** if there is $c > 1$ such that $\gamma(n) \geq c^n$ for all n.

Theorem 2.28 (Gromov [11])**.** *The group G is of polynomial growth if and only if G has a nilpotent subgroup of finite index.*

We denote by M_2 the free semi-group on two generators a, b. This is the semi-group generated by a, b such that every word in a, b corresponds to a unique element in M_2 and the product is defined by concatenation. The definition of growth extends to finitely generated semigroups verbatim.

Lemma 2.29. *Any finitely-generated group which contains a copy of M_2 has exponential growth.*

Proof. Let $A = \{a, b\}$ be the set of generators of M_2, then note that

$$\gamma_A(n) = 2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

since the number of elements of length n is precisely the number of ways to form a n-letter word from a, b. So, if G contains a copy of M_2, let $X \supseteq A$ be a set of generators for G. It follows that the growth function of G has $\gamma_X(n) \gtrsim 2^n$. $\square$

Proposition 2.30. *If G admits a left-order that is not Conradian, then G is of exponential growth.*

Sketch. Fix a left-order $<$ on G that is not Conradian. Let g, h be any two element violating the Conradian property. Then the semigroup $\langle g, h\rangle^+$ is isomorphic to M_2. The conclusion follows from Lemma 2.29. $\square$

It follows from Theorem that if the left-orderable group G is virtually nilpotent then $\mathrm{LO}(G) = \mathrm{CLO}(G)$.

On the other hand, all known examples of left-orderable groups that are not locally indicable contains a nonabelian free subgroup. It is unknown whether containing no copies of $\mathbb{F}_2$ is sufficient.

Conjecture 2.31 (Linnell [2])**.** Let G is a left-orderable group. If G does not contain any copy of $\mathbb{F}_2$, then G is locally indicable.

2.5. **Amenable groups.** Consider the free group $\mathbb{F}_2 = \langle a, b\rangle$, and let us assume that every element of the group starts with \$1. Thus, if $f_0(g)$ denotes the amount of money possessed by element g at time $t = 0$, then $f_0(g) = \$1$, for all $g \in \mathbb{F}_2$. Now, everyone gives their dollar to the person next to them who is closer to the identity. (That is, if $g = x_1 x_2 \cdots x_n$ is a reduced word, with $x_i \in \{a^{\pm 1}, b^{\pm 1}\}$ for each i, then g gives its dollar to $g' = x_1 x_2 \cdots x^{n-1}$. The identity element does not give away any money.)

Then, let f_1 denote the amount of money possessed now (at time $t = 1$), we have $f_1(g) = \$3$ for all g (except that $f_1(e) = \$5$). Thus, everyone has more than doubled their money. Furthermore, this result was achieved by moving the money only a bounded distance. A Ponzi scheme[4] on $\mathbb{F}_2$ is a function describing this arrangement:

Definition 2.32. A **Ponzi scheme** on a group G is a function $M \colon G \to G$, such that:

(1) $M^{-1}(g) \geq 2$ for all $g \in G$;
(2) There is a finite set $S \subseteq G$ such that $M(g) \in gS$ for all $g \in G$.

[4]A Ponzi scheme is a fraudulent investment scheme that pays returns to earlier investors using money from later investors. The scheme eventually fails because the earnings are less than the payments. Charles Ponzi ran a Ponzi scheme in the early 1900s that defrauded 30,000 Americans of about \$10 million.

Roughly speaking (2) guarantees that money is passed to a bounded distance. We saw that there is a Ponzi scheme on $\mathbb{F}_2$.

Exercise 16. Prove that if G admits a Ponzi scheme, then the growth of G is exponential. Deduce that there is no Ponzi scheme on $\mathbb{Z}^n$ for all $n \geq 1$.

Let G be a countable group. Fix a finite subset $S \subseteq G$ and some $\epsilon > 0$. A finite, nonempty subset $F \subseteq G$ is (S, ϵ)**-almost invariant** if and only if

$$|F \cap gF| > (1 - \epsilon)|F| \qquad \text{for all } g \in S.$$

Definition 2.33. A countable group G is **amenable** if and only if G admits an (S, ϵ)-almost invariant finite subset for all finite S and $\epsilon > 0$.

It is straight-forward to see that $\mathbb{Z}$. For any finite $S \subseteq \mathbb{Z}$ and $\epsilon > 0$, let $F = [-n, n]$ for n sufficiently large. (For example, let $n \geq \max\{|s| : s \in S\}\epsilon^{-1}$.)

A direct computation also shows that all $\mathbb{Z}^n$ are amenable. In fact, all countable abelian groups are amenable.

The following characterization of amenable groups is due to Gromov [12].

Theorem 2.34 (Gromov)**.** *Let G be a countable group. There exists a Ponzi scheme on G if and only if G is not amenable.*

Proof. $\Rightarrow$) We shall use the following lemma.

Claim 4. If G is amenable, S is a finite subset of G, and $\epsilon > 0$, then there exists a finite subset $F \subseteq G$, such that $|SF| < (1 + \epsilon)|F|$.

Proof. Let $n = |S|$. Choose F so that $|F \cap gF| > (1 - (\epsilon/n))|F|$ for all $g \in S$. It follows that $|gF \setminus F| < (\epsilon/n)|F|$ for all $g \in S$. Therefore,

$$|SF| - |F| < n(\epsilon/n) = \epsilon. \qquad \square$$

Let M be a Ponzi scheme on G. By (2) there is a finite set $S \subseteq G$ such that $M(g) \in gS$ for all $g \in G$. By the claim there is a finite set F , such that $|FS^{-1}| < 2|F|$. This is impossible because (1) yields that M is at least 2-to-1 and $M^{-1}(F) \subseteq FS^{-1}$. $\square$

An almost-invariant set is also called a "Følner set". Moreover, a sequence $\{F_n\}$ of nonempty, finite subsets of G is said to be a **Følner sequence** if, for every finite subset S of G, every $\epsilon > 0$, and every sufficiently large n, the set F_n is (S, ϵ)-invariant. (The set F_n is often called a "Følner set".) Definition 2.33 can be reformulated by saying that G is amenable if and only if it has a Følner sequence.

Denote by $\ell^\infty(G)$ the Banach space of bounded real-valued functions on G with the sup norm $||f||_\infty = \sup\{|f(x)| : x \in G\}$.

Definition 2.35. A mean on $\ell^\infty(G)$ is a linear functional $A \colon \ell^\infty(G) \to \mathbb{C}$, such that $A(\varphi)$ satisfies two axioms that would be expected of the average value of φ:

- The average value of a constant function is that constant

$$A(c) = c \qquad \text{if } c \text{ is a constant;}$$

- The average value of a positive-valued function cannot be negative

$$A(\varphi) \geq 0 \qquad \text{if } \varphi \geq 0.$$

The mean is left-invariant if $A(\varphi^g) = A(\varphi)$ for all $\varphi \in \ell^\infty(G)$ and all $g \in G$, where $\varphi^g(x) = \varphi(gx)$.

Proposition 2.36. *G is amenable if and only if there exists a left-invariant mean on $\ell^\infty(G)$.*

Proof. $\Rightarrow$) Choose a sequence $\{F_n\}_{n=1}^\infty$ of almost-invariant sets with $\epsilon \to 0$ as $n \to \infty$,and let $A_n(\varphi)$ be the average value of φ on the finite set F_n.

$$A_n(\varphi) = \frac{1}{|F_n|} \sum_{x \in F_n} \varphi(x).$$

It is clear that A_n is a mean on $\ell^\infty(G)$. Since the set F_n is almost invariant, the mean A_n is close to being left-invariant. To obtain left-invariance, we take a limit:

$$A(\varphi) = \lim_{k \to \infty} A_{n_k}(\varphi),$$

where $\{n_k\}$ is a subsequence chosen so that the limit exists. — If we choose different subsequences for different functions φ, then the limit may not be linear or left-invariant, so we need to be consistent in our choice of $A(\varphi)$ for all φ. Here we must use some choice. This can be done using an ultrafilter. □

Corollary 2.37. *Suppose G is amenable, X is a compact metric space and $G \curvearrowright X$ continuously. Then there exists a G-invariant Borel probability measure μ on X.*

Proof. Fix a basepoint $x_0 \in X$. For any $f \in C(X)$, we can define a function $\varphi_f \colon G \to \mathbb{C}$ by restricting f to the G-orbit of x_0. More precisely,

$$\varphi_f(g) = \varphi(gx_0).$$

Since f is continuous and X is compact, we know that f is bounded. So φ_f is also bounded. Therefore, Proposition 3.11 tells us that it has an average value $A(\varphi_f)$, which we call $\mu(f)$.

Since A is a mean, it is easy to see that μ is a positive linear functional of finite norm (in fact, $||\mu|| = 1$)). So the Riesz Representation Theorem tells us that μ is a Borel measure on X. Since A is translation-invariant and $A(1) = 1$, we see that μ is translation-invariant and $\mu(X) = 1$, so μ is a translation-invariant probability measure. □

2.6. **Recurrent left-orders.** In this section we discuss progress on Linnell's conjecture. (See Conjecture 2.31.) First, recall that a countable group G is amenable if whenever G acts by continuous maps on a compact metric space X, there exists a probability measure μ on X such that $\mu(A) = \mu(gA)$ for all $g \in G$ and all measurable sets $A \subseteq X$. We will use the following famous theorem from ergodic theory.

Theorem 2.38 (Poincaré Recurrence Theorem [22]). *Suppose*

- *X is a measure space with probability measure μ.*
- *$T\colon X \to X$ is an invertible, μ-preserving, measurable map.*
- *A is any measurable subset of X.*

Then there is a measurable subset Z of X with $\mu(Z) = 0$, such that, for every $a \in A \setminus Z$, there is a sequence of positive integers $n_0 < n_1 < n_2 < \cdots$, such that $T^{n_i}(a) \in A$ for every i.

Let G act on $\mathrm{LO}(G)$ by conjugation. That is $G \curvearrowright \mathrm{LO}(G), g \cdot P = g^{-1}Pg$. Note that whenever P is the positive cone of the left-order $<$, the set $g^{-1}Pg$ is the positive cone of $<_g$, defined by setting

$$x <_g y \iff gxg^{-1} < gyg^{-1} (\iff xg^{-1} < yg^{-1}).$$

Theorem 2.39 (Morris [16]). *Suppose that G is a countable left-orderable group. If G is amenable, then it is locally indicable.*

Proof. Let G be a left-orderable amenable group, and let μ be a G-invariant probability measure on the compact space $\mathrm{LO}(G)$.

For all $g_1, \ldots, g_n \in G$, let

$$A_{g_1,\ldots,g_n} = \bigcap_{n=1}^{n-1} V_{g_i^{-1} g_{i+1}}.$$

Note that

$$P \in A_{g_1,\ldots,g_n} \iff g_1 <_P g_2 <_P \cdots <_P g_n.$$

Let $g \in G$ and apply Poincaré recurrence theorem with $X = \mathrm{LO}(G)$, $T(P) = gPg^{-1}$, and $A = A_{g_1,\cdots,g_n}$. Then, there is some null $Z_{g,g_1,\ldots,g_n} \subseteq \mathrm{LO}(G)$ such that for all $P \in A \setminus Z_{g,g_1,\ldots,g_n}$, there exists integers $n_1 < n_2 < \cdots$ such that

$$g^{n_i} P g^{-n_i} \in A.$$

This means that for all $i \in \mathbb{N}$,

$$g^{-n_i} g_1 g^{n_i} < g^{-n_i} g_2 g^{n_i} < \cdots < g^{-n_i} g_n g^{n_i}$$

Therefore, for all $i \in \mathbb{N}$,

$$g_1 g^{n_i} < g_2 g^{n_i} < \cdots < g_n g^{n_i}.$$

Now let

$$Z = \bigcup_{\substack{g \in G \\ (g_1,\dots,g_n) \in G^{<\omega}}} Z_{g,g_1,\dots g_n}.$$

Note that $\mu(Z) = 0$ since it is countable union of null sets. In particular, $\mathrm{LO}(G) \setminus Z \neq \emptyset$. So, let P be (the positive cone of) a left-order on G such that $P \notin Z$. Fix, two positive elements $g, h \in G$. That is, $1 <_P g, h$. Clearly, $P \in A_{1,h} = V_h$. Since $P \notin Z$, there exists an increasing sequence of integers $n_1 < n_2 < \cdots$ such that

$$g^{n_i} <_P hg^{n_i}$$

for all $i \in \mathbb{N}$. Clearly, since $1 < g$, we must have $g <_P g^2 <_P \cdots <_P g^n$. Transitivity implies that $g <_P hg^{n_i}$ for all $i \in \mathbb{N}$. This proves that P is Conradian, therefore G is locally indicable. $\square$

The proof of Theorem 2.39 inspires the following definition.

Definition 2.40. A left-order $P \in \mathrm{LO}(G)$ is **recurrent** if for every (finite) increasing sequence $g_1 <_P g_2 <_P \cdots <_P g_n$ of elements of G and for every $g \in G$, there exists a sequence of positive integers $n_1 < n_2 < \cdots$, such that for all $i \in \mathbb{N}$,

$$g_1 g^{n_i} < g_2 g^{n_i} < \cdots < g_n g^{n_i}.$$

Also we can abstract the following:

Proposition 2.41. *Every recurrent order is Conradian.*

Note that the converse of Proposition 2.41 is not true. We discuss a counterexample below.

Example 2.42. Consider the action$\mathbb{Z} \overset{a}{\curvearrowright} \mathbb{Q}^2$ by $k \cdot \vec{v} = T^k(\vec{v})$ with $T = \left(\begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right)$. Then let $G = \mathbb{Q}^2 \rtimes_a \mathbb{Z}$ so that we can sandwich G in the short exact sequence

$$1 \to \mathbb{Q}^2 \to G \to \mathbb{Z} \to 1$$

Let α be an irrational positive slope and let P_α be the positive cone of $\mathbb{Q}^2$ by declaring positive those vectors lying above the line $y = \alpha x$. In particular, all vectors in the second quadrant are positive. Last, define a left-ordering $<$ on G using the sandwich trick from P_α and the reverse bi-order on the integers. Clearly $<$ is Conradian. However, there are $1_G < g, h$ such that there is $N \in \mathbb{N}$ with

$$1_G < h^{-k} g h^k \qquad \text{for all } k > N.$$

To see this, let $h = (\vec{0}, -1)$ and $g = (\vec{u}, 0)$ for some $\vec{u} \in P_\alpha$ lying in the first quadrant. A routine computation in G yields that

$$h^{-k} g h^k = (T^k(\vec{u}), 0).$$

However, for sufficiently large $k \in \mathbb{N}$ then vector $T^k(\vec{u}) \notin P_a$ as it approaches the horizontal axis, therefore $(T^k(\vec{u}), 0) < 1_G$.

3. A CLOSER LOOK AT LO(G)

3.1. There is a characterization of those left-orderable groups for which LO(G) is finite.

Theorem 3.1. *Let G be a left-orderable group. If G admits only finitely many left-orderings, then G admits a unique sequence of normal subgroups*

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each G_{i+1}/G_i is torsion-free Abelian of rank one, and no quotient G_{i+2}/G_i is bi-orderable. Conversely if G admits such a sequence of normal subgroups, then LO(G) *is finite and consists of exactly 2^n points.*

Example 3.2. We note that $\mathbb{Q}$ and the fundamental group of the Klein bottle $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ satisfy the assumption of Theorem 3.1. Recall that

$$\{1\} \lhd \langle b \rangle \lhd K$$

and $K/\langle b \rangle \cong \mathbb{Z}$.

Theorem 3.3. *If a group G is locally indicable, then* LO(G) *is either finite or uncountable. Therefore, $|\mathrm{LO}(G)| = 2^{\alpha}$ for $\alpha = 0, 1, \ldots, \omega$.*

We need extend our discussion before proving Theorem 3.3. We start by presenting the following Lemma.

Lemma 3.4. *Suppose that G is bi-orderable and fits into a short exact sequence*

$$1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$$

where both K and H are torsion-free Abelian of rank one. Then $|\mathrm{LO}(G)| = 2^{\aleph_0}$.

Before proving Lemma 3.4 it is useful to recall an example of such bi-orderable group G that satifies the assumption. Recall form that the solvable Baumslag Solitar's group $BS(1, \ell)$, for $\ell \geq 2$ fits into the short exact sequence

$$1 \to \mathbb{Z}[1/\ell] \to G \to \mathbb{Z} \to 1.$$

For $G = \mathrm{BS}(1, \ell) = \langle a, b \mid aba^{-1} = b^{\ell} \rangle$ we have the embedding

$$\begin{aligned} \rho \colon G &\to \mathrm{Homeo}_+(\mathbb{R}) \\ a &\mapsto \rho_a(x) = \ell x \\ b &\mapsto \rho_b(x) = x + 1, \end{aligned}$$

Then, for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we can define a positive cone $P_\alpha \in \mathrm{LO}(G)$ by setting

$$P_\alpha = \{g \in G \mid \rho(g)(\alpha) > \alpha\}.$$

In the proof of Lemma 3.4 we use the same idea.

Proof of Lemma 3.4. If G is Abelian, then G is a torsion-free abelian group of rank 2, so $|\mathrm{LO}(G)| = 2^{\aleph_0}$.

Suppose that G is not Abelian and fix a bi-order $<$ on G. Since K has rank 1, it can be identified with some subgroup of $\mathbb{Q}$. We may assume that the bi-order K induced by $<$ coincide with the obvious ordering of $\mathbb{Q}$.

Now let the map $\phi\colon H \to \mathrm{Aut}(K), \phi(h)(x) = gxg^{-1}$ for any $q(g) = h$.

Claim 5. The value of ϕ is independent of the choice of $g \in q^{-1}(h)$.

Proof of Claim 5. If $q(g) = q(g') = h$, then $q(g^{-1}g') = 1_H$, therefore $g^{-1}g' \in \ker(q) = K$. But, since K is abelian we have

$$g^{-1}g'x(g^{-1}g')^{-1} = x$$

for all $x \in K$. Therefore, we have $g'x(g')^{-1} = gxg^{-1}$. $\square$

Also note that each $\phi(h)$ preserves the order of K. Therefore, by Hion's Lemma, for all $h \in H$, there is some positive $r_h \in \mathbb{Q}$. Such that $\phi(h)(k) = r_h k$, for all $k \in K$.

We claim[5] that G is isomorphic to the semi-direct product $K \rtimes_\phi H$.

Now we define an embedding $\rho\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ as follows: Fix a nonzero $k \in K$ and set

$$\rho(k)(x) = x + 1.$$

Then for all $k' \in K$, since $K \subseteq \mathbb{Q}$, there are some $m, n \in \mathbb{Z}$ such that $nk' = mk$, so set

$$\rho(k')(x) = x + \tfrac{m}{n}.$$

If $H \cong \langle h \rangle$ with $\phi(h)(k) = rk$ for all $k \in K$, where $r = r_h \in \mathbb{Q}$, set

$$\rho(h)(x) = rx$$

This induces a map $\rho\colon G \to \mathrm{Homeo}_+(\mathbb{R})$. Then we can define a positive cone $P_\alpha \in \mathrm{LO}(G)$ for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by setting

$$P_\alpha = \{g \in G \mid \rho(g)(\alpha) > \alpha\}.$$ $\square$

Proof of Theorem 3.3. Fix a Conradianl left-order $<$ on G and suppose that $\mathrm{LO}(G)$ is countable. By Lemma..., it is clear that there are finitely many convex jumps in $(G, <)$, and each for the jump has an Archimedean quotient. So let

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

be the convex subgroups in $(G, <)$, with G_{i+1}/G_i torsion-free abelian groups of rank 1. Since $\mathrm{LO}(G)$ is countable we must have that G_{i+2}/G_i is not bi-orderable by Lemma 3.4, therefore G is Tararin and $\mathrm{LO}(G) = 2^n$. $\square$

Theorem 3.5 (Linnell)**.** *For all countable left-orderable groups G, we have $|\mathrm{LO}(G)| = 2^n$ for some $n \in \mathbb{N}$ or $|\mathrm{LO}(G)| = 2^{\aleph_0}$.*

[5]It boils down to show that $H \cong \mathbb{Z}$, which is a free group, so the short exact sequence above splits.

Proof. Suppose that G is a left-orderable group such that $\mathrm{LO}(G)$ is countably infinite. To simplify the notation let $X = \mathrm{LO}(G)$.

We have a G-action on X by homeomorphisms defined by $g \cdot P = g^{-1}Pg$. For all countable ordinal α, let X^α be the α-derived space after applying the Cantor-Bendixon derivative with a transfinite induction. This will restrict to G-actions on X^α for all $\alpha < \omega_1$. By Lemma D.3, there is an ordinal β such that X^β is finite and nonempty. Then there is a normal $H \lhd G$ with $[G : H] < \infty$ such that the action of $H \curvearrowright X^\beta$ fixes all positive cones in X^β. In particular, there is a positive cone $P \in \mathrm{LO}(G)$ such that $h^{-1}Ph = P$ for all $h \in H$. Therefore $P \cap H$ is a bi-order on H. (See Exercise 1.) It follows that $P \cap H$ is a Conradian left-order on H.

Lemma 3.6. *Suppose that G is a left-orderable group and $H < G$ with $[G : H] < \infty$. Then if $P \in \mathrm{LO}(G)$ such that $P \cap H$ is a Conradian left-orderable, then P is Conradian.*

Proof. Towards contradiction assume that assume that P is not)the positive cone of) a Conradian left-order. So, there are $a, b \in G$ with $1_G <_P a$ and $1_G <_P b$ and $ba^n <_P a$ for all positive n. Since $1_G <_P a$, we have $b <_G ba$. Next, note that $ba^n < a$ implies that

$$b^2a^n <_P ba <_P a$$

for all $n \geq 0$. Inductively, we can prove that for all $m \geq 1$ and $n \geq 0$

$$b^m a^n <_P a.$$

However, since $[G : H] < \infty$, there is $k > 0$ such that a^k and b^k are in H. It follows that

$$b^k(a^k)^n <_P a <_P a^n.$$

This contradicts the assumption that $P \cap H$ is Conradian. $\square$

So G admits a Conradian order, therefore it is locally indicable by Corollary 2.18. Now the result follows from Theorem 3.3. $\square$

3.2. **The topology of $\mathrm{LO}(G)$.**

Theorem 3.7. $\mathrm{LO}(\mathbb{Z}^2)$ *has no isolated points.*

Proof. The topological space $\mathrm{LO}(G)$ has a sub-basis consisting of the clopen sets $V_g = \{P \in \mathrm{LO}(G) \mid g \in P\}$, for $g \in G$. Given $g_1, \ldots, g_n \in \mathbb{Z}^2$ and a positive cone P in the basic open set $V_{g_1} \cap \cdots \cap V_{g_n}$ it is very easy to find $Q \neq P$ such that $Q \in V_{g_1} \cap \cdots \cap V_{g_n}$. (Argue by picture.) $\square$

Theorem 3.8 (Rivas [20])**.** *If G, H are left-orderable, then* $\mathrm{LO}(G * H)$ *has no isolated points. In particular, no positive cone in* $\mathrm{LO}(\mathbb{F}_2)$ *is isolated.*

Proof. For the case $G \cong H \cong \mathbb{Z}$, see [5, Theorem 10.14]. $\square$

Theorem 3.9 (Brouwer)**.** *Every non-empty perfect, compact, Polish, and zero-dimensional topological space is homeomorphic to* $2^{\mathbb{N}}$.

It follows that $\mathrm{LO}(\mathbb{Z}^2)$ and $\mathrm{LO}(\mathbb{F}_2)$ are homeomorphic. This rather surprising – while we have a complete understanding of the positive cones on $\mathbb{Z}^2$, we do not have any feasible description of $\mathrm{LO}(\mathbb{F}_2)$. There is a genuine difference between the two spaces that cannot be explained by topology.

EXERCISES

Exercise 14. Prove Lemma 2.11 and Lemma 2.11

Exercise 15. An order $<$ of an Abelian group A is Archimedean if and only if there are no nontrivial $<$-convex subgroups in A.

Exercise 16. Prove that if G admits a Ponzi scheme, then the growth of G is exponential. Deduce that there is no Ponzi scheme on $\mathbb{Z}^n$ for all $n \geq 1$.

4. COUNTABLE BOREL EQUIVALENCE RELATIONS

4.1. **Preliminaries in DST.** Recall that a topological space (X, τ) is Polish if and only if it admits a complete separable metric. Whenever (X, τ) is Polish we denote by $\mathcal{B}(\tau)$ the σ-algebra of Borel subset of X.

Definition 4.1. Let $\mathcal{B}$ be a σ-algebra on a set X. Then $(X, \mathcal{B})$ is a **standard Borel space** if and only if there is a Polish topology τ on X such that $\mathcal{B} = \sigma(\tau)$.

To what extent does the topology determine the Borel structure? None, as shown by the following theorem.

Theorem 4.2. *Let (X, τ) be a Polish space and let $Y \in B(T)$. Then there exists a finer Polish topology $T_Y \supseteq T$ on X such that $\mathcal{B}(\tau_Y) = \mathcal{B}(\tau)$ and Y is clopen in (X, τ_Y).*

Corollary 4.3. *If $(X, \mathcal{B})$ is a standard Borel space, and $Y \in \mathcal{B}$, then $(Y\mathcal{B} \restriction Y)$ is a standard Borel space.*

Here, $\mathcal{B} \restriction Y$ just means $\{A \in \mathcal{B} : A \subseteq Y\}$. This follows just because we can make Y into a clopen set, and $Y \subseteq X$ is Polish with the subspace topology.

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be Borel spaces. A map $f \colon X \to Y$ is said to be **Borel** if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$. The following is a consequence of a deep result in descriptive set theory known as Suslin's theorem.

Theorem 4.4. *If X, Y are standard Borel spaces and $f \colon X \to Y$, then the following are equivalent:*

(1) *f is Borel;*
(2) $\mathrm{graph}(f)$ *is a Borel subset of $X \times Y$.*

We also highlight the following important theorem.

Theorem 4.5 (Luzin-Suslin)**.** *Let X, Y be standard Borel spaces and $f \colon X \to Y$ be Borel. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel.*

A *Borel isomorphism* between X, Y is a bijection $f\colon X \to Y$ such that both f, f^{-1} are Borel.

Theorem 4.6 (Kuratowski)**.** *There is a unique uncountable standard Borel space up to Borel isomorphism.*

4.1.1. *Uniformization of Borel sets with small sections.*

Definition 4.7. Given two standard Borel spaces X, Y and $P \subseteq X \times Y$, a *uniformization* of P is a subset $P^* \subseteq P$ such that for all $x \in X$,

$$\exists y\, (x, y) \in P \qquad \Longleftrightarrow \qquad \exists! y\, (x, y) \in P^*.$$

In essence, P^* is the graph of a function f where $f\colon \operatorname{proj}_X P \to Y$ such that $f(x) \in P_x = \{y \in Y : (x, y) \in P\}$ holds. Clearly, the axiom of choice implies that every $P \subseteq X \times Y$ has a uniformization. Now suppose that X, and Y are standard Borel spaces, and $P \subseteq X \times Y$ is Borel. Does P necessarily have a Borel uniformization?

Proposition 4.8. *Let $P \subseteq X \times Y$ with X, Y standard Borel spaces. If P has a Borel uniformization P^*, then* $\operatorname{proj}_X(P)$ *is Borel.*

Proof. Suppose that P^* is a Borel uniformization. Since $\operatorname{proj}_X\colon P^* \to X$ is injective, it follows that $\operatorname{proj}_X(P^*)$ is Borel by Theorem 4.5. But $\operatorname{proj}_X(P) = \operatorname{proj}_X(P^*)$. $\square$

Corollary 4.9. *There exists a Borel $P \subseteq X \times Y$ with X, Y standard Borel spaces, and with no Borel uniformization.*

Theorem 4.10 (Luzin–Novikov)**.** *Suppose X, Y are standard Borel spaces and $P \subseteq X \times Y$ is a Borel subset such that P_x is countable (perhaps empty) for all $x \in X$. Therefore,*

(1) *P has Borel uniformization and thus* $\operatorname{proj}_X(P)$ *is Borel.*
(2) *Moreover, we can express $P = \bigcup_{n \in \mathbb{N}} P_n$, where each P_n is the Borel graph of a partial function; i.e. if $P_n(x, y)$ and $P_n(x, z)$, then $y = z$.*

Corollary 4.11. *Suppose that X, Y are standard Borel spaces and $f\colon X \to Y$ is a countable-to-one Borel map. Then $f(X)$ is Borel, and there exists a Borel map $g\colon f(X) \to X$ such that $f \circ g(y) = y$ for all $y \in f(X)$.*

Proof. Let $P = \operatorname{graph}(f)^- = \{(f(x), x) : x \in X\}$. By Theorem 4.10 there is a Borel uniformization $P^* \subseteq P$ and so $f(X)$ is Borel and $P^* = \operatorname{graph}(g)$ for some partial Borel function $g\colon f(X) \to X$. Then, for any $y \in f(X)$, we have $(y, g(y)) \in \operatorname{graph}(g) \subseteq \operatorname{graph}(f)^-$. It follows that $(g(y), y) \in \operatorname{graph}(f)$, thus $f(g(y)) = y$. $\square$

Theorem 4.12 (Feldman–Moore)**.** *Let E be a countable Borel equivalence relation on X. Then there is a countable group G and a Borel action of G on X such that $E = E_G^X$. Moreover, G and the action can be chosen so that*

$$x\, E\, y \qquad \Longleftrightarrow \qquad \exists g \in G (g^2 = 1 \text{ and } g \cdot x = y).$$

Proof. By Luzin–Novikov theorem we can write E as a disjoint union of (graphs of) Borel partial functions $E = \bigsqcup_{n\in\mathbb{N}} \mathrm{graph}(f_n)$. Without loss of generality we may assume that f_n's are injective. If not, then note that $E^{-1} = \{(y,x) : (x,y) \in E\} (= E)$ can be written as a disjoint union of (graphs of) Borel partial functions $E^{-1} = \bigsqcup_{n\in\mathbb{N}} \mathrm{graph}(h_n)$. So, after possibly replacing $(f_n)_{n\in\mathbb{N}}$ with $(f_n \cap h_m^{-1})_{n,m\in\mathbb{N}}$, we may assume that each f_n is injective. For, if $(x_0, y), (x_1, y) \in f_n \cap g_m^{-1}$, then it must hold that $x_0 = x_1$ because $g_m(y) = x_0$ and $g_m(y) = x_1$.

Next, we can write $E = \Delta(X) \cup \bigsqcup_n \mathrm{graph}(f_n)$ and, moreover, $\mathrm{dom}(f_n)$ and $\mathrm{ran}(f_n)$ are disjoint. Since X (and X^2) are Hausdorff and second countable we can write $X^2 \setminus \Delta(X) = \bigcup U_n \times V_n$ for some disjoint open sets $U_n, V_n \subseteq X$. Then,

$$E = \Delta(X) \cup (E \cap \bigcup U_n \times V_n) = \Delta(X) \cup \bigcup_{n,m} f_n \cap (U_m \times V_m).$$

Therefore, replace $(f_n : n \in \mathbb{N})$ with $(f_n \cap (U_m \times V_m) : n, m \in \mathbb{N})$ if needed.

We extend each partial function f_n to a Borel involution $g_n \colon X \to X$ by setting

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \in \mathrm{dom}(f_n), \\ f_n^{-1}(x) & \text{if } x \in \mathrm{ran}(f_n), \\ x & \text{otherwise.} \end{cases}$$

Let $G = \langle g_n : n \in \mathbb{N} \rangle$. It follows that

$$x \mathrel{E} y \iff \exists n (g_n(x) = y) \iff x \mathrel{E_G^X} y. \qquad \square$$

4.2. **Smooth equivalence relations.**

Theorem 4.13 (Silver [21])**.** *If E is a Borel equivalence relation with uncountable many classes, then $=_{\mathbb{R}} \leq_B E$.*

Definition 4.14. A Borel equivalence relation is smooth if and only if $E \leq_B =_X$ for some (equivalently every) uncountable standard Borel space X.

A **transversal** for E is a set $T \subseteq X$ which intersect each E-class in exactly one point. A **selector** for E is a function $s \colon X \to X$ such that $s(x) \mathrel{E} x$ and $x \mathrel{E} y \implies s(x) = s(y)$.

Theorem 4.15. *Let E be a countable Borel equivalence relation on X. The following are equivalent:*

(i) *E is smooth;*
(ii) *E admits a Borel transversal;*
(iii) *E admits a Borel selector.*

Proof. $(i) \implies (ii)$ Let Y be a Polish space and $f \colon X \to Y$ witness that E is smooth. Clearly f is a countable-to-one function, therefore let $g \colon f(X) \to X$ be a Borel right-inverse of f as in Corollary 4.11. Then the set $g(f(X))$ is a Borel transversal for E. $(ii) \implies (iii)$ Let $T \subseteq X$ be a Borel transversal for T. Define

$s\colon X \to X$ by setting $s(x)$ equal to the unique $t \in T$ such that $t\,E\,x$. It is clear that s is a selector for E and $\mathrm{graph}(s) = \{(x,y) \in E : y \in T\}$ is a Borel set, thus s is Borel.

$(iii) \implies (i)$ Any Borel selector $s\colon X \to X$ for the equivalence relation E is a Borel reduction from E to id_X. □

Note that this applies only to countable Borel equivalence relations. For example, there exists a smooth Borel equivalence relation with no Borel transversal. To see this, let X, Y be such that X and Y are standard Borel spaces and $A \subseteq X \times Y$ is Borel with $\mathrm{proj}_X A$ non-Borel. Then A is also a standard Borel space, and we can define a Borel equivalence relation E on by declaring $(x_1, y_1)\,E\,(x_2, y_2)$ if and only if $x_1 = x_2$. Then map $(x, y) \mapsto x$ is a Borel reduction from E to $=_X$, and so E is smooth. So suppose that $T \subseteq A$ is a Borel transversal for E. But then $\mathrm{proj}_X\colon X \times Y \to X$ is injective on T. Hence $\mathrm{proj}_X(T) = \mathrm{proj}_X A$ is Borel, contradicting the property of A.

Theorem 4.15 has several applications. For example the Vitali equivalence relation defined on $\mathbb{R}$ by setting $x\,E_v\,y \iff x - y \in \mathbb{Q}$ is not smooth because it cannot have a transversal that is Lebesgue measurable and therefore neither a Borel one.

For $x, y \in 2^{\mathbb{N}}$ define

$$x\,E_0\,y \qquad \iff \qquad \exists m \forall n \geq m (x_n = y_n)$$

Proposition 4.16. *E_0 is not smooth.*

Proof. Let μ be the usual uniform product probability measure on $2^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let $\pi_n\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel bijection $(x_0, \dots, x_n, \dots) \mapsto (x_0, \dots, 1 - x_n, \dots)$ flipping just the n-th entry.

Let $G = \bigoplus_{n \in \mathbb{N}} C_n$, where $C_n = \langle \pi_n \rangle$. Then let $G \curvearrowright 2^{\mathbb{N}}$ as a group of measure preserving transformations E_0 is the orbit equivalence relation generated by such action. Furthermore, G acts freely on $2^{\mathbb{N}}$, i.e., if $g \neq 1_G$, then $g \cdot x \neq x$ for all $x \in X$.

Now suppose that E_0 is smooth. Then there exists a Borel transversal $T \subseteq 2^{\mathbb{N}}$ for E_0. Since G acts freely, $2^{\mathbb{N}} = \bigsqcup_{g \in G} gT$.

Note that if $g_1 \neq g_2$, then $g_1 T$ and $g_2 T$ are disjoint because the action is free. To see this, note that if $t_1, t_2 \in T$ with $g_1 t_1 = g_2 t_2$ implies $g_2^{-1} g_1 t_1 = t_2$ and therefore $t_1 = t_2$ because T is a transversal.

Since T is Borel, T is Borel measurable, therefore $\mu(gT) = \mu(T)$ for all $g \in G$. Therefore,

$$\sum_{g \in G} \mu(gT) = \mu(2^N) = 1,$$

which is a contradiction. □

Proposition 4.17. *If a countably infinite group G acts freely on X and preserves a probability measure μ, then E_G^X is not smooth.*

Define the tail equivalence relation on $2^{\mathbb{N}}$ by setting

$$x\, E_t\, y \qquad \iff \qquad \exists m \exists k \forall n (x_{m+n} = y_{k+n})$$

for any two $x, y \in N$.

Proposition 4.18. *Suppose that $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X. If F is smooth, then E is also smooth.*

Proof. Suppose that F is smooth. By Theorem 4.15 there is a Borel transversal $T \subseteq X$ for F. Also, by Feldman-Moore's theorem, there exists a Borel action of a countable group $G = \{g_n : n \in \mathbb{N}\}$ on X such that $F = E^X_G$. Hence we can define a Borel selector $s\colon X \to X$ for E by setting $s(x) = g_N \cdot t$, where $t \in T \cap [x]_F$ and $N = \min\{n \in \mathbb{N} : g_N \cdot t\, E\, x\}$. $\square$

Corollary 4.19. *E_t is not smooth.*

Proof. Note that $E_0 \subseteq E_t$. $\square$

Let E, F be countable Borel equivalence relations defined on the standard Borel spaces X, Y, respectively. We say that E is **weakly Borel reducible** (in symbols, $E \leq_{wB} F$) to F if there is a countable-to-one Borel function $f\colon X \to Y$ such that $x_1\, E\, x_2 \implies f(x_1)\, F\, f(x_1)$.

Exercise 17. Prove that if E, F are countable Borel equivalence relations so that $E \leq_{wB} F$, and F is smooth, then E is smooth as well.

4.3. **The generic obstruction to smoothness.**

Definition 4.20. An equivalence relation E on a Polish space X is *generically ergodic* if every invariant Baire measurable subset of X is either meager or comeager.

Proposition 4.21. *Let E be a generically ergodic equivalence relation on a Polish space X and Y be any Polish space. If $f\colon X \to Y$ be a Baire measurable function such that $x\, E\, y \implies f(x) = f(y)$. Then there is a comeager $C \subseteq X$ such that $f \restriction C$ is constant.*

Proof. Given a countable basis $\{U_n : n \in \mathbb{N}\}$ for Y, the preimage $f^{-1}(U_n)$ has the Baire property and is E-invariant. For each $n \in \mathbb{N}$, denote by C_n the unique comeager set in $\{f^{-1}(U_n), X \setminus f^{-1}(U_n)\}$. Then f is constant on $C := \bigcap_{n \in \mathbb{N}} C_n$. $\square$

Theorem 4.22. *Let E be an equivalence relation on a Polish space X. If E is generically ergodic and every E-class is meager, then E is not smooth.*

Proof.] If $f\colon X \to 2^{\mathbb{N}}$ is Borel and $x\, E\, y \iff f(x) = f(y)$, then the preimage of each point is an E-class, therefore is meager. However, f is Baire measurable and the preimage of some element must be comeager, a contradiction. $\square$

The condition of being generically ergodic looks difficult to handle. However, for orbit equivalence relation induced by continuoius group actions we have a much better understanding of if.

Proposition 4.23. *Let G be a group acting on a Polish space X continuously. The following are equivalent:*

(1) *E_G^X is generically ergodic;*
(2) *There is a dense orbit.*

Corollary 4.24. *If a Polish group G acts continuously on a Polish space X such that every orbit is meager and there is a dense orbit, then E_G is not smooth.*

Corollary 4.25. *If G is a Polish group and $\Gamma < G$ is a dense discrete subgroup. Then the orbit equivalence relation E_Γ^G induced by left (right) multiplication is not smooth.*

It follows in particular that the Vitali equivalence relation is not smooth. (Why?) We conclude this section with the following useful proposition.

Definition 4.26. Let G be a countable (discrete) group and X a topological space. A *continuous action* of G on X is a continuous function

$$\begin{aligned} a\colon G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that $1_G \cdot x = x$, and $g \cdot (h \cdot x) = (hg) \cdot x$, for all $g, h \in G$, $x \in X$. Moreover, for any element $x \in X$ we let $G \cdot x = \{g \cdot x : g \in G\}$ be the *G-orbit* of x.

Proposition 4.27. *Let $G \curvearrowright X$ continuously. If $A \subseteq X$ is G-invariant, then so is $\overline{A}$.*

Proof. Let $g \in G$ and $x \in \overline{A}$. We want to show that show that $g \cdot x \in \overline{A}$. Since $x \in \overline{A}$ there exists a sequence $\{x_n : n \in \mathbb{N}\}$such that $x_n \to x$. Since G acts continuously, $g \cdot x_n \to g \cdot x$. Then since A is G-invariant, $\{g \cdot x_n : n \in \mathbb{N}\}$ is a sequence in A converging to $g \cdot x$. Thus $g \cdot x \in \overline{A}$. $\square$

Let G act on X continuously. A *minimal (invariant) set* for the action is a closed, G-invariant subset $M \subseteq X$ such that: if $C \subseteq X$ is any closed, G-invariant set and $C \subseteq M$ then $C = \emptyset$ or $C = M$.

Proposition 4.28. *Let $\emptyset \neq M \subseteq X$. The following are equivalent:*

(i) *M is minimal;*
(ii) *For every $x \in M$, $G \cdot x$ is dense in M.*

Proof. **(i) $\Rightarrow$ (ii):** Suppose that M is minimal and let $x \in M$. Then $\overline{G \cdot x}$ is closed and invariant set and $\overline{G \cdot x} \subseteq M$. Therefore $\overline{G \cdot x} \subseteq M$, since M is minimal.

(ii) $\Rightarrow$ (i): If M is not minimal, then there is a closed invariant set $\emptyset \neq C \subsetneq M$. Then, if $x \in C$ we have $\overline{G \cdot x} \subseteq C$ and therefore $\overline{G \cdot x} \neq M$, showing that $G \cdot x$ is not dense.

$\square$

Proposition 4.29. *Suppose that G is a countable group acting by homeomorphisms on a compact Polish space X. Then there exist a minimal set $M \subseteq X$ for the action of G.*

Proof. Let $\mathcal{S} := \{A \subseteq X \mid A \text{ is nonempty, closed and } G\text{-invariant}\}$. We note that $\mathcal{S}$ is nonempty because $X \in \mathcal{S}$ and it is partially ordered under inclusion. A consequence of the compactness of X is that every chain $\{A_i\}_{i\in I}$ satisfies the finite intersection property. It follows that $\bigcap_{i\in I} A_i$ is nonempty and necessarily closed and G-invariant. Therefore, every chain in X admits has lower bound. Then, using Zorn's Lemma, there exists a minimal invariant set $M \subseteq X$ for the action $G \curvearrowright X$. □

Proposition 4.30. *Let X be a compact Polish space. Suppose that G is a countable group acting on X continuously. If E_G^X is smooth, then there exists a finite orbit.*

Proof. Suppose that E_G^X is smooth, and let $M \subseteq X$ be a minimal set for the action of G on X. Since E_G^X is smooth, so is the action E_G^M. Note that every G-orbit of every point of M is dense in M, so by Corollary 4.24, there must be $x_0 \in M$ such that $G \cdot x_0$ is nonmeager. However, $G \cdot x_0 = \bigcup_{g\in G}\{g \cdot x_0\}$ can be written as a countable union of nowhere dense set, unless there is some $g_0 \in G$ such that $\{g_0 \cdot x_0\}$ is open in M. Since every orbit is dense in M, it follows that M must in fact consist of a single orbit, precisely $M = G \cdot x_0$. But $M \subseteq X$ is countable and closed. Thus M is a compact space with the relative topology, and every point in M is isolated because G acts continuously. Consequently, M is discrete and compact. This is not possible if M is infinite. □

Let X be a G-space. A point $x_0 \in X$ is condensed if and only if x_0 is an accumulated point of its orbit.

Theorem 4.31. *For a G-space X we have*

(i) *E_G^X is smooth*
(ii) *There are no condensed points.*

Corollary 4.32. *Let G be a Polish group and $\Gamma \leq G$ be a countable discrete subgroup. Then E_Γ^G is smooth.*

4.4. **Condensation.** Let G be a countable discrete group. A Polish G-space is a Polish space X equipped with a continuous action $(g, x) \mapsto g \cdot x$ of G on X. For the orbit equivalence relation induced by$G$-spaces we have the following characterization of smoothness.

Proposition 4.33. *Suppose that G is a countable group and X is a Polish G-space. Then the following are equivalent:*

(1) *E_G^X is smooth.*
(2) *The are no condensed points in X.*

Proof. Suppose that E_G^X is smooth and let x be any element of X. Consider the closed G-invariant set $Y = \overline{G \cdot x}$. If E_G^X is smooth, then E_G^Y is also smooth. As $G \cdot x$ is a dense G-orbit in Y, the action $G \curvearrowright Y$ is generically ergodic, so there must be an isolated point in $x_0 \in Y$. The point x_0 cannot be an element of $Y \setminus G \cdot x)$ since these points are non-isolated by definition, and so $x_0 \in G \cdot x$. Now as the G-action is continuous, every point of $G \cdot x$, and in particular x itself, must be isolated in the subspace topology. It follows that x cannot be a condensed point.

On the other hand, suppose that no $x \in X$ is a condensed point. Then for every $x \in X$, the subspace topology on $G \cdot x$ is discrete, and since $G \cdot x$ is countable it is therefore Polish. By Alexandrov's theorem, $G \cdot x$ must be a G_δ set for all $x \in X$. (See [13, Theorem 3.11].) Further note that the saturation of an arbitrary open set $U \subset X$ is itself open, since the saturation can be written as a union of the sets $g \cdot U$ where $g \in G$, each of which is open since G acts continuously. This implies that the map $X \to F(X), x \mapsto \overline{G \cdot x}$ is Borel showing that E_G^X is Borel reducible to $=_{F(X)}$. (E.g., see [9, Exercise 5.4.8].) □

For a left-orderable group G denote by $\mathrm{LO}((G))$ the *free part of its conjugacy action*. That is, we set

$$\mathrm{LO}((G)) = \{P \in \mathrm{LO}(G) : \forall g \neq 1 (g^{-1}Pg \neq P)\}.$$

Note that for any $P \in \mathrm{LO}((G))$, the orbit $\mathrm{Orb}(P)$ is infinite.

Proposition 4.34. *If* $\mathrm{LO}((G)) \neq \emptyset$, *then* $E_{\mathsf{lo}}(G)$ *is not smooth.*

Proof. Suppose $P \in \mathrm{LO}((G))$. By Proposition 4.33 it suffices to show that P is condensed. Let $P \in \bigcap_{i=1}^n U_{g_i}$, which is a basic open neighborhood of P. And assume that $g_1 <_P \cdots <_P g_n$ without loss of generality. Then, we claim that

$$g_1^{-1} g_i g_1 \in P$$

for $i = 1, \dots, n$. For $i = 1$, it follows from the assumption that $P \in U_{g_1}$. For $i \geq 2$, $g_1 <_P g_i$ implies that $1 <_P g_1^{-1} g_i$, whence $1 <_P g_1^{-1} g_i g_1$ because P is a semigroup. Therefore, for $i = 1, \dots, n$, we have

$$g_i \in g_1 P g_1^{-1}.$$

This shows that $g_1 P g_1^{-1} \in \mathrm{Orb}(P) \cap \bigcap_{i=1}^n U_{g_i}$. Since $P \in \mathrm{LO}((G))$, we conclude that $g_1^{-1} P g_1 \neq P$, therefore P is condensed. □

4.5. **Baumslag-Solitar groups.** Fix an integer n. The Baumslag–Solitar group $\mathrm{BS}(1, n)$ is given by the presentation $\langle a, b \mid bab^{-1} = a^n \rangle$. There is an injective homomorphism $\rho\colon \mathrm{BS}(1, n) \to \mathrm{Homeo}_+(\mathbb{R})$ defined by setting

$$\rho(a)(x) = x + 1,$$
$$\rho(b)(x) = nx.$$

The following construction of left-orderings on $\mathrm{BS}(1, n)$ is due to Smirnov [**?**]. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we can define a corresponding $P_\alpha \in \mathrm{LO}\big(\mathrm{BS}(1, n)\big)$ by declaring

$$g \in P_\alpha \iff \rho(g)(\alpha) > \alpha.$$

Note that the map $\mathbb{R} \setminus \mathbb{Q} \to \mathrm{LO}(\mathrm{BS}(1,n)), \alpha \mapsto P_\alpha$ is injective. In fact, for different irrational numbers $\alpha < \beta$, we can choose some $g \in \mathrm{BS}(1,n)$ such that

$$\rho(g) = n^r x + \frac{s}{n^t}$$

with $r > 0$, and having fixed point $q = \frac{s}{n^t(1-n^r)}$ strictly between α and β. This choice is always possible because the range of ρ consists of precisely those functions of the form $f(x) = n^r x + \frac{s}{n^t}$ with $r, s, t \in \mathbb{Z}$. Moreover, we can always choose t and r so that the denominator $D = n^t(1-n^r)$ satisfies $1/|D| < \beta - \alpha$, therefore the interval (α, β) must contain a point of the form $\frac{m}{D}$, for some $m \in \mathbb{Z}$. Then, since $r > 0$, we have $\rho(g)(\alpha) < \alpha$ for all $\alpha < q$, and $\rho(g)(\beta) > \beta$ for all $\beta > q$. This means that $g \in P_\beta \setminus P_\alpha$, showing that the function $\alpha \mapsto P_\alpha$ is injective.

It is well known that the conjugacy action $\mathrm{BS}(1,n) \curvearrowright \mathrm{LO}(\mathrm{BS}(1,n))$ is not generically ergodic, however, with our new technique, we can easily prove the following:

Corollary 4.35. *For $n > 1$ and $G = \mathrm{BS}(1,n)$, the conjugacy relation $E_{\mathsf{lo}}(G)$ is not smooth.*

Proof. By Proposition 4.34 is suffices to prove that for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the positive cone P_α belongs the free part of the conjugacy action. To see this, assume that $hP_\alpha h^{-1} = P_\alpha$. Note that

$$\begin{aligned} hP_\alpha h^{-1} =& \{hgh^{-1} : g \in P_\alpha\} \\ =& \{x \in G : h^{-1}xh \in P_\alpha\} \\ =& \{x \in G : \rho(h^{-1}xh)(\alpha) > \alpha\} \\ =& \{x \in G : \rho(h)^{-1}\rho(x)\rho(h)(\alpha) > \alpha\} \\ =& \{x \in G : \rho(x)(\rho(h)(\alpha)) > \rho(h)(\alpha)\} = \\ =& P_{\rho(h)(\alpha)}. \end{aligned}$$

Therefore, we have $P_{\rho(h)(\alpha)} = P_\alpha$ and, since the map $\alpha \mapsto P_\alpha$ is injective, it yields that $\rho(h)(\alpha) = \alpha$. However, for every $h \neq 1$, the order-preserving homeomorphism $\rho(h)$ has only rational fixed points. Therefore, it must hold that h is the group identity as desired. □

5. Hyperfiniteness

Definition 5.1. A Borel equivalence relation E is **hyperfinite** if there is an increasing sequence of finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that $E = \bigcup_n F_n$.

Notice that all hyperfinite equivalence relations are in fact countable. It is easy to see that E_0 is hyperfinite, since $E_0 = \bigcup_n F_n$ where F_n is defined by

$$x \, F_n \, y \quad \Longleftrightarrow \quad \forall m \geq n \, (x(m) = y(m)).$$

Proposition 5.2. *If E smooth, then E is hyperfinite.*

Proof. Let $E = E_G^X$ for some countable group G. Suppose that $G = \{g_0, g_1, \dots\}$ with $g_0 = 1$. Since E is smooth, E admits a Borel selector $s \colon X \to X$ by Proposition 4.15–(iii). Then define an sequence of finite Borel equivalence relations F_n, for $n \geq 1$, by

$$x \, F_n \, y \qquad \iff$$
$$x \, E \, y \quad \text{and} \quad \left(x = y \quad \text{or} \quad \left(\bigvee_{i=0}^{n} (x = g_i \cdot s(x)) \text{ and } \bigvee_{i=0}^{n} (y = g_i \cdot s(x)) \right) \right).$$

Note that whenever $x \, E_n \, y$, then $x \, E \, y$ so $s(x) = s(y)$. Therefore, one sees that the F_n's are equivalence relations. Moreover, for any F_n, each class has at most $n + 1$ elements. Hence F_n is a finite Borel equivalence relation and $E = \bigcup_n F_n$. ☐

It is not too hard that the class of hyperfinite equivalence relations satisfies the following closure properties. The proof is left as an exercise.

Exercise 18. (i) If $E \subseteq F$ and F is hyperfinite, then E is hyperfinite.
(ii) If E is hyperfinite and $Y \subseteq X$ is Borel, then $E \restriction Y$ is hyperfinite.

Given an equivalence relation E on X, a subset $A \subseteq X$ is a *complete section* for E is it meets every E-class (i.e., for all $x \in X$, $[x]_E \cap A \neq \emptyset$).

Proposition 5.3. *Let E be a countable Borel equivalence relation on X. If $A \subseteq X$ is a Borel complete section for E such that $E \restriction A$ is hyperfinite, then E is hyperfinite.*

Proof. Suppose that $F_0 \subseteq F_1 \subseteq \cdots$ are finite Borel equivalence relations on A such that $E \restriction A = \bigcup_n F_n$. Let $E = E_G^X$ and $G = \{g_0, g_1, \dots\}$ such that $g_0 = 1_G$. For each $x \in X$, let $\ell(x)$ be the least index such that $g_{\ell(x)} x \in A$. Define R_n by

$$x \, R_n \, y \qquad \iff \qquad x = y \text{ or } \big(\ell(x) \leq n \text{ and } \ell(y) \leq n \text{ and } g_{\ell(x)} \cdot x \, F_n \, g_{\ell(y)} \cdot y \big).$$

Each R_n is a finite Borel equivalence relation. One can check that $E = \bigcup_n \widetilde{F}_n$, so E is hyperfinite. ☐

Proposition 5.4. *If F is hyperfinite and $E \leq_B F$, then E is hyperfinite.*

Proof. Suppose that E, F are countable Borel equivalence relations on X, Y, respectively, and suppose that F is hyperfinite. Let $f \colon X \to Y$ be a Borel reduction from E to F. Then $B = f(X)$ is a Borel subset of Y, since it is a countable-to-one image of X through f. It follows that $F \restriction B$ is a hyperfinite Borel equivalence relation.

Then let g be a Borel right-inverse of f (i.e., $g \colon B \to X$ such that $f(g(y)) = y$ for all $y \in B$) and let $A = g(B)$. Then A is a complete section for E and $E \restriction A \cong F \restriction B$. Since $E \restriction A$ is hyperfinite, so is E by Proposition 5.3. ☐

Example 5.5. The Vitali equivalence relation is hyperfinite. To see that it suffices to show that $E_v \restriction [0,\infty) \leq_B E_0$. Express every $x \in [0,\infty)$ as

$$x = a_0 + \frac{a_1}{2!} + \frac{a_2}{3!} + \cdots + \frac{a_n}{n!} + \cdots$$

where $a_0 = \lfloor x \rfloor$ and for $n > 0$, $a_n \in \{0, 1, \ldots, n\}$. Note that such an expression is uniquely determined. In fact, one can inductively define

$$\begin{aligned} a_0 &= \lfloor x \rfloor \\ a_n &= \left\lfloor \left(x - \sum_{k=0}^{n-1} \frac{a_k}{(k+1)!} \right) (n+1)! \right\rfloor \end{aligned}$$

This allows us to define a Borel function $f \colon \mathbb{R} \to \mathbb{N}^{\mathbb{N}}$ by $f(x)(n) = a_n$. Note that f is not continuous. To see that f is a reduction, suppose that $x \, E_v \, y$. Without loss of generality assume $x > y$ so that $r = x - y \in \mathbb{Q}^+$. Let m be the least integer such that $r = \frac{k}{(m+1)!}$ for some $k \in \mathbb{N}$. Then for all $n > m$, we have $f(x)(n) = f(y)(n)$. Conversely, assume $f(x) \, E_0 \, f(y)$. Then the expressions for x and y differ by only finitely many terms, and this implies that $x - y$ is rational.

Exercise 19. Let $E_0(\mathbb{N})$ denote eventual equality on the Baire space $\mathbb{N}^{\mathbb{N}}$. Prove that $E_0(\mathbb{N})$ is hyperfinite.

[Hint: fix a computable bijection $\langle -, - \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For any x in the Baire space, let $f(x) = f(x)(\langle n, k \rangle)$ be the $(k+1)$-th least significant digit of the binary expansion of $x(n)$.]

Lemma 5.6. *Suppose that E is a countable Borel equivalence relation on the Polish space X. Then the following are equivalent:*

(i) *E is smooth;*
(ii) *There is a sequence $(s_n)_{n \in \mathbb{N}}$ of Borel selectors for E such that $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(s_n)$.*
(iii) *There is a sequence $(T_n)_{n \in \mathbb{N}}$ of Borel transversals for E such that $X = \bigcup_{n \in \mathbb{N}} T_n$.*

Theorem 5.7. *Let E be a countable Borel equivalence relations on the Polish space X. Then the following are equivalent:*

(1) *E is hyperfinite;*
(2) *E is hypersmooth.*

Proof. (1) $\implies$ (2) Obvious, since any finite Borel equivalence relation is smooth.

(2) $\implies$ (1) Fix a increasing smooth equivalence relations $R_0 \subseteq R_1 \subseteq \cdots$ on X whose union is E. Let s_n be a Borel selector for R_n and suppose $R_n = E_{G_n}$ for some countable groups $G_n = \{g_n^{(k)} : k \in \mathbb{N}\}$. Define F_n by setting

$$\begin{aligned} x \, F_n \, y \iff \; & \exists m \leq n (x \, R_n \, y \text{ and } \exists k_0, \ldots, k_m, \ell_0, \ldots, \ell_m \leq n (\\ & x = g_0^{(k_0)} s_0 g_1^{(k_1)} s_1 \cdots g_m^{(k_m)} s_m(x) \text{ and } y = g_0^{(\ell_0)} s_0 g_1^{(\ell_1)} s_1 \cdots g_m^{(\ell_m)} s_m(y))). \end{aligned}$$

We claim that each F_n is an equivalence relations. (Exercise. The tricky part is transitivity.) It is clear from the definition that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$.

Claim 6. E= $\bigcup F_n$

Proof. First note that for all $x \in X$ and $j \in \mathbb{N}$, we have $x \, R_j \, s_j(x)$ and so there is some index $k_j(x)$ such that the corresponding element $g_j^{(k_j(x))} \in G_j$ gives $x = g_j^{(k_j(x))} s_j(x)$. So, if $x \, E \, y$, and $x \, R_m \, y$ for some $m \in N$, then let

$$n = \max\{m, k_0(x), \ldots, k_m(x), k_0(y), \ldots, k_m(y)\}.$$

It follows that $x \, F_n \, y$. $\square$

Moreover, note that all F_n's are finite. If $x \in [y]_{F_n}$, there is $m < n$ with $x \, R_m \, y$ and hence $s_m(x) = s_m(y)$. So x is completely determined by numbers $k_0, , k_m < n$ and $s_m(y)$, so it can only take finitely many distinct values. $\square$

Theorem 5.8. *If E is a countable Borel equivalence relation on Polish space X, then the following are equivalent:*

(a) *E is hyperfinite.*
(b) *There exists a Borel action $\mathbb{Z} \curvearrowright X$ such that $E = E_{\mathbb{Z}}^X$.*

Proof. $(a) \implies (b)$ Fix a Borel linear ordering $<$ on X. We can break X into countably many E-invariant pieces $X = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} X_n$ where each E-class $C \subseteq X_n$ has cardinality n. For each $n \in \mathbb{N}$, define $T \colon X_n \to X_n$ by setting

$$T(x) = \begin{cases} x^+ & \text{if } x \neq \max_< [x]_E \\ \min_< [x]_E & \textit{otherwise}. \end{cases}$$

Next suppose that $X = X_\infty$ without loss of generality. Write $E = \bigcup_{n \in \mathbb{N}} F_n$ as an increasing union $F_0 \subseteq F_1 \subseteq \cdots$ of finite Borel equivalence relation and assume that $F_0 = \mathrm{id}_X$. For each n, let $s_n \colon X \to X$ be a Borel selector for F_n.

Given two distinct $x, y \in X$ with $x \, E \, y$ let $n(x,y)$ be the maximum n such that $s_n(x) \neq s_n(y)$. Then define a binary relation on X by setting

$$x \prec y \iff s_n(x,y)(x) < s_n(x,y)(y)$$

for $x \, E \, y$. One can show that $\prec \restriction C$ is a linear order, for all equivalence class $C \in X/E$, and one can prove that such an ordering is always discrete. Then define $T \colon X \to X$ as the $\prec$-successor map where the order type is $\mathbb{Z}$ and similarly on the classes of order type ω and $-\omega$. (Use the any bijection between $\mathbb{Z}$ and ω.) Then E is the orbit equivalence relation induced on X by the infinite cyclic group $\langle T \rangle$.

$(b) \implies (a)$ Without loss of generality we can assume that E is aperiodic. This is harmless because the restriction of E on the periodic part is smooth, therefore hyperfinite.

A **vanishing sequence of markers** for E is a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$ such that

(1) Each A_n is a complete Borel section for E, i.e., $A_n \cap [x]_E \neq \emptyset$ for all $x \in X$;
(2) $\bigcap A_n = \emptyset$.

Lemma 5.9 (Markers lemma)**.** *Every aperiodic countable Borel equivalence relation admits a sequence of vanishing markers.*

Proof. Without loss of generality, assume that $X = 2^{\mathbb{N}}$. For each $x \in 2^{\mathbb{N}}$, and $n \in \mathbb{N}$, let $s_n(x)$ be the be the least $s \in 2^n$ (in the lexicographic order) such that $[x]_E \cap \mathcal{N}_s$ is infinite. Then define

$$x \in A_n \iff x \restriction n = s_n(x).$$

Then $(A_n : n \in \mathbb{N})$ is a decreasing sequence of Borel sets that intersects each E-class in infinitely many elements. To see this fix $n \in \mathbb{N}$ and $x \in X$. Then the set $[x]_E \cap N_{s_n(x)}$ is infinite. And for every $y \in [x]_E$, we have $s_n(y) = s_n(x)$ therefore $[x]_E \cap N_{s_n(x)} \subseteq A_n \cap [x]_E$. However the descending chain $A_0 \supseteq A_1 \supseteq \cdots$ need not be vanishing, but $A = \bigcap_{n \in \mathbb{N}} A_n$ intersects each equivalence class in at most one point. So $(A_n \setminus A : n \in \mathbb{N})$ forms a vanishing sequence of markers. $\square$

Let $T \colon X \to X$ be a Borel bijection which generates the $\mathbb{Z}$-action and let $<$ be the Borel partial order on X defined by

$$x < y \qquad \iff \qquad \exists n > 0\, (T^n(x) = y).$$

Then $<$ gives a $\mathbb{Z}$-ordering of every E-class. By the markers Lemma there is a vanishing sequence of markers $(A_n : n \in \mathbb{N})$ for E. Define

$$Y = \{x \in X : \text{there is } n \in \mathbb{N} \text{ such that } A_n \cap [x]_E \text{ has a least or greatest element}\}.$$

Note that Y is an E-invarant Borel subset and that $E \restriction Y$ is smooth because we can define a Borel selector. So without loss of generality, $Y = \emptyset$. Thus for each $x \in X$ and each $n \in \mathbb{N}$, $A_n \cap [x]_E$ is unbounded in both directions. Therefore, we can define an increasing union of finite Borel equivalence relations by

$$x\, F_n\, y \iff x = y \text{ or } (x\, E\, y \text{ and } A_n \cap [x, y] = \emptyset).$$

It is clear that every equivalence class is finite since any F_n-equivalent point must be between two elements $a, b \in A_n$ and since the classes of E are discretely ordered by $<$ the interval (a, b) is finite. Since $\bigcap_{n \in \mathbb{N}} A = \emptyset$, it is clear that $\bigcup_{n \in \mathbb{N}} F_n$ as desired. $\square$

6. Measures

Definition 6.1. Let $(X, \mathcal{B})$ be a standard Borel space. A **Borel measure** on $(X, \mathcal{B})$ is a function $\mu \colon \mathcal{B} \to [0, \infty)$ such that

(i) $\mu(\emptyset) = 0$;

(ii) μ is countably additive – given a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\sigma(X)$ we have $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

Further, we say that $M \subseteq X$ is **measurable** if there are Borel sets A, B with $A \subseteq M \subseteq B$, and $\mu(B \setminus A) = 0$. We then let $\mu(M) = \mu(A)$.

Exercise 20. Show that if (X, μ) is a standard Borel probability space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of Borel sets in X, then

(a) $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{N\to\infty} \mu(\bigcup_{n\le N} A_n)$.
(b) $\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{N\to\infty} \mu(\bigcap_{n\le N} A_n)$.

We call a Borel measure μ on X *finite* if $\mu(X) < \infty$. In case $\mu(X) = 1$, we call μ a *probability measure*.

Definition 6.2. Let E be a countable Borel equivalence relation on X. A measure μ on X is said to be E**-ergodic** if for every E-invariant Borel $A \subseteq X$, either $\mu(A) = 0$ or $\mu(X \smallsetminus A) = 0$. A measure μ is called *non-atomic* if every singleton has measure 0. Note that in case μ is non-atomic and E is countable $\mu([x_0]_E) = 0$ for all $x_0 \in X$.

Theorem 6.3. *Let E be an equivalence relation on a standard Borel space X. E is smooth if and only if there is a Borel separating family. I.e., a family $\{B_n : n \in \mathbb{N}\}$ of Borel subset of X such that*

$$x\,E\,y \iff \forall n \in \mathbb{N}\,(x \in B_n \iff y \in B_n).$$

Proof. Suppose that $f\colon X \to \mathbb{R}$ is a Borel function such that $x\,E\,y \iff f(x) = f(y)$. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis for $\mathbb{R}$, then $\{f^{-1}(U_n) : n \in \mathbb{N}\}$ is a Borel separating family.

On the other hand, if $\{B_n : n \in \mathbb{N}\}$ is a Borel separating family, then let

$$\begin{aligned} f\colon X &\to 2^{\mathbb{N}} \\ x &\mapsto \chi_{\{n\in\mathbb{N}: x\in B_n\}} \end{aligned}$$

is Borel and $x\,E\,y \iff f(x) = f(y)$. □

Proposition 6.4. *Let E be a countable Borel equivalence relation on a standard Borel space. If X has an E-ergodic, nonatomic measure μ, then E is not smooth.*

Proof. Suppose that E is smooth towards contradiction. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable Borel separating family for E. (I.e. $x\,E\,y \iff \forall n \in \mathbb{N}\,(x \in B_n \iff y \in B_n)$.) Each B_n is E-invariant, so either $\mu(B_n) = 0$ or $\mu(X \smallsetminus B_n) = 0$ by ergodicity. Then the set

$$C = \bigcap\{B_n : \mu(X \smallsetminus B_n) = 0\} \cap \bigcap\{X \smallsetminus B_n : \mu(B_n) = 0\}$$

has positive measure. However, $C = [x_0]_E$ for some $x_0 \in X$. It follows that $\mu(C) = 0$, since μ is non-atomic. □

Definition 6.5. Let G be a countable group G and X be a standard Borel space. The action $G \curvearrowright X$ is said to be uniquely ergodic if and only if there exists a unique G-invariant probability measure μ on X.

Theorem 6.6. *If $G \curvearrowright X$ is a uniquely ergodic probability measure μ, then μ is E_G^X-ergodic.*

Proof. Suppose that μ is not E_G^X-ergodic. Then there is an E_G^X-invariant Borel $A \subseteq X$ with $0 < \mu(A) < 1$ and $0 < \mu(X \setminus A) < 1$. We can define $\nu_1 \neq \nu_2$ on X by

$$\nu_1(Y) = \frac{\mu(Y \cap A)}{\mu(A)}$$
$$\nu_2(Y) = \frac{\mu(Y \cap (X \setminus A))}{\mu(X \setminus A)}.$$ □

Example 6.7. Let K be a separable, compact group. There exists a unique measure μ on K such that μ is invariant under the left-translation action $K \curvearrowright K$, the Haar measure. If $\Gamma < K$ is countable, dense subgroup, then $\Gamma \curvearrowright (K, \mu)$ is uniquely ergodic.

Example 6.8. Let $K = \prod_{n \in \mathbb{N}} C_n$ with $C_n = \mathbb{Z}/2\mathbb{Z}$. The group K is compact and the usual product probability measure is a Haar measure.

Then let $\Gamma = \bigoplus_{n \in \mathbb{N}} C_n$ the action ΓG is uniquely ergodic.

Corollary 6.9. *E_0 is not smooth.*

Proof. $E_0 = E_\Gamma^G$ in the example above. □

Example 6.10. Let $K = \prod_{n \in \mathbb{N}} C_n$ and $\Delta = \bigoplus_{n \geq 1} C_n$.

Then $A = \prod_{n \geq 1} C_n$ is E_Δ^K-invariant and $\mu(A) = \frac{1}{2}$. (Why? $[G : A] = 2$.)

Theorem 6.11. *Suppose that G acts freely on (X, μ) and μ is a G-invariant probability measure on X. If E_G^X is hyperfinite, then G is amenable.*

6.1. **Universality.**

Definition 6.12. A countable Borel equivalence relation F is said to be **universal** if $E \leq_B F$ for every cber E.

Theorem 6.13. *There is a universal cber E_∞.*

Note that if F is also universal then $F \leq_B E_\infty \leq F$, therefore E_∞ is unique up to Borel reducibility.

Proof of Theorem 6.13 . Consider the Polish space $(2^{\mathbb{N}})^{\mathbb{F}_\infty} = \{f \colon \mathbb{F}_2 \to 2^{\mathbb{N}}\}$ with the obvious product topology. Define E_∞ as the orbit equivalence relation induced by the shift action of $\mathbb{F}_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{F}_\infty}$; that is,

$$g \cdot f(x) = f(g^{-1}x),$$

for all $g \in \mathbb{F}_\infty$ and $f\colon \mathbb{F}_\infty \to 2^{\mathbb{N}}$.

Fix any cber E on a standard Borel space X. Without loss of generality let $E = E^X_{\mathbb{F}_\infty}$ for some Borel action $\mathbb{F}_\infty \curvearrowright X$ which descends from a surjection $\mathbb{F}_\infty \to G$. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of Borel sets $U_n \subseteq X$ which separates points.

Define a map $X \to (2^{\mathbb{N}})^{\mathbb{F}_\infty}$ by

$$f_x(a)(i) = 1 \iff a^{-1}x \in U_i.$$

Clearly f is injective. To see that f is a Borel reduction from $E = E^X_{\mathbb{F}_\infty}$ note that

$$\begin{aligned} g \cdot f_x(a)(i) \quad &\iff \quad f_x(g^{-1}a)(i) \\ &\iff \quad (a^{-1}g) \cdot x \in U_i \\ &\iff \quad a^{-1} \cdot (g \cdot x) \in U_i \\ &\iff \quad f_{gx}(a)(i) = 1. \end{aligned}$$

□

Next, for a Polish space X and a countable group G denote by $E(G, X)$ the countable Borel equivalence relation arising from the shift action of $G \curvearrowright X^G$. Therefore, for $X = 2$, one recovers the cber induced by the shift action of $G \curvearrowright 2^G = \mathcal{P}(G)$. We already showed that every cber $E \leq_B E(\mathbb{F}_\infty 2^{\mathbb{N}}) = E_\infty$. In the remainder of this section we can show the following:

Theorem 6.14. *$E(\mathbb{F}_2, 2)$ is a universal cber.*

The structure of the proof is as follows:

$$\begin{aligned} E(\mathbb{F}_\infty, 2^{\mathbb{N}}) &\leq_B E(\mathbb{F}_\infty, 2^{\mathbb{Z}\setminus\{0\}}) \\ &\leq_B E(\mathbb{F}_\infty \times \mathbb{Z}, 3) \\ &\leq_B E(\mathbb{F}_\infty \times \mathbb{Z} \times \mathbb{Z}_2, 2) \\ &\leq_B E(\mathbb{F}_2, 2) \end{aligned}$$

Proposition 6.15. *If G is a subgroup of H, then $E(G, X) \leq_B E(H, X)$.*

Proof. Define $X^G \to X^H$, $p \mapsto p^*$ by

$$p * (h) = \begin{cases} p(h) & h \in G \\ x_0 & \text{otherwise,} \end{cases}$$

for some fixed $x_0 \in X$. □

Corollary 6.16. $E(\mathbb{F}_\infty, 2) \leq_B E(\mathbb{F}_2, 2)$.

Proof. Let $\mathbb{F}_2 = \langle a, b \rangle$

$$\begin{aligned} h\colon \mathbb{F}_2 &\to \mathbb{Z} \\ a &\mapsto 1 \\ b &\mapsto 0. \end{aligned}$$

Let $H = \ker(h)$. Then $\{a^n : n \in \mathbb{N}\}$ is a set of coset representatives for H in $\mathbb{F}_2$. It follows from the Schreier Reidmaster theorem that $H \cong \mathbb{F}_\infty$ with basis $S = \{a^n b a^{-1} \mid n \in \mathbb{Z}\}$. $\square$

To prove Theorem 6.14

$$\begin{aligned}
E(\mathbb{F}_\infty, 2^{\mathbb{N}}) &\cong_B E(\mathbb{F}_\infty, 2^{\mathbb{Z}\setminus\{0\}}) \\
&\leq_B E(\mathbb{F}_2, 2^{\mathbb{Z}\setminus\{0\}}) \qquad \text{because } \mathbb{F}_\infty \leq \mathbb{F}_2 \\
&\leq_B E(\mathbb{F}_2 \times \mathbb{Z}, 3) \\
&\leq_B E(\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2) \\
&\leq_B E(\mathbb{F}_\infty, 2) \qquad \text{because } \mathbb{F}_\infty \times \mathbb{Z} \times \mathbb{Z}_2 \text{ surjects onto } \mathbb{F}_\infty \\
&\leq_B E(\mathbb{F}_2, 2)
\end{aligned}$$

The following two propositions will conclude the proof.

Proposition 6.17. *For any countable group G, we have $E(G, 2^{\mathbb{Z}\setminus\{0\}}) \leq_B E(G \times \mathbb{Z}, 3)$*

Proof. Define the map

$$\begin{aligned}
(2^{\mathbb{Z}\setminus\{0\}}) &\to 3^{G\times\mathbb{Z}} \\
p &\mapsto p^*,
\end{aligned}$$

where

$$p^*(g,n) = \begin{cases} p(g)(n) & n \neq 0, \\ 2 & \text{otherwise.} \end{cases}.$$

Note that $p^*(g,n) = 2$ precisely when $n = 0$.

Suppose $p, q \in (2^{\mathbb{Z}} \setminus \{0\})^G$. If $p = g \cdot q$, the $q^* = (g,0) \cdot p^*$.

Next, suppose that $q^* = (g,n) \cdot p^*$. If $n = 0$, then we have $q(x)(k) = p(g^{-1}x)(k) = g \cdot p(x)(k)$. If $n \neq 0$, then $q^*(x,k) = (g,n) \cdot p^*(x,k) = p^*(g^{-1}x, k-n)$. This implies that $2 \neq q^*(x,k) = p^*(g^{-1}x, 0) = 2$, a contradiction. $\square$

Proposition 6.18. *For any countable group G, we have $E(G,3) \leq_B E(G \times \mathbb{Z}_2, 2)$*

Denote by $(2)^{\mathbb{F}}_2$ the free part of the shift action $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$. Precisely, $(2)^{\mathbb{F}_2} = \{x \in 2^{\mathbb{F}}_2 : \forall g \neq 1 (g \cdot x \neq x)\}$.

Proposition 6.19. $\mu\big((2)^{\mathbb{F}_2}\big) = 1$.

Proof. We show that $Z = 2^{\mathbb{F}_2} \setminus (2)^{\mathbb{F}_2}$ is a null set. To see this note that

$$Z = \bigcap_{g \neq 1} \{x \in 2^{\mathbb{F}_2} \mid g \cdot x = x\}$$

For any fixed $1 \neq g \in \mathbb{F}_2$, we have $g^n \neq g^m$ because g has infinite order.

Note that

$$\begin{aligned}\mu(\{x : g \cdot x = x\}) \le & \mu(\{x : \forall n \in \mathbb{Z}(g \cdot x(g^{2n+1}) = x(g^{2n+1}))\}) = \\ = & \mu(\{x : \forall n \in \mathbb{Z}(x(g^{2n}) = x(g^{2n+1}))\}) = \\ = & \prod_{n\in\mathbb{Z}} \mu(\{x : x(g^{2n}) = x(g^{2n+1})\}) \\ = & 0\end{aligned}$$

□

It is natural to ask whether $\mathbb{F}_2 \curvearrowright (2)^{\mathbb{F}_2}$ is universal. The answer is negative, since Thomas [23] showed that:

Theorem 6.20 (Thomas)**.** *For all countable group G, if the action $G \curvearrowright X$ is free, then E^X_G is not universal.*

6.2. **Universality of left-orders.**

Definition 6.21. Let G be a left-orderable group. A subgroup $C \le G$ is **relatively convex** if and only if there exists a left-order $<$ on G such that C is convex in $(G, <)$.

Fact 6.22. *$C \le G$ is relatively convex if and only if there is $Q \subseteq G$ such that*

(1) *$QQ \subseteq Q$;*
(2) *$CQC \subseteq Q$;*
(3) *$Q \cup Q^{-1} \cup C = G$.*

In which case, for any $P \in \mathrm{LO}(C)$, then $P \cup Q \in \mathrm{LO}(G)$. Moreover, if $R = P \cup Q$ with $P \in \mathrm{LO}(C)$, than Q satisfies (1)–(3)*.*

Proposition 6.23. *If C is relatively convex in G, and*

$$\forall h \in H \qquad (hCh^{-1} \subseteq C \implies h \in C),$$

then $E_{\mathsf{lo}}(C) \le_B E_{\mathsf{lo}}(\mathbb{F}_n)$.

Corollary 6.24. *For $n \ge 2$, we have $E_{\mathsf{lo}}(\mathbb{F}_2) \le_B E_{\mathsf{lo}}(\mathbb{F}_n)$.*

Sketch. It is well-known that $\mathbb{F}_2$ is relatively convex in $\mathbb{F}_n$. Moreover, $\mathbb{F}_2$ is **malnormal** in $\mathbb{F}_n$; i.e., for all $g \in \mathbb{F}_n \setminus \mathbb{F}_2$, the intersection $\mathbb{F}_2 \cap g\mathbb{F}_2 g^{-1} = \{1\}$. □

Corollary 6.25. $E_{\mathsf{lo}}(\mathbb{F}_\infty) \le_B E_{\mathsf{lo}}(\mathbb{F}_2)$.

Proof. Let $\mathbb{F}_2 = \langle a, b\rangle$. Set $x_n = a^n b a^{-n}$. Note that $axa^{-1} = x_{n+1}$.

Find an infinite $T \subseteq \{x_n : n \in \mathbb{N}\}$ such that

$$|a^{\updownarrow} T a^{-\ell} \cap T| \le 1 \text{ for any nonzero } \ell \in \mathbb{Z}$$

Set $f(0) = 1$ and $f(n+1) = 1 + \sum_{i\le} f(i)$, and let $T = \{x_{f(n)} : n \ge 1\}$. Identify $\mathbb{F}_\infty$ with $\langle T\rangle$ which satisfies the assumption of the above proposition. □

For $f\colon \mathbb{F}_2 \to \mathbb{Z}$ let $S(f) = \{x \in \mathbb{F}_2 : f(x) \neq 0\}$ and define

$$B = \{f\colon \mathbb{F}_2 \to \mathbb{Z} \mid S(f) \text{ is finite}\}$$

with point-wise operation $fg(x) = f(x) + g(x)$ for $f, g \in B$, $x \in \mathbb{F}_2$.

Note that $\mathbb{F}_2 \curvearrowright B$ by shift

$$a \cdot f(x) = f(a^{-1}x)$$

for all $a \in \mathbb{F}_2$ and $f \in B$.

Next denote $W = \mathbb{Z} \operatorname{wr} \mathbb{F}_2 = B \rtimes \mathbb{F}_2$, where the operation is defined by $(f, a)(g, b) = (f(a \cdot g), ab)$. Note that W encodes the shift action of $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$ in the following sense. Let $C_x = \{f \in B : f(y) = 0 \text{ for all } y \neq x\}$. Then $aC_xa^{-1} = A_{ax}$.

Proposition 6.26. $E(\mathbb{F}_2, 2) \leq_B E_{\mathsf{lo}}(W)$.

Sketch. Let $P \subseteq \mathbb{Z}$ be the usual positive cone. Fix a left-order on $\mathbb{F}_2$, say $<$. For any $A \subseteq \mathbb{F}_2$, and $x \in \mathbb{F}_2$ let

$$P_x = \begin{cases} P & \text{if } x \in A \\ P^{-1} & \text{otherwise.} \end{cases}$$

First we define a map:

$$\begin{aligned} \mathcal{P}(\mathbb{F}_2) &\to \mathrm{LO}(B) \\ A &\mapsto R_A \end{aligned}$$

where R_A is the lexicographic order with respect to $<$ and $\{P_x\}_{x \in \mathbb{F}_2}$.

After fixing a bi-order Q on $\mathbb{F}_2$, define a left-oerder on W by the sandwich trick

$$1 \to B \to W \to \mathbb{F}_2 \to 1,$$

i.e., $R_A \cup q^{-1}(Q)$.

Claim 7. The map $A \mapsto R_A \cup q^{-1}(Q)$ is a Borel reduction from $E(\mathbb{F}_2, 2)$ to $E_{\mathsf{lo}}(W)$.

$\square$

Corollary 6.27. *$E_{\mathsf{lo}}(W)$ is universal.*

Lemma 6.28. *Suppose $1 \to K \xrightarrow{i} G \xrightarrow{q} H \to 1$ is a short exact sequence and G is bi-orderable. The the Borel map*

$$\begin{aligned} \mathrm{LO}(H) &\to \mathrm{LO}(G) \\ Q &\mapsto q^{-1}(Q) \cup (P \cap K), \end{aligned}$$

for some fixed bi-order $P \in \mathrm{LO}(G)$, witnesses that $E_{\mathsf{lo}}(H) \leq_B E_{\mathsf{lo}}(G)$.

Theorem 6.29. $E(\mathbb{F}_2, 2) \leq_B E_{\mathsf{lo}}(\mathbb{F}_2)$.

Sketch. $E(\mathbb{F}_2, 2) \leq_B E_{\mathsf{lo}}(W) \leq_B E_{\mathsf{lo}}(\mathbb{F}_\infty) \leq_B E_{\mathsf{lo}}(\mathbb{F}_2)$. $\square$

APPENDIX A. SEMIDIRECT PRODUCT

For any groups N and H and group homomorphism $\phi\colon H \to \mathrm{Aut}(N), h \mapsto \phi_h$, we define a new group $N \rtimes_\phi H$, called the **(outer) semidirect product** of N and H with respect to ϕ, defined as follows. The underlying set is the cartesian product $N \times H$, and the group operation is defined by setting

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1\phi_{h_1}(n_2), h_1h_2).$$

This defines a group in which the identity element is $(1_N, 1_H)$ and the inverse of the element (n, h) is $(\phi_{h^{-1}}(n^{-1}), h^{-1})$. Pairs $(n, 1_H)$ form a normal subgroup of $N \rtimes_\phi H$ isomorphic to N, while pairs $(1_N, h)$ form a subgroup isomorphic to H.

Note that the direct product is a special case of the semidirect product. To see this, let $\phi\colon H \to \mathrm{Aut}(N)$ be the trivial homomorphism $\phi(h) = id_N$ then $N \rtimes_\phi H = N \times H$.

Proposition A.1. *The group G is isomorphic to $N \rtimes_\phi H$ if and only if there exists a short exact sequence*

$$1 \to N \to G \to H \to 1$$

and a group homomorphism $\gamma\colon H \to G$ such that $\alpha \circ \gamma = id_H$, the identity map on H. In this case, $\phi\colon H \to \mathrm{Aut}(N)$ is defined by $\phi_h(n) = \beta^{-1}(\gamma(h)\beta(n)\gamma(h^{-1}))$.

APPENDIX B. COMPACT SPACES

Definition B.1. We say that X satisfies the finite intersection property (or FIP) for closed sets if any collection $\{C_i\}_{i\in I}$ of closed sets in X with all finite intersections

$$C_{i_1} \cap \cdots \cap C_{i_n} \neq \emptyset,$$

the intersection $\bigcap_{i\in I} C_i$ of all C_i 's is non-empty.

Proposition B.2. *The topological space X is compact if and only if X satisfies the FIP for closed sets.*

Proof. For an arbitrary topological space X, to give a collection $\{C_i : i \in I\}$ of closed subsets of X is exactly the same thing as to give a collection $\{U_i : i \in I$ of open sets in X where $U_i = X \setminus C_i$. Then it is easy to check that X X satisfies the open covering criterion for compactness if and only if X satisfies FIP for closed sets. □

APPENDIX C. ABELIANIZATION

Let G be any group. For any two elements $g, h \in G$, the commutator of g, h is the element $[g, h] = g^{-1}h^{-1}gh$. We have $[g, h] = 1_G$ if and only if g and h commute in G. Denote by $G' = \langle [g, h] \mid g, h \in G\rangle$ be the **commutator subgroup** of G.

Proposition C.1. $G' \lhd G$.

Proof. Let $h \in G'$ and $g \in G$. We need to show that $ghg^{-1} \in G'$.

Consider the commutator $ghg^{-1}h^{-1} \in G'$. Since G' is closed under product, we have $(ghg^{-1}h^{-1})h \in G'$. It follows that $ghg^{-1} \in G'$ as desired. □

Definition C.2. Define the **abelianization of** G by $G^{\mathrm{ab}} = G/G'$ and the canonical projection $\pi_{\mathrm{ab}} \colon G \to G^{\mathrm{ab}}$.

Fact C.3. *Let G be any group. For any Abelian group A and any homomorphism $f \colon G \to A$ there exists a unique homomorphism $F \colon G^{ab} \to A$ such that $f = F \circ \pi_{ab}$.*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi_{ab}"'] & A \\ G^{ab} \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Appendix D. Cantor-Bendixson rank

For any topological space X define the Cantor-Bendixson derivative X' of X by setting:

$$X' = \{x \in X : x \text{ is a limit point of } X\}.$$

Note that X' is closed because $X' = X \setminus U$, where U is the union of all singletons consisting of isolated points. Moreover, when X is second countable, the open set U is countable. Using transfinite recursion we define the iterated Cantor–Bendixson derivatives X^α , as follows:

$$\begin{aligned} X^0 &= X \\ X^{\alpha+1} &= (X^\alpha)' \\ X^\lambda &= \bigcap_{\alpha<\lambda} X^\alpha \qquad \text{if } \lambda \text{ is a limit.} \end{aligned}$$

Lemma D.1. *Let X be a Polish space. The Cantor Bendixson derivatives stabilize at some countable stage. There is some $\beta \in \omega_1$ such that $X^{\beta+1} = X^\beta$.*

Proof. Fix a countable basis of X. Suppose otherwise. Then for each $\alpha \in \omega_1$, there is some $x \in X^{\alpha+1} \setminus X^\alpha$, and moreover there is some open set U_α from a countable basis such that $x \in U_\alpha$ and $U_\alpha \cap (X^\alpha \setminus \{x\}) = \emptyset$. But this yields an uncoutable set of distinct basic open sets $\{U_\alpha \mid \alpha < \omega_1\}$, which is a contradiction. □

Lemma D.2. *A Polish space X is countable if and only if $X^\infty = \emptyset$.*

Lemma D.3. *Let X be a nonempty countable compact Hausdorff space. Then there exists an ordinal β such that X^β is finite and nonempty.*

Proof. By Lemma D.2, let α be the least ordinal such that $X^\beta = \emptyset$. Suppose β is a limit ordinal. Then $X^\beta = \bigcap_{\alpha<\beta} X^\alpha$. Since $\{X^\alpha \mid \alpha < \beta\}$ is a descending sequence of closed nonempty subset of the compact space X we have $\bigcap_{\alpha<\beta} X^\alpha \neq \emptyset$, a contradiction. Therefore β must be a successor ordinal and we may write $\beta = \alpha + 1$ for some $\alpha < \omega_1$. Then X^α is a nonempty compact Hausdorff space that consists only of isolated points, because $X^\alpha = \emptyset$. Therefore X^α is finite and nonempty, as desired. $\square$

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