A COURSE ON COUNTABLE BOREL EQUIVALENCE RELATIONS

FILIPPO CALDERONI

Contents

Date: April 29, 2026.

2020 *Mathematics Subject Classification.* Primary: 03E15.

Filippo Calderoni is supported by the NSF Grant DMS – 2348819. These notes were edited while Filippo Calderoni was visiting the University of Turin in Summer 2026. We thank the University of Turin for the hospitality. We thank Roberto Carnevale, Costanza Furone, Gabriele Maiolino, Davide Peccioli, and Davide Quadrellaro for helpful comments.

WARM-UP

Let X be a set.

Definition 0.1. An *algebra* on X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ such that:

(i) $\mathcal{A}$ is closed under complement: $A \in \mathcal{A} \implies A^c \in \mathcal{A}$;
(ii) $\mathcal{A}$ is closed under finite unions: $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

We say that $\mathcal{A}$ is a *σ-algebra* if it is also closed under countable unions.

Note that every σ-algebra on the set X contains $\emptyset$ and X itself.

Definition 0.2. For a Polish space X, denote by $\mathcal{B}(X)$ the σ-algebra generated by the open sets. The elements of $\mathcal{B}(X)$ are called the *Borel* subsets of X.

Borel sets of a topological space are the sets obtained from the open sets using the set-theoretical operations of countable unions and taking the complements.

Definition 0.3. An equivalence relation E on the Polish space X is said to be a *countable Borel equivalence relation (cber)* if and only if E is a Borel subset of $X \times X$ and E is *countable* in the sense that E-classes are countable.

Notice that the word "countable" is a slight abuse of language. Countable equivalence relations on uncountable spaces are actually uncountable! In fact, any equivalence relation contains the diagonal by reflexivity.

Before starting with the theory of countable Borel equivalence relations, it is natural to ask how many equivalence relations there are. In other words, how restrictive is the property of being Borel? The next proposition shows that it is essential.

Proposition 0.4. *There are (many!) countable equivalence relations on $\mathbb{R}$ that are not Borel.*

Proposition 0.4 can be proved by doing the only thing set theorists can really do: counting!

First, recall the following well-known result from descriptive set theory.

Lemma 0.5. *There are at most continuum*[1] *many cbers over $\mathbb{R}$.*

Proof. By a well-known result in descriptive set theory for any uncountable Polish space X, we have $|\mathcal{B}(X)| = \mathfrak{c}$. Then, as there are continuum many Borel subsets of $\mathbb{R} \times \mathbb{R}$, we conclude that the continuum is an upper bound. $\square$

On the other hand, one can prove the following result.

Lemma 0.6. *There are $2^{\mathfrak{c}}$ countable equivalence relations on $\mathbb{R}$.*

[1] In the later sections, we will show that there are actually continuum many different cbers in a very strong sense.

Proof. To simplify the exposition, denote by $\mathrm{CER}(\mathbb{R})$ the set of countable equivalence relations on $\mathbb{R}$.

First, we show that $|\mathrm{CER}(\mathbb{R})| \leq 2^{\mathfrak{c}}$. Note that every countable equivalence relation on $\mathbb{R}$ corresponds to a partition of $\mathbb{R}$ into disjoint subsets of $\mathbb{R}$. Therefore, each partition has size $\mathfrak{c}$. Then, we have

$$|\mathrm{CER}(\mathbb{R})| \leq |[\mathcal{P}(\mathbb{R})]^{\mathfrak{c}}| = (2^{\mathfrak{c}})^{\mathfrak{c}} = 2^{\mathfrak{c}\cdot\mathfrak{c}} = 2^{\mathfrak{c}},$$

where $[A]^{\leq\kappa}$ is the collection of subsets of A of size κ.

Next, let $\mathcal{E}$ be a collection of partitions of $\mathbb{R}$ into classes of size 2.

Claim 0.6.1. $|\mathcal{E}| = 2^{\mathfrak{c}}$.

Proof of Claim 0.6.1. By the previous paragraph $|\mathcal{E}| \leq 2^{\mathfrak{c}}$. So, it suffices to show that $2^{\mathfrak{c}} \leq |\mathcal{E}|$.

Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ be two enumerations of the sets $(-\infty, 0]$ and $(0, \infty)$, respectively. (The only thing that matters about the two sets is the equinumerosity with $\mathbb{R}$.) Let $\mathrm{Sym}(\mathfrak{c})$ be the group of permutations of $\mathfrak{c}$. Since $|\mathrm{Sym}(\mathfrak{c})| = 2^{\mathfrak{c}}$, it suffices to define an injection of $\mathrm{Sym}(\mathfrak{c})$ into $\mathcal{E}$. To do that, consider the function

$$\begin{aligned} \mathrm{Sym}(\mathfrak{c}) &\to \mathcal{E} \\ \varphi &\mapsto \{\{x_\alpha, y_{\varphi(\alpha)}\} : \alpha \in \mathfrak{c}\}. \end{aligned}$$

□

Claim 0.6.1 yields that there are at least $2^{\mathfrak{c}}$ finite equivalence relations on $\mathbb{R}$, therefore $2^{\mathfrak{c}} \leq |\mathrm{CER}(\mathbb{R})|$. It follows by the Cantor-Schröder-Bernstein theorem that $|\mathrm{CER}(\mathbb{R})| = 2^{\mathfrak{c}}$. □

A common mistake about cbers is the following fallacious reasoning: "In any Polish space, finite sets are Borel. Therefore, any equivalence relation with finite classes is Borel". The proof of Claim 0.6.1 reveals that this is far from true.

1. Preliminaries

In this section, we recall some preliminary concepts about descriptive set theory that we will use throughout the course.

1.1. **Polish spaces.**

Recall that a topological space (X, τ) is *Polish* if and only if it is separable and admits a complete compatible metric.

A topological space X is called *second countable* if it has a countable basis. Clearly, $\mathbb{R}$ with the Euclidean topology is second countable – a basis is given by the open intervals with rational endpoints. For metrizable topological spaces, metrizability and second countability are equivalent.

Proposition 1.1. *A metrizable space is separable if and only if it is second countable.*

Proof. Let X be second countable and $\{U_n : n \in \mathbb{N}\}$ a countable base with all U_n's nonempty. Choose[2] $x_n \in U_n$. Clearly, $\{x_n : n \in N\}$ is dense. On the other hand, let (X, d) be a separable metric space and $\{x_n : n \in \mathbb{N}\}$ a countable dense set in X. Then

$$\mathcal{B} = \{B(x_n, r) : r \in \mathbb{Q}, r > 0 \text{ and } n \in \mathbb{N}\}$$

is a countable base for the topology of (X, d). □

Definition 1.2. A subset $A \subseteq X$ is said to be G_δ if it is a countable intersection of open sets. Similarly, A is said to be F_σ if it is a countable union of closed set[3].

Proposition 1.3. *If (X, d) is a metric space, then closed sets are G_δ.*

Proof. Let $C \subseteq X$ be a nonempty closed set. For $\epsilon > 0$, define $U_\epsilon = \{x \in X : d(x, C) < \epsilon\}$, and we claim that $C = \bigcap_n U_{1/n}$ and argue by double inclusion. For the nontrivial direction, let $x \in U_{1/n}$. So, for every $n \in \mathbb{N}$ we can pick $x_n \in C$ with $d(x, x_n) < 1/n$. It follows that $\lim_{n\to\infty} x_n = x$. Therefore, $x \in C$ because C is closed. □

Theorem 1.4 (Alexandrov)**.** *Let X be a Polish space. A subspace $Y \subseteq X$ is Polish if and only if Y is G_δ.*

In particular, open subsets of Polish spaces are Polish with the subspace topology.

Example 1.5. Let $\mathrm{GL}_n(\mathbb{R})$ be the set of invertible $n \times n$ realvalued matrices. We can identify $\mathrm{GL}_n(\mathbb{R})$ as a subset of $\mathbb{R}^{n\times n}$. Note that $\mathrm{GL}_n(\mathbb{R})$ is open. (Why?) It follows that $\mathrm{GL}_n(\mathbb{R})$ is a Polish space.

1.2. **Standard Borel spaces and Borel functions.** Whenever (X, τ) is Polish we denote by $\mathcal{B}(\tau)$ the σ-algebra of Borel subset of X. The σ-algebra $\mathcal{B}(\tau)$ is the smallest σ-algebra containing τ.

Definition 1.6. Let $\mathcal{B}$ be a σ-algebra on a set X. Then $(X, \mathcal{B})$ is a *standard Borel space* if and only if there is a Polish topology τ on X such that $\mathcal{B} = \mathcal{B}(\tau)$.

Obviously, if (X, τ) is a Polish space, then $(X, \mathcal{B}(\tau))$ is standard Borel. To what extent does the topology determine the Borel structure? None... as shown by the following theorem.

Theorem 1.7. *Let (X, τ) be a Polish space and let $Y \in \mathcal{B}(\tau)$. Then there exists a finer Polish topology $\tau_Y \supseteq \tau$ on X such that $\mathcal{B}(\tau_Y) = \mathcal{B}(\tau)$ and Y is clopen in (X, τ_Y).*

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[2]Here we tacitly use the axiom of countable choice (AC_ω). The axiom of countable choice is strictly weaker than the axiom of dependent choice, which is weaker than the (full) axiom of choice. Paul Cohen proved that (AC_ω), is not provable in Zermelo–Fraenkel set theory without the axiom of choice.

[3]It is easier to remember the notation G_δ and F_σ if we recall some foreign languages. For the former we need some German: G as in *gebiet* (area or neighborhood) and δ as in *durchschnitt* (intersection). For the latter, French is your friend, F as in *fermé* (closed) and σ as in *somme* (sum, or union).

Corollary 1.8. *If $(X, \mathcal{B})$ is a standard Borel space, and $Y \in \mathcal{B}$, then $(Y, \mathcal{B} \restriction Y)$ is a standard Borel space.*

Here, $\mathcal{B} \restriction Y$ just means $\{A \in \mathcal{B} : A \subseteq Y\}$. Corollary 1.8 easily follows because we can change the topology of X to turn Y into a clopen set, then $Y \subseteq X$ becomes a Polish space with the subspace topology.

Definition 1.9. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be standard Borel spaces. A map $f\colon X \to Y$ is said to be *Borel* if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$.

The following is a consequence of a deep result in descriptive set theory known as Suslin's theorem.

Theorem 1.10. *If X, Y are standard Borel spaces and $f\colon X \to Y$, then the following are equivalent:*

(1) *f is Borel;*
(2) *$\operatorname{graph}(f)$ is a Borel subset of $X \times Y$.*

A *Borel isomorphism* between X, Y is a bijection $f\colon X \to Y$ such that both f and f^{-1} are Borel.

Theorem 1.11 (Kuratowski)**.** *There is a unique uncountable standard Borel space up to Borel isomorphism.*

It follows from Theorem 1.11 that $\mathbb{R}$, $2^{\mathbb{N}}$, $\mathrm{GL}_n(\mathbb{R})$ and all other uncountable Polish spaces have isomorphic Borel structures. – The terminology "standard" Borel space is well-justified. (The fact that $\mathbb{R}$ and $2^{\mathbb{N}}$ are Borel isomorphic may sound surprising as they are clearly not homeomorphic as topological spaces. This is because $2^{\mathbb{N}}$ is compact, while $\mathbb{R}$ is not.)

We also highlight the following important theorem.

Theorem 1.12 (Luzin-Suslin)**.** *Let X, Y be standard Borel spaces and $f\colon X \to Y$ be Borel. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel.*

1.3. **Uniformization and the Luzin-Novikov theorem.**

Definition 1.13. Given two standard Borel spaces X, Y and $P \subseteq X \times Y$, a *uniformization* of P is a subset $P^* \subseteq P$ such that for all $x \in X$,

$$\exists y\,(x, y) \in P \iff \exists! y\,(x, y) \in P^*.$$

In essence, P^* is the graph of a function $f\colon \operatorname{proj}_X P \to Y$ such that $f(x) \in P_x = \{y \in Y : (x, y) \in P\}$ holds. Clearly, the axiom of choice implies that every $P \subseteq X \times Y$ has a uniformization. Now suppose that X and Y are standard Borel spaces, and $P \subseteq X \times Y$ is Borel. Does P necessarily have a Borel uniformization?

Proposition 1.14. *Let $P \subseteq X \times Y$ with X, Y standard Borel spaces. If P has a Borel uniformization P^*, then $\operatorname{proj}_X(P)$ is Borel.*

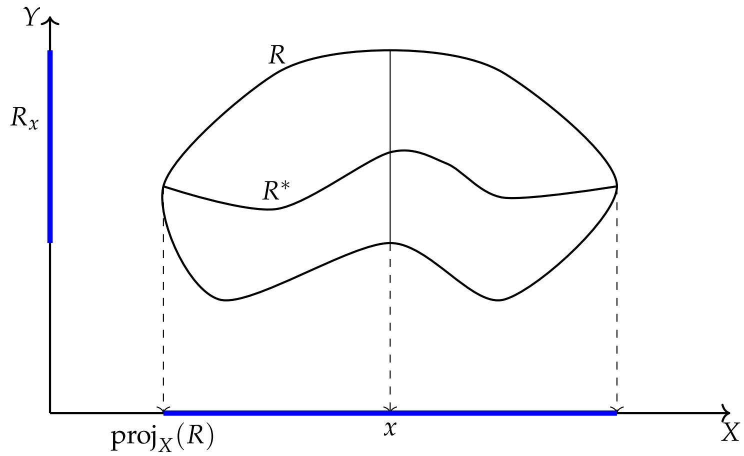

FIGURE 1. Uniformization

Proof. Suppose that P^* is a Borel uniformization. First observe that $\mathrm{proj}_X(P) = \mathrm{proj}_X(P^*)$. Since $\mathrm{proj}_X \colon P^* \to X$ is injective, it follows that $\mathrm{proj}_X(P^*)$ is Borel by Theorem 1.12. □

Corollary 1.15. *There exists a Borel $P \subseteq X \times Y$ with X, Y standard Borel spaces, and with no Borel uniformization.*

The following is a major theorem of descriptive set theory.

Theorem 1.16 (Luzin–Novikov)**.** *Suppose X, Y are standard Borel spaces and $P \subseteq X \times Y$ is a Borel subset such that P_x is countable (perhaps empty) for all $x \in X$. Therefore,*

(1) *P has Borel uniformization and thus $\mathrm{proj}_X(P)$ is Borel.*
(2) *Moreover, we can express $P = \bigcup_{n \in \mathbb{N}} P_n$, where each P_n is the Borel graph of a partial function; i.e. if $(x, y) \in P_n$ and $(x, z) \in P_n$, then $y = z$.*

An important consequence of Luzin-Novikov theorem is that every countable-to-one Borel map between standard Borel spaces admits a right inverse.

Corollary 1.17. *Suppose that X, Y are standard Borel spaces and $f \colon X \to Y$ is a countable-to-one Borel map. Then $f(X)$ is Borel, and there exists a Borel map $g \colon f(X) \to X$ such that $f \circ g(y) = y$ for all $y \in f(X)$.*

Proof. Let $P = \mathrm{graph}(f)^- = \{(f(x), x) : x \in X\}$. By Theorem 1.16 there is a Borel uniformization $P^* \subseteq P$ and so $f(X)$ is Borel and $P^* = \mathrm{graph}(g)$ for some Borel function $g \colon f(X) \to X$. Then, for any $y \in f(X)$, we have $(y, g(y)) \in \mathrm{graph}(g) \subseteq \mathrm{graph}(f)^-$. It follows that $(g(y), y) \in \mathrm{graph}(f)$; thus, $f(g(y)) = y$. □

Let G be a group. Recall that a *(left-)action* of G on the set X is a map $G \times X \to X, (g, x) \mapsto g \cdot x$ such that for all $g, h \in G$, and $x \in X$ we have:

(i) $id_G \cdot x = x$,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$.

Most of the countable Borel equivalence relations that we will consider in this course will be defined as follows. Let G be a countable discrete group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $a\colon G \times X \to X$. We denote by E_a the corresponding orbit equivalence relation on X, whose equivalence classes are the G-orbits. That is, $x \mathrel{E_a} y \iff \exists g \in G(g \cdot x = y)$. It is clear that E_a is a countable Borel equivalence relation.

In our discussion, the G-action will be clear from the context; therefore, we will denote by E_G^X induced by the G-space X.

Theorem 1.18 (Feldman–Moore)**.** *Let E be a countable Borel equivalence relation on X. Then there is a countable group G and a Borel action of G on X such that $E = E_G^X$.*

Proof. We want to express $E \setminus \Delta(X) = \bigcup_{n\in\mathbb{N}} g_n$ where each g_n is the Borel graph of a one-to-one partial function such that $\operatorname{dom}(g_n)$ and $\operatorname{ran}(g_n)$ are disjoint.

Then we extend g_n to the Borel involution $\tilde{g}_n\colon X \to X$ by

$$\tilde{g}_n(x) = \begin{cases} g_n(x) & x \in \operatorname{dom}(g_n); \\ g_n^{-1}(x) & x \in \operatorname{ran}(g_n); \\ x & \text{otherwise.} \end{cases}$$

Now let $G = \langle \tilde{g}_n : n \in \mathbb{N} \rangle$. We define a Borel G-action by letting the generators act in the obvious way $\tilde{g}_n \cdot x = \tilde{g}_n(x)$ for all $x \in X$. It follows that $x \mathrel{E} y \iff x \mathrel{E_G^X} y$ as desired.

To obtain such a sequence of g_n as above we use Luzin–Novikov (Theorem 1.16). First note that E is a Borel subset of $X \times X$ with countable section, so we can write $E = \bigsqcup_{n\in\mathbb{N}} f_n$ where each f_n is the Borel graph of a Borel function.

Without loss of generality we may assume that all f_n are one-to-one. To see this, consider the relations $f_n^- := \{(y, x) : (x, y) \in f_n\}$. Then we can also write $E = \bigcup_{m,n}(f_n \cap f_m^-)$ and $f_n \cap f_m^-$ is the Borel graph of an injective partial function. So we can replace $(f_n)_{n\in\mathbb{N}}$ with $(f_n \cap f_m^-)_{n,m\in\mathbb{N}}$ if needed.

Then fix a compatible Polish topology τ on X. Since (X, τ) is Hausdorff, the diagonal $\Delta(X)$ is closed. Then, since X is second countability, we can write $X^2 \setminus \Delta(X) = \bigcup_{n\in\mathbb{N}}(U_n \times V_n)$ where $U_n, V_n \subseteq X$ are disjoint open sets. It follows that

$$E \setminus \Delta(X) = E \cap \bigcup_{m\in\mathbb{N}} U_m \times V_m = \bigcup_{m,n\in\mathbb{N}} f_n \cap (U_m \times V_m),$$

the partial functions $f_n \cap (U_m \times V_m)$ are one-to-one, and their domains and ranges are disjoint. □

2. SMOOTH EQUIVALENCE RELATIONS

The Feldman-Moore theorem (Theorem 1.18) states that every cber can be realized as an orbit equivalence relation induced by the Borel action of a countable discrete group.

The requirement that the action be Borel is an essential restriction. Using the axiom of choice, it is not hard to see that every equivalence relation with countable classes can be induced by a $\mathbb{Z}$-action. So, a follow up question is: "Is it true that every cber can be realized as the orbit equivalence relation induced by a Borel $\mathbb{Z}$-action?"

The answer to this question is negative. It will follow from the theory of Borel reducibility and it will be discussed in the forthcoming sections.

Definition 2.1. Let E, F be two Borel equivalence relations on the standard Borel spaces X, Y, respectively. We say that $E \leq_B F$ if and only if there is a Borel $f\colon X \to Y$ such that

$$x_0 \, E \, x_1 \iff f(x_0) \, F \, f(x_1).$$

The function f is called a *Borel reduction*, and we say that E is *Borel reducible* to F of E to F.

We also write $E <_B F$, whenever $E \leq_B F$ and $F \not\leq_B E$. And we write $E \sim_B F$ if and only if $E \leq_B F$ and $F \leq_B E$.

The following is a spacial case of a famous theorem of Jack Silver [18].

Theorem 2.2 (Silver)**.** *For any countable Borel equivalence relation E on an uncountable Polish space X, we have $=_{\mathbb{R}} \leq_B E$.*

Definition 2.3. An equivalence relation on a standard Borel space is *smooth* (or *concretely classifiable*) if and only if $E \leq_B =_{\mathbb{R}}$.

Note that every smooth equivalence relation is necessarily Borel. (Why?)

It follows from Silver's theorem that every smooth countable Borel equivalence relation is Borel bi-reducible to one of the following.

$$=_1 <_B =_2 <_B \cdots <_B =_n <_B \cdots <_B =_{\mathbb{N}} <_B =_{\mathbb{R}},$$

where $1, 2, \ldots, \mathbb{N}$ denotes respectively the countable standard spaces $\{0\}$, $\{0,1\}$, $\ldots$, $\{0, 1, \ldots, n \ldots\}$ with the discrete standard Borel structure (i.e., the standard Borel structure induced by the discrete topology, which coincide with the entire power set.)

Now we provide a convenient characterization of smoothness in the context of cbers.

Definition 2.4. Let E be an equivalence relation on X. A *transversal* for E is a set $T \subseteq X$ which intersects each E-class in exactly one point. A *selector* for E is a function $s\colon X \to X$ such that $s(x) \, E \, x$ and $x \, E \, y \implies s(x) = s(y)$.

Theorem 2.5. *Let E be a countable Borel equivalence relation on X. The following are equivalent:*

(i) *E is smooth;*
(ii) *E admits a Borel transversal;*
(iii) *E admits a Borel selector.*

Proof. (i) $\implies$ (ii) Let Y be a Polish space and $f\colon X \to Y$ witness that E is smooth. Clearly f is a countable-to-one function, therefore let $g\colon f(X) \to X$ be a Borel right-inverse of f as in Corollary 1.17. Then the set $g(f(X))$ is a Borel transversal for E.

(ii) $\implies$ (iii) Let $T \subseteq X$ be a Borel transversal for T. Define $s\colon X \to X$ by setting $s(x)$ equal to the unique $t \in T$ such that $t \mathrel{E} x$. It is clear that s is a selector for E and $\operatorname{graph}(s) = \{(x,y) \in E : y \in T\} = E \cap (X \times T)$ is a Borel set, thus s is Borel.

(iii) $\implies$ (i) Any Borel selector $s\colon X \to X$ for the equivalence relation E is a Borel reduction from E to id_X. □

Note that the implication (i) $\implies$ (ii) of Theorem 2.5 applies only to countable Borel equivalence relations. To apply Luzin-Novikov theorem we crucially use that E-classes are countable.

There exist smooth Borel equivalence relations with no Borel transversal. To see this, let X, Y be such that X and Y are standard Borel spaces and $A \subseteq X \times Y$ is Borel with $\operatorname{proj}_X A$ non-Borel. Then A is also a standard Borel space, and we can define a Borel equivalence relation E on A by declaring $(x_1, y_1) \mathrel{E} (x_2, y_2)$ if and only if $x_1 = x_2$. Then map $(x,y) \mapsto x$ is a Borel reduction from E to $=_X$, and so E is smooth. So suppose that $T \subseteq A$ is a Borel transversal for E. But then $\operatorname{proj}_X\colon X \times Y \to X$ is injective on T. Hence $\operatorname{proj}_X(T) = \operatorname{proj}_X A$ is Borel, contradicting the property of A.

In the remainder of this subsection we mention some consequences of Theorem 2.5.

An equivalence relation E on X is said to be *finite* if its equivalence classes are *finite*.

Proposition 2.6. *Any finite Borel equivalence relation is smooth.*

Proof. Let E be a finite equivalence relation on X. Fix a Borel linear ordering $<$ on X. Then every class is obviously well-ordered by $<$. Then define the Borel selector $s\colon X \to X$ by setting $s(x) = \min([x]_E)$. □

An equivalence relation is said to be *aperiodic* if all equivalence classes are infinite. Next example shows that aperiodic cbers can have either behavior.

Example 2.7. Let G be a nontrivial countable subgroup of $\mathbb{R}$. And consider the translation G-action on $\mathbb{R}$. The is, for any $g \in G$ and $x \in X$ define $g \cdot x = x + g$.

Since every subgroup of $\mathbb{R}$ is torsion-free, and the equivalence classes are cosets of G, the induced orbit equivalence relation $E_G^{\mathbb{R}}$ is aperiodic.

(A) Let $G = \mathbb{Z}$. Then the orbit equivalence relation $E_{\mathbb{Z}}^X$ is smooth. The set $[0,1)$ is a Borel transversal.

(B) Whenever $G = \mathbb{Q}$, we denote $E_{\mathbb{Q}}^{\mathbb{R}}$ by E_v. Any transversal for E_v is a Vitali set, therefore it is not Borel (In fact, it is not even Lebesgue measurable).

Let $2^{\mathbb{N}} = \{x \colon \mathbb{N} \to 2\}$ be the Cantor space with the product topology. It is customary to identify each $x \in 2^{\mathbb{N}}$ with the infinite binary sequence $(x_0, x_1, \dots)$.

The Polish space $2^{\mathbb{N}}$ admits a subbase consisting of the sets

$$U_n = \{x \in 2^{\mathbb{N}} : x_n = 1\} \qquad \text{and} \qquad U_n^C = \{x \in 2^{\mathbb{N}} : x_n = 0\},$$

for all $n \in \mathbb{N}$. Then all nonempty open sets can be written as union of finite intersection of the clopen sets U_n's and their complements.

Let $(X, \mathcal{B})$ be a standard Borel space. A *Borel measure* (or simply measure) on X is a *σ-additive* function $\mu \colon \mathcal{B} \to [0, \infty)$. By σ-additivity we mean that for any sequence $A_0, A_1, \dots, A_n, \dots$ of pairwise disjoint Borel set we have

$$\mu\Big(\bigcup_n A_n\Big) = \sum_{n=0}^{\infty} \mu(A_i).$$

Whenever $\mu(X) = 1$ we say that μ is a *probability measure*.

If X is a standard Borel G-space and μ is a (Borel) probability measure on X, we say that the action $G \curvearrowright (X, \mu)$ is *measure preserving* if $\mu(g \cdot A) = \mu(A)$, for any Borel set $A \subseteq X$ and $g \in G$.

Let μ be the product measure on $2^{\mathbb{N}}$, where $2 = \{0, 1\}$ has the $(\frac{1}{2}, \frac{1}{2})$-measure. So, for all subbasic open sets U_n, we have $\mu(U_n) = \mu(U_n^C) = \frac{1}{2}$. Intuitively, we regard each $x \in 2^{\mathbb{N}}$ as an infinite sequence of coin tosses. Each coin toss results in heads (or tails) with $\frac{1}{2}$ probability. It is clear that $\mu(2^{\mathbb{N}}) = 1$, so μ is a probability measure.

Proposition 2.8. *E_0 is not smooth.*

Proof. Let μ be the product $(\frac{1}{2}, \frac{1}{2})$ probability measure on $2^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let $\pi_n \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel bijection $(x_0, \dots, x_n, \dots) \mapsto (x_0, \dots, 1 - x_n, \dots)$ that flips the n-th entry.

Let $G = \bigoplus_{n \in \mathbb{N}} C_n$, where $C_n = \langle \pi_n \rangle$. Then let $G \curvearrowright 2^{\mathbb{N}}$ as a group of measure preserving transformations. Note that E_0 is the orbit equivalence relation generated by such action. Furthermore, G acts freely on $2^{\mathbb{N}}$, i.e., if $g \neq 1_G$, then $g \cdot x \neq x$ for all $x \in X$.

Now suppose that E_0 is smooth. Then there exists a Borel transversal $T \subseteq 2^{\mathbb{N}}$ for E_0.

Note that if $g_1 \neq g_2$, then $g_1 T$ and $g_2 T$ are disjoint because the action is free. To see this, note that if $t_1, t_2 \in T$ with $g_1 t_1 = g_2 t_2$ implies $g_2^{-1} g_1 t_1 = t_2$ and therefore $t_1 = t_2$ because T is a transversal. It follows that $2^{\mathbb{N}} = \bigsqcup_{g \in G} gT$.

Since T is Borel, T is Borel measurable; therefore, $\mu(gT) = \mu(T)$ for all $g \in G$. Therefore,

$$\sum_{g \in G} \mu(gT) = \mu(2^N) = 1,$$

which is a contradiction. □

For the previous argument, derive a proof of the following theorem.

Theorem 2.9. *Let (X, μ) be a Borel probability measure space. If a countably infinite group G acts freely on X and preserves the probability measure μ, then E_G^X is not smooth.*

The proof of Proposition 2.8 mimics the proof that no Vitali set is Borel. There are alternative proofs that will be discussed in the next subsection.

In view of Silver's theorem (see Theorem 4.1) it is immediate that $=_{2^{\mathbb{N}}} \leq_B E_0$. However, we can define an explicit Borel reduction. (The identity map is NOT a Borel reduction. Why?) Let $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, f(x_0, x_1, \dots) = x_0 {}^\frown x_0 x_1 {}^\frown x_0 x_1 x_2 {}^\frown \cdots$. It is easy to check that f is Borel reduction. In particular, if $x \neq y$, then $(f(x), f(y)) \notin E_0$.

2.1. **Closure properties of smooth equivalence relations.** Given an equivalence relation E on X, a subset $A \subseteq X$ is a *complete section* for E is it meets every E-class (i.e., for all $x \in X$, $[x]_E \cap A \neq \emptyset$).

The next proposition collects some closure properties of smooth equivalence relations. We omit the proof, which is straightforward.

Proposition 2.10. *Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y, respectively.*

(i) *If $E \leq_B F$, and F is smooth, then F is smooth.*
(ii) *Let $B \subseteq X$ be a Borel set that meets all E equivalence classes. If $E \restriction B$ is smooth, then E is smooth.*
(iii) *If E and F are smooth then the product*[4] *equivalence relation $E \times F$ is smooth.*

We discuss one more closure property, which will be used later without reference.

Proposition 2.11. *Suppose that $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X. If F is smooth, then E is also smooth.*

Proof. Suppose that F is smooth. By Theorem 2.5, there is a Borel transversal $T \subseteq X$ for F. Also, by Feldman-Moore's theorem, there exists a Borel action of a countable group $G = \{g_n : n \in \mathbb{N}\}$ on X such that $F = E_G^X$. Hence, we can define a Borel

[4]We say that $(x_0, y_0)\ E \times F\ (x_1, y_1) \iff (x_0\ E\ x_1 \text{ and } y_0\ F\ y_1)$.

selector $s\colon X \to X$ for E by setting $s(x) = g_N \cdot t$, where $t \in T \cap [x]_F$ is unique and $N = \min\{n \in \mathbb{N} : g_N \cdot t \mathrel{E} x\}$. □

It is worth pointing out that $E \subseteq F$ does not imply $E \leq_B F$ in general.

Proposition 2.11 has several applications. For example, let E_t denote the *tail equivalence relation* on $2^{\mathbb{N}}$. Precisely, for $x, y \in 2^{\mathbb{N}}$, we define

$$x \mathrel{E_t} y \iff \exists m \exists k \forall n (x_{m+n} = y_{k+n})$$

Corollary 2.12. *E_t is not smooth.*

Proof. It is clear that $E_0 \subseteq E_t$. So, if E_t were smooth, then E_0 would be smooth as well by Proposition 2.11. □

2.2. **The generic obstruction to smoothness.** Let X be a topological space. Recall that $A \subseteq X$ is *meager* if and only if A is a countable union of nowhere dense sets. Equivalently, there are closed sets $C_n \subseteq X$ with empty interior such that $A \subseteq \bigcup_{n \in \mathbb{N}} C_n$.

It is clear from the definitions that meager sets are closed under taking subsets and countable unions.

Further, we say that $A \subseteq X$ is *Baire measurable* if and only if there is an open set $U \subseteq X$ such that $A \Delta U$ is meager. The set of all Baire measurable subsets of X is the σ-algebra generated by the open and meager subsets of X. So, in particular, every Borel subset of X is Baire measurable.

Let X, Y be topological spaces. A function $f\colon X \to Y$ is *Baire measurable* if the preimage of every open set has the Baire property.

The main concept of this section is that of generic ergodicity.

Definition 2.13. An equivalence relation E on a Polish space X is *generically ergodic* if every E-invariant Baire measurable subset of X is either meager or comeager.

Proposition 2.14. *Let E be a generically ergodic equivalence relation on a Polish space X, and Y be any Polish space. If $f\colon X \to Y$ is a Baire measurable function such that $x \mathrel{E} y \implies f(x) = f(y)$. Then there is a comeager $C \subseteq X$ such that $f \restriction C$ is constant.*

Proof. Given a countable basis $\{U_n : n \in \mathbb{N}\}$ for Y, the preimage $f^{-1}(U_n)$ is Baire measurable and is E-invariant. For each $n \in \mathbb{N}$, denote by C_n the unique comeager set in $\{f^{-1}(U_n), X \setminus f^{-1}(U_n)\}$. Then f is constant on $C := \bigcap_{n \in \mathbb{N}} C_n$. □

Theorem 2.15. *Let E be an equivalence relation on a Polish space X. If E is generically ergodic and every E-class is meager, then E is not smooth.*

Proof. If $f\colon X \to 2^{\mathbb{N}}$ is Borel and $x \mathrel{E} y \iff f(x) = f(y)$, then the preimage of each point is an E-class; therefore, it is meager. However, f is Baire measurable, and the preimage of some element must be comeager, which is a contradiction. □

The condition of being generically ergodic looks difficult to handle and verify. However, for the orbit equivalence relations induced by continuous group actions, we have a much better understanding of generic ergodicity.

Proposition 2.16. *Let G be a (countable) Polish group acting on the Polish space X continuously. Then the following are equivalent:*

(1) *E_G^X is generically ergodic;*
(2) *There is a dense orbit.*

Corollary 2.17. *If a Polish group G acts continuously on a Polish space X such that every orbit is meager and there is a dense orbit, then E_G is not smooth.*

It follows, in particular, that the Vitali equivalence relation is not smooth. (Why?) We conclude this section with the following useful proposition.

Let G be a countable (discrete) group and X a topological space. Recall that a *continuous action* of G on X is a continuous function

$$a\colon G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

such that $1_G \cdot x = x$, and $g \cdot (h \cdot x) = (hg) \cdot x$, for all $g, h \in G$, $x \in X$. Moreover, for any element $x \in X$ we let $G \cdot x = \{g \cdot x : g \in G\}$ be the *G-orbit* of x. A subset $A \subseteq X$ is *G-invariant* if it is the union of G-orbits.

Proposition 2.18. *Let $G \curvearrowright X$ continuously. If $A \subseteq X$ is G-invariant, then so is $\overline{A}$.*

Proof. Let $g \in G$ and $x \in \overline{A}$. We want to show that $g \cdot x \in \overline{A}$. Since $x \in \overline{A}$ there exists a sequence $\{x_n : n \in \mathbb{N}\}$such that $x_n \to x$. Since G acts continuously, $g \cdot x_n \to g \cdot x$. Then, since A is G-invariant, $\{g \cdot x_n : n \in \mathbb{N}\}$ is a sequence in A converging to $g \cdot x$. It follows that $g \cdot x \in \overline{A}$. □

Let G act on X continuously. A *minimal (invariant) set* for the action is a nonempty, closed, G-invariant subset $M \subseteq X$ such that: if $C \subseteq X$ is any closed, G-invariant set and $C \subseteq M$, then $C = \emptyset$ or $C = M$.

Proposition 2.19. *Let $\emptyset \neq M \subseteq X$. The following are equivalent:*

(i) *M is minimal;*
(ii) *For every $x \in M$, $G \cdot x$ is dense in M.*

Proof. **(i) ⇒ (ii):** Suppose that M is minimal and let $x \in M$. Then $\overline{G \cdot x}$ is closed and invariant set and $\overline{G \cdot x} \subseteq M$. Therefore $\overline{G \cdot x} = M$, since M is minimal.

(ii) ⇒ (i): If M is not minimal, then there is a closed invariant set $\emptyset \neq C \subsetneq M$. Then, if $x \in C$ we have $\overline{G \cdot x} \subseteq C$ and therefore $\overline{G \cdot x} \neq M$, showing that $G \cdot x$ is not dense.

□

Proposition 2.20. *Suppose that G is a countable group acting by homeomorphisms on a compact Polish space X. Then there exists a minimal set $M \subseteq X$ for the action of G.*

Proof. Let $\mathcal{S} := \{A \subseteq X \mid A \text{ is nonempty, closed and } G\text{-invariant}\}$. We note that $\mathcal{S}$ is nonempty because $X \in \mathcal{S}$ and it is partially ordered under inclusion. A consequence of the compactness of X is that every chain $\{A_i\}_{i\in I}$ satisfies the finite intersection property. It follows that $\bigcap_{i\in I} A_i$ is nonempty and necessarily closed and G-invariant. Therefore, every chain in X admits has lower bound. Then, using Zorn's Lemma, there exists a minimal invariant set $M \subseteq X$ for the action $G \curvearrowright X$. □

Proposition 2.21. *Let X be a compact Polish space. Suppose that G is a countable group acting on X continuously. If E_G^X is smooth, then there exists a finite orbit.*

Proof. Suppose that E_G^X is smooth, and let $M \subseteq X$ be a minimal set for the action of G on X. Since E_G^X is smooth, so is the action E_G^M. Note that every G-orbit of every point in M is dense in M, so by Corollary 2.17, there must be $x_0 \in M$ such that $G \cdot x_0$ is not meager. However, $G \cdot x_0 = \bigcup_{g\in G}\{g \cdot x_0\}$ can be written as a countable union of nowhere dense sets, unless there is some $g_0 \in G$ such that $\{g_0 \cdot x_0\}$ is open in M. Since every orbit is dense in M, it follows that M must, in fact, consist of a single orbit, precisely $M = G \cdot x_0$. But $M \subseteq X$ is countable and closed. Thus, M is a compact space with the relative topology, and every point in M is isolated because G acts continuously. Consequently, M is discrete and compact. This is not possible if M is infinite. □

2.3. **Ergodicity.** Generic ergodicity is closely related to the important measure-theoretic notion of ergodicity. Let E be a countable Borel equivalence relation on X.

Definition 2.22. A measure μ on X is said to be *E-ergodic* if for every E-invariant Borel $A \subseteq X$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. A measure μ is called *non-atomic* if every singleton has measure 0.

Note that when μ is non-atomic, then $\mu([x_0]_E) = 0$ for all $x_0 \in X$.

Proposition 2.23. *Let E be a countable Borel equivalence relation on a standard Borel space. If X has an E-ergodic, nonatomic measure μ, then E is not smooth.*

Proof. Suppose that E is smooth towards contradiction. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable Borel separating family for E. (I.e. $x \mathrel{E} y \iff \forall n \in \mathbb{N}\,(x \in B_n \iff y \in B_n)$.) Each B_n is E-invariant, so either $\mu(B_n) = 0$ or $\mu(X \setminus B_n) = 0$ by ergodicity. Then the set

$$C = \bigcap\{B_n : \mu(X \setminus B_n) = 0\} \cap \bigcap\{X \setminus B_n : \mu(B_n) = 0\}$$

has positive measure. However, $C = [x_0]_E$ for some $x_0 \in X$. It follows that $\mu(C) = 0$, since μ is non-atomic. □

3. Hyperfiniteness

We have already discussed that E_0 is not smooth on several occasions. However, E_0 can be approximated by finite Borel (therefore smooth) equivalence relations. In fact, we can express $E_0 = \bigcup_{n\in\mathbb{N}} F_n$ where F_n is the equivalence relation on $2^{\mathbb{N}}$ by declaring

$$x \, F_n \, y \iff \forall k \geq n (x_k = y_k).$$

It is clear that F_n is Borel and each F_n-class has size 2^n. We call hyperfinite those equivalence relations that admit a similar approximation as the union of an increasing sequence of finite Borel equivalence relations.

Definition 3.1. A Borel equivalence relation E is *hyperfinite* if there is an increasing sequence of finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that $E = \bigcup_{n\in\mathbb{N}} F_n$.

Notice that all hyperfinite equivalence relations are, in fact, countable. Before discussing some closure properties, we prove the following fact as a warm-up.

Proposition 3.2. *If E smooth, then E is hyperfinite.*

Proof. Let $G = \{g_n : n \in \mathbb{N}\}$ be a countable group with $E = E_G^X$ and $g_0 = 1_G$. Since E is smooth, E admits a Borel selector $s\colon X \to X$ by Proposition 2.5–(iii). Then define increasing finite Borel equivalence relations F_n, for $n \geq 0$, by

$$x \, F_n \, y \iff$$
$$x \, E \, y \quad \text{and} \quad \left(x = y \quad \text{or} \quad \left(\bigvee_{i=0}^{n} (x = g_i \cdot s(x)) \text{ and } \bigvee_{i=0}^{n} (y = g_i \cdot s(x)) \right) \right).$$

Note that if $x \, F_n \, y$, then $x \, E \, y$ so $s(x) = s(y)$. Therefore, one sees that the F_n's are equivalence relations. Moreover, for any F_n, each class has at most $n+1$ elements. Hence F_n is a finite Borel equivalence relation and $E = \bigcup_n F_n$. □

Next we discuss some closure properties of hyperfinite equivalence relations. Given an equivalence relation E on X, a subset $A \subseteq X$ is a *complete section* for E is it meets every E-class (i.e., for all $x \in X$, $[x]_E \cap A \neq \emptyset$).

Proposition 3.3. *Let E, F be cbers on X, Y, respectively.*

(a) *If $X = Y$, $E \subseteq F$ and F is hyperfinite, so is E.*
(b) *If E is hyperfinite and $A \subseteq X$ is Borel, then $E \restriction A$ is hyperfinite.*
(c) *Suppose A is a Borel complete section of E, i.e., a Borel subset of X which meets all E-equivalence classes. Then if $E \restriction A$ is hyperfinite, so is E.*
(d) *If F is hyperfinite and $E \leq_B F$, then E is hyperfinite.*

Proof. The proof of (a) and (b) is straightforward, so it is left to the reader.

(c) Let $G = \{g_n : n \in \mathbb{N}\}$ be a countable group with $E = E_G^X$. For every $x \in X$, let $\ell(x)$ be the least $n \in \mathbb{N}$ such that $g_n \cdot x \in A$. The map $\ell \colon X \to \mathbb{N}$ is Borel. Let $E \restriction A = \bigcup_{n\in\mathbb{N}} F_n$, with $F_0 \subseteq F_1 \subseteq \cdots$ finite Borel equivalence relations. Then set

$$x \mathrel{R_n} y \qquad \iff \qquad x = y \quad \text{or} \quad \big(\ell(x), \ell(y) \leq n \quad \text{and} \quad g_{\ell(x)} \cdot x \mathrel{F_n} g_{\ell(y)} \cdot y\big).$$

Clearly, each R_n is reflexive and symmetric by definition. Moreover, it is transitive because F_n is transitive. Then the R_n's are increasing finite Borel equivalence relations, and $E = \bigcup_{n\in\mathbb{N}} R_n$.

(d) Suppose that F is hyperfinite and that $f \colon X \to Y$ witnesses that $E \leq_B F$. Then, since f is countable-to-one, $B = f(X)$ is a Borel subset of Y and f admits a Borel right-inverse $g \colon B \to X$. Let $A = g(B)$. It follows from (b) that the restriction $F \restriction B$ is hyperfinite. Since $E \restriction A$ is Borel isomorphic to $F \restriction B$, we conclude that $E \restriction A$ is hyperfinite as well. Note that A is a complete section for E. It follows that E is hyperfinite by (c). □

Note that it is possible to derive Proposition 3.2 immediately from Proposition 3.3(d) and the definition of smooth equivalence relation.

The following theorem presents two (very useful!) characterization of hyperfiniteness due to Dougherty, Jackson, and Kechris [3].

Theorem 3.4 (Dougherty–Jackson–Kechris)**.** *Let E be a countable Borel equivalence relation. Then the following are equivalent:*

(i) *E is hyperfinite.*
(ii) *E is hypersmooth, i.e., there is an increasing sequence of smooth equivalence relations $R_0 \subseteq R_1 \subseteq \cdots$ such that $E = \bigcup_{n\in\mathbb{N}} R_n$.*
(iii) *There is a Borel action $a \colon \mathbb{Z} \curvearrowright X$ such that $E = E_a$*

Proof. The implication $(1) \implies (2)$ is clear since every finite Borel equivalence relation is smooth (see Proposition 2.6). For $(2) \implies (1)$ we refer the reader to [3].

$(1) \implies (3)$ Fix a Borel linear ordering $<$ on X. We can break X into countably many E-invariant pieces $X = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} X_n$ where each E-class $C \subseteq X_n$ has cardinality n. For each $n \in \mathbb{N}$, define $T \colon X_n \to X_n$ by setting

$$T(x) = \begin{cases} x^+ & \text{if } x \neq \max_<[x]_E \\ \min_<[x]_E & \textit{otherwise}. \end{cases}$$

Next suppose that $X = X_\infty$ without loss of generality. Write $E = \bigcup_{n\in\mathbb{N}} F_n$ as an increasing union $F_0 \subseteq F_1 \subseteq \cdots$ of finite Borel equivalence relation and assume that F_0 is $=_X$. For each n, let $s_n \colon X \to X$ be a Borel selector for F_n.

Given two distinct $x, y \in X$ with $x \mathrel{E} y$ let $n(x,y)$ be the maximum n such that $s_n(x) \neq s_n(y)$. Then define a Borel binary relation on X by

$$x \prec y \qquad \iff \qquad x \neq y \quad \text{and} \quad x \mathrel{E} y \quad \text{and} \quad s_{n(x,y)}(x) < s_{n(x,y)}(y).$$

Claim 3.4.1. *For each E-class C, the relation $\prec \restriction C$ is a discrete linear order.*

Proof. Exercise. □

Then define $T\colon X \to X$ as the $\prec$-successor map where the order type is $\mathbb{Z}$ and similarly on the classes of order type ω and $-\omega$. (Use the any bijection between $\mathbb{Z}$ and ω.) Then E is the orbit equivalence relation induced on X by a Borel action of the infinite cyclic group $\langle T \rangle$.

(3) $\implies$ (1) Without loss of generality we can assume that E is aperiodic. This is harmless because the restriction of E on the periodic part is smooth, therefore hyperfinite.

A *vanishing sequence of markers* for E is a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$ such that

(1) Each A_n is a Borel complete section for E, i.e., $A_n \cap [x]_E \neq \emptyset$ for all $x \in X$;
(2) $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

Lemma 3.5 (Marker lemma)**.** *Every aperiodic countable Borel equivalence relation admits a sequence of vanishing markers.*

Proof. (Slaman–Steel) Without loss of generality, assume that $X = 2^{\mathbb{N}}$. For any $x \in 2^{\mathbb{N}}$, and any $n \in \mathbb{N}$, let $s_n(x)$ be the be the least $s \in 2^n$ (in the lexicographic order) such that $[x]_E \cap \mathcal{N}_s$ is infinite. Here, $\mathcal{N}_s = \{x \in 2^{\mathbb{N}} : x \supseteq s\}$. Then define

$$x \in A_n \iff x \restriction n = s_n(x).$$

Then $(A_n : n \in \mathbb{N})$ is a decreasing sequence of Borel sets that intersects each E-class in infinitely many elements. To see this fix $n \in \mathbb{N}$ and $x \in X$. Then the set $[x]_E \cap N_{s_n(x)}$ is infinite. And for every $y \in [x]_E$, we have $s_n(y) = s_n(x)$ therefore $[x]_E \cap N_{s_n(x)} \subseteq A_n \cap [x]_E$. However, the descending chain $A_0 \supseteq A_1 \supseteq \cdots$ need not be vanishing. Fortunately, the Borel set $A = \bigcap_{n\in\mathbb{N}} A_n$ intersects each equivalence class in at most one point. So $(A_n \setminus A : n \in \mathbb{N})$ forms a vanishing sequence of markers as desired. □

Let $T\colon X \to X$ be a Borel bijection which generates the $\mathbb{Z}$-action and let $<$ be the Borel partial order on X defined by

$$x < y \qquad \iff \qquad \exists n > 0\,(T^n(x) = y).$$

Then $<$ gives a $\mathbb{Z}$-ordering of every E-class. By the marker lemma there is a vanishing sequence of markers $(A_n : n \in \mathbb{N})$ for E. Define

$$Y = \{x \in X : \text{there is } n \in \mathbb{N} \text{ such that } A_n \cap [x]_E \text{ has a least or greatest element}\}.$$

Note that Y is an E-invarant Borel subset and that $E \restriction Y$ is smooth because we can define a Borel selector. So without loss of generality, $Y = \emptyset$. Thus for each $x \in X$

and each $n \in \mathbb{N}$, $A_n \cap [x]_E$ is unbounded in both directions. Therefore, we can define an increasing union of finite Borel equivalence relations by

$$x\, F_n\, y \iff x = y \text{ or } (x\, E\, y \text{ and } A_n \cap [x, y] = A_n \cap [y, x] = \emptyset),$$

where $[x, y] = \{a \in [x]_E : x \leq a \leq y\}$. It is clear that every equivalence class is finite since any F_n-equivalent point must be between two elements $a, b \in A_n$ and since the classes of E are discretely ordered by $<$ the interval is finite. Since $\bigcap_{n\in\mathbb{N}} A = \emptyset$, it follows that $\bigcup_{n\in\mathbb{N}} F_n = E$ as desired. □

Corollary 3.6. *The Vitali equivalence relation E_v is hyperfinite.*

Proof. Let $R_n = E^{\mathbb{R}}_{G_n}$, where $G_n = \mathbb{Z}[\frac{1}{n!}]$ for the G_n-action on $\mathbb{R}$ by translation. Each R_n is smooth as $[0, \frac{1}{n!})$ is a Borel transversal. □

Another example of a hyperfinite non-smooth equivalence relation is the following.

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Consider the Borel isomorphism $T_\alpha \colon \mathbb{S}^1 \to \mathbb{S}^1, x \mapsto e^{\alpha i \pi} x$. For any $x.y \in \mathbb{S}^1$ let $x\, E_\alpha\, y \iff \exists n \in \mathbb{Z}(T^n(x) = y)$. Since α is irrational, T_α is not periodic. So the cyclic group generated by T_α is infinite; thus, it is isomorphic to $\mathbb{Z}$. It is clear how to write E_α as the orbit equivalence relation induced by a Borel $\mathbb{Z}$-action.

3.1. **A gentle introduction to countable amenable groups.** In this subsection we introduce the important concept of countable (discrete) amenable groups. As we will discuss in Subsection 3.2, amenability is closely related to hyperfiniteness.

Consider the free group $\mathbb{F}_2 = \langle a, b\rangle$, and let us assume that every element of the group starts with \$1. Thus, if $f_0(g)$ denotes the amount of money possessed by element g at time $t = 0$, then $f_0(g) = \$1$, for all $g \in \mathbb{F}_2$. Now, everyone gives one dollar to the person next to them who is closer to the identity. (That is, if $g = x_1 x_2 \cdots x_n$ is a reduced word, with $x_i \in \{a^{\pm 1}, b^{\pm 1}\}$ for each i, then g gives its dollar to $g' = x_1 x_2 \cdots x_{n-1}$. The identity element does not give away any money.)

Then, let f_1 denote the amount of money possessed now (at time $t = 1$), we have $f_1(g) = \$3$ for all g (except that $f_1(e) = \$5$). Thus, everyone has more than doubled their money. Furthermore, this result was achieved by moving the money only a bounded distance. A Ponzi scheme[5] on $\mathbb{F}_2$ is a function describing this arrangement.

Definition 3.7. A *Ponzi scheme* on a group G is a function $M \colon G \to G$, such that:

(1) $|M^{-1}(g)| \geq 2$ for all $g \in G$;

[5] A Ponzi scheme is a fraudulent investment scheme that pays returns to earlier investors using money from later investors. The scheme eventually fails because the earnings are less than the payments. Charles Ponzi ran a Ponzi scheme in the early 1900s that defrauded 30,000 Americans of about \$10 million.

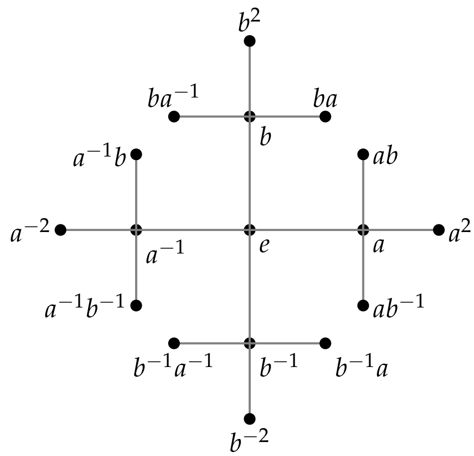

FIGURE 2. The Cayley graph of $\mathbb{F}_2$ with respect to the generating set $\{a^{\pm 1}, b^{\pm b}\}$

(2) There is a finite set $S \subseteq G$ such that $M(g) \in gS$ for all $g \in G$.

Roughly speaking (2) guarantees that money is passed to a bounded distance. We just saw that there is a Ponzi scheme on $\mathbb{F}_2$ — in that example we can let $S = \{e, \pm a, \pm b\}$.

One can prove that if G admits a Ponzi scheme, then the growth of G is exponential. We deduce that there is no Ponzi scheme on $\mathbb{Z}^n$ for all $n \geq 1$. (We do not discuss group growth in the lecture notes. The interested reader is referred to [24].)

Let G be a countable group. Fix a finite subset $S \subseteq G$ and some $\epsilon > 0$. A finite, nonempty subset $F \subseteq G$ is said to be (S, ϵ)-*almost invariant* if and only if

$$|F \cap gF| > (1 - \epsilon)|F| \qquad \text{for all } g \in S.$$

Definition 3.8. A countable group G is *amenable* if and only if G admits an (S, ϵ)-almost invariant finite subset for all finite $S \subseteq G$ and $\epsilon > 0$.

It is straight-forward to see that $\mathbb{Z}$ is amenable. For any finite $S \subseteq \mathbb{Z}$ and $\epsilon > 0$, let $F = [-n, n]$ for n sufficiently large. (For example, let $n \geq \max\{|s| : s \in S\}\epsilon^{-1}$.) A direct computation also shows that all $\mathbb{Z}^n$ are amenable. More generally, all countable abelian groups are amenable.

The following characterization of amenable groups is due to Gromov [6].

Theorem 3.9 (Gromov)**.** *Let G be a countable group. There exists a Ponzi scheme on G if and only if G is not amenable.*

Proof of Theorem 3.9($\Rightarrow$). We only discuss the forward implication. The proof uses the following lemma.

Claim 3.9.1. *If G is amenable, S is a finite subset of G, and $\epsilon > 0$, then there exists a finite subset $F \subseteq G$, such that $|SF| < (1+\epsilon)|F|$.*

Proof of Claim 3.9.1. Let $n = |S|$. Choose F so that $|F \cap gF| > (1 - (\epsilon/n))|F|$ for all $g \in S$. It follows that $|gF \setminus F| < (\epsilon/n)|F|$ for all $g \in S$. Therefore,

$$|SF - F| < n(\epsilon/n)|F| = \epsilon|F|.$$

Since $|SF| - |F| \leq |SF \setminus F|$, we obtain the desired statement. □

Let M be a Ponzi scheme on G. By Definition 3.7(2) there is a finite set $S \subseteq G$ such that $M(g) \in gS$ for all $g \in G$. By the claim there is a finite set F, such that $|SF^{-1}| =< 2|F^{-1}|$. Then $|FS^{-1}| < 2|F^{-1}| = 2|F|$. This is impossible because Definition 3.7(1) yields that M is at least 2-to-1 and $M^{-1}(F) \subseteq FS^{-1}$. □

3.2. **Amenability and hyperfiniteness.** We discuss some consequences of the following major theorem of Jackson, Kechris, and Louveau [9].

Theorem 3.10. *Let G be a countable nonamenable group, let X be a free standard Borel G-space and μ be a G-invariant probability measure on. Then E_G^X is not hyperfinite.*

For an infinite countable group G, let $2^G = \{f\colon G \to \{0,1\}\}$. We equip 2^G with the product topology. The sets

$$U_g = \{f \mid f(g) = 1\} \text{ and } U_g^c = \{f \mid f(g) = 0\}$$

form a subbase of (cl)open sets for this product. Denote by μ the $\{\frac{1}{2}, \frac{1}{2}\}$-product measure on 2^G.

Consider the continuous action $G \curvearrowright 2^G$ by left-shift. That is, for any $g \in G$ and $f \in 2^G$, define

$$g \cdot f(x) = f(g^{-1}x).$$

Note that whenever A is the subset of G such that $x \in A \iff f(x) = 1$, then

$$\begin{aligned} g \cdot f(x) = 1 &\iff f(g^{-1}x) = 1 \\ &\iff g^{-1}x \in A \\ &\iff x \in gA. \end{aligned}$$

We showed that $g \cdot f$ is precisely the characteristic function of the set gA.

Definition 3.11. For any G-action $G \curvearrowright X$ on the set X, let $\mathrm{Fr}(G \curvearrowright X) = \{x \in X : \forall g \in G(g \neq 1_G \implies g \cdot x \neq x)\}$ be the *free part* of the action.

Proposition 3.12. *Let G be a countable group, X be a Polish space, and $G \curvearrowright X$ be a continuous action. Then $\mathrm{Fr}(G \curvearrowright X)$ is a E_G^X-invariant G_δ set.*

Proof. Note that

$$X \setminus \mathrm{Fr}(G \curvearrowright X) = \bigcup_{g \in G \setminus \{1_G\}} (f_g \times id_X)^{-1}(\Delta(X)),$$

where $f_g \times id_X \colon X \to X \times X$ is the continuous map taking x to the pair $(g \cdot x, x)$. □

To simplify our notation, let $E_s(G) = E_G^{2^G}$, and denote by $(2)^G$ the free part of the shift action $\mathrm{Fr}(G \curvearrowright 2^G)$.

Proposition 3.13. $\mu\big((2)^{\mathbb{F}_2}\big) = 1$.

Proof. For $g \in G$, let $X_g = \{x \in 2^G : g \cdot x = x\}$. Note that it suffices to show that $\mu(X_g) = 0$, for all $g \in G \setminus \{1_G\}\}$. Then, it will follow that the complement of $(2)^G$ is a null set, therefore $\mu\big((2)^G\big) = 1$.

Since $\mathbb{F}_2$ is torsion-free, every non-identity $g \in \mathbb{F}$ has infinite order. Then for all $m \neq n$, we have $g^m \neq g^n$. If $x \in X_g$, then observe that $x(g^{2n}) = x(g^{2n+1})$ for all $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned} \mu\big(\{x : g \cdot x = x\}\big) &\leq \mu\big(\{x : \forall n \in \mathbb{Z}\big(x\big(g^{2n}\big) = x\big(g^{2n+1}\big)\big)\}\big) \\ &= \prod_{n \in \mathbb{Z}} \mu\big(\{x : x\big(g^{2n}\big) = x\big(g^{2n+1}\big)\}\big) \\ &= \lim_{n \to \infty} \frac{1}{2^n} = 0 \end{aligned}$$

□

Corollary 3.14. *Let $X = (2)^{\mathbb{F}_2}$ and let $\mathbb{F}_2$ act on X by left shift. The induced equivalence relation $E_{\mathbb{F}_2}^X$ is not hyperfinite. Therefore $E_s(\mathbb{F}_2)$ is not.*

Proof. By Proposition 3.12, X is a Polish. The continuous shift action $\mathbb{F}_2 \curvearrowright X$ is free by definition of X, and preserves the probability measure μ by Proposition 3.13. Therefore $E_{\mathbb{F}_2}^X$ is not hyperfinite by Theorem 3.10 The last statement follows because $E_{\mathbb{F}_2}^X \leq_B E_s(\mathbb{F}_2)$. □

We will see in the forthcoming sections that $E_s(\mathbb{F}_2)$ is not Borel reducible to any countable Borel equivalence relation induced by a free action.

Another aspect we should highlight is that the results of this section use measure theory substantially. This is hard to avoid because there is no analogue of generic ergodidicity to prove non-hyperfiniteness. Because every cber is hyperfinite almost everywhere, any attempt to use Baire category methods is destined to fail. The following result can be found in both [8] and [20].

Theorem 3.15 (Hjorth–Kechris, Sullivan–Weiss–Wright, Woodin)**.** *If E is a cber on a Polish space X, then there is an E-invariant comeager set $C \subseteq X$ so that $E \restriction C$ is hyperfinite.*

Proof. Coming soon. Meanwhile you can read the proof from [11, Theorem 12.1]. □

The next open question is one of the most long-standing open problems about countable Borel equivalence relations. We date back the first formulation to [25].

Question 3.16 (Weiss)**.** *Let G be a countable amenable group and $G \curvearrowright X$ a Borel action. Is E_G^X hyperfinite?*

4. Structural results

In this section we discuss some structural results, i.e., some results about the pre-order $(\mathsf{CBERS}/\sim_B, \leq_B)$.

First we state a theorem of Silver [18].

Theorem 4.1 (Silver's Dichotomy)**.** *For every Borel (in fact, coanalytic) equivalence relation on a standard Borel space, exactly one of the following holds:*

(I) *There are only countably many E-classes, so $E \leq_B =_{\mathbb{N}}$;*
(II) $=_{\mathbb{R}} \leq_B E$.

It follows from Silver's dichotomy that there is no intermediate cber between $=_{\mathbb{N}}$ and $\mathbb{R}$.

Next, recall that $E_=$ is not smooth. (Why[6]?) The following crucial theorem says that reducing E_0 is essentially the only restriction to smoothness.

The original proof of Silver's original result is quite involved and uses a forcing argument. Another proof was found by Ben Miller who deduced Silver's dichotomy from a dichotomy result of Kechris, Solecki, and Todorcevič [13] about Borel combinatorics of Borel colorings for analytic graphs. Another alternative proof of Silver's dichotomy was given by Harrington, who used the so-called Gandy-Harrington topology and methods from recursion theory.

Theorem 4.2 (Harrington-Kechris-Louveau)**.** *Let E be a Borel equivalence relation on a standard Borel space. Then either*

(I) *E is smooth; or*
(II) $E_0 \leq_B E$.

The reason why Theorem 4.2 is named after Glimm and Effros is because it generalizes results of both from the '60s. In fact, Glimm [5] proved a particular case of Theorem 4.2 for equivalence relations induced by continuous action of locally compact Polish groups. Glimm's result was generalized afterwards by Effros [4] those orbit equivalence relations induced by continuous actions of Polish groups that are F_σ.

[6] If you do not remember, review the previous chapters.

Combining Theorem 4.1 with Theorem 4.2 it is clear that an initial segment of the class of countable Borel equivalence relation have the following linear structure

$$\underbrace{=_1 <_B =_2 <_B =_3 <_B \cdots <_B =_{\mathbb{N}} <_B =_{\mathbb{R}}}_{\text{smooth}} <_B E_0 \leq_B \cdots$$

In the remainder of this section we discuss the following result, showing that the pre-order $(\mathsf{CBERS}/\sim_B, \leq_B)$ has a maximum. A cber F is called *universal* if for every cber E, we have $E \leq_B F$.

Theorem 4.3 (Dougherty–Jackson–Kechris)**.** *There is a universal cber E_∞.*

Nowadays, we know several examples of universal cbers. (E.g., see [10, Chapter 12].) Clearly they are all bi-reducible to each other. We first present the first instance of universality as pointed out in Dougherty–Jackson–Kechris [3].

Proof of Theorem 4.3. Consider the Polish space $(2^{\mathbb{N}})^{\mathbb{F}_\infty} = \{f \mid f\colon \mathbb{F}_2 \to 2^{\mathbb{N}}\}$ with the obvious product topology. Define E_∞ as the orbit equivalence relation induced by the shift action of $\mathbb{F}_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{F}_\infty}$; that is,

$$g \cdot f(x) = f(g^{-1}x),$$

for all $g \in \mathbb{F}_\infty$ and $f\colon \mathbb{F}_\infty \to 2^{\mathbb{N}}$.

Fix any cber E on a standard Borel space X. Without loss of generality let $E = E^X_{\mathbb{F}_\infty}$ for some Borel action $\mathbb{F}_\infty \curvearrowright X$. One obtains such an $\mathbb{F}_\infty$-action by lifting the G-action abotained by applying Feldman-Moore's theorem through a surjective homomorphism $\mathbb{F}_\infty \to G$. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of Borel sets $U_n \subseteq X$ which separates points.

Then define a map $X \to (2^{\mathbb{N}})^{\mathbb{F}_\infty}$ by setting

$$f_x(a)(i) = 1 \iff a^{-1}x \in U_i.$$

Since the U_n's separate points, the map f is one-to-one. To see that f is a Borel reduction from $E = E^X_{\mathbb{F}_\infty}$ to E_∞ note that for any $a \in \mathbb{F}_\infty$ and $i \in \mathbb{N}$, we have

$$\begin{aligned} g \cdot f_x(a)(i) &\iff f_x(g^{-1}a)(i) \\ &\iff (a^{-1}g) \cdot x \in U_i \\ &\iff a^{-1} \cdot (g \cdot x) \in U_i \\ &\iff f_{gx}(a)(i) = 1. \end{aligned}$$

Thus $g \cdot f_x = f_{g \cdot x}$. Now, if $y = g \cdot x$ for some $g \in \mathbb{F}_\infty$, then $f_y = f_{g\cdot x} = g \cdot f_x$. And this shows that $x \mathrel{E} y \implies f_x \mathrel{E} f_y$. For the converse, note that if $f_y = g \cdot f_x$ for some $g \in \mathbb{F}_\infty$, then $f_y = f_{g\cdot x}$, which yields that $y = g \cdot x$ by injectiveness. $\square$

A little bit more effort will show that there is a "simpler" universal cber that we already know from previous sections — the induced by the shift-action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$.

Notation 4.4. For any Polish space Y, and countable group G. Let G act on Y^G by shift and let $E(G,Y)$ denote the corresponding orbit equivalence relation on Y^G. That is, $E(G,Y) = E_G^{Y^G}$.

Proposition 4.5. *Suppose that G is a subgroup of H. Then $E(G,Y) \leq_B E(H,Y)$ for any Polish space Y.*

Proof. Fix any $y_0 \in Y$. Then define $Y^G \to Y^H$, $p \mapsto p^*$ by

$$p^*(x) = \begin{cases} p(x) & x \in G \\ y_0 & \text{otherwise.} \end{cases}$$

Checking that $p \mapsto p^*$ is a Borel reduction is left to the reader. □

Proposition 4.6. *Suppose that G is homomorphic image of H. Then $E(G,Y) \leq_B E(H,Y)$ for any Polish space Y.*

Proof. Let $\pi \colon H \to G$ be a surjective homomorphism. Then we define $Y^G \to Y^H$, $p \mapsto p^*$ by setting $p^*(x) = p(\pi(x))$. We claim that the map $p \mapsto p^*$ is a Borel reduction.

First suppose that $q = g \cdot p$ for some $g \in G$. Then

$$\begin{aligned} q^*(x) &= q(\pi(x)) \\ &= p(g^{-1}\pi(x)) \\ &= p(\pi(h^{-1})\pi(x)) \qquad \text{for some } h \in H \\ &= p(\pi(h^{-1}x)) \\ &= p^*(h^{-1}x) \\ &= h \cdot p^*(x) \end{aligned}$$

Conversely, suppose that $q^* = h \cdot p^*$ for some $h \in H$. We claim that $q = \pi(h) \cdot p$. To see this, for any $x \in G$, let h_x be an element of H such that $\pi(h_x) = x$. Then we see that for all $x \in G$, $q(x) = q^*(h_x) = p^*(h^{-1}h_x) = p(\pi(h)^{-1}\pi(h_x)) = \pi(h) \cdot p(x)$. □

Now we are ready to state the following useful result:

Theorem 4.7. *$E(\mathbb{F}_2, 2)$ is a universal cber.*

To prove Theorem 4.7 we produce the following chain of Borel reduction.

$$\begin{aligned}
E(\mathbb{F}_\infty, 2^{\mathbb{N}}) &\cong_B E(\mathbb{F}_\infty, 2^{\mathbb{Z}\setminus\{0\}}) \\
&\leq_B E(\mathbb{F}_2, 2^{\mathbb{Z}\setminus\{0\}}) \qquad \text{because } \mathbb{F}_\infty \hookrightarrow \mathbb{F}_2 \\
&\leq_B E(\mathbb{F}_2 \times \mathbb{Z}, 3) \\
&\leq_B E(\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2) \\
&\leq_B E(\mathbb{F}_\infty, 2) \qquad \text{because } \mathbb{F}_\infty \times \mathbb{Z} \times \mathbb{Z}_2 \text{ surjects onto } \mathbb{F}_\infty \\
&\leq_B E(\mathbb{F}_2, 2) \qquad \text{because } \mathbb{F}_\infty \hookrightarrow \mathbb{F}_2
\end{aligned}$$

Even though it may sound surprising, $\mathbb{F}_2$ has several subgroups isomorphic to $\mathbb{F}_\infty$. For example if $\mathbb{F}_2 = \langle a, b \rangle$, then the (necessarily free) subgroup $\langle a^n b^n : n \geq 1 \rangle$ has infinite rank.

The following two propositions will conclude the proof.

Proposition 4.8. *For any countable group G, we have $E(G, 2^{\mathbb{Z}\setminus\{0\}}) \leq_B E(G \times \mathbb{Z}, 3)$*

Proof. Define the map

$$\begin{aligned}
(2^{\mathbb{Z}\setminus\{0\}})^G &\to 3^{G\times\mathbb{Z}} \\
p &\mapsto p^*,
\end{aligned}$$

by setting

$$p^*(g, n) = \begin{cases} p(g)(n) & n \neq 0, \\ 2 & \text{otherwise.} \end{cases}.$$

First note that $p^*(g, n) = 2$ precisely when $n = 0$.

Now suppose that $p, q \in (2^{\mathbb{Z}} \setminus \{0\})^G$. If $q = g \cdot p$, then a direct computation shows that $q^* = (g, 0) \cdot p^*$.

Next, suppose that $q^* = (g, n) \cdot p^*$. We consider two cases separately.

Case 0: If $n = 0$, then we have $q = g \cdot p$ because

$$\begin{aligned}
g \cdot p(x)(k) &= p(g^{-1}x)(k) \\
&= p^*(g^{-1}x, k) \\
&= (g, 0) \cdot p^*(x, k) \\
&= q(x, k) \\
&= q(x)(k).
\end{aligned}$$

Case 1: If $n \neq 0$, then for any $k \in \mathbb{Z} \setminus \{0\}$ we have $q^*(x, k) = (g, n) \cdot p^*(x, k) = p^*(g^{-1}x, k - n)$. For $k = n$, it follows that $2 \neq q^*(x, k) = p^*(g^{-1}x, 0) = 2$, a contradiction. So this case never happens to be true. □

Proposition 4.9. *For any countable group G, we have $E(G, 3) \leq_B E(G \times \mathbb{Z}_2, 2)$*

Proof. Exercise, or have a look at [3]. □

Corollary 4.10. *If G contains a subgroup isomorphic to $\mathbb{F}_2$, then $E(G,2)$ is universal. In particular, for any $n \geq 2$, $E(\mathbb{F}_n,2)$ is universal.*

Proof. Combine the results of Theorem 4.7 with Proposition 4.5. □

Definition 4.11. A countable group G is said to be *action universal* if there exists a standard Borel X and a Borel action of G on X such that E_G^X is universal.

It follows from Corollary 4.10 that if the countable group G has a nonabelian free subgroup, then G is action universal. No other examples of action universal groups are currently known. On the other hand, it is known that amenable groups are not action universal. Hence, Thomas [22] posed the following question, which, as of today, is still open.

Question 4.12 (Thomas)**.** *Is it true that if G is a countable group, then the following statements are equivalent:*

(i) *G is action universal.*
(ii) *G contains a nonabelian free subgroup.*

Another open problem concerns understanding $\leq_B$ versus inclusion. We include a famous conjecture of Hjorth.

Conjecture 4.13 (Hjorth)**.** If E is a universal countable Borel equivalence relation on the standard Borel space X and F is a countable Borel equivalence relation such that $E \subseteq F$, then F is also universal.

5. TREEABILITY

Let E be a countable Borel equivalence relation on a standard Borel space X. A *Borel graphing* of E is a Borel graph $\mathcal{G} = (X,R)$ whose connected components are the E-classes. A *Borel treeing* of E is a Borel graphing of E whose connected components are acyclic.

Definition 5.1. A Borel equivalence relation E is *treeable* if it admits a Borel treeing.

Example 5.2. It is not hard to see that all hyperfinite equivalence relations are treeable, since they are induced by a Borel $\mathbb{Z}$-action.

Proposition 5.3. *Let $\mathbb{F}_n$ be the free group on n generators. Suppose that $\mathbb{F}_n$ acts freely on a standard Borel space X in a Borel fashion. Then $E_{\mathbb{F}_n}^X$ is treeable.*

Proof. Let $\mathbb{F}_n = \langle g_1, \dots g_n \rangle$ be the free group with generators $g_1, \dots, g_n$ and let $\mathbb{F}_n$ act on X freely so that $E = E_{\mathbb{F}_n}^X$. For each $C \in X/E$ and $x, y \in C$, define

$$(x,y) \in T \iff \exists i \leq n\, (g_i \cdot x = y \text{ or } x = g_i \cdot y). \qquad \square$$

For a countable group G and a Polish space Y, denote by $F(G,Y)$ the restriction of $E(G,2)$ to the free part of the action $G \curvearrowright Y^G$. A consequence of the previous proposition is that $F(\mathbb{F}_2, 2)$ is treeable. This observation yields that

$$\text{hyperfinite} \subsetneq \text{treeable}.$$

Proposition 5.4. *Let E be a countable Borel equivalence relation on a a standard Borel space X.*

(i) *If E is treeable and $A \subseteq X$ is Borel, then $E \restriction A$ is treeable.*
(ii) *If $A \subseteq X$ is a Borel complete section E and $E \restriction A$ is treeable, so is E.*
(iii) *If $E \leq_B F$ and F is treeable, then so is E.*
(iv) *If $E \subseteq F$ and F is treeable, so is E*

Proof. (i) Let $E = E_G^X$ for some countable group $G = \{g_i : i \in \mathbb{N}\}$. Since T is treeable, there is a tree for E with connected components T_C, for $C \in X/E$. Now, for every $x \in X$, we define a lexicographic order $<_x$ on the paths in $T_{[x]_E}$ starting at x defined as follows:

$$\begin{aligned}(x, x_1, \dots, x_m) <_x (x, x_1', \dots, x_n') \quad &\Longleftrightarrow \quad m < n \text{ or} \\ &\big(m = n \text{ and } \exists k \leq m \,\big(\, (x_1, \dots, x_{k-1}) = (x_1', \dots, x_{k-1}') \text{ and} \\ &\qquad \big(\exists i (g_i \cdot x_{k-1}) = x_k \text{ and } \forall j < i (g_j \cdot x_{k-1}' \neq x_k')\big)\big).\end{aligned}$$

We also stipulate that $(x) <_x (x, x_1, \dots, x_m)$ for any $m \in \mathbb{N}_+$. Now fix an equivalence class D of $E \restriction A$ and let C be the E-equivalence class such that $D = C \cap A$. For each $x \in C$ let $\rho(x)$ be the end of the $<_x$-least path from x to (some point in) D. Now $\rho \colon C \to D$ and $\{\rho^{-1}(d) : d \in D\}$ is a partition of C.

Claim 5.4.1. *The restriction $T_C \restriction \rho^{-1}(d)$ of the tree T_C on the Borel set $\rho^{-1}(d)$ is a tree, for each $d \in D$.*

Sketch. Note that if $\rho(x) = d$ and the path $(x, x_1, \dots, x_n)$ is the $<_x$-least path from x to D (thus $x_n = d$), then $\rho(x_i) = d$ for $i = 1, \dots, n$. □

Clearly, we also have $\rho(d) = d$ for all $d \in D$. So D meets all sets $\rho^{-1}(d)$, for $d \in D$. Then we define a tree $\tilde{T}_D$ on D by letting

$$(x, y) \in \tilde{T}_D \quad \Longleftrightarrow \quad \exists a \in \rho^{-1}(x),\, b \in \rho^{-1}(y) \,\big((a, b) \in T_C\big).$$

for any $x, y \in D$.

(ii) For each $C \in X/E$, let $T_{C \cap A}$ be the associated tree on $C \cap A$. Let $f \colon X \to A$ be a Borel map with $f(x) \mathrel{E} x$, and $f(x) = x$ for $x \in A$. For example we can define f in this way. Let $E = E_G^X$, for some countable group G and Borel action $G \curvearrowright X$. Also fix an enumeration $\{g_0, g_1, \dots\}$ for G with $g_0 = \mathrm{id}$. Then, let $f(x) = g_{\ell(x)} \cdot x$ where $\ell(x) = \min\{n \in \mathbb{N} : g_n \cdot x \in A\}$.

For each $C \in X/E$, let $T_{A\cap C}$ be the associated tree. We can define a tree T_C from $T_{A\cap C}$ by adding an edge $(x, f(x))$ from each $x \in X \setminus A$. Precisely, we let

$$(x,y) \in T_C \iff$$
$$(x,y) \in T_{C\cap A} \text{ or } (x \notin A \text{ and } y = f(x)) \text{ or } (y \notin A \text{ and } x = f(y)).$$

(iii) Essentially we repeat the same argument of Proposition 3.3(d). Suppose that E is a cber on X and that F is a treeable cber on Y. Any Borel reduction $f\colon X \to Y$ from E to F is countable-to-one. So $B = f(X)$ is a Borel subset of Y and f has a Borel right inverse g. Let $A = g(B)$. Then, $F \restriction B \cong_B E \restriction A$ is treeable by (i). Moreover, since A is a Borel complete section for E is treeable by (ii).

(iv) Whenever $E \subseteq F$, every F class is partitioned into E classes. So we can mimic the last part of the proof of (i) □

5.1. **A characterization of treeability.**

Definition 5.5. Let X be a standard Borel space and E be a countable Borel equivalence relation. A *Borel cocycle for E into G* is a Borel map $\rho\colon E \to G$ such that

$$x\,E\,y\,E\,z \quad\Longrightarrow\quad \rho(x,z) = \rho(y,z)\rho(x,y).$$

Next, we discuss a property of group actions. As we wish to talk about different actions inducing the same equivalence relation, it will be useful to introduce the notation E_a to denote the equivalence relation induced on X by the Borel action $a\colon G \times X \to X$.

Definition 5.6. An action $a\colon G \curvearrowright X, (g,x) \mapsto g \cdot x$ has the *cocycle property* if there is a Borel cocycle $\rho\colon E_a \to G$ for E_a into G with $\rho(x,y) \cdot x = y$ for all $(x,y) \in E_a$.

If the action $a\colon G \times X \to X$ is free, then we can easily define Borel cocycle $\rho\colon E_a \to G$. For each $x\,E_a\,y$ we set $\rho(x,y)$ to be the unique $g \in G$ such that $g \cdot x = y$. The cocycle property characterizes treeability in the following strong sense:

Theorem 5.7 (Hjorth–Kechris '96; Jackson–Kechris–Louveau 2000)**.** *For a countable Borel equivalence relation E on the standard Borel space X the following are equivalent:*

(1) *E is treeable;*
(2) *For every countable group G, every Borel G-action $a\colon G \curvearrowright X$ such that $E = E_a$ has the cocycle property.*
(3) *For every countable group G, and every Borel G-action $a\colon G \curvearrowright X$ such that $E = E_a$ there is a standard Borel space Y and a free G-action $b\colon G \curvearrowright Y$ such that $E \sim_B E_b$.*

Proof. (3) $\Rightarrow$ (1) Let E be a countable Borel equivalence on X satisfying (3). By Feldman–Moore theorem there is a countable group G and some Borel G-action on X such that $E = E^X_G$. Fix any surjection $\pi\colon \mathbb{F}_\infty \to G$. Then lift the G-action through π to the $\mathbb{F}_\infty$-action $a\colon \mathbb{F}_\infty \times X \to X$ defined by $a(g,x) = \pi(g) \cdot x$. Clearly $E = E_a$. Now by (3) there is an $\mathbb{F}_\infty$-action $b\colon Y \curvearrowright Y$ for some standard Borel space such that $E_a \sim_B E_b$. It follows that E_b is treeable by Proposition 5.3, thus E is treeable by 5.4(iii)

(1) $\Rightarrow$ (2) Assume that $E = E_a$, for some Borel G-action $a\colon G \curvearrowright X$ is treeable. Let (X, T) be a treeing for E. First we "make" T directed, by fixing a Borel ordering $<$ of X and by letting $T^+ = T \cap <$. Then define a Borel function ρ^+ on T^+ such that $\rho^+(x, y) \cdot x = y$ for all $(x, y) \in T^+$. (This is not too hard! If $(x, y) \in T^+$, then the set $G_{x,y} = \{g \in G : g \cdot x = y\} \neq \emptyset$. So we can define $\rho(x, y)$ to be the least element in $G_{x,y}$ according to some fixed enumeration of G.) Next, we extend ρ to a function $\rho\colon E \to G$ by setting:

(i) $\rho(x, x) = 1$;
(ii) If $x\,E\,y$ and $(x, x_1, \ldots, x_{n-1}, y)$ is the unique path from x to y in T, put $\rho(x, y) = \rho^*(x_{n-1}, y)\rho^*(x_{n-2}, x_{n-1}) \cdots \rho^*(x, x_1)$ where

$$\rho^*(u, v) = \begin{cases} \rho^+(x, y) & \text{if } (x, y) \in T^+, \\ \rho^+(x, y)^{-1} & \text{if } (y, x) \in T^+. \end{cases}$$

(2) $\Rightarrow$ (3) We refer the reader to [9]. □

5.2. **Universal treeable equivalence relations.** In this subsection we prove the following main theorem.

Theorem 5.8. *The relation $F(\mathbb{F}_2, 2)$ is a universal treeable countable Borel equivalence relation.*

Before we discuss a proof we need the following lemmas.

Lemma 5.9. *Let G be a countable group and $G \curvearrowright X$ freely. Then $E_G^X \leq F(G, 2^{\mathbb{N}})$.*

Proof. It may be convenient to assume that X is a Borel subset of $2^{\mathbb{N}}$. Define a Borel function $X \to (2^{\mathbb{N}})^G, x \mapsto f_x$, by $f_x(g) = g^{-1} \cdot x$. It is straightforward to check that f is one-to-one and Borel. Morever, f is G-equivariant. To see this, let $g \in G$ and $x \in X$. Then, we have

$$f_{g \cdot x}(h) = h^{-1} \cdot (g \cdot x) = (h^{-1}g) \cdot x = f_x(g^{-1}h) = g \cdot f_x(h).$$

It follows that f is a Borel reduction from E_G^X to $E(G, 2^{\mathbb{N}})$.

Next, we show that $f_x \in ((2^{\mathbb{N}}))^G$, for any $x \in X$. Let $g \in G$ be such that $g \cdot f_x(h) = f_x(h)$ for all $h \in G$. Then

$$f_x(g^{-1}h) = f_x(h) \implies (h^{-1}g) \cdot x = h^{-1} \cdot x \implies g \cdot x = x \implies g = \mathrm{id}_G .$$

This shows that the stabilizer of f_x is trivial, thus $f_x \in ((2^{\mathbb{N}}))^G$. □

Lemma 5.10. *Let $G \leq H$ with $G \neq \{\mathrm{id}_G\}$. Then $F(G, X) \leq_B F(H, X)$.*

Proof. Exercise. □

Lemma 5.11 (Dougherty)**.** *Suppose that $G \leq H$ and there is $a \in H$ with $a^n \in G$, for all $n \geq 1$. Then $F(G, 2^{\mathbb{N}}) \leq_B F(H, 2)$.*

Proof. Let a be such that $a^n \in H \setminus G$ for all $n \geq 1$. Define a map $((2))^{\mathbb{N}} \to (2)^H$, $p \mapsto p^*$ by

$$p^*(x) = \begin{cases} 1 & x \in G \\ p(g)(n) & x = ga^{n+1} \text{ for some } g \in G \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $(p, q) \in F(G, 2^{\mathbb{N}}) \implies (p^*, q^*) \in F(H, 2)$. In fact, if $p = g \cdot q$, then $p^* = g \cdot q^*$.

Conversely, suppose that $p^* = h \cdot q^*$ for some $h \in H$. We want to show that $h \in G$. We immediately note the identity $1 = p^*(\mathrm{id}_G) = q^*(h^{-1})$. Since $q^*(h^{-1}) \neq 0$, this means that

$$h^{-1} = g_1 a^k \tag{5.2.1}$$

for some $g_1 \in G$ and $k \geq 1$.

Next, $p^* = h \cdot q^*$ yields that $p^*(hg_1) = q^*(h^{-1}hg_1) = q^*(g_1) = 1$. With an argument similar to the above, we conclude that

$$hg_1 = g_2 a^\ell, \tag{5.2.2}$$

for some $g_2 \in G$ and $\ell \geq 1$. Combining (5.2.1) and (5.2.2) we obtain

$$g_1 = g_1 a^k g_2 a^\ell,$$

therefore $g_2^{-1} = a^{k+\ell}$. This is only possible if $k + \ell = 0$, and since $k, \ell \geq 0$, we conclude that $k = \ell = 0$ and $h = g_1^{-1} \in G$, as desired. □

Proof of Theorem 5.8. Let E be a treeable equivalence relation. We can assume that $E \leq_B E_a$ for some free action $a \colon \mathbb{F}_\infty \curvearrowright Y$ on some standard Borel space Y. Then we have the following chain of reductions:

$$\begin{aligned} E_a &\leq_B F(\mathbb{F}_\infty, 2^{\mathbb{N}}) \\ &\leq_B F(\mathbb{F}_3, 2^{\mathbb{N}}) \\ &\leq_B F(\mathbb{F}_2, 2) \end{aligned}$$

For the last step we use Lemma 5.11 and the fact that $\mathbb{F}_2$ contains an infinite rank nonabelian free groups $\langle x_0, x_1, x_2, \dots \rangle$. Then it is easy to vrify that the group $G = \langle x_0, x_1, x_2 \rangle \cong \mathbb{F}_3$ and the element $a = x_i$, for any $i > 2$, satisfies the assumption of Lemma 5.11. □

6. Ergodic theory and cocycle superrigidity

Let G be a countable group acting on the standard Borel space X in a Borel fashion and μ be a G-invariant probability measure on X. A function $f \colon X \to Y$ is *G-invariant* if $f(g \cdot x) = f(x)$ for all $x \in X$.

Theorem 6.1. *The following are equivalent:*

(1) *The action $G \curvearrowright (X, \mu)$ is ergodic.*
(2) *Every Borel G-invariant map $X \to Y$ is essentially constant. I.e., there is $X_0 \subseteq X$, with $\mu(X_0) = 1$ such that $f \restriction X_0$ is constant.*

Note that Theorem 6.1 can be used to give an alternative proof of Proposition 2.23.

The following definition is a possible strengthening of the classical notion of ergodicity.

Definition 6.2. Let E and F be cbers on X and Y respectively. Let μ be an E invariant probability measure on X. We say that E is *F-ergodic* if and only if for every $f\colon X \to Y$ such that $x_0 \, E \, x_1 \implies f(x_0) \, F \, f(x_1)$, there is some Borel $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f(X_0)$ is contained in a single F-class.

Below, we explain how to use a famous cocycle reduction theory to prove an interesting ergodicity result that will shed light on the structure of (treeable) cbers up to Borel reducibility. First, we define the notion of cocycle for a Borel action.

Definition 6.3. A *Borel (strict) cocycle* for a into the group H is a Borel map $\alpha\colon G \times X \to H$ such that for all $g, h \in G$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \qquad (\text{for all } x \in X)\mu - \text{a.e.}(x).$$

Example 6.4. (1) Suppose that H acts on Y freely and that $F\colon X \to Y$ is such that $x_0 E_G^X \implies f(x_0) \, E_H^Y \, f(x_1)$.

Then for any $g \in G$ and $x \in X$ there is a unique $\alpha(g, x) \in H$ such that $f(g \cdot x) = \alpha(g, x) \cdot f(x)$. It is clear that α is a strict cocycle. We will refer to such an α as *the cocycle associated with the homomorphism f*.

(2) Any group homomorphism $\rho\colon G \to H$, induces the cocycle $\alpha_\rho(g, x) = \rho(g)$ that does not depend on the second coordinate. With a slight abuse of language we will make no distinction between ρ and α_ρ.

The following proposition will show that, as long as we are working with countable groups only, there is no harm in assuming that all cocycles are strict.

Proposition 6.5. *Let G, H be countable groups. For every cocycle $\alpha\colon G \times X \to H$, there is a strict cocycle $\alpha' : G \times X \to H$ such that*

$$\alpha(g, x) = \alpha'(g, x)$$

μ-a.e.(x).

Proof. For any cocycle $\alpha\colon G \times X \to H$, let $A_{g,h} = \{x \mid \alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x)\}$. Then let $A = \bigcap_{g,h \in G} A_{g,h}$. Then let $X_0 = \bigcap_{g \in G} g \cdot A$. It follows that X_0 is G-invariant and $\mu(A) = \mu(X_0) = 1$. The cocycle $\alpha \restriction G \times X_0$ is strict on X_0 by definition. Then define

$$\alpha'(g,x) = \begin{cases} \alpha(g,x) & x \in X_0 \\ 1_H & \text{otherwise.} \end{cases}$$

It follows that α' is strict and agrees with α on X_0, which is a measure one set. □

Definition 6.6. Let $\alpha, \beta \colon G \times X \to H$ be two cocycles. We say that α and β are equivalent (or cohomologous), written $\alpha \sim \beta$, if and only if there is a G-invariant Borel $X_0 \subseteq X$ with $\mu(X_0) = 1$ and some Borel $B \colon X_0 \to H$

$$\beta(g,x) = B(g \cdot x)\alpha(g,x)B(x)^{-1},$$

for all $g \in G, x \in X_0$.

Definition 6.7. Let G be a countable group. We say that the action $a \colon G \curvearrowright X$ is *cocycle superrigid* if and only if every Borel cocycle for a is equivalent to a group homomorphism.

Fact 6.8. (I) *The free shift action* $\mathrm{SL}_3(\mathbb{Z}) \curvearrowright ((2)^{\mathrm{SL}_3(\mathbb{Z})}, \mu)$ *is cocycle superrigid.*
(II) *Every normal* $N \lhd \mathrm{SL}_3(\mathbb{Z})$ *with* $N \neq \{id\}$, *has finite index in* $\mathrm{SL}_3(\mathbb{Z})$.

Fact 6.8(I) is a consequence of a superrigidity result for property (T) group by Sorin Popa [17]. (See also [21].) While Fact 6.8(II) is a consequence of Margulis' normal subgroup theorem. (E.g. see [26].)

Using cocycle superrigidity, we can now prove the following result:

Theorem 6.9. $F(\mathrm{SL}_3(\mathbb{Z}), 2)$ *is* $F(\mathbb{F}_2, 2)$*-ergodic.*

Proof. Let $G = \mathrm{SL}_3(\mathbb{Z})$ and $X = (2)^{\mathrm{SL}_3(\mathbb{Z})}$. Suppose that $f \colon X \to (2)^{\mathbb{F}_2}$ is a Borel map such that $x_0 \, E^X_G \, x_1 \implies (f(x_0), f(x_1)) \in F(\mathbb{F}_2, 2)$. Let $\alpha \colon \mathrm{SL}_3(\mathbb{Z}) \times X \to \mathbb{F}_2$ be the cocycle associated with f, so that $f(g \cdot x) = \alpha(g,x) \cdot f(x)$, for all $g \in G$, and $x \in X$. By cocycle superrigidity, there is some $\mathrm{SL}_3(\mathbb{Z})$-invariant Borel $X_0 \subseteq X$ with $\mu(X_0) = 1$ and some group homomorphism $\rho \colon G \to \mathbb{F}_2$ such that

$$\rho(g) = B(g \cdot x)\alpha(g,x)B(x)^{-1}$$

for every $g \in \mathrm{SL}_3(\mathbb{Z})$, and $x \in X_0$. Define $f' \colon X \to (2)^{\mathbb{F}_2}$ by slightly adjusting f. For all $x \in X$, let $f'(x) = B(x) \cdot f(x)$. Note that $f(x)$ and $f'(x)$ are in the same $\mathbb{F}_2$-orbit. Therefore the ranges of f and f' intersect the same $F(\mathbb{F}_2, 2)$-classes.

Claim 6.9.1. *The homomorphism* ρ *is the cocycle associated with* f'. *I.e., for every* $x \in X_0$, $g \in G$,

$$f'(g \cdot) = \rho(g) \cdot f'(x).$$

Proof of Claim 6.9.1. For all $g \in G$ and $x \in X_0$ we have

$$\begin{aligned} f(g \cdot x) &= \alpha(g, x) f(x) \\ f(g \cdot x) &= (B(g \cdot x)^{-1} \rho(g) B(x)) \cdot f(x) \\ B(g \cdot x) \cdot f(g \cdot x) &= \rho(g) \cdot (B(x) \cdot f(x)) \\ f'(g \cdot x) &= \rho(g) \cdot f'(x). \end{aligned}$$

□

Then note that $K = \ker(\rho) \neq \{id\}$ because $G)$ does not embed into $\mathbb{F}_2$. This is because G is not a free group. Otherwise observe that G has torsion elements such as $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, while $\mathbb{F}_2$ is torsion-free. Therefore $[G : K] < \infty$. (See Fact6.8(II).) Therefore, the quotient group G/K is isomorphic to some finite subgroup of $\mathbb{F}_2$. We conclude that $K = G$, and ρ is the trivial homomorphism. Therefore, the map f' is G-invariant and, by ergodicity, there is some Borel $X_1 \subseteq X_0$ such that $f' \restriction X_1$ is constant. As f and f' touch the same orbits, we conclude that f maps X_1 into a single $F(\mathbb{F}_2, 2)$-class, and this concludes the proof. □

Corollary 6.10. *$F(\mathbb{F}_2, 2)$ is not a universal cber.*

Proof. The obvious product measure on $(2)^{\mathrm{SL}_3(\mathbb{Z})}$ is nonatomic; therefore, there is no Borel reduction from $F(\mathrm{SL}_3(\mathbb{Z}), 2)$ to $F(\mathbb{F}_2, 2)$, which is universal for all treeable equivalence relations. □

Corollary 6.11. *$F(\mathrm{SL}_3(\mathbb{Z}), 2)$ and E_∞ are not treeable.*

Proof. If any of them were treeable, then it would be Borel reducible to $F(\mathbb{F}_2, 2)$ by Theorem 5.8. This is excluded by Theorem 6.9 and the previous corollary. □

6.1. **More application of cocycle superrigidity.** Now we discuss more applications of cocycle superrigidity. Many structural results about cbers were proved at the end of the '90s using Zimmer's theory of cocycle superrigidity. With the technology of today, we can shorten many of those proofs and provide a more conceptual presentation of older results.

Definition 6.12. Let E, F be equivalence relations on X, Y, respectively. Recall that a Borel map $f \colon X \to Y$ is a *weak Borel reduction* if and only if f is countable-to-one and $x_0 \mathrel{E} x_1 \implies f(x_0) \mathrel{F} f(x_1)$.

We also recall a useful strengthening of ergodicity.

Definition 6.13. We say that the action of G on (X, μ) is *strongly mixing* if and only if for any two Borel subsets $A, B \subseteq X$, if $(g_n)_{n \in \mathbb{N}}$ is a sequence of distinct elements of G, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A)\mu(B).$$

Notice that any strongly mixing action is necessarily ergodic. To see this, let A be a G-invariant Borel subset of X. Then $g(A) \cap A = A$ for all $g \in G$. Hence, for any sequence $(g_n)_{n\in\mathbb{N}}$ of distinct elements of G,

$$\mu(A) = \lim_{n\to\infty} \mu(A) = \lim_{n\to\infty} \mu(g_n(A) \cap A) = \mu(A)\mu(A),$$

therefore $\mu(A)$ is either 0 or 1.

Lemma 6.14. *If G is a countable infinite group and μ is the product measure on $(2)^G$, then the action of G on $((2)^G, \mu)$ is strongly mixing.*

Proof. See [7, Proposition A6.1]. □

Note that if the action of G on (X, μ) is strongly mixing and H is an infinite subgroup of G, then the action of H is also strongly mixing; in particular, the action $H \curvearrowright (X, \mu)$ is ergodic.

We will be using the following lemma more than once.

Lemma 6.15. *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and H be any countable group acting freely and in a Borel way on the standard Borel space H. If $f \colon (2)^\Gamma \to Y$ is a weak Borel reduction from $F(\Gamma, 2)$ to E^Y_G, then there are:*

(i) *a group embedding $\beta \colon \Gamma \to H$, and*
(ii) *a weak Borel reduction $f' \colon (2)^\Gamma \to Y$ such that $f'(g \cdot x) = \beta(g) f'(x)$ almost everywhere.*

Moreover, if f is a Borel reduction, then so is f'.

Proof. Let $f \colon (2)^\Gamma \to Y$ be a countable-to-one Borel map such that $(x_0, x_1) \in F(\Gamma, 2) \implies f(x_0) \, E^Y_H \, f(x_1)$. Also, let μ be the obvious product measure on $(2)^\Gamma$.

Since the H acts freely, we can define a corresponding Borel cocycle $\alpha \colon \Gamma \times (2)^\Gamma \to H$ such that

$$\alpha(g, x) \cdot f(x) = f(g \cdot x),$$

for all $g \in G$ and $x \in X$. Any irreducible lattice in a semisimple By cocycle superrigidity, there are some Γ-invariant Borel $X_0 \subseteq X$, some group homomorphism $\beta \colon \Gamma \to H$ and some Borel function $b \colon X \to \Delta$ such that $\beta(g) = b(g \cdot x)\alpha(g, x)b(x)^{-1}$ for all $x \in X_0$. Then let $f' \colon (2)^\Gamma \to Y$ be the corresponding adjustment of f defined by $f'(x) = b(x) \cdot f(x)$. Observe that $f(x)$ is in the same H-orbit as $f'(x)$. So f' is a weak Borel reduction (resp., Borel reduction) only if f is as well. Also, note that on a full measure set we have $f'(g \cdot x) = \beta(g) \cdot f'(x)$, namely β is the cocycle associated with f'.

Now let $K = \ker(\beta)$. By Fact 6.8(I) it suffices to show that K is finite. To see this, suppose that K is infinite. Then the action $K \curvearrowright (X_0, \mu)$ is ergodic because the action $\Gamma \curvearrowright (2)^\Gamma$ is strongly mixing and the map f' is K-invariant. Therefore, there is some Borel $X \subseteq X_0$ with $\mu(X) = 1$ such that $f' \restriction X$ is constant. It follows that $f(X)$ is contained into a single E_H^Y- class and this is not possible because f is countable-to-one and μ is not atomic. $\square$

6.1.1. *A note on the cocycle property.* We already showed that $F(\mathrm{SL}_3(\mathbb{Z}),2)$ is not treeable in Corollary 6.11. Therefore, by the characterization of Theorem 5.7 there is a Borel action of some countable group inducing $F(\mathrm{SL}_3(\mathbb{Z}),2)$ that does not have the cocycle property. We can find such an action using cocycle superrigidity. The following was proved by Thomas [23].

Theorem 6.16 (Thomas '24)**.** *Let $\pi\colon \mathbb{F}_\infty \to \mathrm{SL}_3(\mathbb{Z})$ be any onto homomorphism. Define the action $a\colon \mathbb{F}_\infty \curvearrowright (2)^{\mathrm{SL}_3(\mathbb{Z})}$ by letting $a(g,x) = \pi(g)\cdot x$. Then a does not have the cocycle property.*

Proof. Suppose otherwise towards a contradiction. Let $\rho\colon E_a \to \mathbb{F}_\infty$ be a Borel function such that $x\, E_a\, y\, e_a\, z \implies \rho(x,z) = \rho(y,z)\rho(x,y)$, and

$$\rho(x,y)\cdot x = y, \tag{6.1.1}$$

for all $x\, E_a\, y$. Define the Borel function $\alpha\colon \mathrm{SL}_3(\mathbb{Z}) \to \mathbb{F}_\infty$ by setting $\alpha(g,x) = \rho(x, g\cdot x)$ for all $x \in X$ and $g \in G$. Note that

$$\alpha(hg,x) = \rho(x,(hg)\cdot x) = \rho(g\cdot x, h\cdot(g\cdot x))\rho(x,g\cdot x) = \alpha(h,g\cdot x)\alpha(g,x),$$

showing that α is a cocycle for the shift action of $\mathrm{SL}_3(\mathbb{Z})$. Also note that $E_a = F(\mathrm{SL}_3(\mathbb{Z}),2)$ and the identity map $(2)^{\mathrm{SL}_3(\mathbb{Z})} \to (2)^{\mathrm{SL}_3(\mathbb{Z})}$ is a Borel reduction from $F(\mathrm{SL}_3(\mathbb{Z}),2)$ to E_a. Arguing as in the proof of Lemma 6.15, we conclude that there is an embedding $\mathrm{SL}_3(\mathbb{Z}) \to \mathbb{F}_\infty$. This is not possible because $\mathrm{SL}_3(\mathbb{Z})$ is not a free group. $\square$

6.1.2. *Essentially free cbers and universality.* We say that E is *essentially free* if there a countable group G and a free standard Borel G-space X such that $E \le_B E_G^X$.

Theorem 6.17. *If E is essentially free, then E is not universal.*

The proof uses the following lemma, which follows for instance from a result of Ol'shanskii [16]. (See also [22, Theorem 4.10 and Remark 4.11].)

Lemma 6.18. *There are continuum many pairwise non-isomorphic finitely generated simple groups $\{H_\eta : \eta < 2^{\aleph_0}\}$ whose shift actions $H_\eta \curvearrowright ((2)^{H_\eta},\mu)$ are cocycle superrigid.*

Proof. Let G be a countable group and X be a free standard Borel G-space such that $E \le_B E_G^X$. Since G has countably many finitely generated subgroup. Let H_η be as in Lemma 6.18 such that $H_\eta \not\hookrightarrow G$. Then it is easy to show that $F(H_\eta,2) \not\le_B E$, using the fact that the only group homomorphism from H_η to G is the trivial one. $\square$

Corollary 6.19. *For any countable group G, the free equivalence relation $F(G,2)$ is not universal.*

6.1.3. *Large anti-chain of cbers.* A groundbreaking result of Adams and Kechris [1] was the existence of a family of pairwise $\leq_B$-incomparable cbers of size continuum. The gist of their argument is showing ergodicity for the shift-action of certain linear groups. Their methods employ a constellation of different cocycle superrigidity results due to Zimmer. We discuss a streamlined proof of the original ergodicity result of Adams and Kechris below, using the same groups.

Let $\mathbb{P}$ be the set of prime numbers. For a subset $S \subseteq \mathbb{P}$, let $\Gamma_S = \mathrm{SO}_7(\mathbb{Z}[1/S])$, i.e., the group of special orthogonal matrices whose entry denominators factor as a product of prime numbers in S. We let $\Gamma_p = \Gamma_{\{p\}}$.

Theorem 6.20 (Adams–Kechris'00)**.** *Let $S, T \subseteq \mathbb{P}$. If $S \not\subseteq T$, then $F(\Gamma_S, 2)$ is $F(\Gamma_T, 2)$-ergodic.*

We will be using the following neat consequence of Magulis' superrigidity for S-arithmetic groups. (See [14].)

Lemma 6.21. *If $p \notin T$, then Every group homomorphism from Γ_p to Γ_T has finite image.*

Proof. Fix $p \in S \setminus T$. Let $f\colon (2)^{\Gamma_S} \to (2)^{\Gamma_T}$ be a Borel homomorphism from $F(\Gamma_S, 2)$ to $F(\Gamma_T, 2)$. Consider the action of $a\colon \Gamma_p \curvearrowright (2^{\Gamma_S})$. Since $E_a \subseteq F(\Gamma_S, 2)$, f is also a Borel homomorphism from E_a to $F(\Gamma_T, 2)$. Let $\alpha\colon \Gamma_p \times (2)^{\Gamma_S}$ be the Borel cocycle associated with f so that $f(g \cdot x) = \alpha(g, x) \cdot f(x)$. Then α is equivalent to some group homomorphism $\rho\colon \Gamma_p \to \Gamma_T$. Therefore, there is a G-invariant Borel subset $X_0 \subseteq (2)^{\Gamma_S}$, and a Borel map $B\colon X \to \Gamma_T$ such that

$$\rho(g) = B(g \cdot x)\alpha(g, x)B(x)^{-1}$$

for all $x \in X_0$. So define $f'\colon X_0 \to (2)^{\Gamma_T}, f'(x) = B(x) \cdot f(x)$. As usual, we have $f'(g \cdot x) = \rho(g) \cdot f'(g \cdot x)$.

Now, for any $x \in X_0$ define

$$\Phi(x) = \{\rho(g) \cdot f'(x) : g \in \Gamma_p\}.$$

By Lemma 6.21 $\Phi(x)$ is a finite subset of $(2)^{\Gamma_T}$. So we can write Φ as a Borel map from (X_0, μ) to some Borel space Z, whose elements are the finite subsets of $(2)^{\Gamma_T}$. Moreover, note that Φ is a Γ_p-invariant map. It follows by ergodicity that there is a Borel $X_1 \subseteq X_0$ with $\mu(X_0) = 1$ such that $\Phi \restriction X_1$ is constant. It follows that $f'(X_1)$ is contained in a single $F(\Gamma_T, 2)$-class. Since the ranges of f and f' intersect the same classes, we are done. $\square$

7. Martin's conjecture

In this section, we discuss a crucial interplay between recursion theory and descriptive set theory.

Definition 7.1. For $r, s \in 2^{\mathbb{N}}$ we say that r is *Turing reducible* to s (written $r \leq_T s$) if and only if there exists an oracle Turing machine that computes r when its oracle tape contains s.

We say that r and s are *Turing equivalent* (written $r \equiv_T s$) if and only if $r \leq_T s$ and $s \leq_T r$.

It is clear from the definition that $\equiv_T$ is a countable Borel equivalence relation. In fact, every point $s \in 2^{\mathbb{N}}$ has countably many $\leq_T$-predecessors.

Definition 7.2. For each $r \in 2^{\mathbb{N}}$, the corresponding *cone* above r is the Borel set $C(r) = \{s \in 2^{\mathbb{N}} \mid r \leq_T s\}$.

The following theorem is a neat application of the Borel determinacy theorem due to Martin.

Theorem 7.3 (Martin). *If $A \subseteq 2^{\mathbb{N}}$ is a Borel $\equiv_T$-invariant set, then either A contains a cone or $2^{\mathbb{N}} \setminus A$ contains a cone.*

Proof. The proof uses Martin's result that every Borel game is determined. Let $A \subseteq 2^{\mathbb{N}}$ be a Borel $\equiv_T$-invariant set. Consider the Gale-Stewart game $G(A)$ with payoff with payoff set A. That is, the game described by the following plays:

I	$s_0 \in \{0,1\}$		$s_2 \in \{0,1\}$	$\cdots$		$s_{2n} \in \{0,1\}$	
II		$s_1 \in \{0,1\}$			$\cdots$		$s_{2n+1} \in \{0,1\}$

where Player I wins if and only if $s = (s_n)_{n \in \mathbb{N}} \in A$. By Borel determinacy theorem, one of the two players has a winning strategy. We show that if Player I has a winning strategy , then A contains a cone. If Player II has a winning strategy, then the argument showing that $2^{\mathbb{N}}$ contains a cone is anapogous.

If $\sigma \colon 2^{<\mathbb{N}} \to 2$ is a winning strategy for Player I, then consider the cone $C = C(\sigma)$. We are going to show that $C \subseteq A$. To see this, let $t \in C$ so we have $\sigma \leq_T t$. Now consider the following round of $G(A)$

- Player I plays some $s_0 \in \{0,1\}$, and then follows the winning strategy σ (i.e., $s_{2n} = \sigma(s_0 {}^\frown \cdots {}^\frown s_{2n-1})$);
- Player II plays $s_{2n+1} = t_n$.

Since σ is a winning strategy for Player I, it follows that $s \in A$. Moreover, it is clear that $t \leq_T s$. On the other hand, we have $\sigma \leq_T t$ and $s \leq \sigma \oplus t$, which implies that $s \leq_T t$. So $s \equiv_T t$. Since A is $\equiv_T$-invariant, we have $t \in A$. □

Corollary 7.4. *If $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel function such that $x \equiv_T y$ implies $f(x) = f(y)$, then there is a cone $C \subseteq 2^{\mathbb{N}}$ such that $f \upharpoonright C$ is constant.*

Proof. Let $U_n = \{x \in 2^{\mathbb{N}} : x_n = 1\}$. It follows by Theorem 7.3 that for each $n \in \mathbb{N}$, either $\{x \in 2^{\mathbb{N}} : f(x)_n = 0\}$ or $\{x \in 2^{\mathbb{N}} : f(x)_n = 1\}$ contains a cone. So, for each $n \in \mathbb{N}$, there is $\epsilon_n \in \{0,1\}$ such that $X_n = \{x \in 2^{\mathbb{N}} : f(x)_n = \epsilon_n\}$ contains a cone C_n. Let $D = \bigcap_n C_n$. Clearly $f \upharpoonright D$ is constant. Moreover, we notice that $\bigcap_n C_n$ contains a cone. To see this, let $r_n \in 2^{\mathbb{N}}$ be such that $C_n = C(r_n) = \{s \in 2^{\mathbb{N}} : r_n \leq_T s\}$. Then let $\bigoplus_n r_n$ be the join of the r_n's. Clearly, we have $r_n \leq_T \bigoplus_n r_n$ for all $n \in \mathbb{N}$. It follows that $C(\bigoplus_n r_n) = \{s \in 2^{\mathbb{N}} : \bigoplus_n r_n \leq_T s\} \subseteq D$. □

We point out another nice consequence of Martin's theorem. We say that $A \subseteq 2^{\mathbb{N}}$ is $\leq_T$*-cofinal* if for every $s \in 2^{\mathbb{N}}$ there is $s' \in A$ such that $s \leq_T s'$.

Exercise 7.5. Let $A \subseteq 2^{\mathbb{N}}$ be a $\equiv_T$-invariant Borel subset. If A is $\leq_T$-cofinal then A contains a cone.

Definition 7.6. A *Turing invariant function* is a Borel homomrphism $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\equiv_T$ to itself.

The following open problem is the Borel version of the first part of a conjecture of Martin [12].

Conjecture 7.7. If $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Turing invariant Borel function, then exactly one of the following holds:

(I) There is a cone $C \subseteq 2^{\mathbb{N}}$, f maps C into a single $\equiv_T$-class;
(II) There is a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$.

In the sequel, we will refer to Conjecture 7.7 as (MC). **We will discuss some results that are true relative to** (MC).

Remark 7.8. Without the requirement that f be Borel, the statement above is false in ZFC. In fact, the axiom of choice implies the existence of several pathological Turing invariant functions. The original version of Martin's conjecture[7] is formulated in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. (See also the excellent introduction of [15].)

The only known progress towards (MC) is the following result by Slaman and Steel.

Theorem 7.9 (Slaman-Steel [19])**.** *Suppose that* $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *is a Turing invariant Borel function. If there is a cone* $C \subseteq 2^{\mathbb{N}}$ *such that* $f(x) <_T x$ *for all* $x \in C$*, then there is a cone* $D \subseteq C$ *such that* f *maps* D *into a single* $\equiv_T$*-class.*

We also emphasize the following variant of MC:

(MC') If $f \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Turing invariant Borel function, then exactly one of the following holds:
- (I) For all $x \in 2^{\mathbb{N}}$, there is $x \leq_T y$ such that $f(y) <_T y$.
- (II) For all $x \in 2^{\mathbb{N}}$, there is $x \leq_T y$ such that $y \leq_T f(y)$.

Exercise 7.10. (MC) is equivalent to (MC')

[7] Martin's conjecture is one of the few unsolved problems entitled to Victoria Delfino. More on the character of Victoria Delfino and the list of problems entitled to her can be found in [2].

7.1. **The Borel complexity of $\equiv_T$.** In this subsection we investigate some consequences of (MC). First it is natural to ask what is the Borel complexity of $\equiv_T$.

Theorem 7.11 (MC)**.** *If $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel $\equiv_T$-invariant function, then either*

(i) *There is a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single $\equiv_T$-class;*
(ii) *There is a cone $C \subseteq 2^{\mathbb{N}}$ such that $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$. Moreover, for every cone $D \subseteq 2^{\mathbb{N}}$ the saturation $[f(D)]_{\equiv_T}$ contains a cone.*

Proof. If (i) fails, then (MC) implies the existence of $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$. Since every $x \in 2^{\mathbb{N}}$ has countably many predecessors $f \restriction C$ is necessarily countable-to-one. So, $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$.

Moreover, let $D \subseteq 2^{\mathbb{N}}$ be any cone and let $D_0 = D \cap C$. Since $f \restriction C$ is countable-to-one, so is $f \restriction D_0$; it follows that $f(D_0)$ is Borel. (See Corollary 1.17.) Therefore, the saturation $[f(D_0)]_{\equiv_T}$ is a Borel $\equiv_T$-invariant set. Moreover, $[f(D_0)]_{\equiv_T}$ is $\leq_T$-cofinal. It follows by Exercise 7.5 that $[f(D_0)]_{\equiv_T}$ contains a cone. □

Let E be an equivalence relation on X. We denote by $E \oplus E$ the equivalence relation defined on $X \sqcup X = X \times \{0,1\}$ by

$$(x,i)\ E \oplus E\ (y,j) \qquad \Longleftrightarrow \qquad x\ E\ y \text{ and } i = j.$$

Theorem 7.12 (MC)**.** *Let E be $\equiv_T \oplus \equiv_T$. Then E is not Borel reducible to $\equiv_T$. Thus, $\equiv_T$ is not universal.*

Proof. Suppose that $f\colon 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel reduction from $\equiv_T \oplus \equiv_T$ to $\equiv_T$. Then define $f_0\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, s \mapsto f(s,0)$ and $f_1\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, s \mapsto f(s,1)$.

Note that for any $s,t \in 2^{\mathbb{N}}$, the element $(s,0)$ and $t,1$ are not $\equiv_T \oplus \equiv_T$, thus $f_0(s) \not\equiv_T f_1(t)$. Therefore, $[f_0(s^{\mathbb{N}})]_{\equiv_T}$ and $[f_1(s^{\mathbb{N}})]_{\equiv_T}$ are disjoint $\equiv_T$-invariant Borel set. However, this creates a contradiction because both contains a cone by Theorem 7.11(ii). □

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