Inverse scattering theory is a major theme in applied mathematics, with applications to such diverse areas as medical imaging, geophysical exploration, and nondestructive testing. The inverse scattering problem is both nonlinear and ill-posed, thus presenting challenges in the development of efficient inversion algorithms. A further complication is that anisotropic materials cannot be uniquely determined from given scattering data.

In the first edition of *Inverse Scattering Theory and Transmission Eigenvalues*, the authors discussed methods for determining the support of inhomogeneous media from measured far field data and the role of transmission eigenvalue problems in the mathematical development of these methods.

In this second edition, three new chapters describe recent developments in inverse scattering theory. In particular, the authors

- explore the use of modified background media in the nondestructive testing of materials and methods for determining the modified transmission eigenvalues that arise in such applications from measured far field data,
- examine nonscattering wave numbers—a subset of transmission eigenvalues—using techniques taken from the theory of free boundary value problems for elliptic partial differential equations, and
- discuss the dualism of scattering poles and transmission eigenvalues that has led to new methods for the numerical computation of scattering poles.

This book will be of interest to research mathematicians and engineers and physicists working on problems in target identification. It will also be useful to advanced graduate students in many areas of applied mathematics.

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Inverse Scattering Theory and Transmission Eigenvalues

Second Edition

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# Inverse Scattering Theory and Transmission Eigenvalues

## Second Edition


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# Contents

# Preface to the Second Edition

In the first edition of this book, we discussed methods for determining the support of an inhomogeneous medium from measured far field data as well as an extensive study of the central role played by the transmission eigenvalue problem in the mathematical development of these methods. In particular, we introduced the generalized linear sampling method (GLSM) and showed that this method provides a mathematical explanation of why it is permissible to use Tikhonov regularization to obtain an approximate solution of the far field equation associated with the linear sampling method.

In the six years since the first edition of our book appeared, there has been considerable progress in both the development of GLSM as well as the theory of transmission eigenvalues. In this second edition, in addition to correcting typos in the first edition, we have added several highlights taken from these new developments. In particular, we have included new chapters on (1) the use of modified background media in the nondestructive testing of materials and in particular methods for determining the modified transmission eigenvalues that arise in such applications from measured far field data, (2) a study of a subset of transmission eigenvalues, called nonscattering wave numbers, through the use of techniques taken from the theory of free boundary value problems for elliptic partial differential equations, and (3) the duality between scattering poles and transmission eigenvalues which, in addition to their intrinsic mathematical interest, leads to new methods for the numerical computation of scattering poles.

We hope that this new edition will attract many newcomers to this intriguing new area in applied mathematics.

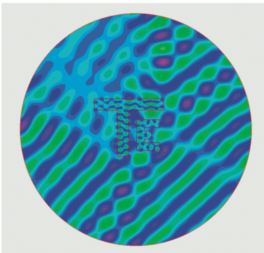
# Preface to the First Edition

In the past thirty years the field of inverse scattering theory has become a major theme of applied mathematics with applications to such diverse areas as medical imaging, geophysical exploration, and nondestructive testing. The growth of this field has been characterized by the realization that the inverse scattering problem is both nonlinear and ill-posed, thus presenting particular problems in the development of efficient inversion algorithms. Although linearized models continue to play an important role in many applications, the increased need to focus on problems in which multiple scattering effects can no longer be ignored has led to the nonlinearity of the inverse scattering problem playing a central role. In addition, the possibility of collecting large amounts of data over limited regions of space has led to the situation where the ill-posed nature of the inverse scattering problem becomes a problem of central importance.

Initial efforts to deal with the nonlinear and ill-posed nature of the inverse scattering problem focused on the use of nonlinear optimization methods, in particular Newton's method and various versions of what are now called decomposition methods. For a discussion of this approach to the inverse scattering problem together with numerous references, we refer the reader to [69]. Although efficient in many situations, the use of nonlinear optimization methods suffers from the need for strong a priori information in order to implement such an approach. Hence, in order to circumvent this difficulty, a recent trend in inverse scattering theory has focused on the development of a qualitative approach in which the amount of a priori information needed is drastically reduced but at the expense of obtaining only limited information of the scatterer such as the connectivity, support, and an estimate on the values of the constitutive parameters. Examples of such an approach are the linear sampling method, the factorization method, and the theory of transmission eigenvalues. It is these topics that are the theme of this monograph, focusing on their use in the inverse acoustic scattering problem for inhomogeneous media.

The qualitative approach to inverse scattering theory was initiated by Colton and Kirsch in 1996 [66]. In this paper they introduced a linear integral equation of the first kind, called the far field equation, whose solution could be used as an indicator function to determine the support of the scattering obstacle. This method is called the linear sampling method. The mathematical difficulties inherent in this approach were subsequently resolved by the factorization method of Kirsch and Grinberg [113], and further clarification of the relationship between the linear sampling and factorization methods was obtained by Arens and Lechleiter [6] and Audibert and Haddar [13]. Having determined the support of the scatterer, the next step in the qualitative approach is to determine estimates on the material properties of the scatterer. This was accomplished by Cakoni, Gintides, and Haddar [44] through the use of transmission eigenvalues first introduced by Kirsch [108] and Colton and Monk [73]. The development of the above themes is the subject matter of the chapters that follow. This book is intended for mathematicians, physicists, and engineers who either are actively involved in problems arising in scattering theory or have an interest in this field and wish to know more about recent developments in this area. It will also be of interest to advanced graduate students wishing to become more informed about new ideas in inverse scattering theory. On the other hand, for those unfamiliar with classical scattering theory, Chapter 1 provides a basic introduction to this area and also serves as an introduction to the chapters which follow.

This monograph is based on lectures given by David Colton and Fioralba Cakoni at the CBMS–NSF sponsored summer school "Inverse Scattering Theory and Transmission Eigenvalues" held at the University of Texas in Arlington during the week of May 27 – May 31, 2014. Special thanks are given to the National Science Foundation for their financial support as well as to Professor Tuncay Aktosun, whose expert skills in organizing and running the summer school made it so successful. We would also like to thank Dr. Arje Nachman of the Air Force Office of Scientific Research (AFOSR) for his long term support of Professors Cakoni and Colton as well as both AFOSR and L'Institut National de Recherche en Informatique et en Automatique (INRIA) for supporting exchange visits between Professors Cakoni and Colton and Professor Haddar which has been indispensable for our long term research efforts. We would also like to thank Dr. Richard Albanese, USAF (retired), for his continuous interest and encouragement of our research. Finally, we thank the editorial office at SIAM for their expert handling of our manuscript through the publishing process.



# Chapter 1

# Scattering Theory

In this introductory chapter we provide an overview of the basic ideas of scattering theory for inhomogeneous media of compact support and in particular the associated inverse scattering problems, which will become the major theme of this monograph. In addition to introducing the concept of the far field operator and the basic theory of ill-posed problems, we also establish uniqueness results for inverse scattering problems for both isotropic and anisotropic media. The results presented here are basic to the chapters that follow which develop the qualitative approach to inverse scattering theory.

## 1.1 ▪ The Helmholtz Equation

The starting point of any discussion of classical scattering theory is the Helmholtz equation and in particular spherical Bessel functions and spherical harmonics which arise when separation of variables is implemented in spherical coordinates. More specifically, we look for solutions of the Helmholtz equation in $\mathbb{R}^3$,

$$\Delta u + k^2 u = 0$$

for $k > 0$, in the form

$$u(x) = f(k\,|x|)Y_n^m(\hat{x}),$$

where $x \in \mathbb{R}^3$, $\hat{x} := x/\,|x|$, and $Y_n^m(\hat{x})$ is a *spherical harmonic* defined by

$$Y_n^m(\theta,\phi) := \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\,P_n^m(\cos\theta)e^{im\phi},$$

where $m = -n,\ldots,n$, $n = 0,1,2,\ldots$, $(\theta,\phi)$ are the spherical angles of $\hat{x}$, and $P_n^m$ is an *associated Legendre polynomial*. We note here that $\{Y_n^m\}$ is a complete orthonormal system in $L^2(S^2)$, where

$$S^2 := \{x : |x| = 1\}$$

and $Y_0^0 = \frac{1}{\sqrt{4\pi}}$. Then $f$ is a solution of the *spherical Bessel equation*

$$t^2 f''(t) + 2tf'(t) + \left[t^2 - n(n-1)\right] f(t) = 0 \tag{1.1}$$

with two linearly independent solutions,

$$
\begin{aligned}
j_n(t) &:= \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p!\, 1 \cdot 3 \cdots (2n+2p+1)}, \\
y_n(t) &:= \frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p! (-2n+1)(-2n+3) \cdots (-2n+2p-1)},
\end{aligned} \tag{1.2}
$$

called, respectively, the *spherical Bessel function* and the *spherical Neumann function* of order $n$. We note that

$$
j_0(t) = \frac{\sin t}{t}, \quad y_0(t) = -\frac{\cos t}{t}. \tag{1.3}
$$

The functions

$$
\begin{aligned}
h_n^{(1)}(t) &:= j_n(t) + i y_n(t), \\
h_n^{(2)}(t) &:= j_n(t) - i y_n(t)
\end{aligned}
$$

are called, respectively, the *spherical Hankel functions* of the first and second kind of order $n$. From (1.2) and (1.3) we have that for $f_n = j_n$ or $f_n = y_n$ that

$$
f_{n+1}(t) = -t^n \frac{d}{dt} \left\{ t^{-n} f_n(t) \right\}
$$

for $n = 0, 1, 2, \dots$ and

$$
h_0^{(1)}(t) = \frac{e^{it}}{it}, \quad h_0^{(2)}(t) = -\frac{e^{-it}}{it}.
$$

From this we see that the spherical Hankel functions have the asymptotic behavior

$$
\begin{aligned}
h_n^{(1)}(t) &= \frac{1}{t} e^{i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \\
h_n^{(2)}(t) &= \frac{1}{t} e^{-i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}
\end{aligned} \tag{1.4}
$$

as $t$ tends to infinity. In particular, $h_n^{(1)}(kr)$ satisfies the *Sommerfeld radiation condition*

$$
\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0,
$$

i.e., if $u(x) = h_n^{(1)}(kr) Y_n^m(\hat{x})$, then from the above asymptotic behavior of the spherical Hankel functions we see that $u(x)e^{-i\omega t}$ (where $\omega$ is the frequency and $t$ is time) is an *outgoing* wave. In particular this implies that energy is radiated out to infinity as required by physical considerations. Solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition uniformly in $\hat{x}$ are called *radiating*. An equivalent condition for a solution of the Helmholtz equation to be radiating is that

$$
\lim_{r \to \infty} \int_{|x|=r} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds = 0. \tag{1.5}
$$

The Wronskian of $h_n^{(1)}(t)$ and $h_n^{(2)}(t)$ is given by

$$
\begin{aligned}
W\left(h_n^{(1)}, h_n^{(2)}\right) &:= h_n^{(1)}(t) h_n^{(2)\prime}(t) - h_n^{(2)}(t) h_n^{(1)\prime}(t) \\
&= -\frac{2i}{t^2}.
\end{aligned} \tag{1.6}
$$

For later use we also quote the following identity for the modulus of $h_n^{(1,2)}$ that can be found in [139]:

$$|h_n^{(1,2)}(t)|^2 = \frac{1}{t^2} + \sum_{\ell=1}^{n} \frac{\alpha_\ell^n}{t^{2(\ell+1)}}, \quad \alpha_\ell^n = \frac{(2n)!(n+\ell)!}{(n!2^n)^2(n-\ell)!}. \tag{1.7}$$

Now let $D$ be a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and assume that $\partial D$ is Lipschitz with unit outward normal $\nu$ directed into the exterior of $D$. Let

$$\Phi(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y, \tag{1.8}$$

be the radiating fundamental solution to the Helmholtz equation, and let $H^2(D)$ be the usual Sobolev space (correspondingly, $H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$). For further reference we define

$$H_0^2(D) = \left\{ u \in H^2(D) : \ u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \right\}. \tag{1.9}$$

Then using Green's second identity

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds$$

we can deduce *Green's formula* for functions $u \in H^2(D)$ [69]:

$$\begin{aligned} u(x) = & \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu} \Phi(x,y) - u \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right\} ds(y) \\ & - \int_D \left\{ \left(\Delta u + k^2 u\right) \Phi(x,y) \right\} dy, \quad x \in D. \end{aligned} \tag{1.10}$$

**Theorem 1.1.** *Let $u \in H^2(D)$ be a solution of the Helmholtz equation in $D$. Then $u$ is analytic in $D$, i.e., $u$ can be locally expanded in a power series for each point $x \in D$.*

***Proof.*** Let $x \in D$, and choose a closed ball contained in $D$ with center $x$. Apply Green's formula to the ball and note that for $x \neq y$ we have that $\Phi(x,y)$ is real analytic in $x$. □

**Theorem 1.2 (Holmgren's Theorem).** *Let $u \in H^2(D)$ be a solution to the Helmholtz equation in $D$ such that*

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \textit{on } \Gamma$$

*for some open subset $\Gamma \subset \partial D$. Then $u$ is identically zero in $D$.*

***Proof.*** Using (1.10) we can extend $u$ by setting

$$u(x) := \int_{\partial D \setminus \Gamma} \left\{ \frac{\partial u}{\partial \nu} \Phi(x,y) - u \frac{\partial}{\partial \nu(y)} \Phi(x,y) \right\} ds(y)$$

for $x \in (\mathbb{R}^3 \setminus \overline{D}) \cup \Gamma$. By Green's second identity applied to $u$ and $\Phi(x,\cdot)$ we see that $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. But $u$ is a solution of the Helmholtz equation in $(\mathbb{R}^3 \setminus \partial D) \cup \Gamma$ and hence by the analyticity of $u$ we have that $u = 0$ in $D$. □

We now derive a representation formula analogous to (1.10) for radiating solutions of the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$. Part of the proof of this theorem will also be used at the end of this section in order to provide a uniqueness theorem for radiating solutions of the Helmholtz equation.

**Theorem 1.3.** *Let $u \in H^2_{loc}(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation. Then we have* Green's formula

$$u(x) = \int_{\partial D} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D}.$$

***Proof.*** Let $S_r := \{x : |x| = r\}$. Then the Sommerfeld radiation condition implies that

$$\begin{aligned} &\int_{S_r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 + 2k \Im \left( u \frac{\partial \overline{u}}{\partial \nu} \right) \right\} ds \\ &\qquad = \int_{S_r} \left| \frac{\partial u}{\partial \nu} - iku \right|^2 ds \to 0 \end{aligned} \tag{1.11}$$

as $r$ tends to infinity. We now assume that $r$ is large enough such that $D$ is contained in the ball bounded by $S_r$ and apply Green's first identity

$$\int_D (u \Delta v + \nabla u \cdot \nabla v) \, dx = \int_{\partial D} u \frac{\partial v}{\partial \nu} \, ds$$

to $D_r := \left\{ x \in \mathbb{R}^3 \setminus \overline{D} : |x| < r \right\}$ to obtain

$$\int_{S_r} u \frac{\partial \overline{u}}{\partial \nu} \, ds = \int_{\partial D} u \frac{\partial \overline{u}}{\partial \nu} \, ds - k^2 \int_{D_r} |u|^2 \, dy + \int_{D_r} |\nabla u|^2 \, dy. \tag{1.12}$$

Taking the imaginary part of (1.12) and substituting this into (1.11) gives

$$\lim_{r \to \infty} \int_{S_r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 \right\} ds = -2k \Im \left( \int_{\partial D} u \frac{\partial \overline{u}}{\partial \nu} \, ds \right), \tag{1.13}$$

which implies that

$$\int_{S_r} |u|^2 \, ds = O(1), \quad r \to \infty.$$

Using the Cauchy–Schwarz inequality and the Sommerfeld radiation condition, we now have that

$$\begin{aligned} &\int_{S_r} \left\{ u \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu} \Phi(x,y) \right\} ds(y) \\ &\qquad = \int_{S_r} u \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik \Phi(x,y) \right\} ds(y) \\ &\qquad\qquad - \int_{S_r} \Phi(x,y) \left\{ \frac{\partial u}{\partial \nu} - iku \right\} ds(y) \to 0 \end{aligned}$$

as $r$ tends to infinity. Hence, applying Green's formula (1.10) to $D_r$ and letting $r$ tend to infinity gives the theorem. □

**Corollary 1.4.** *An entire solution to the Helmholtz equation satisfying the Sommerfeld radiation condition must vanish identically.*

***Proof.*** This follows immediately from Green's formula and Green's second identity. □

**Corollary 1.5.** *Every radiating solution $u \in H^2_{loc}(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave,*

$$u(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \tag{1.14}$$

*uniformly in all directions $\hat{x} = x/|x|$. The function $u_\infty$ defined on the unit sphere $S^2$ is called the* far field pattern *of $u$ and can be expressed as*

$$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \left\{ u(y) \frac{\partial}{\partial \nu(y)} e^{-ik\hat{x}\cdot y} - \frac{\partial u}{\partial \nu}(y) e^{-ik\hat{x}\cdot y} \right\} ds(y), \quad \hat{x} \in S^2. \tag{1.15}$$

***Proof.*** From

$$|x-y| = \sqrt{|x|^2 - 2x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right)$$

we obtain

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right) \right\}$$

and

$$\frac{\partial}{\partial \nu(y)} \frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ \frac{\partial}{\partial \nu(y)} e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$ uniformly for all $y \in \partial D$. The corollary now follows by substituting into Green's formula. □

The next result is a cornerstone of scattering theory and will be used repeatedly in what follows.

**Lemma 1.6 (Rellich's Lemma).** *Let $u \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 \, ds(x) = 0.$$

*Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.*

***Proof.*** For $|x|$ sufficiently large we have that

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(r) Y_n^m(\hat{x}),$$

where

$$a_n^m(r) = \int_{S^2} u(r\hat{x})\overline{Y_n^m(\hat{x})}\, ds(\hat{x}) \tag{1.16}$$

and

$$\int_{|x|=r} |u(x)|^2\ ds = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m(r)|^2 .$$

The assumption of the theorem implies that

$$\lim_{r\to\infty} r^2 |a_n^m(r)|^2 = 0. \tag{1.17}$$

But from (1.16) and the fact that $u$ is a solution of the Helmholtz equation we can deduce that the $a_n^m(r)$ are solutions of the spherical Bessel equation (1.1), i.e.,

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(kr) + \beta_n^m h_n^{(2)}(kr), \tag{1.18}$$

where $\alpha_n^m$ and $\beta_n^m$ are constants. Substituting (1.18) into (1.17) and using the asymptotic formulae (1.4) now implies that $\alpha_n^m = \beta_n^m = 0$ for all $n$, $m$ and hence $u = 0$ outside a sufficiently large ball. This implies that $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ by analyticity (Theorem 1.1). □

**Corollary 1.7.** *Assume $u \in H^2_{loc}(\mathbb{R}^3 \setminus D)$ is a radiating solution to the Helmholtz equation such that*

$$\Im\left(\int_{\partial D} u\frac{\partial \overline{u}}{\partial \nu}\, ds\right) \geq 0.$$

*Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.*

***Proof.*** From (1.13) and the assumption of the theorem we have that the assumption of Rellich's Lemma is valid. □

## 1.2 ▪ The Scattering Problem for Inhomogeneous Isotropic Media

We will now present the simplest scattering problem that will serve as a model for the inverse problems which will be discussed in this book. It is related to the propagation of sound waves of small amplitude in $\mathbb{R}^3$ viewed as a problem in fluid dynamics. Let $v(x,t)$, $x \in \mathbb{R}^3$, be the velocity potential of a fluid particle in an inviscid fluid, and let $p(x,t)$ be the pressure, $\rho(x,t)$ the density, and $S(x,t)$ the specific entropy. Then, if there are no external forces, we have that

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v\cdot\nabla)v + \frac{1}{\rho}\nabla\rho = 0 &\quad \text{(Euler's equation)},\\
\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 &\quad \text{(equation of continuity)},\\
p = f(\rho, s) &\quad \text{(equation of state)},\\
\frac{\partial s}{\partial t} + v\cdot\nabla s = 0 &\quad \text{(adiabatic hypothesis)},
\end{aligned}$$

where $f$ is a function depending on the fluid. Assuming that $v(x,t)$, $p(x,t)$, $\rho(x,t)$, and $S(x,t)$ are small we perturb around the static case $v=0$, $p=p_0=$ constant, $\rho=\rho_0(x)$, $S=S_0(x)$ with $p_0=f(\rho_0,S_0)$:

$$\begin{aligned}
v(x,t) &= \epsilon v_1(x,t)+O(\epsilon^2),\\
p(x,t) &= p_0+\epsilon p_1(x,t)+O(\epsilon^2),\\
\rho(x,t) &= \rho_0(x)+\epsilon\rho_1(x,t)+O(\epsilon^2),\\
S(x,t) &= S_0(x)+\epsilon S_1(x,t)+O(\epsilon^2),
\end{aligned}$$

where $0<\epsilon\ll 1$. Substituting the above into the equations of motion and equating the coefficients of $\epsilon$, we arrive at

$$\begin{aligned}
&\frac{\partial v_1}{\partial t}+\frac{1}{\rho_0}\nabla p_1=0,\\
&\frac{\partial \rho_1}{\partial t}+\nabla(\rho_0 v_1)=0,\\
&\frac{\partial p_1}{\partial t}+c^2(x)\left(\frac{\partial \rho_1}{\partial t}+v_1\cdot\nabla\rho_0\right),
\end{aligned}$$

where the sound speed $c$ is defined by

$$c^2(x)=\frac{\partial}{\partial\rho}f\left(\rho_0(x),S_0(x)\right).$$

Hence

$$\frac{\partial^2 p_1}{\partial t^2}=c^2(x)\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla p_1\right).$$

If $p_1(x,t)=\Re\left\{u(x)e^{-iwt}\right\}$, we have that $u$ satisfies

$$\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla u\right)+\frac{w^2}{c^2(x)}u=0.$$

Making the further assumption that $\nabla\rho_0$ can be ignored, we arrive at

$$\Delta u+\frac{w^2}{c^2(x)}u=0. \tag{1.19}$$

We now assume that the slowly varying inhomogeneous medium is of compact support and is embedded in $\mathbb{R}^3$ where the sound speed is $c(x)=c_0=$ constant. If the wave motion is caused by an *incident field* $u^i$ satisfying (1.19) with $c(x)=c_0$, we arrive at the *scattering problem* of determining $u$ such that

$$\Delta u+k^2 n(x)u=0 \quad \text{in } \mathbb{R}^3, \tag{1.20}$$

$$u=u^i+u^s, \tag{1.21}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r}-iku^s\right)=0, \tag{1.22}$$

where $n(x)=1$ outside the inhomogeneous medium,

$$n(x)=\frac{c_0^2}{c^2(x)}$$

inside the inhomogeneous medium, $r = |x|$, the radiation condition (1.22) is valid uniformly with respect to $\hat{x} = x/|x|$, $k = w/c_0 > 0$ is the *wave number*, $u^i$ is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$, $u^s$ is the *scattered field*, and we refer to the function $n(x)$ as the *refractive index* (in the engineering and physics literature $c_0/c(x)$ is the refractive index). The scattering problem (1.20)–(1.22) is the simplest model in which to introduce the basic ideas of inverse scattering theory. However, we shall later consider more physically realistic models in which we no longer ignore $\nabla \rho_0$ and allow $u$ to have jump discontinuities across the boundary of the inhomogeneous media (cf. Section 1.4). Moreover, in order to take into account possible attenuation in the media, we consider a complex valued refractive index.

We now assume that $n \in L^\infty(\mathbb{R}^3)$ with nonnegative imaginary part, set $m := 1 - n$, and let $D$ be a bounded domain with Lipschitz boundary $\partial D$ such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and $m(x) = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. We again let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y.$$

A proof of the following theorem can be found in [69].

**Theorem 1.8.** *Given two bounded domains $D$ and $G$, the* volume potential

$$(V\varphi)(x) := \int_D \Phi(x, y)\varphi(y)\, dy, \quad x \in \mathbb{R}^3,$$

*defines a bounded operator $V\colon L^2(D) \to H^2(G)$, where $H^2(G)$ denotes a Sobolev space.*

A classical approach to solving the scattering problem is based on reformulating the problem as a volume integral equation known as the *Lippmann–Schwinger integral equation*. An alternative variational approach will also be discussed later in this chapter. We now show that the scattering problem (1.20)–(1.22) is equivalent to solving

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y)\, dy, \quad x \in \mathbb{R}^3. \tag{1.23}$$

Due to the fact that $\mathrm{supp}(m) \subseteq \overline{D}$, (1.23) can be viewed as an integral equation over $D$ for $u \in L^2(D)$.

**Theorem 1.9.** *If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of* (1.20)–(1.22)*, then $u$ is a solution of* (1.23) *in $L^2(D)$. Conversely, if $u \in L^2(D)$ is a solution of* (1.23)*, then $u \in H^2_{loc}(\mathbb{R}^3)$ and $u$ is a solution of* (1.20)–(1.22).

***Proof.*** Let $u \in H^2_{loc}(\mathbb{R}^3)$ be a solution of (1.20)–(1.22). Let $x \in \mathbb{R}^3$ and $B$ a ball containing $x$ and $D$. Then Green's formula implies that

$$\begin{aligned} u(x) = &\int_{\partial B} \left\{ \frac{\partial u}{\partial \nu}(y)\Phi(x, y) - u(y)\frac{\partial}{\partial \nu(y)}\Phi(x, y) \right\} ds(y) \\ &- k^2 \int_B \Phi(x, y) m(y) u(y)\, dy \end{aligned}$$

and

$$u^i(x) = \int_{\partial B} \left\{ \frac{\partial u^i}{\partial \nu}(y)\Phi(x,y) - u^i(y)\frac{\partial}{\partial \nu(y)}\Phi(x,y) \right\} ds(y).$$

Furthermore, Green's second identity and the Sommerfeld radiation condition imply that

$$\int_{\partial B} \left\{ \frac{\partial u^s}{\partial \nu}(y)\Phi(x,y) - u^s(y)\frac{\partial}{\partial \nu(y)}\Phi(x,y) \right\} ds(y) = 0.$$

Adding these equations together gives the Lippmann–Schwinger integral equation (1.23), noting that since $m$ has compact support the integral over $B$ can be replaced by an integral over $\mathbb{R}^3$.

Conversely, let $u \in L^2(D)$ be a solution of (1.23) and define

$$u^s(x) := -k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy, \quad x \in \mathbb{R}^3.$$

Then $u^s$ satisfies the Sommerfeld radiation condition and $u^s \in H^2_{loc}(\mathbb{R}^3)$ satisfies $\Delta u^s + k^2 u^s = k^2 mu$. Since $\Delta u^i + k^2 u^i = 0$ we have that $u = u^i + u^s$ satisfies $\Delta u + k^2 nu = 0$ in $\mathbb{R}^3$. □

The existence of a unique solution to the scattering problem (1.20)–(1.22) is now equivalent to showing the existence of a unique solution to the Lippmann–Schwinger integral equation. For the wave number $k$ sufficiently small, this can be done by the method of successive approximations.

**Theorem 1.10.** *Suppose that $m(x) = 0$ for $|x| \geq a$ and $k^2 < 2/Ma^2$, where $M := \max_{|x|\leq a} |m(x)|$. Then there exists a unique solution to the Lippmann–Schwinger integral equation.*

***Proof.*** It suffices to solve (1.23) in $C(\overline{B})$ with $B := \{x \in \mathbb{R}^3 : |x| < a\}$. On $C(\overline{B})$ define

$$(T_m u)(x) := \int_B \Phi(x,y)m(y)u(y)\,dy, \quad x \in \overline{B}.$$

By the method of successive approximations, the theorem will be proved if $\|T_m\|_\infty \leq Ma^2/2$. To this end, we have

$$|(T_m u)(x)| \leq \frac{M\,\|u\|_\infty}{4\pi} \int_B \frac{dy}{|x-y|}, \quad x \in \overline{B}.$$

Now note that

$$h(x) := \int_B \frac{dy}{|x-y|}, \quad x \in \overline{B},$$

satisfies $\Delta h = -4\pi$ and is a function only of $r = |x|$. Hence

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dh}{dr}\right) = -4\pi$$

and thus $h(r) = -\frac{2}{3}\pi r^2 + \frac{c_1}{r} + c_2$ where $c_1$ and $c_2$ are constants. Since $h$ is continuous at the origin, $c_1 = 0$, and letting $r$ tend to zero shows that

$$c_2 = h(0) = \int_B \frac{dy}{|y|} = 4\pi \int_0^a \rho \, d\rho = 2\pi a^2.$$

Hence $h(r) = 2\pi(a^2 - r^2/3)$ and thus $\|h\|_\infty = 2\pi a^2$. We now have that

$$|(T_m u)(x)| \leq \frac{Ma^2}{2} \|u\|_\infty, \quad x \in \overline{B},$$

and the theorem follows. $\square$

From (1.23) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y) \, dy, \quad x \in \mathbb{R}^3,$$

and hence

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

where the *far field pattern* $u_\infty$ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u(y) \, dy, \quad \hat{x} = \frac{x}{|x|}. \tag{1.24}$$

Assuming that $k$ is sufficiently small and replacing $u$ by the first term in solving (1.23) by iteration (the *weak scattering* assumption) gives the *Born approximation* to the far field pattern

$$u_\infty(\hat{x}) \approx -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u^i(y) \, dy.$$

The Born approximation has been used extensively in inverse scattering where the weak scattering assumption is valid; for details of such an approach, see [81].

The proof of the existence of a unique solution to the Lippmann–Schwinger integral equation for arbitrary $k > 0$ is more delicate than for $k > 0$ sufficiently small and is based on the *unique continuation principle*. This principle is a basic result in the theory of linear elliptic partial differential equations and in the case of elliptic equations in $\mathbb{R}^3$ dates back to Müller [135], [136].

**Unique Continuation Principle.** *Let $G$ be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of $\Delta u + k^2 n(x) u = 0$ in $G$ for $n \in L^2(G)$. Then if $u$ vanishes in a neighborhood of some point in $G$, $u$ is identically zero in $G$.*

For a proof of the above unique continuation principle, see [69]. We can now use this principle to show that for each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem (1.20)–(1.22) (or equivalently the Lippmann–Schwinger integral equation).

**Theorem 1.11.** *For each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem* (1.20)–(1.22).

***Proof.*** The integral operator appearing in the Lippmann–Schwinger integral equation has a weakly singular kernel and hence this operator is compact on $L^2(\overline{D})$, where $\overline{D}$ is the support of $m$. Hence by the Riesz–Fredholm theory it suffices to show the uniqueness of a solution to (1.23), i.e., that the only solution of

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.25}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{1.26}$$

is $u = 0$. To this end, Green's first identity and (1.25) imply that

$$\int_{\partial D} u \frac{\partial \overline{u}}{\partial \nu}\, ds = \int_D \left\{ |\nabla u|^2 - k^2 \bar{n}\, |u|^2 \right\} dx$$

and hence

$$\Im\left( \int_{\partial D} u \frac{\partial \overline{u}}{\partial \nu}\, ds \right) = \int_D k^2 \Im(n)\, |u|^2 \; dx \geq 0.$$

By Corollary 1.7 $u(x) = 0$ for $x \in \mathbb{R}^3 \setminus \overline{D}$ and hence by the unique continuation principle $u(x) = 0$ for all $x \in \mathbb{R}^3$. □

The following theorem can be viewed as a generalization of the Riesz–Fredholm Theorem and will often be referred to in what follows (for a proof see Theorem 8.26 in [69]). Let $\mathcal{L}(X)$ denote the Banach space of bounded linear operators mapping the Banach space $X$ into itself, and let $I$ be the identity operator in $\mathcal{L}(X)$.

**Theorem 1.12 (Analytic Fredholm Theorem).** *Let $D$ be a domain in $\mathbb{C}$, and let $\mathcal{A} : D \to \mathcal{L}(X)$ be an operator valued analytic function such that $\mathcal{A}(z)$ is compact for each $z \in D$. Then either*

(a) $(\mathcal{I} - \mathcal{A}(z))^{-1}$ *does not exist for any $z \in D$ or*

(b) $(\mathcal{I} - \mathcal{A}(z))^{-1}$ *exists for all $z \in D \setminus S$, where $S$ is a discrete subset of $D$.*

## 1.2.1 ▪ The Far Field Operator

The *far field operator* plays a central role in inverse scattering theory and will appear in many of the remaining chapters of this monograph. Hence in this section we will introduce this operator and derive its most important analytic properties. In the course of our analysis we will also encounter the *transmission eigenvalue problem*, which will be seen to play an important role in all of our subsequent investigations.

In order to proceed we will need to be more specific on the nature of the incident field $u^i$. In particular, from now on we will assume that $u^i(x) = e^{ikx\cdot d}$, where $|d| = 1$. Then the solution of the scattering problem

$$\Delta u + k^2 n(x) u = 0, \tag{1.27}$$

$$u(x) = e^{ikx\cdot d} + u^s(x), \tag{1.28}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.29}$$

will depend on $d$, and in particular the far field pattern $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right)$$

now depends on $d$. The following *reciprocity principle* is basic to our investigations.

**Theorem 1.13 (Reciprocity Principle).** *Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to* (1.27)–(1.29). *Then $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$.*

***Proof.*** Let $D \subset \{x\colon\ |x| < a\}$, where again $D := \{x\colon m(x) \neq 0\}$. Then Green's second identity implies that

$$\int\limits_{|y|=a} \left\{ u^i(y,d) \frac{\partial}{\partial\nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial\nu} u^i(y,d) \right\} ds(y) = 0,$$

$$\int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial\nu} u^s(y,-\hat{x}) - u^s(y,-\hat{x}) \frac{\partial}{\partial\nu} u^s(y,d) \right\} ds(y) = 0,$$

where $u^i(x,d) = e^{ikx\cdot d}$. Corollary 1.5 shows that

$$\int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial\nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial\nu} u^s(y,d) \right\} ds(y) = 4\pi u_\infty(\hat{x},d),$$

$$\int\limits_{|y|=a} \left\{ u^s(y,-\hat{x}) \frac{\partial}{\partial\nu} u^i(y,d) - u^i(y,d) \frac{\partial}{\partial\nu} u^s(y,-\hat{x}) \right\} ds(y) = 4\pi u_\infty(-d,-\hat{x}).$$

Subtracting the last of these equations from the sum of the first three gives

$$\begin{aligned} 4\pi\left[u_\infty(\hat{x},d) - u_\infty(-d,-\hat{x})\right] &= \int\limits_{|y|=a} \left\{ u(y,d) \frac{\partial}{\partial\nu} u(y,-\hat{x}) - u(y,-\hat{x}) \frac{\partial}{\partial\nu} u(y,d) \right\} ds(y) \\ &= 0 \end{aligned}$$

by Green's second identity. $\square$

We now define the *far field operator* $F\colon\ L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) := \int\limits_{S^2} u_\infty(\hat{x},d) g(d)\, ds(d).$$

Since $u_\infty(\hat{x},d)$ is infinitely differentiable with respect to each of its variables, $F$ is clearly compact. The corresponding *scattering operator* $S\colon\ L^2(S^2) \to L^2(S^2)$ is defined by

$$S := I + \frac{ik}{2\pi} F. \tag{1.30}$$

We now want to prove some properties of these operators. To this end we define a *Herglotz wave function* to be a function of the form

$$v_g(x) = \int\limits_{S^2} e^{ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{1.31}$$

where $g \in L^2(S^2)$. The function $g$ is called the *Herglotz kernel* of $v_g$. Herglotz wave functions are clearly entire solutions of the Helmholtz equation. We note that for a given $g \in L^2(S^2)$ the function

$$\int\limits_{S^2} e^{-ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

is also a Herglotz wave function. Furthermore, if a Herglotz wave function vanishes in some open subset of $\mathbb{R}^3$, then its kernel must be identically zero [69]. In what follows $(\cdot,\cdot)$ is the inner product in $L^2(S^2)$.

**Theorem 1.14.** *Let $g, h \in L^2(S^2)$, and let $v_g$ and $v_h$ be the Herglotz wave functions with kernels $g$ and $h$, respectively. Then if $w_g$ and $w_h$ are the solutions of the scattering problem* (1.27)–(1.29) *corresponding to the incident field $e^{ikx\cdot d}$ being replaced by the incident fields $v_g$ and $v_h$, respectively, we have that*

$$k^2 \int\limits_D \Im(n)\, w_g\, \overline{w_h}\, dx = 2\pi(Fg,h) - 2\pi(g,Fh) - ik(Fg,Fh).$$

***Proof.*** ([68], [69]) Let $w_g^s = w_g - v_g$ and $w_h^s = w_h - v_h$ denote the scattered fields with far field patterns $w_g^\infty$ and $w_h^\infty$ respectively. Then by linearity $w_g^\infty = Fg$ and $w_h^\infty = Fh$ and Green's second identity implies that, for $a$ sufficiently large such that $D \subset \{x \in \mathbb{R}^3;\ |x| \le a\}$,

$$\int\limits_{|x|=a} \left\{ w_g \frac{\partial \overline{w_h}}{\partial \nu} - \overline{w_h} \frac{\partial w_g}{\partial \nu} \right\} ds = 2k^2 \int\limits_D \Im(n)\, w_g\, \overline{w_h}\, dx \tag{1.32}$$

and

$$\int\limits_{|x|=a} \left\{ v_g \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h} \frac{\partial v_g}{\partial \nu} \right\} ds = 0.$$

Furthermore, for $R > a$ we have that

$$\int\limits_{|x|=a} \left\{ w_g^s \frac{\partial \overline{w_h^s}}{\partial \nu} - \overline{w_h^s} \frac{\partial w_g^s}{\partial \nu} \right\} ds = \int\limits_{|x|=R} \left\{ w_g^s \frac{\partial \overline{w_h^s}}{\partial \nu} - \overline{w_h^s} \frac{\partial w_g^s}{\partial \nu} \right\} ds$$

$$\to -2ik \int\limits_{S^2} w_g^\infty \overline{w_h^\infty}\, ds = -2ik(Fg,Fh)$$

as $R$ tends to infinity. Finally, we have that

$$\begin{aligned}
&\int\limits_{|x|=a} \left\{ v_g \frac{\partial \overline{w_h^s}}{\partial \nu} - \overline{w_h^s} \frac{\partial v_g}{\partial \nu} \right\} ds \\
&= \int\limits_{S^2} g(d) \int\limits_{|x|=a} \left\{ e^{ikx\cdot d} \frac{\partial \overline{w_h^s}}{\partial \nu} - \overline{w_h^s} \frac{\partial}{\partial \nu} e^{ikx\cdot d} \right\} ds(x) ds(d) \\
&= -4\pi \int\limits_{S^2} g(d) \overline{w_h^\infty(d)}\, ds(d) = -4\pi(g,Fh)
\end{aligned}$$

and similarly

$$\int\limits_{|x|=a} \left\{ w_g^s \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h} \frac{\partial w_g^s}{\partial \nu} \right\} ds = 4\pi (Fg, h).$$

Substituting the above identities into (1.32) now implies the theorem. ∎

**Theorem 1.15.** *Assume that $\Im(n) = 0$. Then the far field operator is normal, i.e., $F^*F = FF^*$, and the scattering operator $S$ is unitary, i.e., $SS^* = S^*S = I$.*

***Proof.*** Theorem 1.14 implies that

$$ik(Fg, Fh) = 2\pi \left[ (Fg, h) - (g, Fh) \right] \tag{1.33}$$

for $g, h \in L^2(S^2)$. By reciprocity we have that

$$\begin{aligned} (F^* g)(\hat{x}) &= \int\limits_{S^2} \overline{u_\infty(d, \hat{x})} g(d) \, ds(d) \\ &= \int\limits_{S^2} \overline{u_\infty(-\hat{x}, -d)} g(d) \, ds(d) \\ &= \overline{\int\limits_{S^2} u_\infty(-\hat{x}, -d) \overline{g(d)} \, ds(d)}, \end{aligned}$$

i.e., $F^* g = \overline{RFR\overline{g}}$, where $(Rh)(\hat{x}) := h(-\hat{x})$. Since $(Rg, Rh) = (g, h)$, we have from (1.33) that

$$\begin{aligned} ik(F^*h, F^*g) &= ik(RFR\overline{g}, RFR\overline{h}) \\ &= ik(FR\overline{g}, FR\overline{h}) \\ &= 2\pi (FR\overline{g}, R\overline{h}) - 2\pi (R\overline{g}, FR\overline{h}) \\ &= 2\pi (RFR\overline{g}, \overline{h}) - 2\pi (\overline{g}, RFR\overline{h}) \\ &= 2\pi (h, F^*g) - 2\pi (F^*h, g) \\ &= 2\pi (Fh, g) - 2\pi (h, Fg) \\ &= ik(Fh, Fg) \end{aligned}$$

and hence $F^*F = FF^*$. Finally, (1.33) implies that

$$-(g, ikF^*Fh) = 2\pi \left( g, (F^* - F)h \right),$$

i.e., $ikF^*F = 2\pi(F - F^*)$. This, together with $F^*F = FF^*$, implies that $S^*S = SS^* = I$ by direct substitution. ∎

We now introduce the *transmission eigenvalue problem*: Determine $k \in \mathbb{C}$ and $v, w \in L^2(D)$, $v - w \in H_0^2(D)$, such that $v \neq 0$, $w \neq 0$, and

$$\begin{gathered} \Delta w + k^2 n(x) w = 0 \quad \text{in } D, \\ \Delta v + k^2 v = 0 \quad \text{in } D, \\ v = w \quad \text{on } \partial D, \\ \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D. \end{gathered}$$

Such values of $k$ are called *transmission eigenvalues*. Recall that $D \supset \{x\colon n(x) \neq 1\}$ and it is assumed that $D$ is bounded with Lipschitz boundary $\partial D$ such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. If the solution $v$ of the transmission eigenvalue problem is a Herglotz wave function, we then call the transmission eigenvalue $k$ a *nonscattering wave number*. Obviously the concept of nonscattering wave numbers is much more restrictive than the concept of transmission eigenvalues. The transmission eigenvalues along with the nonhomogeneous interior transmission problem are more precisely introduced in Chapter 2 and are extensively investigated in Chapters 3 and 4. Nonscattering wave numbers, more specifically their existence as well as their connection to transmission eigenvalues and regularity of the inhomogeneity, are discussed in Chapter 7.

**Theorem 1.16.** *Let $F$ be the far field operator corresponding to the scattering problem* (1.27)–(1.29)*. Then $F$ is injective if and only if $k$ is not a nonscattering wave number.*

***Proof.*** ([73], [108]) Suppose $Fg = 0$. Then the far field pattern $w_g^\infty$ of the scattered field $w_g^s$ corresponding to the incident field

$$v_g(x) := \int_{S^2} e^{ikx\cdot d} g(d)\, ds(d)$$

vanishes. By Rellich's Lemma $w_g^s = w_g - v_g$ vanishes outside $D$. Then $w_g = v_g + w_g^s$ satisfies $\Delta w_g + k^2 n w_g = 0$ in $\mathbb{R}^3$ and $w_g - v_g = 0$ on $\partial D$ and $\frac{\partial}{\partial \nu}(w_g - v_g) = 0$ on $\partial D$. If $k$ is not a transmission eigenvalue, then $v_g = w_g = 0$ and hence $g = 0$, i.e., $F$ is injective. □

**Corollary 1.17.** *Let $F$ be the far field operator corresponding to the scattering problem* (1.27)–(1.29)*. Then $F$ has dense range if and only if $k$ is not a nonscattering wave number.*

***Proof.*** ([73], [108]) From a well-known theorem in functional analysis, the orthogonal complement of the range of $F$ is equal to the null space of its adjoint $F^*$. Hence we must show that if $F^*h = 0$, then $h = 0$. To this end, we have that if $F^*h = 0$, i.e.,

$$\int_{S^2} \overline{u_\infty(d, \hat{x})} h(d)\, ds(d) = 0,$$

then

$$\int_{S^2} u_\infty(d, -\hat{x}) \overline{h(d)}\, ds(d) = 0$$

and hence, using reciprocity,

$$\int_{S^2} u_\infty(\hat{x}, d) \overline{h(-d)}\, ds(d) = 0.$$

Since $F$ is injective by Theorem 1.16, we can now conclude that $h = 0$ as desired. □

## 1.2.2 ▪ The Inverse Scattering Problem

We again consider the scattering problem (1.27)–(1.29). It has previously been shown that

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$, where

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int\limits_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u(y)\, dy$$

and $m := 1 - n$. The inverse scattering problem is to determine $n(x)$ (or some properties of $n(x)$) from $u_\infty(\hat{x}, d)$. We begin our discussion with the uniqueness. As motivation we first prove a simple result for harmonic functions.

**Theorem 1.18.** *The set of products $h_1 h_2$ of entire harmonic functions $h_1$ and $h_2$ is complete in $L^2(D)$ for any bounded domain $D \subset \mathbb{R}^3$.*

***Proof.*** ([58]) Given $y \in \mathbb{R}^3$ choose a vector $b \in \mathbb{R}^3$ with $b \cdot y = 0$ and $|b| = |y|$. Then for $z := y + ib \in \mathbb{C}^3$ we have $z \cdot z = 0$, which implies that $h_z := e^{iz\cdot x}$, $x \in \mathbb{R}^3$, is harmonic. Now assume $\varphi \in L^2(D)$ is such that

$$\int\limits_D \varphi h_1 h_2\, dx = 0$$

for all pairs of entire harmonic functions $h_1$ and $h_2$. Our theorem will be proved if we can show that $\varphi = 0$. But for $h_1 = h_z$, $h_2 = h_{\overline{z}}$ we have that

$$\int\limits_D \varphi(x) e^{2iy\cdot x}\, dx = 0$$

for $y \in \mathbb{R}^3$, which implies that $\varphi = 0$ almost everywhere by the Fourier integral theorem. □

To prove uniqueness for the inverse scattering problem of determining $n(x)$ from $u_\infty(\hat{x}, d)$ we need a property corresponding to the above theorem for products $v_1 v_2$ of solutions to $\Delta v_1 + k^2 n_1 v_1 = 0$ and $\Delta v_2 + k^2 n_2 v_2 = 0$ for two different refractive indices $n_1$ and $n_2$. Such a result was first established by Sylvester and Uhlmann [161]. The proofs of the following two lemmas can be found in [69] and [112].

**Lemma 1.19.** *Let $B$ be an open ball centered at the origin and containing the support of $m := 1 - n$. Then there exists a constant $C > 0$ such that for each $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|\Re z| > 2k^2 \|n\|_\infty$ there exists a solution $v \in H^2(D)$ of $\Delta v + k^2 n v = 0$ in $B$ of the form*

$$v(x) = e^{iz\cdot x} \left[1 - w(x)\right],$$

*where*

$$\|w\|_{L^2(D)} \le \frac{C}{|\Re(z)|}.$$

**Lemma 1.20.** *Let $B$ and $B_0$ be two open balls centered at the origin and containing the support of $m := 1 - n$ such that $\overline{B} \subset B_0$. Then the set of total fields $\{u(\cdot, d) \colon d \in S^2\}$ satisfying* (1.27)–(1.29) *is complete in the closure of*

$$\left\{v \in H^2(B_0) \colon \Delta v + k^2 n v = 0 \text{ in } B_0\right\}$$

*with respect to the $L^2(B)$ norm.*

We are now ready to prove the following uniqueness result for the inverse scattering problem due to Nachman [137], Novikov [142], and Ramm [150].

**Theorem 1.21.** *The index of refraction $n$ is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $x, d \in S^2$ and a fixed wave number $k$.*

***Proof.*** Assume that $n_1$ and $n_2$ are two refractive indices such that

$$u_{1,\infty}(\cdot, d) = u_{2,\infty}(\cdot, d), \quad d \in S^2,$$

and let $B$ and $B_0$ be two open balls centered at the origin and containing the support of $1-n_1$ and $1-n_2$ such that $\overline{B} \subset B_0$. By Rellich's Lemma we have that $u_1(\cdot, d) = u_2(\cdot, d)$ in $\mathbb{R}^3 \setminus \overline{B}$ for all $d \in S^2$. Hence $u := u_1 - u_2$ satisfies

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B \tag{1.34}$$

and

$$\Delta u + k^2 n_1 u = k^2 (n_2 - n_1) u_2 \quad \text{in } B.$$

From this and the partial differential equation for $\tilde{u_1} := u_1(\cdot, \tilde{d})$ we have that

$$k^2 \tilde{u_1} u_2 (n_2 - n_1) = \tilde{u_1}(\Delta u + k^2 n_1 u) = \tilde{u_1} \Delta u - u \Delta \tilde{u_1}.$$

Green's second identity and (1.34) now imply that

$$\int_B u_1(\cdot, \tilde{d}) u_2(\cdot, d)(n_2 - n_1)\, dx = 0$$

for all $d, \tilde{d} \in S^2$. Hence, from Lemma 1.20, it follows that

$$\int_B v_1 v_2 (n_1 - n_2)\, dx = 0 \tag{1.35}$$

for all solutions $v_1, v_2 \in H^2(B_0)$ of $\Delta v_1 + k^2 n_1 v_1 = 0$, $\Delta v_2 + k^2 n_2 v_2 = 0$ in $B_0$.

Given $y \in \mathbb{R}^3 \setminus \{0\}$ and $\rho > 0$ we now choose vectors $a, b \in \mathbb{R}^3$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ and $|a| = 1$, $|b|^2 = |y|^2 + \rho^2$. Then for $z_1 := y + \rho a + ib$, $z_2 := y - \rho a - ib$ we have that

$$\begin{aligned} z_j \cdot z_j &= |\Re z_j|^2 - |\Im z_j|^2 + 2i \Re z_j \cdot \Im z_j \\ &= |y|^2 + \rho^2 - |b|^2 = 0 \end{aligned}$$

and

$$|\Re z_j|^2 = |y|^2 + \rho^2 \geq \rho^2.$$

In (1.35) we now insert the solutions $v_1$ and $v_2$ constructed in Lemma 1.19 for the indices of refraction $n_1$ and $n_2$ and the vectors $z_1$ and $z_2$, respectively. Since $z_1 + z_2 = 2y$ this yields

$$\int_B e^{2iy\cdot x} \left[1 + w_1(x)\right] \left[1 + w_2(x)\right] \left[n_1(x) - n_2(x)\right]\, dx = 0$$

and passing to the limit as $\rho$ tends to infinity gives

$$\int\limits_B e^{2iy\cdot x}\,[n_1(x) - n_2(x)]\,dx = 0.$$

By the Fourier integral theorem we now have that $n_1 = n_2$. $\square$

Although nonlinear optimization methods are not the focus of this monograph, we will briefly show how, in principle, $n(x)$ can be constructed from $u_\infty(\hat{x}, d)$ through the use of Newton type methods. To this end, we define the operator $\mathcal{F}\colon m \mapsto u_\infty$ for $u_\infty = u_\infty(\hat{x}, d)$, which we just showed is injective but is obviously nonlinear. Letting $B$ be a ball containing the (unknown) support of $m$, we interpret $\mathcal{F}$ as an operator from $L^2(B)$ into $L^2(S^2 \times S^2)$. From the Lippmann–Schwinger integral equation we can write

$$(\mathcal{F}m)(\hat{x}, d) = -\frac{k^2}{4\pi}\int\limits_B e^{-ik\hat{x}\cdot y} m(y)u(y)\,dy, \tag{1.36}$$

where $u(\cdot, d)$ is the unique solution of

$$u(x, d) + k^2\int\limits_B \Phi(x, y)m(y)u(y, d)\,dy = e^{ikx\cdot d}, \tag{1.37}$$

where again

$$\Phi(x, y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y.$$

Note that $\mathcal{F}$ is a nonlinear operator.

Recall now that a mapping $T\colon X \to Y$ of a normal space $X$ into a normal space $Y$ is called *Fréchet differentiable* if there exists a bounded linear operator $\mathcal{A}\colon X \to Y$ such that

$$\lim_{h\to 0}\frac{1}{\|h\|}\,\|T(x+h) - T(x) - \mathcal{A}h\| = 0$$

and we write $T'(x) = \mathcal{A}$. In particular, from (1.37) it can be seen that the Fréchet derivative $v := u'_m h$ of $u$ with respect to $m$ (in the "direction" h) satisfies the Lippmann–Schwinger integral equation

$$v(x, d) + k^2\int\limits_B \Phi(x, y)\,[m(y)v(y, d) + h(y)u(y, d)]\,dy = 0, \quad x \in B, \tag{1.38}$$

and from (1.36) we have that

$$(\mathcal{F}'_m h)(\hat{x}) = -\frac{k^2}{4\pi}\int\limits_B e^{-ik\hat{x}\cdot y}\,[m(y)v(y, d) + h(y)u(y, d)]\,dy$$

for $\hat{x}, d \in S^2$. Hence $(\mathcal{F}'_m h)(\hat{x})$ coincides with the far field pattern of the solution $v(\cdot, d) \in H^2_{loc}(\mathbb{R}^3)$ of (1.38). Note also that $\mathcal{F}'_m\colon L^2(B) \to L^2(S^2 \times S^2)$ is compact. We have proven the following theorem [69].

**Theorem 1.22.** *The operator $\mathcal{F}\colon m \mapsto u_\infty$ is Fréchet differentiable. The derivative is given by $\mathcal{F}'_m h = v_\infty$, where $v_\infty$ is the far field pattern of the radiating solution $v \in H^2_{loc}(\mathbb{R}^3)$ to $\Delta v + k^2 nv = -k^2 uh$ in $\mathbb{R}^3$.*

**Theorem 1.23.** *The operator $\mathcal{F}'_m\colon L^2(B) \to L^2(S^2 \times S^2)$ is injective.*

***Proof.*** ([69]) Assume that $h \in L^2(B)$ satisfies $\mathcal{F}'_m h = 0$. We want to show that $h = 0$. Since $\mathcal{F}'_m h = 0$ we have that for each $d \in S^2$ the far field pattern of the solution $v$ of (1.38) vanishes and Rellich's Lemma implies that $v(\cdot, d) = \frac{\partial}{\partial \nu} v(\cdot, d) = 0$ on $\partial B$. Hence Green's second identity implies that

$$k^2 \int_B h u(\cdot, d) w \, dx = 0$$

for all $d \in S^2$ and any solution $w \in H^2(B_0)$ of $\Delta w + k^2 n w = 0$ in $B_0$. By Lemma 1.20 we can now conclude that

$$\int_B h w \tilde{w} \, dx = 0$$

for all $w$, $\tilde{w}$ satisfying $\Delta w + k^2 n w = 0$ and $\Delta \tilde{w} + k^2 n \tilde{w} = 0$ in $B_0 \supset \overline{B}$. The proof can now be completed as in the proof of Theorem 1.21. □

We can now apply Newton's method to the nonlinear equation $\mathcal{F}(m) = u_\infty$. However, to implement this procedure we must solve a direct scattering problem at each step of the iteration procedure. We furthermore have the possible problem of local minima and need to solve an "ill-posed" compact operator equation of the first kind at each step. How to solve this last problem will be dealt with in the next section.

## 1.3 ▪ Ill-Posed Problems

In the previous sections we have introduced two different methods for solving the inverse scattering problem: the Born approximation and Newton's method applied to the nonlinear equation $\mathcal{F}(m) = u_\infty$. Both methods involve the solution of an integral equation of the first kind over a bounded region with a smooth kernel. In particular, in both cases the integral operator is compact. As we shall see shortly, the problem of inversion of such an operator is ill-posed in the sense that the solution does not depend continuously on the given (measured) data. The same problem will also arise later when we use the factorization method or the linear sampling method to determine the support of the scattering object. In short, all the available methods for solving the inverse scattering problem involve the solution of ill-posed integral equations of the first kind. Hence in this section we shall give a brief survey of how to solve such equations. For a more comprehensive study we refer the reader to [85], [112], and [119].

**Definition 1.24.** *Let $\mathcal{A}\colon X \to V \subset Y$ be an operator from a normal space $X$ into a subset $V$ of a normed space $Y$. The equation $\mathcal{A}\varphi = f$ is called* well-posed *if $\mathcal{A}\colon X \to V$ is bijective and the inverse operator $\mathcal{A}^{-1}\colon V \to X$ is continuous. Otherwise the equation is called* ill-posed.

**Theorem 1.25.** *Let $\mathcal{A}\colon X \to V \subset Y$ be a linear compact operator. Then $\mathcal{A}\varphi = f$ is ill-posed if $X$ is not finite-dimensional.*

***Proof.*** If $\mathcal{A}^{-1}\colon V \to X$ exists and is continuous, then $I = \mathcal{A}^{-1}\mathcal{A}$ is compact, which implies that $X$ is finite-dimensional. □

We now assume that $\mathcal{A}$ is a linear compact operator and wish to approximate the solution $\varphi$ to $\mathcal{A}\varphi = f$ from a knowledge of a perturbed right-hand side $f^\delta$ with a known error level $\|f^\delta - f\| \le \delta$. We will always assume that $\mathcal{A}\colon X \to Y$ is injective and want the approximate solution $\varphi^\delta$ to depend continuously on $f^\delta$.

**Definition 1.26.** *Let $\mathcal{A}\colon X \to Y$ be an injective compact linear operator. Then a family of bounded linear operators $R_\alpha\colon Y \to X$ with the property that*

$$R_\alpha f \to \mathcal{A}^{-1} f, \quad \alpha \to 0, \tag{1.39}$$

*for all $f \in \mathcal{A}(X)$ is called a* regularization scheme *for $\mathcal{A}$. The parameter $\alpha$ is called the* regularization parameter.

It is easily verified that if $X$ is infinite-dimensional, then the operator $R_\alpha$ cannot be uniformly bounded with respect to $\alpha$ and the operators $R_\alpha \mathcal{A}$ cannot be norm convergent as $\alpha \to 0$ [69]. A regularization scheme approximates the solution $\varphi$ of $\mathcal{A}\varphi = f$ by the *regularized solution* $\varphi_\alpha^\delta := R_\alpha f^\delta$. Hence

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha \mathcal{A}\varphi - \varphi,$$

which implies that

$$\|\varphi_\alpha^\delta - \varphi\| \le \delta \|R_\alpha\| + \|R_\alpha \mathcal{A}\varphi - \varphi\|.$$

The error consists of two parts. The first term reflects the error in the data and the second term the error between $R_\alpha$ and $\mathcal{A}^{-1}$. From the above discussion we see that the first term will be increasing as $\alpha \to 0$ due to the ill-posed nature of the problem, whereas the second term will be decreasing as $\alpha \to 0$ according to (1.39).

**Definition 1.27.** *A* strategy *for a regularization scheme $R_\alpha$, $\alpha > 0$, i.e., the choice of the regularization parameter $\alpha = \alpha(\delta, f^\delta)$, is called* regular *if for all $f \in \mathcal{A}(X)$ and $f^\delta \in Y$ with $\|f^\delta - f\| \le \delta$ we have that*

$$R_{\alpha(\delta, f^\delta)} f^\delta \to \mathcal{A}^{-1} f, \quad \delta \to 0.$$

A natural strategy is the *Morozov discrepancy principle* based on the idea that the residual should not be smaller than the accuracy of the measurements, i.e., $\|\mathcal{A}R_\alpha f^\delta - f^\delta\| \le \gamma\delta$ for some parameter $\gamma \le 1$.

From now on let $X$ and $Y$ be Hilbert spaces and $\mathcal{A}\colon X \to Y$ be a compact linear operator with adjoint $\mathcal{A}^*\colon Y \to X$. The nonnegative square roots of the eigenvalues of $\mathcal{A}^*\mathcal{A}\colon X \to X$ are called the *singular values* of $\mathcal{A}$. We always assume that $A \neq 0$. For a proof of the following theorem, see [29] or [69].

**Theorem 1.28.** *Let $(\mu_n)$, $\mu_1 \ge \mu_2 \ge \cdots$, be the singular values of $\mathcal{A}$. Then there exist orthonormal sequences $(\varphi_n)$ in $X$ and $(g_n)$ in $Y$ such that*

$$\mathcal{A}\varphi_n = \mu_n g_n, \quad \mathcal{A}^* g_n = \mu_n \varphi_n,$$

*and for all $\varphi \in X$*

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi,$$

$$\mathcal{A}\varphi = \sum_{n=1}^{\infty} \mu_n (\varphi, \varphi_n) g_n,$$

*where* $Q\colon X \to N(\mathcal{A})$ *is the orthogonal projection operator. The system* $(\mu_n, \varphi_n, g_n)$ *is called a* singular system *of* $\mathcal{A}$.

**Theorem 1.29 (Picard's Theorem).** *Let* $\mathcal{A}\colon X \to Y$ *be a compact linear operator with singular system* $(\mu_n, \varphi_n, g_n)$. *Then* $\mathcal{A}\varphi = f$ *is solvable if and only if* $f \in N(\mathcal{A}^*)^\perp$ *and satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left|(f, g_n)\right|^2 < \infty. \tag{1.40}$$

*In this case a solution is given by*

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n}(f, g_n)\varphi_n. \tag{1.41}$$

***Proof.*** The necessity of $f \in N(\mathcal{A}^*)^\perp$ follows from $N(\mathcal{A}^*)^\perp = \overline{\mathcal{A}(X)}$. If $\mathcal{A}\varphi = f$, then

$$\mu_n(\varphi, \varphi_n) = (\varphi, \mathcal{A}^* g_n) = (\mathcal{A}\varphi, g_n) = (f, g_n)$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left|(f, g_n)\right|^2 = \sum_{n=1}^{\infty} \left|(\varphi, \varphi_n)\right|^2 \le \|\varphi\|^2$$

and the necessity of (1.40) follows.

Conversely, if $f \in N(\mathcal{A}^*)^\perp$ and (1.40) is satisfied, then (1.41) converges in $X$. Applying $\mathcal{A}$ to (1.41) gives

$$\mathcal{A}\varphi = \sum_{n=1}^{\infty}(f, g_n)g_n = f$$

since $f \in N(\mathcal{A}^*)^\perp$. □

Picard's Theorem shows that the ill-posedness of $\mathcal{A}\varphi = f$ comes from the fact that $\mu_n \to 0$. This suggests filtering out the influence of $1/\mu_n$ in the solution of (1.41). To this end we have the following theorem.

**Theorem 1.30.** *Let* $\mathcal{A}\colon X \to Y$ *be an injective compact linear operator with singular system* $(\mu_n, \varphi_n, g_n)$, *and let* $q\colon (0, \infty) \times (0, \|\mathcal{A}\|) \to R$ *be a bounded function such that for each* $\varphi > 0$ *there exists a positive constant* $c(\alpha)$ *with*

$$|q(\alpha, \mu)| \le c(\alpha)\mu, \quad 0 < \mu \le \|\mathcal{A}\|, \tag{1.42}$$

*and*

$$\lim_{\alpha \to 0} q(\alpha, \mu) = 1, \quad 0 < \mu \le \|\mathcal{A}\|. \tag{1.43}$$

*Then the bounded operators* $R_\alpha\colon Y \to X$, $\alpha > 0$, *defined by*

$$R_\alpha f\colon \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha, \mu_n)(f, g_n)\varphi_n, \quad f \in Y,$$

*describe a regularization scheme with* $\|R_\alpha\| \le c(\alpha)$.

***Proof.*** Since for all $f \in Y$ we have that

$$\|f\|^2 = \sum_{n=1}^{\infty} |(f, g_n)|^2 + \|Qf\|^2 ,$$

we have from (1.42) that

$$\begin{aligned} \|R_\alpha f\|^2 &= \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |q(\alpha, \mu_n)|^2 |(f, g_n)|^2 \\ &\leq |c(\alpha)|^2 \sum_{n=1}^{\infty} |(f, g_n)|^2 \\ &\leq |c(\alpha)|^2 \|f\|^2 \end{aligned}$$

for all $f \in Y$ and hence $\|R_\alpha\| \leq c(\alpha)$. With the aid of

$$\begin{aligned} (R_\alpha \mathcal{A}\varphi, \varphi_n) &= \frac{1}{\mu_n} q(\alpha, \mu_n)(\mathcal{A}\varphi, g_n) \\ &= q(\alpha, \mu_n)(\varphi, \varphi_n) \end{aligned}$$

and the singular value decomposition for $R_\alpha \mathcal{A}\varphi - \varphi$ we obtain

$$\begin{aligned} \|R_\alpha \mathcal{A}\varphi - \varphi\|^2 &= \sum_{n=1}^{\infty} |(R_\alpha \mathcal{A}\varphi - \varphi, \varphi_n)|^2 \\ &= \sum_{n=1}^{\infty} [q(\alpha, \mu_n) - 1]^2 |(\varphi, \varphi_n)|^2 , \end{aligned} \tag{1.44}$$

where we have used the fact that $A$ is injective.

Now let $\varphi \in X$ with $\varphi \neq 0$ and let $\epsilon > 0$ be given. Let $|q(\alpha, \mu)| \leq M$. Then there exists $N = N(\epsilon)$ such that

$$\sum_{n=N+1}^{\infty} |(\varphi, \varphi_n)|^2 < \frac{\epsilon}{2(M+1)^2} .$$

By (1.43) there exists $\alpha_0 = \alpha_0(\epsilon) > 0$ such that

$$[q(\alpha, \mu_n) - 1]^2 < \frac{\epsilon}{2\|\varphi\|^2}$$

for all $n = 1, 2, \ldots, N$ and $0 < \alpha < \alpha_0$. Splitting the series (1.44) into two parts now yields

$$\|R_\alpha \mathcal{A}\varphi - \varphi\|^2 < \frac{\epsilon}{2\|\varphi\|^2} \sum_{n=1}^{N} |(\varphi, \varphi_n)|^2 + \frac{\epsilon}{2} \leq \epsilon$$

for $0 < \alpha \leq \alpha_0$. Hence $R_\alpha \mathcal{A}\varphi \to \varphi$ as $\alpha \to 0$ for all $\varphi \in X$ and the proof is complete. □

The special choice

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2}$$

leads to *Tikhonov regularization*, which is arguably the most popular method for solving ill-posed operator equations of the first kind.

**Theorem 1.31.** *Let $\mathcal{A}\colon X \to Y$ be a compact linear operator. Then for each $\alpha > 0$ the operator $\alpha I + \mathcal{A}^*\mathcal{A}\colon X \to X$ is bijective and has a bounded inverse. Furthermore, if $\mathcal{A}$ is injective, then $R_\alpha := (\alpha I + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^*$ describes a regularization scheme with $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$.*

***Proof.*** From $\alpha \|\varphi\|^2 \leq (\alpha\varphi + \mathcal{A}^*\mathcal{A}\varphi, \varphi)$ for all $\varphi \in X$ we conclude that for $\alpha > 0$ the operator $\alpha I + \mathcal{A}^*\mathcal{A}$ is injective. Let $(\mu_n, \varphi_n, g_n)$ be a singular system for $A$, and let $Q\colon X \to N(\mathcal{A})$ denote the orthogonal projection operator. Then $T\colon X \to X$ defined by

$$T\varphi := \sum_{n=1}^{\infty} \frac{1}{\alpha + \mu_n^2}(\varphi, \varphi_n)\varphi_n + \frac{1}{\alpha}Q(\varphi)$$

is bounded and $(\alpha I + \mathcal{A}^*\mathcal{A})T = T(\alpha I + \mathcal{A}^*\mathcal{A}) = I$, i.e., $T = (\alpha I + \mathcal{A}^*\mathcal{A})^{-1}$.

If $\mathcal{A}$ is injective, then for the unique solution $\varphi_\alpha$ of

$$\alpha\varphi_\alpha + \mathcal{A}^*\mathcal{A}\varphi_\alpha = \mathcal{A}^* f$$

we deduce from the above expression for $(\alpha I + \mathcal{A}^*\mathcal{A})^{-1}$ and the identity $(\mathcal{A}^* f, \varphi_n) = \mu_n(f, g_n)$ that

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{\mu_n}{\alpha + \mu_n^2}(f, g_n)\varphi_n.$$

Hence

$$R_\alpha f = \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha, \mu_n)(f, g_n)\varphi_n, \quad f \in Y,$$

with $q(\alpha, \mu) = \frac{\mu^2}{\alpha+\mu^2}$. The function $q$ satisfies the conditions of Theorem 1.30 with $c(\alpha) = 1/2\sqrt{\alpha}$ due to the fact that

$$\sqrt{\alpha}\mu \leq \frac{\alpha + \mu^2}{2}.$$

The proof of the theorem is now complete (note that $\varphi_\alpha$ can also be determined as the unique minimizer of the Tikhonov functional $\|A\varphi - f\|^2 + \alpha\|\varphi\|^2$ — cf. Theorem 4.14 in [69]). ◻

It can be shown that the Morozov discrepancy principle is a regular strategy for choosing $\alpha$ [69], [119]. Regularization methods can also be developed for the case when the operator $\mathcal{A}$ is perturbed with a known error level [69].

## 1.4 ▪ The Scattering Problem for Anisotropic Media

We now consider a more general scattering problem where the scattering media can exhibit *anisotropic* behavior when interrogated by incident waves. The corresponding direct problem can be formulated as finding $u$ and the scattered field $u^s$ such that

$$\nabla\cdot A\nabla u + k^2 nu = 0 \qquad \text{in } D, \tag{1.45}$$
$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^3\setminus\overline{D}, \tag{1.46}$$
$$u - u^s = u^i \qquad \text{on } \partial D, \tag{1.47}$$
$$\frac{\partial u}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} \qquad \text{on } \partial D, \tag{1.48}$$
$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.49}$$

where $u^i$ is the incident field (to become precise later on), $D$ is the support of the inhomogeneity which is assumed to be a bounded Lipschitz domain such that $\mathbb{R}^3\setminus\overline{D}$ is connected, and $A$ is a $3\times 3$ symmetric matrix with $L^\infty(D)$-entries such that

$$\overline{\xi}\cdot\Re(A)\xi \ge \gamma\,|\xi|^2 \quad \text{and} \quad \overline{\xi}\cdot\Im(A)\xi \le 0$$

for all $\xi\in\mathbb{C}^3$ and almost every $x\in\overline{D}$ and some constant $\gamma>0$. The assumptions on $n$ are the same as in Section 1.2. Here $\partial u/\partial\nu_A$ denotes the conormal derivative, i.e.,

$$\frac{\partial u}{\partial \nu_A} := \nu\cdot A\nabla u.$$

Assuming that $u^i$ is an entire solution to the Helmholtz equation, one can easily see that the function $w$ defined as

$$w(x) = u(x) - u^i(x)\ x\in D \quad \text{and} \quad w(x) = u^s(x), \quad x\in\mathbb{R}^3\setminus\overline{D}$$

satisfies

$$\nabla\cdot A\nabla w + k^2 nw = \nabla\cdot(I-A)\nabla u^i + k^2(1-n)u^i \quad \text{in } \mathbb{R}^3 \tag{1.50}$$

together with the Sommerfeld radiation condition where the matrix $A$ and the index $n$ have been, respectively, extended by the identity matrix and 1 in the whole $\mathbb{R}^3$. Note that (1.50) also holds for $u^i := \Phi(\cdot,z)$, $z\notin D$, where $\Phi(\cdot,z)$ is the fundamental solution of the Helmholtz equation given by (1.8).

Our aim in this section is to establish the existence of a unique solution $w\in H^1_{loc}(\mathbb{R}^3)$ to (1.50). To this end we will rely on a variational approach; hence in the following we lay out the analytical framework for such an approach.

**Definition 1.32.** *Let $X$ be a Hilbert space. A mapping $a(\cdot,\cdot)\colon X\cdot X\to\mathbb{C}$ is called a* sesquilinear form *if*

$$a(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 a(u_1,v) + \lambda_2 a(u_2,v)$$

*for all $\lambda_1,\lambda_2\in\mathbb{C}$, $u_1,u_2\in X$ and*

$$a(u, \mu_1 v_1 + \mu_2 v_2) = \overline{\mu_1} a(u,v_1) + \overline{\mu_2} a(u,v_2)$$

*for all $\mu_1,\mu_2\in\mathbb{C}$, $v_1,v_2\in X$.*

**Definition 1.33.** *A mapping $F\colon X\to\mathbb{C}$ is called a* conjugate linear functional *if*

$$F(\mu_1, v_1 + \mu_2, v_2) = \overline{\mu_1}F(v_1) + \overline{\mu_2}F(v_2), \quad \mu_1,\mu_2\in\mathbb{C}, v_1,v_2\in X.$$

**Lemma 1.34 (Lax–Milgram Lemma).** *Assume that $a\colon X \times X \to \mathbb{C}$ is a sesquilinear form (not necessarily symmetric) for which there exist constants $\alpha, \beta > 0$ such that*

$$|a(u,v)| \le \alpha \, \|u\| \, \|v\| \quad \textit{for all } u, v \in X$$

*and*

$$|a(u,u)| \ge \beta \, \|u\|^2 \quad \textit{for all } u \in X. \tag{1.51}$$

*Then for every bounded conjugate linear functional $F\colon X \to \mathbb{C}$ there exists a unique element $u \in X$ such that*

$$a(u,v) = F(v) \quad \textit{for all } v \in X.$$

*Furthermore $\|u\| \le C \, \|F\|$, where $C > 0$ is a constant independent of $F$.*

**Remark 1.35.** Note that the Lax–Milgram Lemma is a generalization of the *Riesz Representation Theorem.*

**Remark 1.36.** A sesquilinear form satisfying (1.51) is said to be *coercive*.

**Definition 1.37.** *The* Dirichlet-to-Neumann map $\Lambda$ *is defined by*

$$\Lambda\colon v \to \frac{\partial v}{\partial \nu} \quad \textit{on } S_R,$$

*where $v$ is a radiating solution to the Helmholtz equation $\Delta v + k^2 v = 0$, $S_R$ is the boundary of some ball $B_R :=: \{x\colon |x| < R\}$, and $\nu$ is the outward unit normal to $S_R$.*

From the definition, we see that $\Lambda$ maps

$$v = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m$$

with coefficients $a_n^m$ onto

$$\Lambda v = \sum_{n=0}^{\infty} \gamma_n \sum_{m=-n}^{n} a_n^m Y_n^m,$$

where

$$\gamma_n := \frac{k h_n^{(1)\prime}(kR)}{h_n^{(1)}(kR)}, \quad n = 0, 1, \dots .$$

Noting that spherical Hankel functions and their derivatives do not have real zeros since otherwise the Wronskian of $h_n^{(1)}$ and $h_n^{(2)}$ would vanish, we see that $\Lambda$ is bijective. Furthermore, using the results of Section 1.1, it can easily be shown that

$$c_1(n+1) \le |\gamma_n| \le c_2(n+1)$$

for all $n \ge 0$ and some constants $0 < c_1 < c_2$. From this if follows that $\Lambda\colon H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is bounded. We remark that

$$\Re(\gamma_n) = \frac{1}{2} \frac{k(|h_n^{(1)}|^2)'(kR)}{|h_n^{(1)}|^2(kR)} \le 0$$

since the modulus of $h_n^{(1)}(r)$ is decreasing with respect to $r$ (see (1.7)), while

$$\Im(\gamma_n) = -\frac{k}{2i}\frac{W\left(h_n^{(1)}, h_n^{(2)}\right)(kR)}{|h_n^{(1)}|^2(kR)} \geq 0$$

according to (1.6). These properties show in particular that

$$\Im\langle\Lambda v, v\rangle \geq 0 \quad \text{and} \quad \Re\langle\Lambda v, v\rangle \leq 0 \quad \text{for all } v \in H^{1/2}(S_R), \tag{1.52}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(S_R)$ and $H^{1/2}(S_R)$ with respect to the $L^2(S_R)$ scalar product for regular functions.

**Remark 1.38.** If we define $\Lambda_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ by

$$\Lambda_0 v := -\frac{1}{R}\sum_{n=0}^{\infty}(n+1)\sum_{m=-n}^{n} a_n^m Y_n^m,$$

we clearly have that

$$-\langle\Lambda_0 v, v\rangle = R\sum_{n=0}^{\infty}(n+1)\sum_{m=-n}^{n}|a_n^m|^2.$$

Hence

$$-\langle\Lambda_0 v, v\rangle \geq c\,\|v\|^2_{H^{1/2}(S_R)}$$

for some constant $c > 0$, i.e., $-\Lambda_0$ is coercive. From the series expansion for $h_n^{(1)}$ we have that

$$\gamma_n = -\frac{n+1}{R}\left\{1 + O\left(\frac{1}{n}\right)\right\}, \quad n \to \infty,$$

which implies that $\Lambda - \Lambda_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is compact since it is bounded from $H^{1/2}(S_R)$ into $H^{1/2}(S_R)$ and the embedding from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$ is compact.

Setting $\boldsymbol{\varphi} = \nabla u^i|_D$ and $\psi = u^i|_D$, we can now replace the scattering problem (1.45)–(1.49) or (1.50) by an equivalent problem for a bounded domain: Find $w \in H^1(B_R)$ such that

$$\nabla \cdot A\nabla w + k^2 n w = \nabla \cdot (I - A)\boldsymbol{\varphi} + k^2(1-n)\psi \quad \text{in } B_R, \tag{1.53}$$

$$\frac{\partial w}{\partial \nu} = \Lambda w \quad \text{on } S_R. \tag{1.54}$$

Multiply (1.53) by a test function $\overline{v} \in H^1(B_R)$ and use Green's first identity to arrive at the following equivalent *variational formulation* of problem (1.53)–(1.54): Find $w \in H^1(B_R)$ such that

$$a_1(w, v) + a_2(w, v) = F(v) \quad \text{for all } v \in H^1(B_R), \tag{1.55}$$

where

$$a_1(\phi, v) := \int_{B_R} \nabla\overline{v} \cdot A\nabla\phi\, dx + \int_{B_R} \overline{v}\phi\, dx - \langle\Lambda\phi, v\rangle,$$

$$a_2(\phi, v) := -\int_{B_R}\left(nk^2 + 1\right)\overline{v}\phi\, dx,$$

$$F(v) := -\int_D \nabla\overline{v} \cdot (I - A)\boldsymbol{\varphi}\, dx + k^2\int_D (1-n)\overline{v}\psi dx.$$

**Theorem 1.39.** *Assume that $\boldsymbol{\varphi} \in L^2(D)^3$ and $\psi \in L^2(D)$ and in addition that $A$ is continuously differentiable in $D$. Then there exists a unique solution to* (1.55).

***Proof.***

1. From the assumption $\overline{\xi} \cdot \Re(A)\xi \geq \gamma \left|\xi\right|^2$ and (1.52) we can conclude that $a_1(\cdot, \cdot)$ is coercive.

2. Using the Riesz representation theorem we can now define the operator $\mathcal{A}\colon H^1(B_R) \to H^1(B_R)$ by $a_1(w, v) = (\mathcal{A}w, v)_{H^1(B_R)}$ and from 1 and the Lax–Milgram lemma we have that $\mathcal{A}^{-1}$ exists and is bounded.

3. Similarly, we can define a bounded linear operator $\mathcal{B}\colon H^1(B_R) \to H^1(B_R)$ by $a_2(w, v) = (\mathcal{B}w, v)_{H^1(B_R)}$ and due to the compact embedding of $H^1(B_R)$ into $L^2(B_R)$ we have that $\mathcal{B}$ is compact. Note that

$$\begin{aligned}\|\mathcal{B}u\|^2_{H^1(B_R)} = (\mathcal{B}u, \mathcal{B}u)_{H^1(B_R)} &\leq c\|u\|_{L^2(B_R)}\|\mathcal{B}u\|_{L^2(B_R)} \\ &\leq c\|u\|_{L^2(B_R)}\|\mathcal{B}u\|_{H^1(B_R)},\end{aligned}$$

   implying that

$$\|\mathcal{B}u\|_{H^1(B_R)} \leq c\|u\|_{L^2(B_R)}.$$

4. The theorem now follows if we can show that $\mathcal{A}+\mathcal{B}$ is boundedly invertible. But this follows from items 2 and 3 by the Fredholm alternative provided we have uniqueness of a solution to (1.45)–(1.49). Under the assumption that $A$ is continuously differentiable, this follows from Rellich's Lemma and the unique continuation principle for solutions to (1.53) in a similar way to the isotropic case discussed in Section 1.2 (cf. [98]). $\square$

Since (1.55) is equivalent to the scattering problem (1.45)–(1.49), the above theorem establishes the well-posedness of the direct scattering problem for anisotropic media.

For further use in Chapter 2, we will need the following formulas:

$$\begin{aligned}w_\infty(\hat{x}) = -\frac{1}{4\pi}\int_D \big(ik\hat{x}\cdot(I-A)(\boldsymbol{\varphi}(y) + \nabla w(y)) \\ +\ k^2(1-n)(y)(\psi(y) + w(y))\big)\, e^{-ik\hat{x}\cdot y}\, dy,\end{aligned} \tag{1.56}$$

$$u_\infty(\hat{x}) = -\frac{1}{4\pi}\int_D \big(ik\hat{x}\cdot(I-A)\nabla u(y) + k^2(1-n)(y)u(y)\big)\, e^{-ik\hat{x}\cdot y}\, dy. \tag{1.57}$$

Since (1.57) follows immediately from (1.56), it suffices to prove (1.56). To this end, we note that

$$\begin{aligned}\Delta w + k^2 w &= \nabla\cdot(I-A)\nabla w + k^2(1-n)w + (\nabla\cdot A\nabla w + k^2 n w) \\ &= \nabla\cdot(I-A)(\nabla w + \boldsymbol{\varphi}) + k^2(1-n)(w+\psi).\end{aligned} \tag{1.58}$$

From Green's formula we immediately have that

$$w(x) = -\int_D \Phi(x,y)(\Delta w + k^2 w)dy, \tag{1.59}$$

where the integral is understood as the convolution of the fundamental solution with the compactly supported distribution $\Delta w + k^2 w$. Then from (1.58) and (1.59) we have that

$$w(x) = -\int_D (I - A)(\nabla w + \boldsymbol{\varphi}) \cdot \nabla_x \Phi(x,y) + k^2(1-n)(w+\psi)\Phi(x,y)\, dy.$$

Finally letting $x$ tend to infinity, now yields (1.56).

### 1.4.1 ▪ The Far Field Operator

If we consider plane wave incident fields, i.e., $u^i(x) := e^{ikx\cdot d}$, where $|d| = 1$, similarly to the isotropic case, we have that the scattered field corresponding to (1.45)–(1.49) satisfies

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right)\right\}.$$

The following reciprocity principle can be proven exactly in the same way as Theorem 1.13 where using the symmetry of $A$ and with the help of Green's theorem the integral over $\partial D$ is moved to the integral over $|y| = a$.

**Theorem 1.40.** *Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to* (1.45)–(1.49). *Then $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$.*

The reciprocity principle states that the far field pattern is unchanged if the direction of the incident field and observation directions are interchanged. It can be generalized to a relationship between the scattering of point sources and plane waves, which is refered to as the *mixed reciprocity principle*. The following theorem can be proven in a similar way to that of Theorem 1.13 (see, for details, Theorem 3.16 in [69]).

**Theorem 1.41.** *Let $u_\infty(\hat{x}, z)$ be the far field pattern of the scattered field $u^s(x, z)$ for the scattering of a point source $u^i := \Phi(x, z)$ located at $z \in \mathbb{R}^3 \setminus \overline{D}$, and let $u^s(x, d)$ be the scattered field due to a plane wave $u^i := e^{ikx\cdot d}$. Then*

$$4\pi u_\infty(-d, z) = u(z, d), \qquad z \in \mathbb{R}^3 \setminus \overline{D},\ d \in S.$$

We can define the *far field operator* $F\colon L^2(S^2) \to L^2(S^2)$ corresponding to (1.45)–(1.49) by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d)\, ds(d)$$

with the corresponding scattering operator given by (1.30).

**Theorem 1.42.** *Let $g, h \in L^2(S^2)$, and let $v_g$ and $v_h$ be the Herglotz wave functions with kernels $g$ and $h$, respectively. Then if $(u_g, u^s_g)$ and $(u_h, u^s_h)$ are the solutions of the scattering problem* (1.45)–(1.49) *corresponding to the incident field $u^i := v_g$ and $u^i := v_h$, respectively, we have that*

$$-\int_D \Im(A)\nabla u_g \cdot \nabla\overline{u_h}\, dx + k^2\int_D \Im(n) u_g \overline{u_h}\, dx = 2\pi(Fg, h) - 2\pi(g, Fh) - ik(Fg, Fh).$$

***Proof.*** Let $u_g = u_g^s + v_g$ and $u_h = u_h^s + v_h$ be the total fields in $\mathbb{R}^3 \setminus \overline{D}$. Then using transmission conditions, the divergence theorem along with the symmetry of $A$, and the equations in $D$ we have

$$\begin{aligned}
&\int\limits_{|x|=a} \left( u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds = \int\limits_{\partial D} \left( u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
&= \int\limits_{\partial D} \left( u_g \overline{A \nabla u_h} \cdot \nu - \overline{u_h} A \nabla u_g \cdot \nu \right) ds = \int\limits_{D} \left( \nabla \cdot (u_g \overline{A \nabla u_h}) - \nabla \cdot (\overline{u_h} A \nabla u_g) \right) dx \\
&= \int\limits_{D} \left( \nabla u_g \cdot \overline{A \nabla u_h} - \nabla \overline{u_h} \cdot A \nabla u_g \right) dx + \int\limits_{D} \left( u_g \nabla \cdot \overline{A \nabla u_h} - \overline{u_h} \cdot \nabla A \nabla u_g \right) dx \\
&= \int\limits_{D} \left( \nabla u_g \cdot \overline{A \nabla u_h} - \nabla \overline{u_h} \cdot A \nabla u_g \right) dx + k^2 \int\limits_{D} \left( \overline{u_h} n u_g - u_g \overline{n u_h} \right) dx.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
&\int\limits_{|x|=a} \left( u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
&\qquad = -2i \int\limits_{D} \Im(A) \nabla u_g \cdot \nabla \overline{u_h}\, dx + 2ik^2 \int\limits_{D} \Im(n) u_g \overline{u_h}\, dx.
\end{aligned} \tag{1.60}$$

Proceeding exactly as in the proof of Theorem 1.14 where $w_g$ and $w_h$ are replaced by the fields outside $D$, $u_g$ and $u_h$, we obtain that

$$\begin{aligned}
&\int\limits_{|x|=a} \left( u_g \frac{\partial \overline{u_h}}{\partial \nu} - \overline{u_h} \frac{\partial u_g}{\partial \nu} \right) ds \\
&\qquad = 4\pi (Fg, h) - 4\pi (g, Fh) - 2ik (Fg, Fh).
\end{aligned} \tag{1.61}$$

Combining (1.60) and (1.61) yields the result. □

Now Theorem 1.42 implies the following property of the far field operator.

**Theorem 1.43.** *Assume that both* $\Im(A) = 0$ *and* $\Im(n) = 0$. *Then the far field operator corresponding to the scattering problem* (1.45)–(1.49) *is normal.*

***Proof.*** The proof is exactly the same as the proof of Theorem 1.15. □

Finally, the proofs of Theorem 1.16 and Corollary 1.17 carry through for the far field operator corresponding to the scattering problem for anisotropic media. More precisely, the following theorem holds (see also Theorem 6.2 in [29]).

**Theorem 1.44.** *Let* $F : L^2(S^2) \to L^2(S^2)$ *be the far field operator corresponding to the scattering problem* (1.27)–(1.29). *Then* $F$ *is injective and has dense range if and only if there does not exist a Herglotz wave function* $v_g$ *such that the pair* $u, v := v_g$ *is a solution* *to the transmission eigenvalue problem*

$$\nabla \cdot A\nabla u + k^2 nu = 0 \qquad \text{in } D, \tag{1.62}$$
$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{1.63}$$
$$v = u \qquad \text{on } \partial D, \tag{1.64}$$
$$\frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu_A} \qquad \text{on } \partial D. \tag{1.65}$$

Values of $k > 0$ for which (1.62)–(1.65) has nontrivial solutions such that $v := v_g$, i.e., $v$ is a Herglotz wave function, are called *nonscattering wave numbers*. In particular $F$ is injective and has dense range if and only $k$ is not a nonscattering wave number. We also mention that values of $k$ for which (1.62)–(1.65) has nontrivial solutions are referred to as *transmission eigenvalues*.

### 1.4.2 ▪ The Inverse Scattering Problem

Similarly to the inverse medium problem for isotropic inhomogeneities (Section 1.2.2), the inverse problem is to determine $A(x)$ and $n(x)$ (or some properties of $A(x)$ and $n(x)$) from a knowledge of the far field $u^\infty(\hat{x}, d)$ corresponding to the scattering problem (1.27)–(1.29). Unfortunately, in the general case of matrix valued functions $A(x)$, the far field patterns $u^\infty(\cdot, d)$ do not uniquely determine $A$ and $n$ even if they are known for all $d \in S^2$ and all wave numbers $k$ [92]. Hence in general for anisotropic media, only the uniqueness of the support $D$ of the inhomogeneity can be expected . The idea of the uniqueness proof for the inverse medium scattering problem originates from [101], [102] in which it is shown that the shape of a penetrable, inhomogeneous, isotropic medium is uniquely determined by its far field pattern for all incident plane waves. The case of an anisotropic medium is due to Hähner [94] (see also [41]), the proof of which is based on the existence of a solution to the modified interior transmission problem. To proceed further let us define the (nonhomogeneous) *interior transmission problem* corresponding to (1.62)–(1.65): Given $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$, find $u \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\nabla \cdot A\nabla u + k^2 nu = 0 \qquad \text{in } D, \tag{1.66}$$
$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{1.67}$$
$$u - v = f \qquad \text{on } \partial D, \tag{1.68}$$
$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on } \partial D, \tag{1.69}$$

This problem will be analyzed in Chapter 3 in this book. The uniqueness result is based on the following assumption on the interior transmission problem.

**Assumption 1.1.** *$A$, $n$ are such that the modified interior transmission problem—given $f \in H^{1/2}(\partial D)$, $h \in H^{-1/2}(\partial D)$, $\ell_1 \in L^2(D)$, and $\ell_2 \in L^2(D)$, find $u \in H^1(D)$ and $v \in H^1(D)$ satisfying*

$$\nabla \cdot A\nabla u + \gamma_1 nu = \ell_1 \qquad \text{in } D, \tag{1.70}$$
$$\Delta v + \gamma_2 v = \ell_2 \qquad \text{in } D, \tag{1.71}$$
$$u - v = f \qquad \text{on } \partial D, \tag{1.72}$$
$$\frac{\partial u}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on } \partial D \tag{1.73}$$

*for some constants $\gamma_1$ and $\gamma_2$—has a unique solution which satisfies*

$$\|u\|_{H^1(D)} + \|v\|_{H^1(D)} \le C\left(\|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)} + \|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)}\right).$$

Note that the interior transmission problem (1.66)–(1.69) is a compact perturbation of (1.70)–(1.73). This implies the following lemma, which will be used in the proof of uniqueness, in order to obtain the result without assuming that $k$ is not a transmission eigenvalue.

**Lemma 1.45.** *Assume that Assumption* 1.1 *holds, and let $\{v_n,\, u_n\} \in H^1(D) \times H^1(D)$, $j \in \mathbb{N}$, be a sequence of solutions to the interior transmission problem* (1.66)–(1.69) *with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$, $h_n \in H^{-\frac{1}{2}}(\partial D)$. If the sequences $\{f_n\}$ and $\{h_n\}$ converge in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$, respectively, and if the sequences $\{v_n\}$ and $\{u_n\}$ are bounded in $H^1(D)$, then there exists a subsequence $\{v_{n_k}\}$ which converges in $H^1(D)$.*

***Proof.*** Thanks to the compact embedding of $H^1(D)$ into $L^2(D)$ we can select $L^2$-convergent subsequences $\{v_{n_k}\}$ and $\{u_{n_k}\}$, which satisfy

$$\begin{aligned}
\nabla\cdot A\nabla u_{n_k} + \gamma_1 u_{n_k} &= (\gamma - k^2 n)u_{n_k} && \text{in} \quad D,\\
\Delta v_{n_k} + \gamma_2\, v_{n_k} &= (\gamma_2 - k^2)v_{n_k} && \text{in} \quad D,\\
u_{n_k} - v_{n_k} &= f_{n_k} && \text{on} \quad \partial D,\\
\frac{\partial u_{n_k}}{\partial \nu_A} - \frac{\partial v_{n_k}}{\partial \nu} &= h_{n_k} && \text{on} \quad \partial D.
\end{aligned}$$

Then the result follows from Assumption 1.1. □

We are now ready to prove the uniqueness theorem.

**Theorem 1.46.** *Let the domains $D_1$ and $D_2$, the matrix-valued functions $A_1$ and $A_2$, and the functions $n_1$ and $n_2$ be such that Assumption* 1.1 *holds. If the far field patterns $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$ corresponding to $D_1, A_1, n_1$ and $D_2, A_2, n_2$, respectively, coincide for all $\hat{x} \in S^2$ and $d \in S^2$, then $D_1 = D_2$.*

***Proof.*** Denote by $G$ the unbounded connected component of $\mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2)$ and define $D_1^e := \mathbb{R}^3 \setminus \bar{D}_1$, $D_2^e := \mathbb{R}^3 \setminus \bar{D}_2$. By the analyticity of the far field patterns and Rellich's Lemma we conclude that the scattered fields $u_1^s(\cdot, d)$ and $u_2^s(\cdot, d)$ which are solutions of (1.45)–(1.49) with $D_1, A_1, n_1$ and $D_2, A_2, n_2$, respectively, and $u^i = e^{ikx\cdot d}$, coincide in $G$ for all $d \in S^2$. For the incident field $u^i := \Phi(x, z)$ we denote by $u_1^s(\cdot, z)$ and $u_2^s(\cdot, z)$ the corresponding scattered solutions. The mixed reciprocity relation in Theorem 1.41 with another application of Rellich's Lemma implies that $u_1^s(\cdot, z)$ and $u_2^s(\cdot, z)$ also coincide for all $z \in G$. In terms of notation (1.50), this means that $w_1(\cdot, z) = w_2(\cdot, z)$ for all $z \in G$.

Let us now assume that $\bar{D}_1$ is not included in $\bar{D}_2$. Since $D_2^e$ is connected, we can find a point $z \in \partial D_1$ and $\epsilon > 0$ with the following properties, where $\Omega_\delta(z)$ denotes the ball of radius $\delta$ centered at $z$:

1. $\Omega_{8\epsilon}(z) \cap \bar{D}_2 = \emptyset$.

2. The intersection $\bar{D}_1 \cap \Omega_{8\epsilon}(z)$ is contained in the connected component of $\bar{D}_1$ to which $z$ belongs.

3. There are points from this connected component of $\bar{D}_1$ to which $z$ belongs that are not contained in $\bar{D}_1 \cap \bar{\Omega}_{8\epsilon}(z)$.

4. The points $z_n := z + \frac{\epsilon}{n}\nu(z)$ lie in $G$ for all $n \in \mathbb{N}$, where $\nu(z)$ is the unit normal to $\partial D_1$ at $z$.

Due to the singular behavior of $\Phi(\cdot,\, z_n)$, it is easy to show that $\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$. We now define

$$v^n(x) := \frac{1}{\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)}}\Phi(x,\, z_n), \qquad x \in \bar{D}_1 \cup \bar{D}_2,$$

and let $w_1^n$ and $w_2^n$ be the scattered fields solving the scattering problem (1.50) with $u^i := v^n$ corresponding to $D_1, A_1, n_1$ and $D_2, A_2, n_2$, respectively. Note that for each $n$, $v^n$ is a solution of the Helmholtz equation in $D_1$ and $D_2$. Our aim is to prove that if $\bar{D}_1 \not\subset \bar{D}_2$, then the equality $w_1(\cdot, z) = w_2(\cdot, z)$ for $z \in G$ allows the selection of a subsequence $\{v^{n_k}\}$ from $\{v^n\}$ that converges to zero with respect to $H^1(D_1)$. This certainly contradicts the definition of $\{v^n\}$ as a sequence of functions with $H^1(D_1)$-norm equal to one. Note that $w_1(\cdot, z) = w_2(\cdot, z)$ obviously implies that $w_1^n = w_2^n$ in $G$.

We begin by noting that since the functions $\Phi(\cdot,\, z_n)$ together with their derivatives are uniformly bounded in every compact subset of $\mathbb{R}^3 \setminus \Omega_{2\epsilon}(z)$, and since $\|\Phi(\cdot,\, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$, then $\|v^n\|_{H^1(D_2)} \to 0$ as $n \to \infty$. Hence, if $\Omega_R$ is a large ball containing $\bar{D}_1 \cup \bar{D}_2$, then $\|w_2^n\|_{H^1(\Omega_R \cap G)} \to 0$ also as $n \to \infty$ from the well-posedness of the direct scattering problem . Since $w_1^n = w_2^n$ in $G$ then $\|w_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ as $n \to \infty$ as well. Now, with the help of a cutoff function $\chi \in C_0^\infty(\Omega_{8\epsilon}(z))$ satisfying $\chi(x) = 1$ in $\Omega_{7\epsilon}(z)$ we see that $\|w_1^n\|_{H^1(\Omega_R \cap G)} \to 0$ implies that

$$(\chi w_1^n) \to 0, \qquad \frac{\partial(\chi w_1^n)}{\partial \nu} \to 0, \qquad \text{as } n \to \infty \tag{1.74}$$

on $\partial D_1$, with respect to the $H^{\frac{1}{2}}(\partial D_1)$-norm and $H^{-\frac{1}{2}}(\partial D_1)$-norm, respectively. Indeed, for the first convergence we simply apply the trace theorem, while for the convergence of $\partial(\chi w_1^n)/\partial\nu$, we first deduce the convergence of $\Delta(\chi w_1^n)$ in $L^2(\Omega_R \cap D_1^e)$, which follows from $\Delta(\chi w_1^n) = \chi \Delta w_1^n + 2\nabla\chi \cdot \nabla w_1^n + w_1^n \Delta\chi$, and then apply Green's Theorem. Note here that we need conditions 2 and 4 on $z$ to ensure $\Omega_{8\epsilon}(z) \cap D_1^e = \Omega_{8\epsilon}(z) \cap G$.

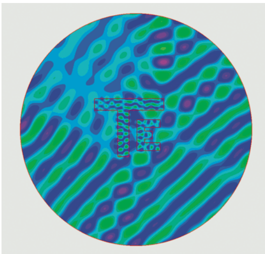
We next note that in the exterior of $\Omega_{2\epsilon}(z)$ the $H^2(\Omega_R \setminus \Omega_{2\epsilon}(z))$-norms of $v^n$ remain uniformly bounded. Then, thanks to the smoothness of $A$ and $n$, regularity results for (1.50) [88] imply that $w_1^n$ is uniformly bounded with respect to the $H^2((\Omega_R \cap D_1^e) \setminus \Omega_{4\epsilon}(z))$-norm. Therefore, using the compact embedding of $H^2(\Omega_R \cap D_1^e)$ into $H^1(\Omega_R \cap D_1^e)$, we can select an $H^1(\Omega_R \cap D_1^e)$ convergent subsequence $\{(1-\chi)w_1^{n_k}\}$ from $\{(1-\chi)w_1^n\}$. Hence, $\{(1-\chi)w_1^{n_k}\}$ is a convergent sequence in $H^{\frac{1}{2}}(\partial D_1)$, and similarly to the above reasoning we also have that $\{\partial((1-\chi)w_1^{n_k})/\partial\nu\}$ converges in $H^{-\frac{1}{2}}(\partial D_1)$. This, together with (1.74), implies that the sequences

$$\{w_1^{n_k}\} \qquad \text{and} \qquad \left\{\frac{\partial w_1^{n_k}}{\partial \nu}\right\}$$

converge in $H^{\frac{1}{2}}(\partial D_1)$ and $H^{-\frac{1}{2}}(\partial D_1)$, respectively.

Finally, since the functions $w_1^{n_k} + v^{n_k}$ and $v^{n_k}$ are solutions to the interior transmission problem (1.66)–(1.69) for the domain $D_1$ with boundary data $f = w_1^{n_k}$ and $h = \partial w_1^{n_k}/\partial\nu$, and since the $H^1(D_1)$-norms of $w_1^{n_k} + v^{n_k}$ and $v^{n_k}$ remain uniformly bounded, according to Lemma 1.45 we can select a subsequence of $\{v^{n_k}\}$, denoted again by $\{v^{n_k}\}$, which converges in $H^1(D_1)$ to a function $v \in H^1(D_1)$. As a limit of weak solutions to the Helmholtz equation, $v \in H^1(D_1)$ is a weak solution to the Helmholtz equation. We also have that $v|_{D_1 \setminus \Omega_{2\epsilon}(z)} = 0$ because the functions $v^{n_k}$ converge uniformly to zero in the exterior of $\Omega_{2\epsilon}(z)$. Hence, $v$ must be zero in all of $D_1$ (here we make use of condition 3, namely, the fact that the connected component of $D_1$ containing $z$ has points which do not lie in the exterior of $\bar{\Omega}_{2\epsilon}(z)$). This contradicts the fact that $\|v^{n_k}\|_{H^1(D_1)} = 1$. Hence the assumption $\bar{D}_1 \not\subset \bar{D}_2$ is false.

Since we can derive the analogous contradiction for the assumption $\bar{D}_2 \not\subset \bar{D}_1$, we have proved that $D_1 = D_2$. $\square$



# Chapter 2
# The Determination of the Support of Inhomogeneous Media

We now introduce and analyze a class of inversion methods, often referred to as qualitative methods, that solve the inverse problem of finding $D$ from the measured far field data $u_\infty(\hat{x}, d)$ for $(\hat{x}, d) \in S^2 \times S^2$ without reconstructing the index of refraction $n$ or other medium physical parameters. These methods are based on a careful analysis of the range of the far field operator $F : L^2(S^2) \to L^2(S^2)$ defined by

$$(Fg)(\hat{x}) := \int\limits_{S^2} u_\infty(\hat{x}, d)g(d)\, ds(d). \tag{2.1}$$

The analysis of these methods does not require weak scattering approximations. In addition, the associated algorithms do not require a forward solver of the scattering problem, and hence they are faster to implement.

We start in Section 2.1 with the Linear Sampling Method (LSM) that has been introduced in [66] to solve the aforementioned inverse problem and that was further analyzed in a number of subsequent works [26], [65], and [76]. We refer the reader to [29] for an extensive presentation of this method and its various applications. This method has the simplest formulation and can be easily adapted to different settings of the data (near field data, data available on a limited aperture) and the scattering problem (inhomogeneous background). However, the theoretical foundation of the method does not fully justify why it numerically works. For instance, the theory does not provide a regularization scheme that constructs the predicted indicator function of the domain $D$. We provide in Section 2.1 a complete analysis of this method in the simple isotropic case.

A new formulation of LSM, referred to as Generalized Linear Sampling Method (GLSM), was proposed in [13] in order to circumvent the above-mentioned weak point. It gives an exact characterization of the domain $D$ in terms of the range of $F$. Moreover, it yields a numerically tractable indicator function. A detailed presentation of this method is given in Section 2.2 and follows the one given in [12] and [13]. This method has also been extended to the case of Maxwell's equations [93]. We provide in Section 2.2.1 the theoretical foundation of the GLSM in an abstract framework that can then be applied to various inverse scattering problems. We confine ourselves to the theory adapted to data available on a full aperture and refer the reader to [7] for more elaborate formulations that can apply to near field data, data available on a limited aperture, and inhomogeneous backgrounds. Sections 2.2.2 and 2.2.3 address the issue of noisy operators. Although important from a practical point of view, these sections can be skipped in a first reading. The application of the abstract theory to the isotropic inverse problem is then presented in Section 2.2.4.

Another exact characterization of $D$ in terms of the far field operator can be obtained using the so-called inf-criterion. This method is presented in Section 2.3. The main drawback of this characterization is that it is numerically less attractive than other sampling methods. However, this criterion can be used to justify other methods like the factorization method presented in Section 2.4. The latter was first introduced by Kirsch in [109], and we refer the reader to [113] for a detailed analysis of this method. We give here a self-contained and slightly different presentation of the abstract theory related to this method for both versions, the $(F^*F)^{1/4}$ and $F_\sharp$ methods. We also discuss for each version the application to the isotropic inverse problem. The factorization method requires (in principle) stronger assumptions than the other sampling methods. For instance, the generalization to the case of limited aperture is an open problem as well as for inhomogeneous backgrounds that contain absorption.

Section 2.5 complements the picture on sampling methods by addressing some link between them. We explain, for instance, how the $(F^*F)^{1/4}$ method can be used to provide precise information on the behavior of the Tikhonov regularized solution of the LSM equation. We also explain how the factorization method can complement the GLSM to solve the imaging problem where one would like to identify a change in the background using differential measurements. Some simple comparative numerical illustrations of these methods are given in Section 2.5.3. Application to the case of differential measurements is discussed in Section 2.5.4 in a simplified configuration. This section does not intend to give a full presentation of this important problem but rather a glimpse on potential new applications of sampling methods.

We close this chapter with Section 2.6, where the application of all previously introduced sampling methods is discussed in the case of anisotropic media. This provides a unified presentation of the analysis of these methods for a particular problem.

## 2.1 ▪ The Linear Sampling Method (LSM)

We consider here the first class of qualitative methods that was introduced in [66] and that was further analyzed in a number of subsequent works [26], [65], [76]. Roughly speaking, the idea of the method is to consider approximate solutions to (2.1) (in a sense that will be made precise later), i.e., $g_z \in L^2(S^2)$ satisfying

$$Fg_z \simeq \Phi_\infty(\cdot, z)$$

with $\Phi_\infty(\hat{x}, z) := \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$ being the far field pattern associated with the fundamental solution $\Phi(\cdot, z)$ and then use $z \mapsto 1/\|g_z\|_{L^2(S^2)}$ as an indicator function for the domain $D$. We shall first give a presentation of the method in the special case where $u_\infty(\cdot, d)$ is the far field pattern associated with the scattered field $u^s(\cdot, d) \in H^1_{loc}(\mathbb{R}^3)$ solution to (1.27)–(1.29). The index of refraction $n \in L^\infty(\mathbb{R}^3)$ is such that $\Re(n) > 0$, $\Im(n) \geq 0$, $n = 1$ outside the support $\overline{D}$ of $m := 1 - n$, and assume that $D$ contains the origin, has Lipschitz boundary $\partial D$, and connected complement in $\mathbb{R}^3$. According to Theorem 1.11, let us define for $u_0 \in L^2(D)$ the unique function $w \in H^1_{loc}(\mathbb{R}^3)$ satisfying

$$\begin{cases} \Delta w + k^2 n w = k^2(1-n)u_0 \quad \text{in } \mathbb{R}^3, \\ \displaystyle\lim_{R\to\infty} \int_{|x|=R} |\partial w/\partial |x| - ik\, w|^2 \, ds = 0. \end{cases} \tag{2.2}$$

Obviously, if $u_0(x) = e^{ikd\cdot x}$, then $w = u^s(\cdot, d)$, and therefore the far field pattern $w_\infty$ of $w$ coincides with $u_\infty(\cdot, d)$. Let us consider the (compact) operator $\mathcal{H} : L^2(S^2) \to L^2(D)$ defined by

$$\mathcal{H}g := v_g|_D, \tag{2.3}$$

where the Herglotz wave function $v_g$ is defined by (1.31), namely,

$$v_g(x) := \int\limits_{S^2} e^{ikd\cdot x} g(d) ds(d),\ x \in \mathbb{R}^3.$$

Let us denote by $H_{\rm inc}(D)$ the closure of the range of $\mathcal{H}$ in $L^2(D)$. We then consider the (compact) operator $G : H_{\rm inc}(D) \to L^2(S^2)$ defined by

$$G(u_0) := w_\infty, \tag{2.4}$$

where $w_\infty$ is the far field pattern of $w \in H^1_{loc}(\mathbb{R}^3)$ satisfying (2.2). One therefore easily observes that $F$ can be factorized as

$$F = G\mathcal{H}. \tag{2.5}$$

The justification of the Linear Sampling Method (LSM) is mainly based on the characterization of $D$ in terms of the range of the operator $G$. This characterization uses the solvability of the *interior transmission problem*: Find $(u, u_0) \in L^2(D) \times L^2(D)$ such that $u - u_0 \in H^2(D)$ and

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta u_0 + k^2 u_0 = 0 & \text{in } D, \\ u - u_0 = f & \text{on } \partial D, \\ \partial(u - u_0)/\partial\nu = h & \text{on } \partial D \end{cases} \tag{2.6}$$

for given $(f, h) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$, where $\nu$ denotes the outward normal on $\partial D$. Values of $k$ for which this problem is not well-posed are referred to as *transmission eigenvalues*. We consider in this chapter only real transmission eigenvalues.

The analysis of the interior transmission problem and of transmission eigenvalues will be conducted in the next two chapters. We need in this chapter only the well-posedness of this problem (as well as the well-posedness of the direct problem (2.2)) for data $u_0 \in L^2(D)$. At this point we formulate this statement in the following assumption (the solvability of the interior transmission problem is the subject of Chapter 3).

**Assumption 2.1.** *We assume that the refractive index $n$ and the real wave number $k$ are such that* (2.6) *defines a well-posed problem.*

We recall from Theorem 1.11 that (2.2) is well-posed if $n \in L^\infty(\mathbb{R}^3)$, $\Re(n) > 0$, $\Im(n) \geq 0$, and $n = 1$ in $\mathbb{R}^3 \setminus D$. The well-posedness of (2.6) requires at least that $n \neq 1$ in a neighborhood of $\partial D$ and that $k$ is outside a countable set without finite accumulation points (see Chapter 3).

A first step towards the justification of LSM is the characterization of the closure of the range of $\mathcal{H}$.

**Lemma 2.1.** *The operator $\mathcal{H}$ is compact and injective. Let $H_{\rm inc}(D)$ be the closure of the range of $\mathcal{H}$ in $L^2(D)$. Then*

$$H_{\rm inc}(D) = \{v \in L^2(D) : \quad \Delta v + k^2 v = 0 \textit{ in } D\}.$$

***Proof.*** For the first part, assume that $\mathcal{H}g = 0$ in $D$. Since

$$\Delta \mathcal{H}g + k^2 \mathcal{H}g = 0 \quad \text{in } \mathbb{R}^3,$$

by the unique continuation principle, $\mathcal{H}g = 0$ in $\mathbb{R}^3$. This implies (using the Jacobi–Anger expansion [69]) that $g = 0$.

For the second part of the lemma, we give a slightly different proof from the original one in [154]. Set $\widetilde{H_{\mathrm{inc}}}(D) := \{v \in L^2(D) : \Delta v + k^2 v = 0 \text{ in } D\}$. Then obviously $H_{\mathrm{inc}}(D) \subset \widetilde{H_{\mathrm{inc}}}(D)$. To prove the theorem it is then sufficient to prove that $\mathcal{H}^* : L^2(D) \to L^2(S^2)$, the adjoint of the operator $\mathcal{H}$ given by

$$\mathcal{H}^* \varphi(\hat{x}) := \int_D e^{-ik\hat{x}\cdot y} \varphi(y) dy, \; \varphi \in L^2(D), \; \hat{x} \in S^2, \tag{2.7}$$

is injective on $\widetilde{H_{\mathrm{inc}}}(D)$. Let $u_0 \in \widetilde{H_{\mathrm{inc}}}(D)$ and set

$$u(x) := \int_D \Phi(x, y)\, u_0(y)\, dy, \quad x \in \mathbb{R}^3.$$

From the regularity properties of volume potentials (Theorem 1.8), we infer that $u \in H^2_{loc}(\mathbb{R}^3)$ and satisfies

$$\begin{cases} \text{(i)} & \Delta u + k^2 u = -u_0 & \text{in } D, \\ \text{(ii)} & \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}. \end{cases} \tag{2.8}$$

Since by construction $4\pi u_\infty = \mathcal{H}^*(u_0)$, then $\mathcal{H}^*(u_0) = 0$ implies that $u_\infty = 0$ and therefore $u = 0$ in $\mathbb{R}^3 \setminus D$ by Rellich's Lemma. The regularity $u \in H^2_{loc}(\mathbb{R}^3)$ then implies $u \in H^2_0(D)$. Now take the $L^2(D)$ scalar product of (2.8)(i) with $u_0$ to obtain

$$\int_D \left(\Delta u + k^2 u\right) \overline{u_0}\, dx = - \|u_0\|^2_{L^2(D)}.$$

The left-hand side of this equality is zero since $\Delta u_0 + k^2 u_0 = 0$ in the distributional sense and $u \in H^2_0(D)$. This implies $u_0 = 0$. ☐

The following reciprocity lemma will also be useful.

**Lemma 2.2.** *Let $u_0, u_1 \in L^2(D)$, and let $w_0$ and $w_1 \in H^1_{loc}(\mathbb{R}^3)$ be the corresponding solutions satisfying* (2.2). *Then*

$$\int_D (1-n) w_0 \cdot u_1 dx = \int_D (1-n) w_1 \cdot u_0 dx. \tag{2.9}$$

***Proof.*** We have

$$\begin{cases} \text{(i)} & \Delta w_0 + k^2\, n\, w_0 = k^2(1-n)u_0 & \text{in } \mathbb{R}^3, \\ \text{(ii)} & \Delta w_1 + k^2\, n\, w_1 = k^2(1-n)u_1 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.10}$$

Let $B_R$ be an open ball with radius $R$ that contains $\overline{D}$. Multiplying (2.10)(i) by $w_1$ and (2.10)(ii) by $w_0$ yields, after integrating over $B_R$ and taking the difference,

$$\int_{B_R} (\Delta w_0\, w_1 - \Delta w_1\, w_0)\, dx = k^2 \int_D ((1-n)u_0\, w_1 - (1-n)u_1\, w_0)\, dx.$$

Integrating by parts, we obtain

$$\int\limits_{\partial B_R} ((\partial w_0/\partial r)w_1 - (\partial w_1/\partial r)w_0)\, ds(x)$$
$$= k^2 \int\limits_D ((1-n)u_0 \cdot w_1 - (1-n)u_1 \cdot w_0)\, dx. \tag{2.11}$$

Since $w_0$ and $w_1$ satisfy the Sommerfeld radiation condition

$$\lim_{R\to\infty} \int\limits_{\partial B_R} |\partial w_\ell/\partial r - i\, k w_\ell|^2 \, ds(\hat{x}) = 0$$

and

$$\lim_{R\to\infty} \int\limits_{\partial B_R} |w_\ell|^2 \, ds(x) = \int\limits_{S^2} \left|w_\infty^\ell\right|^2 ds(\hat{x})$$

for $\ell = 1, 2$, therefore

$$\lim_{R\to\infty} \int\limits_{\partial B_R} ((\partial w_0/\partial r)w_1 - (\partial w_1/\partial r)w_0)\, ds(x) = 0.$$

The lemma follows by letting $R \to \infty$ in (2.11). □

We now prove the main ingredient for the justification of the LSM.

**Theorem 2.3.** *Assume that Assumption* 2.1 *holds. Then the operator* $G : H_{\mathrm{inc}}(D) \to L^2(S^2)$ *defined by* (2.4) *is injective with dense range. Moreover,*

$$\Phi_\infty(\cdot, z) \in \mathcal{R}(G) \text{ if and only if } z \in D.$$

***Proof.*** We start by proving that $G : H_{\mathrm{inc}}(D) \to L^2(S^2)$ is injective with dense range. Let $u_0$ and $w$ satisfy (2.2). From (1.24), we get

$$w^\infty(\hat{x}) = -\frac{k^2}{4\pi} \int\limits_D e^{-ik\hat{x}\cdot y}(1-n)(u_0(y) + w(y))dy.$$

Therefore, for $g \in L^2(S^2)$,

$$(G(u_0), g)_{L^2(S^2)} = -\frac{k^2}{4\pi} \int\limits_D (1-n)(u_0 + w)\overline{\mathcal{H}g}\, dx. \tag{2.12}$$

Assume that $u_0 = \overline{\mathcal{H}\varphi}$ for some $\varphi \in L^2(S^2)$ and set $w(\varphi) \equiv w$. Then the previous equality can be written as

$$(G(\overline{\mathcal{H}\varphi}), g)_{L^2(S^2)} = k^2 \int\limits_D (1-n)(\overline{\mathcal{H}\varphi} + w(\varphi))\, \overline{\mathcal{H}g}\, dx. \tag{2.13}$$

From Lemma 2.2 we get

$$\int\limits_D (1-n)(\overline{\mathcal{H}\varphi} + w(\varphi))\, \overline{\mathcal{H}g}\, dx = \int\limits_D (1-n)(\overline{\mathcal{H}g} + w(g))\, \overline{\mathcal{H}\varphi}\, dx.$$

Therefore, the identity (2.13) implies the reciprocity relation

$$(G(\overline{\mathcal{H}\varphi}), g)_{L^2(S^2)} = (G(\overline{\mathcal{H}g}), \varphi)_{L^2(S^2)} \quad \text{for all } g, \varphi \in L^2(S^2). \tag{2.14}$$

Now assume that $(G(\overline{\mathcal{H}\varphi}), g)_{L^2(S^2)} = 0$ for all $\varphi \in L^2(S^2)$. We deduce from (2.14) that $G(\overline{\mathcal{H}g}) = 0$. Using Rellich's Lemma and the unique continuation principle we deduce that $w(g) = 0$ in $\mathbb{R}^3 \setminus D$. Consequently, if we set $u := w(g) + \overline{\mathcal{H}g}$, then the pair $(u, \overline{\mathcal{H}g})$ is a solution to (2.6) with zero data. Our hypothesis ensures that $\mathcal{H}g = 0$ in $D$ and consequently $g = 0$ (by Lemma 2.1). This proves the denseness of the range of $G$.

We now prove the injectivity of $G$. Let $u_0 \in H_{\text{inc}}(D)$ and let $w \in H^1_{loc}(\mathbb{R}^3)$ be the associated scattered field via (2.2). As observed earlier, $w \in H^2(B_R)$ for all balls $B_R$ centered at the origin of radius $R$. Assume that $G(u_0) = 0$. From Rellich's Lemma we deduce that

$$w = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}.$$

Consequently, if we set $u := w + u_0$, then the pair $(u, u_0)$ is a solution to (2.6) with zero data. Assumption 2.1 then ensures that $u_0 = 0$, which proves the injectivity of $G$.

We now prove the last part of the theorem. We first observe that $\Phi_\infty(\cdot, z)$ is the far field pattern of $u_e = \Phi(\cdot, z)$ satisfying $\Delta u_e + k^2 u_e = -\delta_z$ in $\mathbb{R}^3$ and the Sommerfeld radiation condition. Let $z \in D$. We consider $(u, u_0) \in L^2(D) \times L^2(D)$ as being the solution to (2.6) with

$$f(x) = u_e(x; z) \text{ and } h(x) = \partial u_e(x; z)/\partial \nu(x) \quad \text{for } x \in \partial D. \tag{2.15}$$

We then define $w$ by

$$\begin{aligned} w(x) &= u(x) - u_0(x) \quad &&\text{in } D, \\ w(x) &= u_e(x; z) &&\text{in } \mathbb{R}^3 \setminus D. \end{aligned}$$

Due to (2.15), we have that $w \in H^2_{loc}(\mathbb{R}^3)$ and satisfies (2.2). Hence $Gu_0 = \Phi_\infty(\cdot, z)$.

Now let $z \in \mathbb{R}^3 \setminus D$. Assume that there exists $u_0 \in H_{\text{inc}}(D)$ such that $Gu_0 = \Phi_\infty(\cdot, z)$. By Rellich's Lemma we deduce that $w = u_e(\cdot; z)$ in $\mathbb{R}^3 \setminus D$, where $w$ is the solution to (2.2). This gives a contradiction since $w \in H^1_{loc}(\mathbb{R}^3 \setminus D)$ while $u_e(\cdot; z) \notin H^1_{loc}(\mathbb{R}^3 \setminus D)$. $\square$

Since the operator $\mathcal{H}$ is compact, the characterization of $D$ in terms of the range of $G$ in Theorem 2.3 does not imply a similar characterization in terms of the range of $F$. However, one can deduce the following.

**Theorem 2.4.** *Assume that assumption* (2.1) *holds; then the operator $F$ is injective with dense range. Moreover,*

- *if $z \in D$, then there exists a sequence $g_z^\alpha \in L^2(S^2)$ such that* $\lim_{\alpha \to 0} \|Fg_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} = 0$ *and* $\lim_{\alpha \to 0} \|\mathcal{H}g_z^\alpha\|_{L^2(D)} < \infty$;
- *if $z \notin D$, then for all $g_z^\alpha \in L^2(S^2)$ such that* $\lim_{\alpha \to 0} \|Fg_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} = 0$ *and* $\lim_{\alpha \to 0} \|\mathcal{H}g_z^\alpha\|_{L^2(D)} = \infty$.

***Proof.*** The injectivity and the denseness of the range of $F$ directly follow from the same properties satisfied by $\mathcal{H}$ (Lemma 2.1) and $G$ (Theorem 2.3). See also Theorem 1.16 and Corollary 1.17 for a direct proof of these properties.

If $z \in D$, let $u_0 \in H_{\text{inc}}(D)$ be such that $Gu_0 = \Phi_\infty(\cdot, z)$, which exists by Theorem 2.3. From Lemma 2.1 there exists a sequence $g_z^\alpha \in L^2(S^2)$ such that $\mathcal{H}g_z^\alpha \to u_0$ as $\alpha \to 0$, and the first statement follows from the fact that $F = G\mathcal{H}$.

Let $z \notin D$ and $g_z^\alpha \in L^2(S^2)$ be such that $\lim\limits_{\alpha\to 0} \|Fg_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} = 0$. Assume that $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ is bounded as $\alpha \to 0$. Without loss of generality we can assume that $\mathcal{H}g_z^\alpha$ weakly converges to some $u_0 \in H_{\text{inc}}(D)$. Since $G\mathcal{H} = F$, we get the limit $Gu_0 = \Phi_\infty(\cdot, z)$, which contradicts the last part of Theorem 2.3. □

The main weak point in this theorem is that it does not indicate how to construct the sequence $g_z^\alpha$ when $z \in D$. In practice one relies on the use of Tikhonov regularization and considers $\tilde{g}_z^\alpha \in L^2(S^2)$ satisfying

$$(\alpha + F^*F)\, \tilde{g}_z^\alpha = F^*\left(\Phi_\infty(\cdot, z)\right). \tag{2.16}$$

Since $F$ has dense range, $\lim\limits_{\alpha\to 0} \|F\tilde{g}_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} = 0$. However, one cannot guarantee in general that $\lim\limits_{\alpha\to 0} \|\mathcal{H}\tilde{g}_z^\alpha\|_{L^2(D)} < \infty$ if $z \in D$. In the case $\Im(n) = 0$, the latter has been proved in [5, 6], based on the so-called $(F^*F)^{1/4}$ method (see Section 2.5.1). A second weak point of Theorem 2.4 is that one cannot compute $\|\mathcal{H}g_\alpha(\cdot; z)\|_{L^2(D)}$ since $D$ is not known. In practice one uses $\|g_\alpha(\cdot; z)\|_{L^2(S^2)}$ as an indicator function for $D$. We refer the reader to [65], [64] for numerical examples of the performance of this method on synthetic data.

**Remark 2.5.** A possible method to fix the Tikhonov regularization parameter $\alpha$ in (2.16) is to use the Morozov discrepancy principle. Assume that $F^\delta$ is the noisy operator corresponding to noisy measurements $u_\infty^\delta$ such that

$$\|u_\infty^\delta - u_\infty\|_{L^2(S^2)\times L^2(S^2)} \le \delta.$$

Then for each sampling point $z$, the parameter $\alpha$ is chosen such that

$$\|F^\delta g_\alpha(\cdot; z) - \Phi_\infty(\cdot, z)\|_{L^2(S^2)} = \delta \|g_\alpha(\cdot; z)\|_{L^2(S^2)}.$$

This leads to a nonlinear equation that determines $\alpha$ in terms of the noise level $\delta$ [76].

## 2.2 ▪ A Generalized Version of LSM (GLSM)

In order to overcome the weak points mentioned above, a new formulation of LSM was proposed in [13]. It gives an exact characterization of the domain $D$ in terms of the range of $F$. Moreover, it provides a numerically tractable indicator function, but at the expense of additional numerical cost. The key idea behind the new formulation is to replace the penalty term in the Tikhonov formulation (2.16) by a term that controls $\|\mathcal{H}g_\alpha(\cdot; z)\|_{L^2(D)}$. This is possible due to the second factorization of the far field operator $F$ that has been used in [113] to design a different family of sampling methods, namely, factorization methods (see Section 2.4). More precisely, since $\mathcal{H}^* : L^2(D) \to L^2(S^2)$, the adjoint of the operator $\mathcal{H}$, is given by (2.7) and since, from (1.24),

$$w^\infty(\hat{x}) = -\frac{k^2}{4\pi} \int\limits_D e^{-ik\hat{x}\cdot y}(1-n)(u_0(y) + w(y))dy,$$

we get that $G = \mathcal{H}^*T$, where $T$: $L^2(D) \to L^2(D)$ is defined by

$$Tu_0 := -\frac{k^2}{4\pi}(1-n)(u_0 + w) \tag{2.17}$$

with $w$ being the solution of (2.2). We then end up with

$$F = \mathcal{H}^* T \mathcal{H}. \tag{2.18}$$

We observe that if $T$ is coercive on the range of the operator $\mathcal{H}$, then $|(Fg, g)_{L^2(S^2)}|$ is equivalent to $\|\mathcal{H}g\|^2_{L^2(D)}$. One can therefore use $|(Fg, g)_{L^2(S^2)}|$ as a penalty term in the Tikhonov functional. However, this cannot be treated as a regular penalty term since it does not define a norm for $g$ which is equivalent to the $L^2(S^2)$ norm (the operator $F$ is compact). This term is also nonconvex in general, which induces difficulties in the analysis and from the numerical point of view. Other alternatives would be, at the expense of possibly more restrictions on the index of refraction $n$, to replace this term with $|(Bg, g)|$, where the operator $B : L^2(S^2) \to L^2(S^2)$ is a self-adjoint and nonnegative operator expressed in terms of $F$. For instance, $B = \Im(F) := \frac{1}{2i}(F - F^*)$ if the imaginary part of $n$ is positive definite in $D$ or $B = F_\sharp := |\Re(F)| + |\Im(F)|$, where $\Re(F) := \frac{1}{2}(F + F^*)$ in a more general case. We shall investigate all these possibilities in an abstract form in the following section.

### 2.2.1 ▪ Theoretical Foundation of GLSM in the Noise Free Case

We follow here the presentation given in [12] and [13]. Let $X$ and $Y$ be two (complex) reflexive Banach spaces with duals $X^*$ and $Y^*$, respectively, and denote by $\langle , \rangle$ a duality product that refers to $\langle X^*, X\rangle$ or $\langle Y^*, Y\rangle$ duality. We consider two linear bounded operators $F : X \to X^*$ and $B : X \to X^*$ for which the following factorizations hold:

$$F = GH \quad \text{and} \quad B = H^* T H, \tag{2.19}$$

where the operators $H : X \to Y$, $T : Y \to Y^*$, and $G : H_{\text{inc}} := \overline{\mathcal{R}(H)} \subset Y \to X^*$ are bounded and where $\overline{\mathcal{R}(H)}$ is the closure of the range of $H$ in $Y$. Let $\alpha > 0$ be a given parameter and $\phi \in X^*$. The GLSM is based on considering minimizing sequences of the functional $J_\alpha(\phi; \cdot) : X \to \mathbb{R}$, where

$$J_\alpha(\phi; g) := \alpha |\langle Bg, g\rangle| + \|Fg - \phi\|^2 \quad \text{for all } g \in X. \tag{2.20}$$

This functional does not have a minimizer in general since the operator $B$ is typically chosen to be compact. However, since $J_\alpha(\phi; \cdot) \geq 0$ one can define

$$j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g). \tag{2.21}$$

A first simple observation is the following.

**Lemma 2.6.** *Assume that $F$ has dense range. Then for all $\phi \in X^*$, $j_\alpha(\phi) \to 0$ as $\alpha \to 0$.*

***Proof.*** Let $\epsilon > 0$. The denseness of the range of $F$ implies the existence of $g_\epsilon$ such that $\|Fg_\epsilon - \phi\| < \sqrt{\frac{\epsilon}{2}}$. One can choose a sufficiently small $\alpha_0(\epsilon)$ such that for all $\alpha \leq \alpha_0(\epsilon)$, $\alpha |\langle Bg_\epsilon, g_\epsilon\rangle| < \frac{\epsilon}{2}$. Consequently $j_\alpha(\phi) \leq J_\alpha(\phi; g_\epsilon) < \epsilon$, which proves the claim. ∎

The central theorem for noise free GLSM is the following characterization of the range of $G$ in terms of $F$ and $B$.

**Theorem 2.7.** *We assume in addition to* (2.19) *that*

- *$G$ is compact and $F = GH$ has dense range;*

- *$T$ satisfies the coercivity property*

$$|\langle T\varphi,\, \varphi\rangle| > \mu\, \|\varphi\|^2 \quad \textit{for all } \varphi \in \mathcal{R}(H), \tag{2.22}$$

*where $\mu > 0$ is a constant independent of $\varphi$. Let $C > 0$ be a given constant (independent of $\alpha$) and consider for $\alpha > 0$ and $\phi \in X^*$, an element $g_\alpha \in X$ such that*

$$J_\alpha(\phi; g_\alpha) \le j_\alpha(\phi) + C\,\alpha. \tag{2.23}$$

*Then the following hold:*

- *If $\phi \in \mathcal{R}(G)$, then* $\limsup\limits_{\alpha\to 0} |\langle Bg_\alpha,\, g_\alpha\rangle| < \infty$.
- *If $\phi \notin \mathcal{R}(G)$, then* $\liminf\limits_{\alpha\to 0} |\langle Bg_\alpha,\, g_\alpha\rangle| = \infty$.

***Proof.*** Assume that $\phi \in \mathcal{R}(G)$. Then by definition one can find $\varphi \in \overline{\mathcal{R}(H)}$ such that $G\varphi = \phi$. For $\alpha > 0$, there exists $g_0 \in X$ such that $\|Hg_0 - \varphi\|^2 < \alpha$. Then by continuity of $G$, $\|Fg_0 - \phi\|^2 < \|G\|^2\alpha$. On the other hand, the continuity of $T$ implies

$$|\langle Bg_0,\, g_0\rangle| = |\langle THg_0,\, Hg_0\rangle| \le \|T\|\, \|Hg_0\|^2 < 2\,\|T\|\,(\alpha + \|\varphi\|^2).$$

From the definitions of $j_\alpha(\phi)$ and $g_\alpha$ we have

$$\alpha|\langle Bg_0,\, g_0\rangle| + \|Fg_0 - \phi\|^2 > j_\alpha(\phi) > J_\alpha(\phi, g_\alpha) - C\alpha.$$

We then deduce from the definition of $J_\alpha$ and previous inequalities that

$$\alpha|\langle Bg_\alpha,\, g_\alpha\rangle| \le J_\alpha(\phi, g_\alpha) \le C\alpha + 2\alpha\,\|T\|\,(\alpha + \|\varphi\|^2) + \alpha\|G\|^2.$$

Therefore $\limsup\limits_{\alpha\to 0} |\langle Bg_\alpha,\, g_\alpha\rangle| < \infty$, which proves the first claim.

Now assume that $\phi \notin \mathcal{R}(G)$ and, contrary to the theorem, that

$$\liminf_{\alpha\to 0} |\langle Bg_\alpha,\, g_\alpha\rangle| < \infty.$$

Then (for some extracted subsequence $g_\alpha$) $|\langle Bg_\alpha,\, g_\alpha\rangle| < A$, where $A$ is a constant independent of $\alpha \to 0$. The coercivity of $T$ implies that $\|Hg_\alpha\|$ is also bounded. Since $Y$ is reflexive and $\overline{\mathcal{R}(H)}$ is closed, one can assume that, up to an extracted subsequence, $Hg_\alpha$ weakly converges to some $\varphi$ in $\overline{\mathcal{R}(H)}$. Compactness of $G$ implies that $GHg_\alpha$ strongly converges to $G\varphi$ as $\alpha \to 0$. On the other hand, Lemma 2.6 and the definition of $J_\alpha(\phi, g_\alpha)$ show that $\|Fg_\alpha - \phi\|^2 \le J_\alpha(\phi, g_\alpha) \le j_\alpha(\phi) + C\alpha \to 0$ as $\alpha \to 0$. Since $Fg_\alpha = GHg_\alpha$ we get $G\varphi = \phi$, which is a contradiction. □

As indicated in the previous section, the range of the operator $G$ characterizes the inhomogeneity $D$. Therefore this theorem leads to a characterization of $D$ in terms of the operators $F$ and $B$ (and therefore a uniqueness result for the reconstruction of $D$ in terms of $F$ and $B$). It also stipulates that an indicator function is given by $|\langle Bg_\alpha,\, g_\alpha\rangle|$ for small values of $\alpha$. Let us note that the parameter $\alpha$ does not play the role of a regularization parameter, since in applications the operator $B$ is, in general, compact. However, constructing a sequence $(g_\alpha)$ satisfying (2.23) for fixed $\alpha > 0$ may be viewed as a regularization of the minimization of $J_\alpha(\phi; \cdot)$ that can be used for numerics. A different regularization procedure that would be more suited for noisy operators is introduced in the following subsection. For this version and particular choices of the operator $B$ one can construct a minimizer by solving a simple linear system (see Remark 2.15).

For the natural choice $B = F$ one can state the following straightforward corollary.

**Corollary 2.8.** *Assume that $G(\varphi) = H^*T(\varphi)$ for all $\varphi \in \mathcal{R}(H)$ and assume in addition that $H$ is compact, $F$ has dense range, and $T$ satisfies the coercivity property* (2.22)*. Let $C > 0$ be a given constant (independent of $\alpha$) and consider for $\alpha > 0$ and $\phi \in X^*$, $g_\alpha \in X$ such that*

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + C\,\alpha.$$

*Then $\phi \in \mathcal{R}(G)$ if and only if $\limsup_{\alpha\to 0} |\,\langle F g_\alpha,\, g_\alpha\rangle\,| < \infty$ and we also have $\phi \notin \mathcal{R}(G)$ if and only if $\liminf_{\alpha\to 0} |\,\langle F g_\alpha,\, g_\alpha\rangle\,| = \infty$.*

In Theorem 2.7 and the case $\phi \in \mathcal{R}(G)$ one only knows that the quantity $|\,\langle B g_\alpha,\, g_\alpha\rangle\,|$ is bounded as $\alpha \to 0$ and nothing is said on the (strong) convergence of the sequence $H g_\alpha$. In order to ensure the strong convergence of this sequence one possibility would be to add a convexity property for $|\,\langle B g_\alpha,\, g_\alpha\rangle\,|$ as in the following theorem.

**Theorem 2.9.** *We assume, in addition to the hypotheses of Theorem* 2.7*, that $F$ is injective and that $h \mapsto \sqrt{|\,\langle Th,\, h\rangle\,|}$ is a uniformly convex function on $H_{\mathrm{inc}}$. Consider for $\alpha > 0$ and $\phi \in X^*$, $g_\alpha \in X$ such that*

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha), \tag{2.24}$$

*where $\frac{p(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$.*

*Then $\phi \in \mathcal{R}(G)$ if and only if $\lim_{\alpha\to 0} |\,\langle B g_\alpha,\, g_\alpha\rangle\,| < \infty$. Moreover, in the case $\phi = G\varphi$, the sequence $H g_\alpha$ strongly converges to $\varphi$ in $Y$.*

***Proof.*** According to Theorem 2.7 we only need to prove the convergence of $H g_\alpha$ to $\varphi$ when $\phi = G\varphi$ for $\varphi \in Y$. The coercivity of $T$ combined with the first part of the proof of Theorem 2.7 imply that $\|H g_\alpha\|^2$ is bounded. Second, from Lemma 2.6, equation (2.24), and the injectivity of $G$ we infer that the only possible weak limit of (any subsequence of) $H g_\alpha$ is $\varphi$. Thus the whole sequence $H g_\alpha$ weakly converges to $\varphi$. Since $\varphi \in \overline{\mathcal{R}(H)}$ we have

$$j_\alpha(\phi) = \inf_{g\in X)} J_\alpha(g,\phi) = \inf_{h\in\overline{\mathcal{R}(H)}} \left(\alpha|\,\langle Th,\, h\rangle\,| + \|Gh - \phi\|^2\right) \leq \alpha|\,\langle T\varphi,\, \varphi\rangle\,|.$$

Thus

$$|\,\langle B g_\alpha,\, g_\alpha\rangle\,| \leq |\,\langle T\varphi,\, \varphi\rangle\,| + \frac{p(\alpha)}{\alpha},$$

which implies (as $\frac{p(\alpha)}{\alpha} \to 0$)

$$\limsup_{\alpha\to 0} |\,\langle T H g_\alpha,\, H g_\alpha\rangle\,| \leq |\,\langle T\varphi,\, \varphi\rangle\,|. \tag{2.25}$$

The uniform convexity of $h \mapsto \sqrt{|\,\langle Th,\, h\rangle\,|}$ and the continuity and coercivity properties of $T$ ensure that $\overline{\mathcal{R}(H)}$ equipped with $\sqrt{|\,\langle Th,\, h\rangle\,|}$ is a uniformly convex Banach space. We deduce from (2.25) and the weak convergence of the sequence $H g_\alpha$ that $H g_\alpha$ strongly converges to $\varphi$ (see, for instance, [27, Chapter 3, Proposition 3.32]). ∎

We remark that the additional hypothesis of Theorem 2.9 is automatically satisfied as soon as the operator $B$ or, equivalently, the operator $T$ is self-adjoint. We refer the reader to Section 2.5.2 for possible choices of such an operator. Let us notice that one can avoid this assumption by adding an extra term in the cost functional, as indicated in the following remark.

**Remark 2.10.** In the case $B = F$, one can avoid the extra assumption on the operator $T$ in Theorem 2.9 by replacing the cost functional $J_\alpha$ with

$$J_\alpha(\phi; g) := \alpha| \langle Fg,\, g\rangle | + \alpha^{1-\eta}| \langle Fg - \phi,\, g\rangle | + \|Fg - \phi\|^2 \quad \text{for all } g \in X \qquad (2.26)$$

with $\eta \in ]0, 1]$ being a fixed parameter. We refer the reader to [7, Chapter 4] for the analysis of this type of function that is also more suited for limited aperture data.

An important application of Theorem 2.9 is the design of a method capable of imaging defects in an unknown multiply connected background from so-called differential measurements (i.e., measurements for the cases with and without defects) as sketched in Section 2.5.4.

## 2.2.2 ▪ Regularized Formulation of GLSM

As will be clearer later, the above formulation of GLSM has to be adapted to the case of noisy operators since in general a noisy operator $B$ does not satisfy a factorization of the form (2.19) (with a middle operator satisfying a coercivity property similar to (2.22)). In order to cope with this issue we introduce a regularized version of $J_\alpha$ which allows a similar range characterization and where one controls both the noisy criteria and the noisy misfit term. Following [13], consider for $\alpha > 0$ and $\epsilon > 0$ (that will later be linked with the noise level) and for $\phi \in X^*$, the functional $J_\alpha^\varepsilon(\phi; \cdot) : X \to \mathbb{R}$ defined by

$$J_\alpha^\varepsilon(\phi; g) = \alpha(| \langle Bg,\, g\rangle | + \varepsilon \|g\|^2) + \|Fg - \phi\|^2 . \qquad (2.27)$$

**Lemma 2.11.** *Assume that $B$ is compact. Then for all $\alpha > 0$, $\epsilon > 0$, and $\phi \in X^*$ the functional $J_\alpha^\varepsilon(\phi; \cdot)$ has a minimizer $g_\alpha^\varepsilon \in X$. If we assume in addition that $F$ has dense range, then*

$$\lim_{\alpha\to 0} \lim_{\varepsilon\to 0} J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) = \lim_{\varepsilon\to 0} \limsup_{\alpha\to 0} J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) = 0.$$

***Proof.*** The existence of a minimizer is clear: for fixed $\alpha > 0$, $\epsilon > 0$, and $\phi \in X^*$, any minimizing sequence $(g^n)$ of $J_\alpha^\varepsilon(\phi; \cdot)$ is bounded and therefore one can assume that it is weakly convergent in $X$ to some $g_\alpha^\varepsilon \in X$. The lower semicontinuity of the norm with respect to weak convergence and the compactness property of $B$ then imply

$$J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) \le \liminf_{n\to\infty} J_\alpha^\varepsilon(\phi; g^n) \le \inf_{g\in X} J_\alpha^\varepsilon(\phi; g),$$

which proves that $g_\alpha^\varepsilon$ is a minimizer of $J_\alpha^\varepsilon(\phi; \cdot)$ on $X$.

Now assume in addition that $F$ has dense range. By Lemma 2.6, $j_\alpha(\phi) \to 0$ as $\alpha \to 0$. Showing that $\lim\limits_{\varepsilon\to 0} J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) = j_\alpha(\phi)$ will then prove that $\lim\limits_{\alpha\to 0} \lim\limits_{\varepsilon\to 0} J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) = 0$. We observe that

$$J_\alpha^\varepsilon(\phi; g) = J_\alpha(\phi; g) + \alpha\varepsilon\|g\|^2 \qquad (2.28)$$

and therefore $|J_\alpha^\varepsilon(\phi; g) - J_\alpha(\phi; g)| \to 0$ as $\varepsilon \to 0$. For $\eta > 0$ one can choose $g$ such that $|J_\alpha(\phi; g) - j_\alpha(\phi)| \le \eta/2$. For this $g$ one then has for $\varepsilon$ sufficiently small that

$|J_\alpha^\varepsilon(\phi;g) - J_\alpha(\phi;g)| < \eta/2$. We obtain by the triangle inequality that for $\varepsilon$ sufficiently small $J_\alpha^\varepsilon(\phi;g) \le j_\alpha(\phi) + \eta$. We now observe from the definitions of $g_\alpha^\varepsilon$ and $j_\alpha$ and from (2.28) that

$$j_\alpha(\phi) \le J_\alpha(\phi; g_\alpha^\varepsilon) \le J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) \le J_\alpha^\varepsilon(\phi; g),$$

which proves the claim.

We now prove $\lim_{\varepsilon\to 0} \limsup_{\alpha\to 0} J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) = 0$. First let $g_\varepsilon$ be a minimizer on $X$ of the Tikhonov functional $\varepsilon^2 \|g\|^2 + \|Fg - \phi\|^2$ and set $j^\varepsilon = \varepsilon^2 \|g_\varepsilon\|^2 + \|Fg_\varepsilon - \phi\|^2$, which goes to zero as $\varepsilon$ goes to zero (see Lemma 2.6, which is valid for any bounded operator $B$). We have that, for $\alpha \le \varepsilon$, $J_\alpha^\varepsilon(g) \le \varepsilon^2 \|g\|^2 + \|Fg - \Phi\|^2 + \alpha(|(Bg,g)|$. By taking the upper limit,

$$\limsup_{\alpha\to 0} J_\alpha^\varepsilon(g_\alpha^\varepsilon) \le \limsup_{\alpha\to 0} J_\alpha^\varepsilon(g_\varepsilon) = j^\varepsilon,$$

which concludes the proof. ∎

**Theorem 2.12.** *Under the assumptions of Theorem* 2.7 *and the additional assumption that* $B$ *is compact the following hold. If* $g_\alpha^\varepsilon$ *denotes the minimizer of* $J_\alpha^\varepsilon(\phi;\cdot)$ *(defined by* (2.27)*) for* $\alpha > 0$, $\varepsilon > 0$, *and* $\phi \in X^*$, *then*

- $\phi \in \mathcal{R}(G) \Longrightarrow \limsup_{\alpha\to 0} \limsup_{\varepsilon\to 0} |\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle| < \infty$*;*
- $\phi \notin \mathcal{R}(G) \Longrightarrow \liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} |\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle| = \infty$.

***Proof.*** The proof is similar to the proof of Theorem 2.7. Assume that $\phi = G(\varphi)$ for some $\varphi \in \overline{\mathcal{R}(H)}$. We consider the same $g_0$ as in the first part of the proof of Theorem 2.7 (that depends on $\alpha$ but is independent from $\varepsilon$). Then we choose $\varepsilon$ such that $\varepsilon\|g_0\|^2 \le 1$. Then

$$J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) \le J_\alpha^\varepsilon(\phi; g_0) \le J_\alpha(\phi; g_0) + \alpha. \tag{2.29}$$

Consequently

$$\alpha|\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle| \le J_\alpha^\varepsilon(\phi; g_\alpha^\varepsilon) \le \alpha + 2\alpha\|T\|(\alpha + \|\varphi\|^2) + \alpha\|G\|^2,$$

which proves $\limsup_{\alpha\to 0} \limsup_{\varepsilon\to 0} |\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle| < \infty$.

Now assume $\phi \notin \mathcal{R}(G)$ and assume that $\liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} |\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle|$ is finite. The coercivity of $T$ implies that $\liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} \|Hg_\alpha^\varepsilon\|^2$ is also finite. This means the existence of a subsequence $(\alpha', \varepsilon(\alpha'))$ such that $\alpha' \to 0$ and $\varepsilon(\alpha') \to 0$ as $\alpha' \to 0$ and $\left\|Hg_{\alpha'}^{\varepsilon(\alpha')}\right\|^2$ is bounded independently from $\alpha'$. On the other hand, the second part of Lemma 2.11 (namely, the first limit), indicates that one can choose this subsequence such that $J_{\alpha'}^{\varepsilon(\alpha')}(g_{\alpha'}^{\varepsilon(\alpha')}) \to 0$ as $\alpha' \to 0$ and therefore $\left\|Fg_{\alpha'}^{\varepsilon(\alpha')} - \phi\right\| \to 0$ as $\alpha' \to 0$. The compactness of $G$ implies that a subsequence of $GHg_{\alpha'}^{\varepsilon(\alpha')}$ converges to some $G\varphi$ in $X^*$. The uniqueness of the limit implies that $G\varphi = \phi$, which is a contradiction. ∎

In this theorem $\varepsilon$ should be viewed as the regularization parameter (and not $\alpha$, which is instead used to construct an indicator function with a limiting process). As indicated by (2.29), this regularization parameter serves in the construction of the minimizing sequence of Theorem 2.7.

This theorem with regularization stipulates that a criterion to localize the target is given by $|\langle Bg_\alpha^\varepsilon, g_\alpha^\varepsilon\rangle|$ for small values of $\epsilon$ and $\alpha$. The reader can easily see from the first part of the proof that the result holds true if we replace this by $(|\langle Bg_\alpha^\varepsilon,\, g_\alpha^\varepsilon\rangle| + \varepsilon\|g_\alpha^\varepsilon\|^2)$. This latter criterion is more suited to the case of noisy measurements as indicated in the section below.

### 2.2.3 ▪ The GLSM for Noisy Data

We consider in this section the case where there may be noise in the data. More precisely, we shall assume that one has access to two noisy operators $B^\delta$ and $F^\delta$ such that

$$\left\|F^\delta - F\right\| \le \delta\|F^\delta\| \quad \text{and} \quad \left\|B^\delta - B\right\| \le \delta\|B^\delta\|$$

for some $\delta > 0$. We also assume in this section that the operators $B$, $B^\delta$ $F^\delta$, and $F$ are compact. We then consider for $\alpha > 0$ and $\phi \in X^*$ the functional $J_\alpha^\delta(\phi;\cdot) : X \to \mathbb{R}$ defined by

$$J_\alpha^\delta(\phi; g) := \alpha(|\left\langle B^\delta g,\, g\right\rangle| + \delta\|B^\delta\|\,\|g\|^2) + \left\|F^\delta g - \phi\right\|^2 \quad \text{for all } g \in X, \tag{2.30}$$

which coincides with a regularized noisy functional $J_\alpha^\varepsilon$ with a regularization parameter $\epsilon = \delta\|B^\delta\|$. According to Lemma 2.11 one can consider $g_\alpha^\delta$ to be a minimizer of $J_\alpha^\delta(\phi; g)$. We first observe (similarly to the second part of the proof of Lemma 2.11) the following lemma.

**Lemma 2.13.** *Assume in addition to our previous assumptions that $F$ has dense range. Then for all $\phi \in X^*$,*

$$\lim_{\alpha\to 0}\limsup_{\delta\to 0} J_\alpha^\delta(\phi; g_\alpha^\delta) = 0.$$

***Proof.*** We observe that for all $g \in X$,

$$J_\alpha^\delta(\phi; g) \le J_\alpha(\phi; g) + (2\alpha\delta\|B^\delta\| + \delta^2\|F^\delta\|^2)\,\|g\|^2\,. \tag{2.31}$$

Since $(2\alpha\delta\|B^\delta\| + \delta^2\|F^\delta\|^2) \to 0$ as $\delta \to 0$, then as in the proof of Lemma 2.11, for any $\eta > 0$ ($\alpha$ fixed), one can choose $g \in X$ such that for sufficiently small $\delta$,

$$J_\alpha^\delta(\phi; g) \le j_\alpha(\phi) + \eta.$$

Consequently, from the definition of $g_\alpha^\delta$,

$$J_\alpha^\delta(g_\alpha^\delta; \phi) \le j_\alpha(\phi) + \eta.$$

This proves the claim, since $j_\alpha(\phi) \to 0$ as $\alpha \to 0$ (by Lemma 2.6). ∎

**Theorem 2.14.** *Assume that the assumptions of Theorem* 2.7 *and the additional assumptions of this subsection hold true. Let $g_\alpha^\delta$ be the minimizer of $J_\alpha^\delta(\phi;\cdot)$ (defined by* (2.30)*) for $\alpha > 0$, $\delta > 0$, and $\phi \in X^*$. Then*

- $\phi \in \mathcal{R}(G) \Longrightarrow \limsup\limits_{\alpha\to 0}\limsup\limits_{\delta\to 0}\left(\left|\left\langle B^\delta g_\alpha^\delta,\, g_\alpha^\delta\right\rangle\right| + \delta\|B^\delta\|\,\left\|g_\alpha^\delta\right\|^2\right) < \infty;$
- $\phi \notin \mathcal{R}(G) \Longrightarrow \liminf\limits_{\alpha\to 0}\liminf\limits_{\delta\to 0}\left(\left|\left\langle B^\delta g_\alpha^\delta,\, g_\alpha^\delta\right\rangle\right| + \delta\|B^\delta\|\,\left\|g_\alpha^\delta\right\|^2\right) = \infty.$

***Proof.*** The proof of this theorem follows along the lines of the proof of Theorem 2.12.

First consider the case where $\phi = G(\varphi)$ for some $\varphi \in \overline{\mathcal{R}(H)}$ and introduce the same $g_0$ as in the first part of the proof of Theorem 2.7 (that depends on $\alpha$ but is independent of $\delta$). Choosing $\delta$ sufficiently small such that

$$(2\alpha\delta\|B^\delta\| + \delta^2\|F^\delta\|^2)\,\|g_0\|^2 \leq \alpha$$

we get

$$J^\delta_\alpha(\phi; g^\delta_\alpha) \leq J^\delta_\alpha(\phi; g_0) \leq J_\alpha(\phi; g_0) + \alpha. \tag{2.32}$$

Consequently

$$\alpha\left(|\left\langle Bg^\delta_\alpha,\, g^\delta_\alpha\right\rangle| + \delta\|B^\delta\|\,\left\|g^\delta_\alpha\right\|^2\right) \leq J^\delta_\alpha(\phi; g^\delta_\alpha) \leq \alpha + 2\alpha\,\|T\|\,(\alpha + \|\varphi\|^2) + \alpha\|G\|^2,$$

which proves that $\limsup_{\alpha\to 0}\limsup_{\delta\to 0}(|\langle B^\delta g^\delta_\alpha,\, \rangle| + \delta\|B^\delta\|\|g^\delta_\alpha\|^2) < \infty$. This proves the first part of the theorem.

Now let $\phi \notin \mathcal{R}(G)$ and assume that $\liminf_{\alpha\to 0}\liminf_{\varepsilon\to 0}(|\langle B^\delta g^\delta_\alpha,\, \rangle| + \delta\|B^\delta\|\|g^\delta_\alpha\|^2)$ is finite. The coercivity of $T$ implies that

$$\mu\left\|Hg^\delta_{\alpha(\delta)}\right\|^2 \leq |\left\langle Bg^\delta_\alpha,\, g^\delta_\alpha\right\rangle| \leq |\left\langle B^\delta g^\delta_\alpha,\, g^\delta_\alpha\right\rangle| + \delta\|B^\delta\|\,\left\|g^\delta_\alpha\right\|^2.$$

Therefore $\liminf_{\alpha\to 0}\liminf_{\delta\to 0}\left\|Hg^\delta_\alpha\right\|^2$ is also finite. This means the existence of a subsequence $(\alpha', \delta(\alpha'))$ such that $\alpha' \to 0$, $\delta(\alpha') \to 0$ as $\alpha' \to 0$, and $\left\|Hg^{\delta(\alpha')}_{\alpha'}\right\|^2$ is bounded independently from $\alpha'$. One can also choose $\delta(\alpha')$ such that $\delta(\alpha') \leq \alpha'$.

On the other hand, Lemma 2.13 indicates that one can choose this subsequence such that $J^{\delta(\alpha')}_{\alpha'}(g^{\delta(\alpha')}_{\alpha'}) \to 0$ as $\alpha' \to 0$ and therefore $\left\|F^\delta g^{\delta(\alpha')}_{\alpha'} - \phi\right\| \to 0$ as $\alpha' \to 0$ and $\alpha'\delta(\alpha')\|g^{\delta(\alpha')}_{\alpha'}\|^2 \to 0$ as $\alpha' \to 0$. By the triangle inequality and $\delta(\alpha') \leq \alpha'$ we then deduce that $\left\|Fg^{\delta(\alpha')}_{\alpha'} - \phi\right\| \to 0$ as $\alpha' \to 0$. The compactness of $G$ implies that a subsequence of $GHg^{\delta(\alpha')}_{\alpha'}$ converges to some $G\varphi$ in $X^*$. The uniqueness of the limit implies that $G\varphi = \phi$, which is a contradiction. $\square$

It is clear from the proof of the previous theorem that any strategy of regularization $\varepsilon(\delta)$ satisfying $\epsilon(\delta) \geq \delta\|B^\delta\|$ and $\epsilon(\delta) \to 0$ as $\delta \to 0$ would be convenient to obtain a similar result. From the numerical perspective this theorem indicates that a criterion to localize the object would be the magnitude of

$$|\left\langle B^\delta g^\delta_\alpha,\, g^\delta_\alpha\right\rangle| + \delta\|B^\delta\|\,\left\|g^\delta_\alpha\right\|^2$$

for small values of $\alpha$. Indeed the theorem only says that this criterion would be efficient for sufficiently small noise. Building an explicit link between the value of $\alpha$ and the noise level $\delta$ (in the fashion of a posteriori regularization strategies) would be of valuable theoretical interest but this seems to be challenging (due to the compactness of the operator $B$). One can see from the proof that adding the term $\delta\|B^\delta\|\,\left\|g^\delta_\alpha\right\|^2$ is important to conclude when $\phi$ is not in the range of $G$. This means that this term is important for correcting the behavior of the indicator function outside the inclusion, which is corroborated by the numerical experiments in [13] for the scalar case.

**Remark 2.15.** If $B^\delta$ is a positive self-adjoint operator (see Section 2.5.2), one can directly compute the minimizer $g^\delta_\alpha$ of $J^\delta_\alpha(\phi; \cdot)$ (defined by (2.30)) for $\alpha > 0$, $\delta > 0$, and $\phi \in X^*$ as the solution of

$$(\alpha B^\delta + \alpha\delta\|B^\delta\|I + (F^\delta)^*F^\delta)g^\delta_\alpha = (F^\delta)^*\phi. \tag{2.33}$$

## 2.2.4 ▪ Application of GLSM to the Inverse Scattering Problem

We return to our model problem and consider the notation and assumptions of Section 2.1. We shall apply GLSM with $B = F$. The central additional theorem needed for this case is the following coercivity property of the operator $T$.

**Assumption 2.2.** *We assume that* $n \in L^\infty(\mathbb{R}^3)$, $\operatorname{supp}(1-n) \subset \overline{D}$, *and* $\Im(n) \geq 0$. *Furthermore, we assume either that* $\Re(1-n) + \alpha\Im(n)$ *or* $\Re(n-1) + \alpha\Im(n)$ *is positive definite on* $D$ *for some constant* $\alpha \geq 0$.

We remark that if $\Im(n)$ is positive definite on $D$, then the last part of Assumption 2.2 is automatically verified.

**Theorem 2.16.** *Assume that Assumptions* 2.1 *and* 2.2 *hold. Then the operator* $T$ *defined by* (2.17) *satisfies the coercivity property* (2.22) *with* $Y = Y^* = L^2(D)$ *and the operator* $H = \mathcal{H}$ *defined by* (2.3).

***Proof.*** We start by proving a useful identity related to the imaginary part of $T$. With $(\,,\,)$ denoting the $L^2(D)$ scalar product, for $\psi \in L^2(D)$ and $w \in H^2_{loc}(\mathbb{R}^3)$ the solution of (2.2) we have that

$$(T\psi, \psi) = -\frac{k^2}{4\pi} \int_D (1-n)(\psi + w)\overline{\psi}\, dx. \tag{2.34}$$

Multiplying (2.2) by $\overline{w}$ and integrating by parts over $B_R$, a ball of radius $R$ with center at the origin containing $D$, we have that

$$k^2 \int_D (1-n)(\psi + w)\overline{w}\, dx = -\int_{B_R} (|\nabla w|^2 - k^2|w|^2)\, dx + \int_{|x|=R} \frac{\partial w}{\partial r}\overline{w}\, ds.$$

The Sommerfeld radiation condition indicates that

$$\lim_{R\to\infty} \Im\left( \int_{|x|=R} \frac{\partial w}{\partial r}\overline{w}\, ds \right) = k \int_{\mathbb{S}^2} |w_\infty|^2 ds.$$

Therefore, taking the imaginary part and then letting $R \to \infty$ yields

$$k^2 \Im\left( \int_D (1-n)(\psi + w)\overline{w}\, dx \right) = k \int_{\mathbb{S}^2} |w_\infty|^2 ds.$$

Consequently, decomposing $(\psi + w)\overline{\psi} = |\psi + w|^2 - (\psi + w)\overline{w}$, we obtain the important identity,

$$4\pi\Im(T\psi, \psi) = \int_D k^2 \Im(n)|\psi + w|^2 dx + k \int_{\mathbb{S}^2} |w_\infty|^2 ds. \tag{2.35}$$

We are now in position to prove the coercivity property using a contradiction argument. Assume, for instance, the existence of a sequence $\psi_\ell \in \mathcal{R}(\mathcal{H})$ such that

$$\|\psi_\ell\|_{L^2(D)} = 1 \quad \text{and} \quad |(T\psi_\ell, \psi_\ell)| \to 0 \text{ as } \ell \to \infty.$$

We denote by $w_\ell \in H^2_{loc}(\mathbb{R}^3)$ the solution of (2.2) with $\psi = \psi_\ell$. Elliptic regularity implies that $\|w_\ell\|_{H^2(D)}$ is bounded uniformly with respect to $\ell$. Then up to changing the initial sequence, one can assume that $\psi_\ell$ weakly converges to some $\psi$ in $L^2(D)$ and $w_\ell$ converges weakly in $H^2_{loc}(\mathbb{R}^3)$ and strongly in $L^2(D)$ to some $w \in H^2_{loc}(\mathbb{R}^3)$. It is then easily seen (using the distributional limit) that $w$ and $\psi$ satisfy (2.2), and since $\psi_\ell \in \mathcal{R}(\mathcal{H})$

$$\Delta\psi + k^2\psi = 0 \quad \text{in } D. \tag{2.36}$$

Identity (2.35) and $|(T\psi_\ell, \psi_\ell)| \to 0$ imply that $w^\ell_\infty \to 0$ in $L^2(\mathbb{S}^2)$ and therefore $w_\infty = 0$. Rellich's Lemma implies $w = 0$ outside $D$ and consequently $w \in H^2_0(D)$. With the help of equation (2.36) we get $u = w + \psi \in L^2(D)$ and $v = \psi \in L^2(D)$ are such that $u - v \in H^2(D)$ and satisfy the interior transmission problem (2.6) with $f = g = 0$. We then infer that $w = \psi = 0$. Identity (2.34) applied to $\psi_\ell$ and $w_\ell$ implies

$$|(T\psi_\ell, \psi_\ell)| \geq \frac{k^2}{4\pi}\left|\int_D (1-n)|\psi_\ell|^2 dx\right| - k^2\left|\int_D (1-n) w_\ell \overline{\psi}_\ell \, dx\right| .$$

Therefore, since $\int_D (1-n) w_\ell \overline{\psi}_\ell dx \to \int_D (1-n) w\overline{\psi} dx = 0$, and using the assumptions on $n$,

$$\lim_{\ell\to\infty} |(T\psi_\ell, \psi_\ell)| \geq \theta\|\psi_\ell\|^2_{L^2(D)} = \theta$$

for some positive constant $\theta$, which is a contradiction. □

**Remark 2.17.** A different proof of this theorem can be obtained as a combination of Lemmas 2.23 and 2.31 below. The proof given here can be adapted to prove the same results under the hypothesis that Assumption 2.2 holds only in a neighborhood of the boundary $\partial D$ (see [7, Chapter 4]).

Set $\phi_z := \Phi_\infty(\cdot, z)$ and denote by $(\ ,\ )$ the $L^2(S^2)$ scalar product and by $\|.\|$ the associated norm. Let $C > 0$ be a given constant (independent of $\alpha$) and consider for $\alpha > 0$ and $z \in \mathbb{R}^3$, $g^z_\alpha \in L^2(S^2)$ such that

$$\alpha|(Fg^z_\alpha, g^z_\alpha)| + \|Fg^z_\alpha - \phi_z\|^2 \leq j_\alpha(\phi_z) + C\,\alpha, \tag{2.37}$$

where

$$j_\alpha(\phi_z) = \inf_{g\in L^2(S^2)} \left(\alpha|(Fg,g)| + \|Fg - \phi_z\|^2\right).$$

Combining the results of Theorems 2.16 and 2.3 and the first claim of Theorem 2.4, we obtain the following as a straightforward application of Corollary 2.8.

**Theorem 2.18.** *Assume that Assumptions* 2.1 *and* 2.2 *hold. Then* $z \in D$ *if and only if* $\limsup_{\alpha\to 0} |(Fg^z_\alpha, g^z_\alpha)| < \infty$.

*We also have* $z \notin D$ *if and only if* $\liminf_{\alpha\to 0} |(Fg^z_\alpha, g^z_\alpha)| = \infty$.

This theorem gives, for instance, a uniqueness result for the reconstruction of $D$ from the far field operator.

Let us remark that in the case where $\Im(n)$ is positive definite on $D$ one can use $B = \Im(F)$. This is justified by the fact that $\Im(T)$ is coercive and positive, as indicated by identity (2.35). In that case one can replace the term $|(Fg^z_\alpha, g^z_\alpha)|$ with $(\Im(F)g^z_\alpha, g^z_\alpha)$ in the definition of $g^z_\alpha$ and in Theorem 2.18.

For practical applications, it is important to use the criterion provided in Theorem 2.14. Consider $F^\delta : L^2(S^2) \to L^2(S^2)$ a compact operator such that

$$\left\|F^\delta - F\right\| \le \delta,$$

and consider for $\alpha > 0$ and $\phi \in L^2(S^2)$ the functional $J^\delta_\alpha(\phi;\cdot) : L^2(S^2) \to \mathbb{R}$ defined by

$$J^\delta_\alpha(\phi; g) := \alpha(|(F^\delta g, g)| + \delta \left\|g\right\|^2) + \left\|F^\delta g - \phi\right\|^2 \quad \text{for all } g \in L^2(S^2). \tag{2.38}$$

Then as a direct consequence of Theorem 2.14, we have the following characterization of $D$.

**Theorem 2.19.** *Assume that Assumptions* 2.1 *and* 2.2 *hold. For $z \in \mathbb{R}^3$ denote by $g^z_{\alpha,\delta}$ the minimizer of $J^\delta_\alpha(\phi_z;\cdot)$ over $L^2(S^2)$. Then*

$$z \in D \;\; \textit{if and only if} \;\; \limsup_{\alpha\to 0} \limsup_{\delta \to 0} \left( \left|(F^\delta g^z_{\alpha,\delta}, g^z_{\alpha,\delta})\right| + \delta \left\|g^z_{\alpha,\delta}\right\|^2 \right) < \infty$$

*and we also have*

$$z \notin D \;\; \textit{if and only if} \;\; \liminf_{\alpha\to 0} \liminf_{\delta \to 0} \left( \left|(F^\delta g^z_{\alpha,\delta}, g^z_{\alpha,\delta})\right| + \delta \left\|g^z_{\alpha,\delta}\right\|^2 \right) = \infty.$$

## 2.3 ▪ The Inf-Criterion

Another exact characterization of $D$ in terms of the far field operator can be obtained using the so-called inf-criterion [113], [138]. For this characterization one basically needs the same coercivity property as in Theorem 2.7.

### 2.3.1 ▪ The Main Theorem

Let $X$ and $Y$ be two (complex) reflexive Banach spaces with duals $X^*$ and $Y^*$, respectively, and denote by $\langle\;,\;\rangle$ a duality product that refers to $\langle X^*,\, X\rangle$ or $\langle Y^*,\, Y\rangle$ duality. We consider three bounded operators $F : X \to X^*$, $H : X \to Y$, and $T : Y \to Y^*$ such that

$$F = H^* T H.$$

We then have the following theorem.

**Theorem 2.20.** *Assume that there exists a constant $\alpha > 0$ such that*

$$|\langle T\varphi, \varphi\rangle| \ge \alpha \|\varphi\|^2_Y \quad \textit{for all } \varphi \in \mathcal{R}(H). \tag{2.39}$$

*Then one has the following characterization of the range of $H^*$:*

$$\{\psi^* \in \mathcal{R}(H^*) \textit{ and } \psi^* \ne 0\} \quad \textit{if and only if} \quad \inf\{|\langle F\psi, \psi\rangle|\,, \psi \in X, \langle \psi^*, \psi\rangle = 1\} > 0.$$

***Proof.*** We first observe that

$$|\langle F\psi, \psi\rangle| = |\langle H^* T H\psi, \psi\rangle| = |\langle TH\psi, H\psi\rangle|\,.$$

Hence,

$$\alpha \|H\psi\|^2_Y \le |\langle F\psi, \psi\rangle| \le \|T\|\|H\psi\|^2_Y \quad \text{for all } \psi \in X. \tag{2.40}$$

Let $\psi^* \in \mathcal{R}(H^*)$ and $\psi^* \neq 0$. Then $\psi^* = H^*(\varphi^*)$ for some $\varphi^* \in Y^*$ and $\varphi^* \neq 0$. Let $\psi \in X$ such that $\langle \psi^*, \psi \rangle = 1$. Then

$$\begin{aligned}\|H\psi\|_Y &= \frac{1}{\|\varphi^*\|_{Y^*}} \|H\psi\|_Y \|\varphi^*\|_{Y^*} \\ &\geq \frac{1}{\|\varphi^*\|_{Y^*}} \langle \varphi^*, H\psi \rangle = \frac{1}{\|\varphi^*\|_{Y^*}} > 0.\end{aligned}$$

We then deduce, using the first inequality in (2.40), that

$$\inf\{|\langle F\psi, \psi\rangle|\,, \psi \in X, \langle \psi^*, \psi \rangle = 1\} \geq \frac{\alpha}{\|\varphi^*\|^2_{Y^*}} > 0.$$

Now assume that $\psi^* \notin \mathcal{R}(H^*)$ and let us show that

$$\inf\{|\langle F\psi, \psi\rangle|\,, \psi \in X, \langle \psi^*, \psi \rangle = 1\} = 0.$$

From the second inequality in (2.40) it is sufficient to prove the existence of a sequence $\psi_n \in X$ such that $\langle \psi^*, \psi_n \rangle = 1$ and $\|H\psi_n\|_Y \to 0$ as $n \to \infty$. Since $\psi^* \neq 0$ and $X$ is reflexive, there exists $\hat{\psi} \in X$ such that $\left\langle \psi^*, \hat{\psi} \right\rangle = 1$. Setting $\hat{\psi}_n = \hat{\psi} - \psi_n$, we see that it is sufficient to show the existence of a sequence $\hat{\psi}_n \in X$ such that

$$\left\langle \psi^*, \hat{\psi}_n \right\rangle = 0 \quad \text{and} \quad H\hat{\psi}_n \to H\hat{\psi} \text{ in } Y. \tag{2.41}$$

Set $V = \{\psi \in X;\ \left\langle \psi^*, \hat{\psi} \right\rangle = 0\} = \{\psi^*\}^\perp$ (where the orthogonality is to be understood in the sense of the $X^*, X$ duality product). Since $H\hat{\psi} \in \mathcal{R}(H)$, in order to prove (2.41) it is sufficient to prove that $H(V)$ is dense in $\mathcal{R}(H)$, and for the latter it is sufficient to prove (since $Y$ is reflexive) that $H(V)^\perp = \mathcal{R}(H)^\perp$ (where the orthogonality is to be understood in the sense of the $Y^*, Y$ duality product). But this equality follows from

$$\varphi^* \in H(V)^\perp \text{ if and only if } H^*\varphi^* \in V^\perp = \operatorname{Span}\{\psi^*\},$$

and the latter is equivalent to $H^*\varphi^* = 0$ (since $\psi^* \notin \mathcal{R}(H^*)$), which means $\varphi^* \in \mathcal{N}(H^*) = \mathcal{R}(H)^\perp$. $\square$

### 2.3.2 ▪ Application to the Inverse Scattering Problem

We turn back to our model problem and consider the notation and assumptions of Section 2.1. We first have the following characterization of $D$ in terms of the operator $\mathcal{H}^*$ where once again $\phi_z := \Phi_\infty(\cdot, z)$.

**Lemma 2.21.** *For $z \in \mathbb{R}^3$ we have that $z \in D$ if and only if $\phi_z$ is in the range of $\mathcal{H}^*$.*

***Proof.*** For $z \in D$ choose a cutoff function $\rho \in C^\infty(\mathbb{R}^3)$ which vanishes near $z$ and equals one in $\mathbb{R}^3 \setminus D$. Then $v(x) = \rho(x)\Phi(x, z)$ has $\phi_z$ as its far field pattern. Note that $f := (\Delta v + k^2 v)$ has compact support in $D$ and $f \in L^2(D)$. Since $v$ satisfies the Sommerfeld radiation condition,

$$v(x) = -\int_D \Phi(x, y) f(y) dy. \tag{2.42}$$

Hence

$$\phi_z = v_\infty = -\frac{1}{4\pi}\mathcal{H}^* f.$$

Now assume that $z \notin D$ and $\phi_z = \mathcal{H}^* f$ for some $f \in L^2(D)$. By Rellich's Lemma $\Phi(\cdot, z) = -4\pi v$ in the exterior of $D \cup \{z\}$, where $v$ is defined by (2.42). This gives a contradiction since $v$ is smooth near $z$ but $\Phi(\cdot, z)$ is singular at $z$. □

Applying Theorem 2.20 to the operator $F$ given by (2.1) and in view of Theorem 2.16 and Lemma 2.21, one can state the following corollary.

**Corollary 2.22.** *Assume that Assumptions* 2.1 *and* 2.2 *hold. Then for* $z \in \mathbb{R}^3$ *we have that* $z \in D$ *if and only if*

$$\inf\{\left|(Fg, g)_{L^2(S^2)}\right|; g \in L^2(S^2), (g, \phi_z)_{L^2(S^2)} = 1\} > 0.$$

The main drawback of this characterization is that it is numerically less attractive than other sampling methods. From the analysis of GLSM one also expects that this procedure would be very sensitive to noise in the operator $F$. Another typical difference with GLSM is that in this characterization one loses the link with the interior transmission problem. For the application and implementation of this method in the case of weakly nonlinear materials we refer the reader to [129]. A nice feature of this criterion is that it can be used to justify other sampling methods like the factorization method presented below.

## 2.4 ▪ The Factorization Method

In this section we present two versions of the *factorization method* for solving the inverse scattering problem for inhomogeneous media. The factorization method was first introduced by Kirsch in [109]. We refer the reader to [113] for a detailed analysis of both of these versions.

### 2.4.1 ▪ The $(F^*F)^{1/4}$ Method

We start with the first version of the factorization method, which relies on the factorization

$$F = H^* T H, \tag{2.43}$$

where now $F : X \to X$, $H : X \to Y$, and $T : Y \to Y^*$ are bounded operators with $X$ being an infinite-dimensional separable Hilbert space (we identify $X^*$ with $X$) and $Y$ a reflexive Banach space. We shall assume the following properties for the operator $T$. We denote by $\langle\ ,\ \rangle$ the $Y*, Y$ duality product.

**Assumption 2.3.** *We assume that* $T : Y \to Y^*$ *satisfies*

$$\Im \langle T\varphi,\, \varphi\rangle \neq 0$$

*for all* $\varphi \in \overline{\mathcal{R}(H)}$ *with* $\varphi \neq 0$ *and* $T = T_0 + C$, *where* $C$ *is compact on* $\overline{\mathcal{R}(H)}$ *and*

$$\langle T_0\varphi,\, \varphi\rangle \in \mathbb{R} \quad \textit{and} \quad \langle T_0\varphi,\, \varphi\rangle \geq \alpha \left\|\varphi\right\|_X^2$$

*for all* $\varphi \in \overline{\mathcal{R}(H)}$ *and some* $\alpha > 0$.

These assumptions are stronger than the coercivity property (2.39), as indicated in the following lemma.

**Lemma 2.23.** *Assume that $T : Y \to Y^*$ satisfies Assumption* 2.3. *Then it also satisfies the coercivity property* (2.39).

***Proof.*** Assume by contradiction that (2.39) is not satisfied. Then one can find a sequence $\varphi_j \in \overline{\mathcal{R}(H)}$ such that $\|\varphi_j\|_X = 1$ and is weakly convergent to $\varphi$ in $\overline{\mathcal{R}(H)}$ and also $|\langle T\varphi_j, \varphi_j\rangle| \to 0$ as $j \to \infty$. By our assumptions,

$$\Im \langle T\varphi_j,\, \varphi_j\rangle = \Im \langle C\varphi_j,\, \varphi_j\rangle \to \Im \langle T\varphi,\, \varphi\rangle$$

as $j \to 0$ since $C$ is compact. This implies that $\Im \langle T\varphi,\, \varphi\rangle = 0$ and therefore $\varphi = 0$. Consequently, by the triangle inequality,

$$0 < \alpha \le \langle T_0\varphi_j,\, \varphi_j\rangle \le |\, \langle T\varphi_j,\, \varphi_j\rangle\,| + |\, \langle C\varphi_j,\, \varphi_j\rangle\,|,$$

where $|\, \langle T\varphi_j,\, \varphi_j\rangle\,| \to 0$ by assumption and $|\, \langle C\varphi_j,\, \varphi_j\rangle\,| \to |\, \langle C\varphi,\, \varphi\rangle\,| = 0$ by the compactness of $C$. This gives a contradiction and proves the lemma. □

We now state and prove the main theorem of this section.

**Theorem 2.24.** *Assume that $F : X \to X$ is compact, injective, and that $I + i\gamma F$ is unitary for some $\gamma > 0$. In addition, assume that $T$ satisfies Assumption* 2.3. *Then the ranges $\mathcal{R}(H^*)$ and $\mathcal{R}((F^*F)^{1/4})$ coincide.*

***Proof.*** The proof follows the one given in [113]. Since $I + i\gamma F$ is unitary for some $\gamma > 0$ this implies that $F$ is normal. Since it is compact and injective, we deduce the existence of an orthonormal complete basis $(g_j)_{j=1,+\infty}$ of $X$ such that $Fg_j = \lambda_j g_j$, where $\lambda_j \neq 0$ forms a sequence of complex numbers that goes to 0 as $j \to \infty$. We remark that by assumption, $\lambda_j$ lies in the circle or radius $1/\gamma$ and center $i/\gamma$ which means in particular that $\Im(\lambda_j) \ge 0$. The operator $\tilde{H} := (F^*F)^{1/4} : X \to X$ is defined by $\tilde{H} g_j = \sqrt{|\lambda_j|} g_j$, and we introduce the operator $\tilde{T} : X \to X$ defined by

$$\tilde{T} g_j = \hat{\lambda}_j g_j, \quad \hat{\lambda}_j = \lambda_j / |\lambda_j|.$$

We then easily observe that $\tilde{H}^* = \tilde{H}$ and

$$F = \tilde{H}^* \tilde{T} \tilde{H}. \tag{2.44}$$

Consequently, in view of the inf-criterion (Theorem 2.20), the original factorization (2.43), and Lemma 2.23, it is sufficient to prove that $\tilde{T}$ is coercive on $X$ to obtain that the ranges of $\tilde{H}^*$ and $H^*$ coincide. Let $g \in X$ such that $\|g\| = 1$. We need to prove the existence of a positive constant $\beta$ independent from $g$ such that

$$0 < \beta \le |(\tilde{T} g, g)_X| = \left| \sum_{j=1}^{\infty} \hat{\lambda}_j |(g, g_j)_X|^2 \right|. \tag{2.45}$$

Since $\sum_{j=1}^{\infty} |(g, g_j)_X|^2 = 1$, the complex number $\sum_{j=1}^{\infty} \hat{\lambda}_j |(g, g_j)_X|^2$ lies in $C$: the closure of the convex hull of the sequence $(\hat{\lambda}_j)$. Giving that $\Im(\hat{\lambda}_j) \ge 0$, in order to prove the coercivity property, one only needs to prove that $0 \notin C$. Observe that, since $\lambda_j$ (for all $j$) lies in the circle or radius $1/\gamma$ and center $i/\gamma$ and $\lambda_j \to 0$ as $j \to \infty$, the only possible accumulation points of the sequence $(\hat{\lambda}_j)$ are $-1$ and $+1$. We shall prove that $-1$ is not an accumulation point, which is sufficient to get $0 \notin C$. Assume the existence of a subsequence, which we denote by $\hat{\lambda}_j$ for convenience, such that $\hat{\lambda}_j \to -1$ and set

$$\varphi_j := \frac{1}{\sqrt{|\lambda_j|}} H g_j.$$

Then using (2.43), clearly

$$\langle T\varphi_j, \varphi_j\rangle = \hat{\lambda}_j (g_j, g_j)_X = \hat{\lambda}_j \to -1. \tag{2.46}$$

From Lemma 2.23 we deduce that the sequence $\varphi_j$ is bounded in $Y$ and then can assume, up to the extraction of a subsequence, that $\varphi_j$ weakly converges to some $\varphi$ in $\overline{\mathcal{R}(H)}$. Taking the imaginary part of (2.46) implies

$$\Im \langle T\varphi_j, \varphi_j\rangle = \Im \langle C\varphi_j, \varphi_j\rangle \to \Im(-1) = 0,$$

which implies that $\Im \langle T\varphi, \varphi\rangle = 0$ and therefore $\varphi = 0$. By the definition of $T_0$ and the corresponding coercivity property we get

$$0 \le \langle T_0\varphi_j, \varphi_j\rangle \le \langle T\varphi_j, \varphi_j\rangle - \langle C\varphi_j, \varphi_j\rangle \to -1$$

since $\langle C\varphi_j, \varphi_j\rangle \to \langle C\varphi, \varphi\rangle = 0$ by compactness of $C$. This gives a contradiction and finishes the proof. $\square$

### 2.4.2 ▪ Application to the Inverse Scattering Problem for Nonabsorbing Media

We turn back to our model problem and consider the notation and assumptions of Section 2.1. According to Theorem 1.15, the normality of the operator $F$ holds if (and only if) $\Im(n) = 0$. Given the characterization of $D$ in terms of the range of $\mathcal{H}^*$ (see Lemma 2.21), we only need to check when Assumption 2.3 for the operator $T$ defined by (2.17) is satisfied.

**Lemma 2.25.** *Assume that* $\Im(n) = 0$ *and* $\Re(n-1) \ge \alpha > 0$ *(respectively,* $\Re(1-n) \ge \alpha > 0$*) in* $D$ *for some constant* $\alpha$ *and that Assumption* 2.1 *holds (i.e.,* $k$ *is not a transmission eigenvalue). Then the operator* $T : L^2(D) \to L^2(D)$ *(respectively,* $-T$*) defined by* (2.17) *satisfies Assumption* 2.3 *with* $Y = Y^* = L^2(D)$.

***Proof.*** Recall that

$$T(\psi) = -\frac{k^2}{4\pi}(1-n)(\psi + w),$$

where $w \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (2.2). Consider the case $n - 1 \ge \alpha > 0$ (the case $1 - n \ge \alpha > 0$ is similar). Let $T_0 : L^2(D) \to L^2(D)$ be defined by

$$T_0\psi = \frac{k^2}{4\pi}(n-1)\psi.$$

Then obviously $T_0$ is real and coercive as in Assumption 2.3. Moreover $T - T_0 : L^2(D) \to L^2(D)$ is compact by the compact embedding of $H^2(D)$ into $L^2(D)$.

Let $\psi \in \overline{\mathcal{R}(\mathcal{H})}$. From the identity (2.35), $\Im(T\psi, \psi) = 0$ implies $w_\infty = 0$ and by Rellich's Lemma $w = 0$ in $\mathbb{R}^3 \backslash D$. Consequently $u = w + \psi \in L^2(D)$ and $v = \psi \in L^2(D)$ are such that $u - v \in H^2(D)$ and are solutions of the interior transmission problem (2.6) with $f = g = 0$. We then infer that $w = \psi = 0$. $\square$

In view of Theorems 1.16, 1.15 and Lemmas 2.21 and 2.25 one can apply Theorem 2.24 to the factorization (2.18) and derive the following characterization of $D$ in terms of the range of the operator $(F^*F)^{1/4}$.

**Theorem 2.26.** *Assume the assumptions of Lemma* 2.25 *hold. Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of $(F^*F)^{1/4}$.*

A method to determine the support $\overline{D}$ of $m = 1 - n$ using Theorem 2.26 is to use Tikhonov regularization to find a regularized solution of

$$(\alpha I + (F^*F)^{1/2}) g_z^\alpha = (F^*F)^{1/4} \Phi_\infty(\cdot, z) \tag{2.47}$$

and note that the regularized solution $g_z^\alpha$ of (2.47) converges in $L^2(S^2)$ as $\alpha \to 0$ if and only if $z \in D$ (see Theorem 1.31). An alternative method to construct $D$ is to let $\lambda_n$ and $\psi_n$ be the eigenvalues and eigenfunctions of $F$ and note that $(F^*F)^{1/4}$ has the singular system $(\sqrt{|\lambda_n|}, \psi_n, \psi_n)$. Then by Picard's Theorem (Theorem 1.29) and Theorem 2.26, $z \in D$ if and only if

$$\sum_{n=1}^{\infty} \frac{|(\psi_n, \Phi_\infty(\cdot, z))|^2}{|\lambda_n|} < \infty. \tag{2.48}$$

For details of the numerical implementation of the factorization method we refer the reader to [113].

Let us now define the operator

$$F_\sharp := |\Re F| + |\Im(F)|, \tag{2.49}$$

where $\Re(F) := \frac{1}{2}(F + F^*)$ and $\Im(F) := \frac{1}{2i}(F - F^*)$. Let $\sigma_n = |\Re(\lambda_n)| + |\Im(\lambda_n)|$. Obviously $F_\sharp$ is a positive self-adjoint compact operator with $(\sigma_n, \psi_n, \psi_n)$ as a singular system. Since

$$|\lambda_n| \le \sigma_n \le \sqrt{2} |\lambda_n|$$

we get, from Picard's Theorem, that the range of $F_\sharp^{1/2}$ and the range of $(F^*F)^{1/4}$ coincide. One therefore can replace $(F^*F)^{1/4}$ by $F_\sharp^{1/2}$ in Theorem 2.26 and $|\lambda_n|$ by $\sigma_n$ in (2.48). The main advantage of the use of $F_\sharp^{1/2}$ is that it can be extended to cases where $F$ is no longer normal (for instance, when $\Im(n) \neq 0$), as indicated in the following section.

### 2.4.3 ▪ The $F_\sharp$ Method

This method was originally proposed in [109] as a generalization of the $(F^*F)^{1/4}$ method. It also relies on the factorization (2.43) of the far field operator, namely,

$$F = H^* T H, \tag{2.50}$$

where $F : X \to X$, $H : X \to Y$, and $T : Y \to Y^*$ are bounded operators with $X$ being an infinite-dimensional separable Hilbert space and $Y$ a reflexive Banach space. We assume in addition that there exists a pivot separable Hilbert space $U$ such that $Y \subset U \subset Y^*$ with dense inclusions (the triple $(Y, U, Y^*)$ is then called a Gelfand triple). The analysis given here follows mainly the one given in [113] but with slight modifications in the presentation and the hypothesis. We denote by $\langle \, , \, \rangle$ the $Y^*, Y$ duality product and by $\| \cdot \|$ the norm in $Y$.

The conditions on $T$ are summarized in the following assumption.

**Assumption 2.4.** *We assume that $T : Y \to Y^*$ satisfies*

$$\Im \langle T\varphi, \, \varphi \rangle \geq 0 \quad \textit{or} \quad \Im \langle T\varphi, \, \varphi \rangle \leq 0 \tag{2.51}$$

*for all $\varphi \in \overline{\mathcal{R}(H)}$ and $\Re T = T_0 + C$, where $C$ is compact on $\overline{\mathcal{R}(H)}$ and*

$$\langle T_0\varphi, \, \varphi \rangle \geq \alpha \, \|\varphi\|^2 \tag{2.52}$$

*for all $\varphi \in \overline{\mathcal{R}(H)}$ and some $\alpha > 0$. Moreover, we assume that if one of the following holds for some $v \in \overline{\mathcal{R}(H)}$, then $v = 0$:*

(i) $\langle Tv, \, \varphi \rangle = 0$ *for all $\varphi \in \mathcal{R}(H)$;*

(ii) $\langle \Im(T)v, \, v \rangle = 0$.

**Remark 2.27.** It is worth noticing that item (i) in Assumption 2.4 is equivalent to the injectivity of the operator $G = H^*T$ on $\overline{\mathcal{R}(H)}$. In the previous edition of this book, this condition was replaced by the injectivity of $T$ on $\overline{\mathcal{R}(H)}$ as in [128], which is not correct. (This mistake was also present in the first edition of [113].)

We now state and prove an intermediate result that will allow us to prove the main theorem.

**Theorem 2.28.** *Let $F = H^*TH : X \to X$, where $H : X \to U$ is compact and injective and has dense range, $T : U \to U$ is self-adjoint, $T = T_0 + C$, where $C$ is compact, and $T_0$ is self-adjoint and satisfies* (2.52)*. Then there exists a finite rank operator $P : U \to U$ such that $I + P : U \to U$ is an isomorphism and*

$$|F| = H^*T(I + P)H.$$

*Moreover, the operator $T(I + P) : U \to U$ is self-adjoint and nonnegative.*

***Proof.*** Since there is no risk of confusion, the scalar product in $X$ or in $U$ is indicated by using the same symbol $( \, , \, )$. The operator $F$ is compact and self-adjoint. Let $\lambda_n \in \mathbb{R}$ and $\psi_n \in X$ be the eigenvalues and eigenfunctions of $F$ such that $\{\psi_n, n \geq 1\}$ form an orthonormal basis of $X$. Then $|F|$ is the operator having $(|\lambda_n|, \psi_n)$ as a singular system. Let us decompose

$$X = X^+ \oplus X^-$$

with $X^+ := \operatorname{span}\{\psi_n; \lambda_n > 0\}$ and $X^- := \operatorname{span}\{\psi_n; \lambda_n \leq 0\}$. Obviously $\overline{HX^+} + \overline{HX^-}$ is dense in $U$. However, there is no guarantee in general that this sum is closed. We shall prove that it is the case by proving that $\overline{HX^-}$ is finite-dimensional. Consider $(\sigma_n, \phi_n)$ an eigenvalue decomposition of $T$ (a self-adjoint and Fredholm operator of index 0). We decompose $U = U^+ \oplus U^-$ with $U^+ := \operatorname{span}\{\phi_n; \sigma_n > 0\}$ and $U^- := \operatorname{span}\{\phi_n; \sigma_n \leq 0\}$. Since $T_0$ is positive, the space $U^-$ is finite-dimensional. Denote by $Q^\pm$ the orthogonal projection on $U^\pm$. Let $\phi \in HX^-$, i.e., $\phi \in H\psi$ for some $\psi \in X^-$. Then

$$0 \geq (F\psi, \psi) = (TH\psi, H\psi) \geq c_1 \|Q^+H\psi\|^2 - c_2 \|Q^-H\psi\|^2$$

with $c_1 = \min\{\sigma_n, \sigma_n > 0\} > 0$ and $c_2 = \max\{|\sigma_n|, \sigma_n \le 0\}$. Consequently

$$\|\phi\|^2 \le (1 + c_2/c_1)\|Q^-\phi\|^2 \quad \text{for all } \phi \in HX^-.$$

This proves that $Q^-$ is a bijection from $\overline{HX^-}$ into $U^-$ and therefore $\overline{HX^-}$ is finite-dimensional and $V^- := HX^- = \overline{HX^-}$. We then obtain that $\overline{HX^+} + V^-$ is a closed dense subspace of $U$ and therefore $U = \overline{HX^+} + V^-$. Let us set

$$V^0 := \overline{HX^+} \cap V^- \quad \text{and} \quad V^+ := (V^0)^\perp \cap \overline{HX^+}.$$

Then $V^+$ is closed and $V^+ \oplus V^0 = \overline{HX^+}$ (since $V^0$ is closed). We then deduce the (nonorthogonal) direct sum decomposition

$$U = V^+ + V^-.$$

Since $V^+$ and $V^-$ are closed and $V^+ \cap V^- = \{0\}$, then the projectors $P^+$ and $P^-$ associated with this sum are continuous operators. We can now conclude the proof by proving that $V^0 \subset \ker T$. Let $\phi \in V^0$. Then $(T\phi, \phi) \ge 0$ since $\phi \in \overline{HX^+}$ and $(T\phi, \phi) \le 0$ since $\phi \in V^-$. Hence $(T\phi, \phi) = 0$. Let $\psi \in \overline{HX^+}$ and $t \in \mathbb{R} \cup i\mathbb{R}$. Then

$$0 \le (T(t\phi + \psi), (t\phi + \psi)) = 2\Re(tT\phi, \psi) + (T\psi, \psi).$$

The latter holds for all $t \in \mathbb{R} \cup i\mathbb{R}$ if and only if $(T\phi, \psi) = 0$. Similar reasoning implies that $(T\phi, \psi) = 0$ for all $\psi \in V^-$. We then obtain $(T\phi, \psi) = 0$ for all $\psi \in U$, which gives $T\phi = 0$.

We are now in position to prove the desired factorization for $|F|$. For $\psi \in X$, $\psi = \psi^+ + \psi^-$ with $\psi^\pm \in X^\pm$ and

$$TH\psi^+ = TP^+H\psi^+ + TP^-H\psi^+ = TP^+H\psi^+ = TP^+H\psi \tag{2.53}$$

since $P^-H\psi^+ \in V^0$ and $P^+H\psi^- \in V^0$. Similarly

$$TH\psi^- = TP^+H\psi^- + TP^-H\psi^- = TP^-H\psi^- = TP^-H\psi. \tag{2.54}$$

Consequently

$$|F|(\psi) = F(\psi^+) - F(\psi^-) = H^*TH\psi^+ - H^*T\psi^- = H^*T(P^+ - P^-)H\psi,$$

which is the desired factorization with $P = -2P^-$. Indeed $I + P = P^+ - P^-$ is an isomorphism ($I + P$ is in fact an involution, $(I + P)^2 = I + P$) and $\tilde{T} := T(I + P)$ is self-adjoint since

$$(|F|\psi, \varphi) = (\tilde{T}H\psi, H\varphi) = (\psi, |F|\varphi) = (H\psi, \tilde{T}H\varphi)$$

and $H$ has dense range in $U$. $\square$

The following lemma will be useful.

**Lemma 2.29.** *Assume that $T : U \to U$ is a self-adjoint nonnegative operator. Then*

$$\|T(\phi)\|^2 \le \|T\|(T\phi, \phi). \tag{2.55}$$

***Proof.*** Let $\phi$ and $\psi \in U$ and let $t \in \mathbb{R}$. Then

$$0 \le (T(\phi + t\psi), (\phi + t\psi)) = (T\phi, \phi) + 2t\Re(T\phi, \psi) + t^2(T\psi, \psi).$$

The latter holds for all $t \in \mathbb{R}$ if and only if

$$\Re(T\phi, \psi)^2 \leq (T\phi, \phi)(T\psi, \psi).$$

Taking $\psi = T\phi$ implies

$$\|T(\phi)\|^4 \leq (T\phi, \phi)(TT\phi, T\phi) \leq (T\phi, \phi)\|T\|\|T(\phi)\|^2,$$

which proves the lemma. $\square$

**Theorem 2.30.** *Let $F$ be given by* (2.50) *and assume that there exists an isomorphism $J : Y \to U$. Assume that $H : X \to Y$ is compact, injective, and that $T$ satisfies Assumption* 2.4. *Then*

$$F_\sharp = H^* T_\sharp H, \tag{2.56}$$

*where $T_\sharp : Y \to Y^*$ is self-adjoint and satisfies the coercivity property* (2.39) *on $\overline{\mathcal{R}(H)}$. Moreover, the ranges $\mathcal{R}(H^*)$ and $\mathcal{R}((F_\sharp)^{1/2})$ coincide.*

***Proof.*** We shall first transform the problem so that it fits the assumptions of Theorem 2.28. The factorization (2.50) can also be written as

$$F = H_1^* T_1 H_1$$

with $H_1 = JH : X \to U$ and $T_1 = (J^*)^{-1} T J^{-1} : U \to U$, which gives a factorization that involves only Hilbert spaces $X$ and $U$. Let us denote by $\tilde{U} = \overline{\mathcal{R}(H_1)}$ and $Q$ the projection operator from $U$ onto $\tilde{U}$. Then using that $QH = H$ we get

$$F = \tilde{H}^* \tilde{T} \tilde{H}$$

with $\tilde{H} := QH_1 : X \to \tilde{U}$, $\tilde{T} := QT_1Q^* : \tilde{U} \to \tilde{U}$. From the assumptions of the theorem it is clear that $\tilde{H}$ is injective with dense range and that if $T$ satisfies Assumption 2.4, then $\Re(\tilde{T}) : \tilde{U} \to \tilde{U}$ is self-adjoint and is the sum of a self-adjoint coercive operator and a compact operator. From Theorem 2.28 we get the existence of an isomorphism $I + P : \tilde{U} \to \tilde{U}$ such that $P$ is a finite rank operator and

$$|\Re(F)| = \tilde{H}^* (\Re(\tilde{T}))(I + P)\tilde{H},$$

where $(\Re(\tilde{T}))(I + P) : \tilde{U} \to \tilde{U}$ is a self-adjoint and nonnegative operator. Assumption 2.4 implies in addition that

$$|\Im(F)| = \tilde{H}^* |\Im(\tilde{T})| \tilde{H},$$

where $|\Im(\tilde{T})| : \tilde{U} \to \tilde{U}$ is a self-adjoint nonnegative operator and $|\Im(\tilde{T})| = \pm\Im(\tilde{T})$ depending on the sign of $\Im(\tilde{T})$. We therefore end up with the factorization

$$F_\sharp = \tilde{H}^* \tilde{T}_\sharp \tilde{H}$$

with $\tilde{T}_\sharp = (\Re(\tilde{T}))(I + P) + |\Im(\tilde{T})|$. We shall now prove that $\tilde{T}_\sharp$ is coercive. Since $|\Im(\tilde{T})|$ is a nonnegative operator then $\Re(\tilde{T}_0) + |\Im(\tilde{T})|$ is a coercive operator on $\tilde{U}$ and therefore $\tilde{T}_\sharp$ is a Fredholm operator of index 0.

Using Assumption 2.4(i) or (ii) we now prove that $\tilde{T}_\sharp$ is injective. $\tilde{T}_\sharp \phi = 0$ implies

$$(\Re(\tilde{T})(I + P)\phi, \phi) = 0 \quad \text{and} \quad (\Im(\tilde{T})\phi, \phi) = 0.$$

From Lemma 2.29, we deduce that $\Re(\tilde{T})(I+P)\phi = 0$ and $\Im(\tilde{T}) = 0$. Since $\Re(\tilde{T})(I+P)$ is self-adjoint, and $(I+P)$ is an isomorphism, $\Re(\tilde{T})(I+P)\phi = 0$ implies $\Re(\tilde{T})\phi = 0$. If condition (ii) of Assumption 2.4 holds, then we immediately get from $\left\langle \Im(T)J^{-1}\phi,\, J^{-1}\phi\right\rangle = (\Im(\tilde{T})\phi, \phi) = 0$ that $\phi = 0$. If condition (i) holds then we also have $\phi = 0$ since $\tilde{T}\phi = \Re(\tilde{T})\phi + i\Im(\tilde{T})\phi = 0$ and therefore $\left\langle TJ^{-1}\phi,\, \psi\right\rangle = 0$ for all $\psi \in \mathcal{R}(H)$.

The injectivity of $\tilde{T}_\sharp$ proves that $\tilde{T}_\sharp$ is invertible. Applying Lemma 2.29 to $\tilde{T}_\sharp^{-1}$ and choosing $\phi = \tilde{T}_\sharp\psi$ in (2.55) implies that

$$\|\psi\|^2 \leq \|\tilde{T}_\sharp^{-1}\|(\tilde{T}_\sharp\psi, \psi),$$

which gives the coercivity of $\tilde{T}_\sharp$ on $\tilde{U}$. The factorization of the theorem follows by setting

$$T_\sharp = J^* Q^* \tilde{T}_\sharp Q J.$$

Using the definition of $J$ and $Q$ we easily get that $T_\sharp$ is coercive on the closure of the range of $H$. We now can apply Theorem 2.20 to the factorizations (2.56) and $F_\sharp = (F_\sharp)^{1/2}((F_\sharp)^{1/2})^*$ to get that the ranges $\mathcal{R}(H^*)$ and $\mathcal{R}((F_\sharp)^{1/2})$ coincide. □

### 2.4.4 ▪ Application to the Inverse Scattering Problem for Absorbing Media

We turn back to our model problem and consider the notation and assumptions of Section 2.1. Consider $F$ satisfying the factorization (2.18) and set, for $\theta \in [0, 2\pi[$,

$$F^\theta := \Re(e^{i\theta}F) + i\Im(F)$$

and

$$F_\sharp^\theta := |\Re(e^{i\theta}F)| + |\Im(F)|. \tag{2.57}$$

Obviously

$$F^\theta = \mathcal{H}^* T^\theta \mathcal{H} \text{ with } T^\theta := \Re(e^{i\theta}T) + i\Im(T),$$

where $T : L^2(D) \to L^2(D)$ is defined by (2.17). We then have the following lemma.

**Lemma 2.31.** *Let $\theta \in [0, \pi]$. Assume that $\Im(n) \geq 0$ and $\Re(e^{i\theta}(n-1)) \geq \alpha > 0$ in $D$ for some constant $\alpha$ and that Assumption* 2.1 *holds (i.e., $k$ is not a transmission eigenvalue). Then the operator $T^\theta : L^2(D) \to L^2(D)$ satisfies Assumption* 2.4 *with $Y = Y^* = L^2(D)$.*

***Proof.*** Recall that

$$T(\psi) = -\frac{k^2}{4\pi}(1-n)(\psi + w(\psi)), \tag{2.58}$$

where $w(\psi) \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (2.2). Let $T_0 : L^2(D) \to L^2(D)$ be defined by

$$T_0\psi = -\frac{k^2}{4\pi}\Re(e^{i\theta}(1-n))\psi.$$

Then obviously $T_0$ is real and coercive as in Assumption 2.4. Moreover $\Re T^\theta - T_0 : L^2(D) \to L^2(D)$ is compact by the compact embedding of $H^2(D)$ into $L^2(D)$. From identity (2.35), $\Im(T^\theta\psi, \psi) = \Im(T\psi, \psi) \geq 0$. We now can conclude as in the proof of Lemma 2.25 that $\Im(T^\theta)$ is injective on the range $\overline{\mathcal{R}(\mathcal{H})}$ since $k$ is not a transmission eigenvalue. Assumption 2.4(ii) is then verified. □

In view of the previous lemma and Lemmas 2.1 and 2.21, we now can state the straightforward application of Theorem 2.30 to the operator $F^\theta$.

**Theorem 2.32.** *Assume $\Im(n) \geq 0$ and there exists $\theta \in [0,\pi]$ such that $\Re(e^{i\theta}(n-1)) \geq \alpha > 0$ in $D$ for some constant $\alpha$. Assume in addition that Assumption 2.1 holds (i.e., $k$ is not a transmission eigenvalue). Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of $(F_\sharp^\theta)^{1/2}$.*

As for $(F^*F)^{1/4}$, the numerical implementation of Theorem 2.32 can rely on either a Tikhonov regularization as in (2.47) or the Picard series as in (2.48).

## 2.5 ▪ Link between Sampling Methods

The assumptions required by the GLSM method are weaker than the ones required by the factorization method but are similar to the inf-criterion. Indeed the main advantage of GLSM with respect to the inf-criterion is that it leads to a more tractable numerical inversion algorithm. In some special configurations there is a direct link between GLSM and the factorization method as explained below. Moreover, the $(F^*F)^{1/4}$ method can be used to provide precise information on the behavior of the Tikhonov regularized solution of the LSM equation.

### 2.5.1 ▪ LSM versus the $(F^*F)^{1/4}$ Method

Let us consider the case where the hypothesis of Theorem 2.24 holds (this corresponds in particular to the case when $\Im(n) = 0$). We shall prove that in this case the Tikhonov solution of (2.16) satisfies $\limsup_{\alpha\to 0} \|\mathcal{H}\tilde{g}_z^\alpha\|_{L^2(D)} < \infty$ if $z \in D$ (see also [5]). This is a direct consequence of the following general result together with Theorem 2.24.

**Theorem 2.33.** *Assume that $F : X \to X$ is as in Theorem 2.24. Let $\phi \in X$, and let $g^\alpha \in X$ be a solution to*

$$(\alpha + F^*F)\, g^\alpha = F^*\phi.$$

*Then $\phi$ is in the range of $(F^*F)^{1/4}$ if and only if $\limsup_{\alpha\to 0} |(Fg^\alpha, g^\alpha)| < \infty$, which is also equivalent to $\limsup_{\alpha\to 0} \|Hg^\alpha\| < \infty$.*

***Proof.*** Using the eigensystem $(\lambda_j, \psi_j)_{j\geq 1}$ of the normal operator $F$, we observe that

$$g^\alpha = \sum_j \frac{\overline{\lambda_j}}{\alpha + |\lambda_j|^2} (\phi, \psi_j)\psi_j.$$

Therefore

$$|(Fg^\alpha, g^\alpha)| = \left| \sum_j \frac{|\lambda_j|^2 \overline{\lambda_j}}{(\alpha + |\lambda_j|^2)^2} |(\phi, \psi_j)|^2 \right| \leq \sum_j \frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} |(\phi, \psi_j)|^2.$$

On the other hand, from the coercivity property (2.45), we also have

$$|(Fg^\alpha, g^\alpha)| \geq \beta \sum_j \frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} |(\phi, \psi_j)|^2.$$

The Picard criterion implies that $\phi$ is in the range of $(F^*F)^{1/4}$ if and only if

$$\sum_j \frac{1}{|\lambda_j|} |(\phi, \psi_j)|^2 < +\infty.$$

Consequently, since

$$\frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} \to \frac{1}{|\lambda_j|} \text{ as } \alpha \to 0 \quad \text{and} \quad \frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} \leq \frac{1}{|\lambda_j|},$$

we get that $\phi$ is in the range of $(F^*F)^{1/4}$ if and only if

$$\limsup_{\alpha \to 0} |(Fg^\alpha, g^\alpha)| < +\infty.$$

We conclude the proof by using the coercivity property (2.39) and the continuity of $T$ to obtain

$$\beta \|Hg^\alpha\|^2 \leq |(Fg^\alpha, g^\alpha)| \leq \|T\| \|Hg^\alpha\|^2$$

for some $\beta > 0$. $\square$

## 2.5.2 ▪ GLSM versus the Factorization Method

We now briefly relate the generalized linear sampling method to both versions of the factorization method.

### GLSM versus the $(F^*F)^{1/4}$ Method

Let us again consider the case where the hypothesis of Theorem 2.24 holds (this corresponds in particular to the case when $\Im(n) = 0$). According to the factorization (2.44) one can apply GLSM with $F = F$, $B = \tilde{H}^* \tilde{H} = (F^*F)^{\frac{1}{2}}$, and $G = \tilde{H}^* \tilde{T}$. In this case, the operator $B$ is positive self-adjoint, and therefore one can say more than in Theorem 2.9. Using the eigensystem $(\lambda_j, \psi_j)_{j \geq 1}$ of the normal operator $F$, we observe that

$$\begin{aligned} J_\alpha(\phi; g) &= \alpha((F^*F)^{\frac{1}{2}} g, g) + \|Fg - \phi\|^2 \\ &= \alpha \sum_i |\lambda_i| |(g, \psi_i)|^2 + \sum_i (\lambda_i (g, \psi_i) - (\phi, \psi_i))^2. \end{aligned}$$

Hence $J_\alpha(\phi; \cdot)$ has a minimizer given by

$$g_\alpha = \sum_j \frac{\overline{\lambda_j} (\phi, \psi_j)}{\alpha |\lambda_j| + |\lambda_j|^2} \psi_j.$$

This minimizer clearly satisfies (2.23). Let us now define

$$g_\alpha^{\mathrm{FM}} = \sum_j \frac{|\lambda_j|^{\frac{1}{2}}}{|\lambda_j| + \alpha} (\phi, \psi_j) \psi_j,$$

which is the minimizer of the Tikhonov functional $\alpha \|g\|^2 + \|(F^*F)^{\frac{1}{4}} g - \phi\|^2$. One then observes that the GLSM indicator function satisfies

$$|((F^*F)^{\frac{1}{2}} g_\alpha, g_\alpha)| = \sum_j \frac{|\lambda_j| (\phi, \psi_j)^2}{(\alpha + |\lambda_j|)^2} = \left\| g_\alpha^{\mathrm{FM}} \right\|^2.$$

This means that the GLSM indicator function (in the noise free case) coincides with the indicator function given by the $(F^*F)^{1/4}$ method when Tikhonov regularization is used, e.g., (2.47). In principle, nothing can be deduced on the boundedness of $Hg_\alpha$ from the analysis of GLSM. However, this information can be obtained from Theorem 2.33.

#### GLSM versus the $F_\sharp$ Method

The factorization method allows one to use for GLSM an operator $B$ that satisfies the assumptions of Theorem 2.9 which is important for the some applications like imaging in unknown backgrounds (see Section 2.5.4). Let $F = H^*TH$ be as in Theorem 2.30. Let us set for $\phi \in X$

$$J_\alpha(\phi; g) := \alpha(F_\sharp g, g) + \|Fg - \phi\|^2$$

and

$$j_\alpha(\phi) = \inf_{g \in X} J_\alpha(\phi; g).$$

Combining Theorems 2.9 and 2.30 we have the following theorem.

**Theorem 2.34.** *Let $F = H^*TH$ be as in Theorem 2.30, set $G = H^*T : \overline{\mathcal{R}(H)} \subset Y \to X$, and assume in addition that $F$ is injective with dense range.*

*Consider for $\alpha > 0$ and $\phi \in X^*$, $g_\alpha \in X$ such that*

$$J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \text{ with } 0 < \frac{p(\alpha)}{\alpha} \to 0 \text{ as } \alpha \to 0.$$

*Then $\phi \in \mathcal{R}(G)$ if and only if $\lim_{\alpha \to 0} (F_\sharp g_\alpha, g_\alpha) < \infty$.*

*Moreover, in the case $\phi = G\varphi$, the sequence $Hg_\alpha$ strongly converges to $\varphi$ in $Y$.*

### 2.5.3 ▪ Some Numerical Examples

We report here some two-dimensional numerical examples from [13]. They correspond to two separate inhomogeneities with different index of refractions, respectively, equal to $n = 2 + 0.5i$ and $2 + 0.1i$ (see Figure 2.1). The frequency is $k = 1$ and 100 equidistant incident directions and observation points have been used. The data have been generated synthetically by solving the forward scattering problem using a standard finite element method. In Figure 2.1 the output of four indicator functions are compared. Let $g_z^\alpha$ be the Tikhonov regularized solution of (2.16), where the regularization parameter is computed using the Morozov discrepancy principle (see Remark 2.5). We define

$$\mathcal{I}^{\mathrm{LSM}}(z) = 1/\|g_z^\alpha\|^2, \tag{2.59}$$

$$\mathcal{I}^{\mathrm{GLSM0}}(z) = 1/|(F^\delta g_z^\alpha, g_z^\alpha)|, \tag{2.60}$$

$$\mathcal{I}^{\mathrm{GLSM}}(z) = 1/\left(|(F^\delta g_z^\alpha, g_z^\alpha)| + \delta\|F^\delta\|\|g_z^\alpha\|^2\right), \tag{2.61}$$

where the noise level $\delta$ is such that

$$\|F - F^\delta\| \leq \delta\|F^\delta\|.$$

Let $g_{z,\sharp}^\alpha$ be the Tikhonov regularized solution of (2.47) (with $(F^*F)^{1/4}$ replaced by $F_\sharp^{1/2}$), where the regularization parameter is computed using the Morozov discrepancy principle. We define

$$\mathcal{I}^{\mathrm{F}_\sharp}(z) = 1/\|g_{z,\sharp}^\alpha\|^2. \tag{2.62}$$

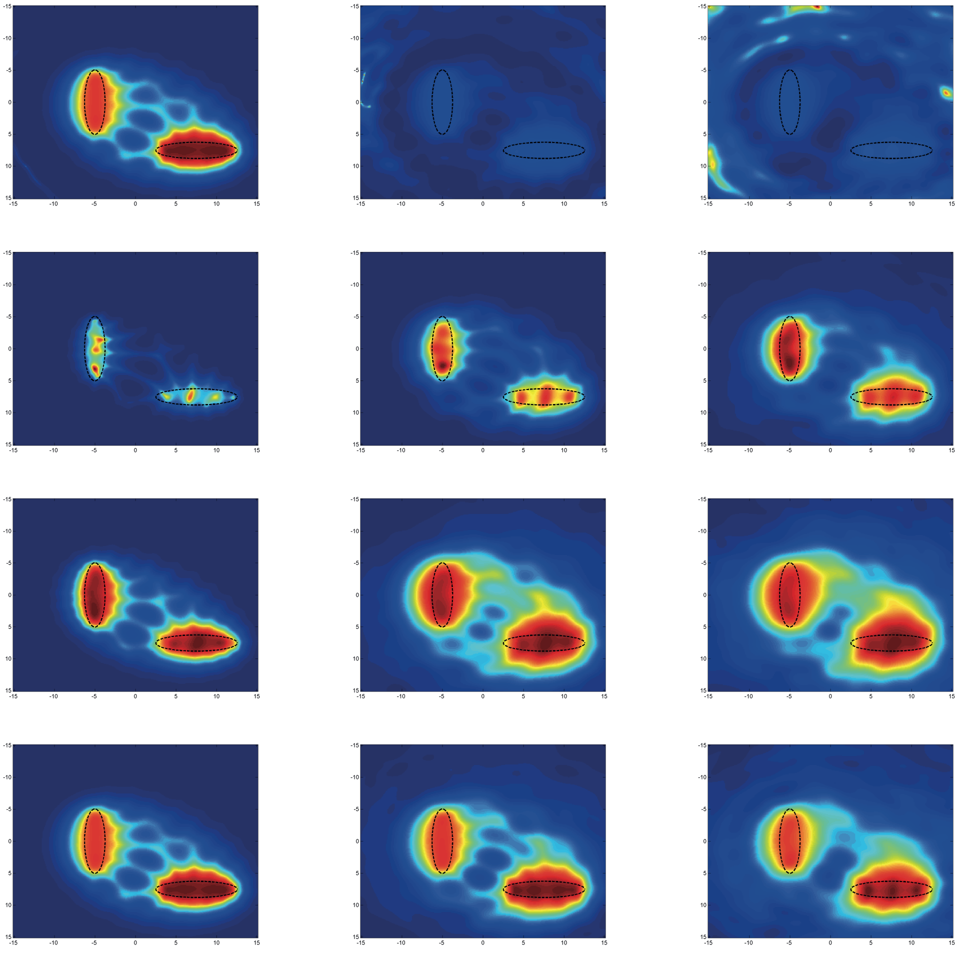



**Figure 2.1.** *Output of four different imaging functions. First row:* $\mathcal{I}^{\mathrm{GLSM0}}$*; second row:* $\mathcal{I}^{\mathrm{LSM}}$*; third row:* $\mathcal{I}^{\mathrm{F}\sharp}$*; and fourth row:* $\mathcal{I}^{\mathrm{GLSM}}$*. The columns correspond to different noise levels* $\delta$*: from left to right,* $\delta = 0, 1,$ *and* $5\%$*. Reproduced from* [13] *with permission.*

In the spirit of the GLSM algorithm one can improve the reconstruction provided by $\mathcal{I}^{\mathrm{GLSM}}$ by using $g_z^\alpha$ as an initial guess to compute a minimizing sequence of (2.38). Figure 2.2 shows how one can obtain better resolutions after applying some gradient descent iterations. For these numerical results the parameter $\alpha$ in (2.38) is taken as $\alpha = \alpha_M/(\|F^\delta\|(1+\delta))$, where $\alpha_M$ is the Morozov parameter used in (2.16). The function $\mathcal{I}^{\mathrm{GLSMoptim}}$ has the same expression as $\mathcal{I}^{\mathrm{GLSM}}$ but with $g_z^\alpha$ being the computed minimizing sequence.

## 2.5.4 ▪ Application to Differential Measurements

We here present an application of the GLSM method to the imaging problem where one would like to identify a change in the background using differential measurements. Assume, for instance, that a reference medium is defined by an index of refraction $n_0$, and let us denote by $F_0$ the far field operator associated with this medium. Applying any of the

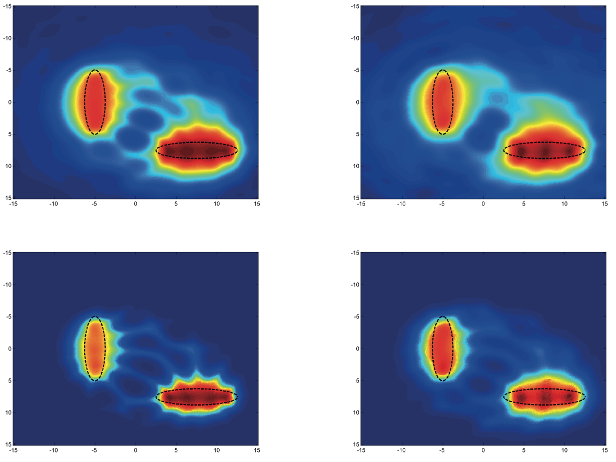


**Figure 2.2.** *First row:* $\mathcal{I}^{\mathrm{GLSM}}$*; second row:* $\mathcal{I}^{\mathrm{GLSMoptim}}$*. The columns correspond to different noise levels* $\delta = 1\%$ *left and* $\delta = 5\%$ *right. Reproduced from* [13] *with permission.*

algorithms above would provide an approximation of $D_0$, the support of $n_0 - 1$. Assume that a change occurred in the medium modifying $n_0$ locally and denote by $n$ the new refractive index. Let $F$ be the far field operator associated with $n$, and let $D$ be the support of $n-1$. The inverse problem we would like to address here is the identification of $D \setminus D_0$ from the knowledge of $F$ and $F_0$ (without reconstructing $n$ and $n_0$ or $D$ and $D_0$). We present here the method proposed in [12] in the simple case where $D = D_0 \cup D_1$ with $D_0 \cap D_1 = \emptyset$ and $n = n_0$ in $D_0$. The inverse problem is then to reconstruct $D_1$ from $F_0$ and $F$. For the analysis of more complex configurations we refer the reader to [7] and [12]. The differential approach has been also applied for crack inverse problems in [148].

Denoting by $\mathrm{itp}(n, D)$ the interior transmission problem (2.6), we assume here that $\mathrm{itp}(n, D)$ and $\mathrm{itp}(n_0, D_0)$ are both well-posed. We shall exploit in the following that the solutions of $\mathrm{itp}(n, D)$ and $\mathrm{itp}(n_0, D_0)$ coincide in $D_0$ if the boundary data coincide on $\partial D_0$. This is easily verified given the special configuration of $D$.

We also assume that there exists $\theta \in [0, \pi]$ such that the assumptions of the refractive index in Theorem 2.32 hold for $n$ in $D$ and $n_0$ in $D_0$. We then set

$$B = F_\sharp^\theta \quad \text{and} \quad B_0 = F_{0,\sharp}^\theta.$$

(See (2.57) for the definition of $F_\sharp^\theta$. The operator $F_{0,\sharp}^\theta$ is defined similarly.) Consider

$$J_\alpha(z; g) := \alpha (Bg, g)_{L^2(S^2)} + \|Fg - \Phi(\cdot, z)\|^2_{L^2(S^2)}$$

and

$$J_{0,\alpha}(z; g) := \alpha (B_0 g, g)_{L^2(S^2)} + \|F_0 g - \Phi(\cdot, z)\|^2_{L^2(S^2)}$$

and $g_\alpha^z$ and $g_{0,\alpha}^z$ in $L^2(S^2)$ such that

$$J_\alpha(z; g_\alpha^z) \le \inf_{g \in L^2(S^2)} J_\alpha(z; g) + p(\alpha)$$

and

$$J_{0,\alpha}(z; g_{0,\alpha}^z) \le \inf_{g \in L^2(S^2)} J_{0,\alpha}(z; g) + p(\alpha)$$

with $0 < \frac{p(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. Application of Theorem 2.34 to $F$ and $F_0$ in combination with the arguments at the end of the proof of Theorem 2.3 show that if $z$ is in $D_0$, then $v_{g^z_\alpha}$ and $v_{g^z_{0,\alpha}}$ converge in $L^2(D_0)$ to the same function $v$ (the fact that it is the same function comes form the considerations above on the solutions of $\mathrm{itp}(n, D)$ and $\mathrm{itp}(n_0, D_0)$). Therefore, if $z$ is in $D_0$, then

$$(B_0(g^z_\alpha - g^z_{0,\alpha}), (g^z_\alpha - g^z_{0,\alpha}))_{L^2(S^2)} \le C\|\mathcal{H}_0 g^z_\alpha - \mathcal{H}_0 g^z_{0,\alpha}\|^2_{L^2(D_0)} \to 0 \tag{2.63}$$

as $\alpha \to 0$. Let us set

$$\mathcal{A}(g) := (Bg, g)_{L^2(S^2)} \quad \text{and} \quad \mathcal{D}(g, g_0) := (B_0(g - g_0), g - g_0)_{L^2(S^2)}$$

and introduce the indicator function

$$\mathcal{I}(g, g_0) := \frac{1}{\mathcal{A}(g)(1 + \mathcal{A}(g)\mathcal{D}(g, g_0)^{-1})}.$$

**Theorem 2.35.** *Let $z \in \mathbb{R}^3$. Then $z \in D_1$ if and only if $\lim\limits_{\alpha\to 0} \mathcal{I}(g^z_\alpha, g^z_{0,\alpha}) > 0$.*

***Proof.*** If $z \notin D$, then from Theorem 2.34 we get that $\mathcal{A}(g^z_\alpha) \to +\infty$ as $\alpha \to 0$ and therefore $\lim\limits_{\alpha\to 0} \mathcal{I}(g^z_\alpha, g^z_{0,\alpha}) = 0$.

Consider now the case of $z \in D_0$. Theorem 2.34 implies that $\mathcal{A}(g^z_\alpha)$ is bounded and converges to $(T_\sharp u_0, u_0)_{L^2(D)}$ where $(u, u_0)$ is the solution of $\mathrm{itp}(n, D)$ with $\Phi(\cdot, z)$ and $\frac{\partial\Phi}{\partial\nu}(\cdot, z)$ as boundary data. Since $z \in D_0$ and $D_0 \cap D_1 = \emptyset$ then $u_0 = \Phi(\cdot, z)$ (and $u = 0$) in $D_1$. Consequently $(T_\sharp u_0, u_0)_{L^2(D)} > 0$. Combining this fact with (2.63) implies $\lim\limits_{\alpha\to 0} \mathcal{I}(g^z_\alpha, g^z_{0,\alpha}) = 0$.

We now treat the case of $z \in D_1$. From Theorem 2.34 applied to $F_0$, we get that $(B_0 g^z_{0,\alpha}, g^z_{0,\alpha})_{L^2(S^2)}$ is unbounded as $\alpha \to 0$, while the same theorem applied to $F$ implies that $(B_0 g^z_\alpha, g^z_\alpha)_{L^2(S^2)}$ is bounded. Consequently $\mathcal{D}(g^z_\alpha, g^z_{0,\alpha})$ is unbounded as $\alpha \to 0$. On the other hand, Theorem 2.34 implies that $\mathcal{A}(g^z_\alpha)$ is bounded. We then get $\lim\limits_{\alpha\to 0} \mathcal{I}(g^z_\alpha, g^z_{0,\alpha}) > 0$, which finishes the proof. □

Indeed, as for GLSM, in the case of a noisy operator $B^\delta$ such that $\|B^\delta - B\| \le \delta\|B^\delta\|$, the indicator function has to be modified by replacing $\mathcal{A}(g)$ with

$$\mathcal{A}^\delta(g) := (B^\delta g, g)_{L^2(S^2)} + \delta\|B^\delta\|\|g\|^2_{L^2(S^2)},$$

while $\mathcal{D}(g)$ is simply replaced with

$$\mathcal{D}^\delta(g, g_0) := (B^\delta_0(g - g_0), g - g_0)_{L^2(S^2)}.$$

For the analysis of the noisy case we refer the reader to [7] and [12].

We now give a two-dimensional numerical example due to Audibert illustrating the performance of the indicator function described above. The medium configuration is described in Figure 2.3 where the solid line indicates the boundary of $D_0$ while the dashed line indicates the boundary of $D_1$. The index of refraction in $D_0$ is $n_0 = 2 + 0.5i$ and the index of refraction in $D_1$ is equal to 3. The wave number is $k = 2\pi$. Figure 2.4 indicates the reconstructions obtained using the GLSM algorithm with optimization as described in the previous section for $D_0$ and $D$ using, respectively, $F_0$ and $F$. The reconstruction of $D_1$ using directly $F$ and $F_0$ as suggested by Theorem 2.35 (i.e., without relying on the reconstruction of $D_0$ and $D$) is shown on the right of Figure 2.4 and clearly indicates that the proposed indicator function provides satisfactory results. We again refer the reader to [7] for a more extensive discussion of numerical issues related to this type of indicator function and applications to imaging in a randomly fluctuating background.

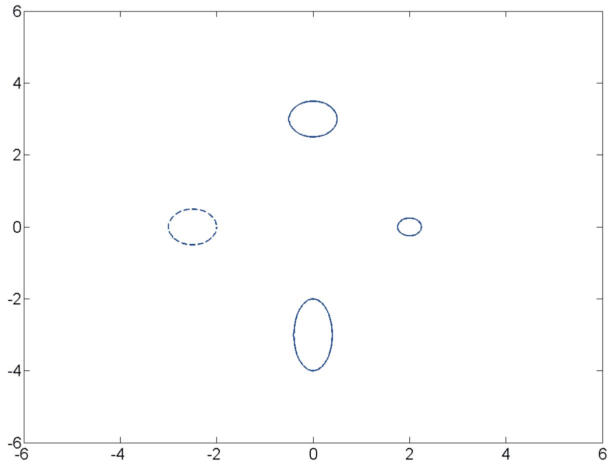



**Figure 2.3.** *The medium configuration: $D_1$ dashed line, $D_0$ solid line.*

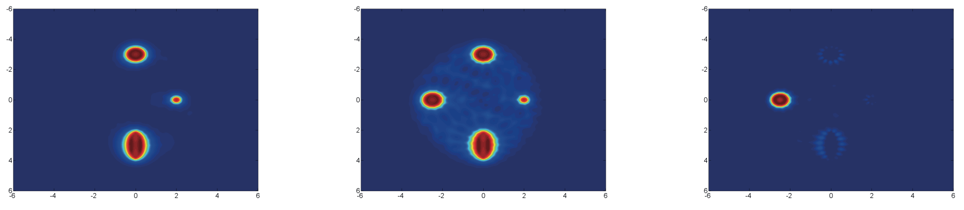


**Figure 2.4.** *Left: Reconstruction of $D_0$ using GLSM. Middle: Reconstruction of $D$ using GLSM. Right: Reconstruction of $D_1$ using differential measurements. The data are corrupted with $1\%$ random noise.*

## 2.6 ▪ Application of Sampling Methods to Anisotropic Media

We now consider the inverse scattering problem associated with the model discussed in Section 1.4 that corresponds to an anisotropic medium characterized by a $3 \times 3$ symmetric matrix with $L^\infty(D)$-entries such that

$$\overline{\xi} \cdot \Re(A)\xi \geq \gamma\, |\xi|^2 \quad \text{and} \quad \overline{\xi} \cdot \Im(A)\xi \leq 0$$

for all $\xi \in \mathbb{C}^3$ and almost every $x \in \overline{D}$ and some constant $\gamma > 0$. Here $D$ is the support of the inhomogeneity which is assumed to be a bounded Lipschitz domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. The assumptions on $n$ are the same as in Section 1.2. See Section 1.4.1 for the definition of the far field operator and some basic properties associated with this operator.

Using Theorem 1.39, let us define for $\boldsymbol{\varphi} \in L^2(D)^3$ and $\psi \in L^2(D)$ the unique function $w \in H^1_{loc}(\mathbb{R}^3)$ satisfying

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = \nabla \cdot (I - A)\boldsymbol{\varphi} + k^2(1-n)\psi \text{ in } \mathbb{R}^3, \\ \displaystyle\lim_{R\to\infty} \int\limits_{|x|=R} |\partial w/\partial |x| - ik\, w|^2 \, ds = 0 \end{cases} \tag{2.64}$$

so that if $\psi(x) = e^{ikd\cdot x}$ and $\boldsymbol{\varphi} = \nabla\psi$, then $w = u^s(\cdot, d)$ and the far field pattern $w_\infty$ of $w$ coincides with $u_\infty(\cdot, d)$. The Herglotz operator is now defined as $\mathcal{H} : L^2(S^2) \to L^2(D)^3 \times L^2(D)$ with

$$\mathcal{H}g := (\nabla v_g|_D, v_g|_D), \tag{2.65}$$

where the Herglotz wave function $v_g$ is defined by (1.31). Setting $H_{\text{inc}}(D)$ to be the closure of the range of $\mathcal{H}$ in $L^2(D)^3 \times L^2(D)$ we then consider the operator $G : H_{\text{inc}}(D) \to L^2(S^2)$ defined by

$$G(\boldsymbol{\varphi}, \psi) := w_\infty, \tag{2.66}$$

where $w_\infty$ is the far field pattern of $w \in H^1_{loc}(\mathbb{R}^3)$ satisfying (2.64). This ensures the first factorization $F = G\mathcal{H}$.

We now proceed with giving the main ingredients for the justification of the LSM. We again rely on the solvability of the interior transmission problem. In the present setting this problem is phrased as $(u, u_0) \in H^1(D) \times H^1(D)$ such that

$$\begin{cases} \nabla \cdot (A\nabla u) + k^2 n u = 0 \quad \text{in } D, \\ \Delta u_0 + k^2 u_0 = 0 \quad \text{in } D, \\ u - u_0 = f \quad \text{on } \partial D, \\ \partial u/\partial \nu_A - \partial u_0/\partial \nu = h \quad \text{on } \partial D \end{cases} \tag{2.67}$$

for given $(f, h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$, where $\nu$ denotes the outward normal on $\partial D$. Values of $k$ for which this problem is not well-posed are referred to as *transmission eigenvalues*. We refer the reader to the next two chapters for the analysis of this problem and content ourselves here with the following assumption.

**Assumption 2.5.** *We assume that the matrix $A$, the index $n$, and the wave number $k$ are such that* (2.67) *defines a well-posed problem.*

**Lemma 2.36.** *The operator $\mathcal{H}$ defined by* (2.65) *is compact and injective. Let $H_{\text{inc}}(D)$ be the closure of the range of $\mathcal{H}$ in $L^2(D)^3 \times L^2(D)$. Then*

$$H_{\text{inc}}(D) = \{(\boldsymbol{\varphi}, \psi) = (\nabla v, v);\ v \in H^1(D);\ \Delta v + k^2 v = 0 \textit{ in } D\}.$$

***Proof.*** The first part follows from the same arguments as in Lemma 2.1. For the second part of the lemma, we also proceed similarly to the proof of Lemma 2.1. Set $\widetilde{H_{\text{inc}}}(D) := \{(\boldsymbol{\varphi}, \psi) = (\nabla v, v);\ v \in H^1(D);\ \Delta v + k^2 v = 0 \text{ in } D\}$. Then obviously $H_{\text{inc}}(D) \subset \widetilde{H_{\text{inc}}}(D)$. To prove the theorem it is then sufficient to prove that $\mathcal{H}^* : L^2(D)^3 \times L^2(D) \to L^2(S^2)$, the adjoint of the operator $\mathcal{H}$, which is given by

$$\mathcal{H}^*(\boldsymbol{\varphi}, \psi)(\hat{x}) := \int_D (-ik\hat{x} \cdot \boldsymbol{\varphi}(y) + \psi(y)) e^{-ik\hat{x}\cdot y}\, dy, \tag{2.68}$$

is injective on $\widetilde{H_{\text{inc}}}(D)$. Let $(\boldsymbol{\varphi}, \psi) = (\nabla u_0, u_0)$ with $u_0 \in H^1(D)$ satisfying $\Delta u_0 + k^2 u_0 = 0$ in $D$. We set

$$u(x) := \int_D (\nabla_y \Phi(x, y) \cdot \nabla u_0(y) + \Phi(x, y) u_0(y))\, dy, \quad x \in \mathbb{R}^3.$$

From the regularity of volume potentials (Theorem 1.8), we infer that $u \in H^1_{loc}(\mathbb{R}^3)$ and satisfies

$$\int_{\mathbb{R}^3} (-\nabla u \cdot \nabla v + k^2 uv)\, dx = -\int_D (\nabla u_0 \cdot \nabla v + u_0 v)\, dx \tag{2.69}$$

for all $v \in H^1_{loc}(\mathbb{R}^3)$ with compact support (together with the Sommerfeld radiation condition). Since by construction $4\pi u_\infty = \mathcal{H}^*(\boldsymbol{\varphi}, \psi)$, then $\mathcal{H}^*(\boldsymbol{\varphi}, \psi) = 0$ implies that $u_\infty = 0$ and therefore $u = 0$ in $\mathbb{R}^3 \setminus D$ by Rellich's Lemma. The regularity $u \in H^1_{loc}(\mathbb{R}^3)$ then implies $u \in H^1_0(D)$. Equation (2.69) then gives

$$\int_D (-\nabla u \cdot \nabla v + k^2 uv)\, dx = -\int_D (\nabla u_0 \cdot \nabla v + u_0 v)\, dx$$

for all $v \in H^1(D)$. Taking $v = \overline{u_0}$ implies

$$\|u_0\|^2_{H^1(D)} = -\int_D (-\nabla u \cdot \nabla u_0 + k^2 u u_0)\, dx = 0,$$

where the last equality follows from $\Delta u_0 + k^2 u_0 = 0$ in $D$ and $u \in H^1_0(D)$. □

We remark that $H_{\mathrm{inc}}(D)$ can be identified with $H^1_{\mathrm{inc}}(D) \subset H^1(D)$ defined by

$$H^1_{\mathrm{inc}}(D) := \{v \in H^1(D);\ \Delta v + k^2 v = 0 \text{ in } D\}$$

through the isomorphism

$$\mathcal{I} : H^1_{\mathrm{inc}}(D) \to H_{\mathrm{inc}}(D);\ \mathcal{I}(v) = (\nabla v, v).$$

Setting, for $g \in L^2(S^2)$,

$$\mathcal{H}^1(g) := \mathcal{I}^{-1}\mathcal{H}(g) = v|_D,$$

we also have the following lemma as an immediate corollary of Lemma 2.36.

**Lemma 2.37.** *The operator $\mathcal{H}^1 : L^2(S^2) \to H^1_{\mathrm{inc}}(D) \subset H^1(D)$ is compact and injective with dense range.*

Setting

$$G^1 = G\mathcal{I}, \tag{2.70}$$

one observes that $F = G\mathcal{H} = G^1\mathcal{H}^1$ and the subsequent analysis can indeed be done with either factorization. We prefer the first one since it leads to explicit expressions for the middle operator in the second factorization introduced below. We now state the following reciprocity lemma, which can be proved in exactly the same way as in Lemma 2.2.

**Lemma 2.38.** *Let $(\boldsymbol{\varphi}_0, \psi_0)$ and $(\boldsymbol{\varphi}_1, \psi_1)$ be as in $L^2(D)^3 \times L^2(D)$, and let $w_0$ and $w_1 \in H^1_{loc}(\mathbb{R}^3)$ be the corresponding solutions satisfying* (2.64). *Then*

$$\int_D ((I-A)\nabla w_0 \cdot \boldsymbol{\varphi}_1 - k^2(1-n) w_0 \psi_1)\, dx = \int_D ((I-A)\nabla w_1 \cdot \boldsymbol{\varphi}_0 - k^2(1-n) w_1 \psi_0)\, dx.$$

The following theorem gives one of the main ingredients for the justification of LSM and GLSM.

**Theorem 2.39.** *Assume that Assumption* 2.5 *holds. Then the operator* $G : H_{\mathrm{inc}}(D) \to L^2(S^2)$ *defined by* (2.66) *is injective with dense range. Moreover,* $\Phi_\infty(\cdot, z) \in \mathcal{R}(G)$ *if and only if* $z \in D$. *The same holds for* $G^1 : H^1_{\mathrm{inc}}(D) \to L^2(S^2)$ *defined by* (2.70).

***Proof.*** We prove the result for $G^1$. The result for $G$ then directly follows from (2.70). The proof is very similar to the proof of Theorem 2.3, and we give here a short outline. Let $(\boldsymbol{\varphi}, \psi) = \mathcal{I}(u_0)$ with $u_0 \in H^1_{\mathrm{inc}}(D)$ and $w$ satisfying (2.2). From (1.56) we get

$$w_\infty(\hat{x}) = -\frac{1}{4\pi} \int\limits_D \big(ik\hat{x} \cdot (I - A)\nabla(u_0 + w) + k^2(1 - n)(u_0 + w)\big)\, e^{-ik\hat{x}\cdot y}\, dy.$$

It is then easy to deduce from Lemma 2.38 that

$$(G^1(\overline{\mathcal{H}^1\varphi}), g)_{L^2(S^2)} = (G^1(\overline{\mathcal{H}^1 g}), \varphi)_{L^2(S^2)} \quad \text{for all } g, \varphi \in L^2(S^2). \tag{2.71}$$

Using this identity, the remainder of the proof can be copied line by line from the proof of Theorem 2.3 after identity (2.14), replacing $G$ and $\mathcal{H}$ by $G^1$ and $\mathcal{H}^1$, respectively, and substituting references to the interior transmission problem (2.6) with references to the interior transmission problem (2.67) with appropriate changes of solution spaces. ∎

We proceed now with the second factorization of the far field operator. From (1.56) we obtain

$$G(\boldsymbol{\varphi}, \psi) = -\frac{1}{4\pi} \int\limits_D \big(ik\hat{x} \cdot (I - A)(\boldsymbol{\varphi} + \nabla w) + k^2(1 - n)(\psi + w)\big)\, e^{-ik\hat{x}\cdot y}\, dy.$$

Using (2.68) we get that $G = \mathcal{H}^* T$, where $T$: $L^3(D) \times L^2(D) \to L^3(D) \times L^2(D)$ is defined by

$$T(\boldsymbol{\varphi}, \psi) := -\frac{1}{4\pi} \big((A - I)(\boldsymbol{\varphi} + \nabla w),\, k^2(1 - n)(\psi + w)\big) \tag{2.72}$$

with $w$ being the solution of (2.64). One then ends up with the second factorization

$$F = \mathcal{H}^* T \mathcal{H}. \tag{2.73}$$

We now give the final additional theorem needed for GLSM and the inf-criterion, which is the following coercivity property of the operator $T$.

**Assumption 2.6.** *We assume that* $n \in L^\infty(\mathbb{R}^3)$, $\Im(n) \geq 0$, $A \in L^\infty(\mathbb{R}^3)^6$, *and* $\Im(A) \leq 0$. *Furthermore, we assume that either of the following conditions applies:*

- $\Re(A - I) - \alpha\Im(A)$ *is positive definite on* $D$ *for some constant* $\alpha \geq 0$.
- $\Re(A)$ *is positive definite on* $D$ *and there exist constants* $\alpha \geq 0$, $0 < \eta \leq 1$, *and* $\theta > 0$ *such that*

$$(I - \Re(A))X \cdot \overline{X} + (1 - \eta)\Re(A) Y \cdot \overline{Y} - \alpha\Im(A)(X + Y)\cdot(\overline{X} + \overline{Y}) \geq \theta |X|^2 \tag{2.74}$$

*on* $D$ *for all* $X$ *and* $Y$ *in* $\mathbb{C}^3$.

**Theorem 2.40.** *Assume that Assumptions* 2.5 *and* 2.6 *hold. Then the operator* $T$ *defined by* (2.72) *satisfies the coercivity property*

$$|(T\mathcal{I}(v), \mathcal{I}(v))_{L^2(D)^4}| \geq \theta \|v\|^2_{H^1(D)} \text{ for all } v \in H^1_{\mathrm{inc}}(D) \tag{2.75}$$

*for some positive constant $\theta$. This implies in particular that $T$ satisfies* (2.22) *with $Y = Y^* = L^3(D) \times L^2(D)$ and the operator $H = \mathcal{H}$ defined by* (2.65).

***Proof.*** With $(\ ,\ )$ denoting the $L^2(D)^4$ scalar product, for $(\boldsymbol{\varphi}, \psi) = \mathcal{I}(v)$, $v \in H^1_{\mathrm{inc}}(D)$, and $w \in H^1_{loc}(\mathbb{R}^3)$ a solution of (2.64), we have that

$$(T\mathcal{I}(v), \mathcal{I}(v)) = -\frac{1}{4\pi} \int_D ((A-I)\nabla(v+w) \cdot \nabla\overline{v} + k^2(1-n)(v+w)\overline{v})\, dx. \quad (2.76)$$

From the variational formulation of (2.64) (see, for instance, (1.55)) with test function equal to $w$ and $B_R$ a ball of radius $R$ containing $D$, we get that

$$\int_D ((A-I)\nabla(v+w) \cdot \nabla\overline{w} + k^2(1-n)(v+w)\overline{w})\, dx$$
$$= -\int_{B_R} (|\nabla w|^2 - k^2|w|^2)\, dx + \int_{|x|=R} \frac{\partial w}{\partial r}\overline{w}\, ds. \quad (2.77)$$

We recall that, due to the Sommerfeld radiation condition,

$$\lim_{R\to\infty} \Im\left( \int_{|x|=R} \frac{\partial w}{\partial r}\overline{w}\, ds \right) = k \int_{\mathbb{S}^2} |w_\infty|^2 ds.$$

Therefore, taking the imaginary part and letting $R \to \infty$ yields

$$\Im \int_D ((A-I)\nabla(v+w) \cdot \nabla\overline{w} + k^2(1-n)(v+w)\overline{w})\, dx = k \int_{\mathbb{S}^2} |w_\infty|^2 ds.$$

Consequently, using the identities

$$(v+w)\overline{v} = |v+w|^2 - (v+w)\overline{w},$$

$$(A-I)\nabla(v+w) \cdot \nabla\overline{v} = (A-I)\nabla(v+w) \cdot \nabla\overline{(v+w)} - (A-I)\nabla(v+w) \cdot \nabla\overline{w}$$

in (2.76) and taking the imaginary part implies (the general form of (2.35))

$$4\pi\Im(T\mathcal{I}(v), \mathcal{I}(v)) = \int_D -\Im(A)\nabla(v+w) \cdot \nabla(\overline{v}+\overline{w})dx$$
$$+ k^2 \int_D \Im(n)|v+w|^2 dx + k \int_{\mathbb{S}^2} |w_\infty|^2 ds. \quad (2.78)$$

We are now in position to prove the desired coercivity property using a contradiction argument. Assume, for instance, the existence of a sequence $v_\ell \in \mathcal{R}(H)$ such that

$$\|v_\ell\|_{H^1(D)} = 1 \quad \text{and} \quad |(T\mathcal{I}(v_\ell), \mathcal{I}(v_\ell))| \to 0 \text{ as } \ell \to \infty.$$

We denote by $w_\ell \in H^1_{loc}(\mathbb{R}^3)$ the solution of (2.64) with $(\boldsymbol{\varphi}, \psi) = \mathcal{I}(v_\ell)$. Elliptic regularity implies that $\|w_\ell\|_{H^2(B\setminus\overline{D})}$ is bounded uniformly with respect to $\ell$ for all bounded domains $B$ containing $D$. Then up to changing the initial sequence, one can assume that $v_\ell$ weakly converges to some $v$ in $H^1(D)$ and $w_\ell$ converges weakly in $H^1_{loc}(\mathbb{R}^3) \cap H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$ to some $w \in H^1_{loc}(\mathbb{R}^3) \cap H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$. It is then easily seen that $w$ and $(\boldsymbol{\varphi}, \psi) = \mathcal{I}(v)$ satisfy (2.64), and

$$\Delta v + k^2 v = 0 \quad \text{in } D. \tag{2.79}$$

Identity (2.78) and $|(T\mathcal{I}(v_\ell), \mathcal{I}(v_\ell))| \to 0$ imply that $w^\ell_\infty \to 0$ in $L^2(\mathbb{S}^2)$ and therefore $w_\infty = 0$. Rellich's Lemma implies $w = 0$ outside $D$. Consequently, $u = w + v \in H^1(D)$ and $v \in H^1(D)$ form a solution to the interior transmission problem (2.67) with $f = g = 0$. This implies that $w = v = 0$. Identity (2.76) applied to $v_\ell$ and $w_\ell$, the fact that $|(T\mathcal{I}(v_\ell), \mathcal{I}(v_\ell))| \to 0$, and the Rellich compact embedding theorem imply that

$$\int_D (A - I)\nabla(v_\ell + w_\ell) \cdot \nabla \overline{v}_\ell dx \to 0 \tag{2.80}$$

as $\ell \to \infty$. From (2.77) applied to $v_\ell$ and $w_\ell$ and the Rellich compact embedding theorem we get

$$\int_D (A - I)\nabla(v_\ell + w_\ell) \cdot \nabla \overline{w}_\ell + \int_{B_R} |\nabla w_\ell|^2 dx \to 0 \tag{2.81}$$

as $\ell \to \infty$. We now consider two separate cases. Consider first the case when $\Re(A - I) - \alpha\Im(A)$ is positive definite on $D$ for some constant $\alpha \geq 0$. Taking the sum of (2.80) and (2.81) we get

$$\int_D (A - I)\nabla(v_\ell + w_\ell) \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) dx + \int_{B_R} |\nabla w_\ell|^2 dx \to 0 \tag{2.82}$$

as $\ell \to \infty$. On the other hand, using the assumption on $A$ (after adding and subtracting $\alpha\Im(A)$ to $\Re(A - I)$), we easily observe that

$$\theta \left( \int_D |\nabla(v_\ell + w_\ell)|^2 dx + \int_{B_R} |\nabla w_\ell|^2 dx \right)$$
$$\leq \left| \int_D (A - I)\nabla(v_\ell + w_\ell) \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) dx + \int_{B_R} |\nabla w_\ell|^2 \right| dx$$

for some positive constant $\theta$ independent of $\ell$. We then obtain using the triangle inequality that $\|\nabla v_\ell\|_{L^2(D)} \to 0$. Combined with the Rellich compact embedding theorem, this implies that $v_\ell \to 0$ strongly in $H^1(D)$, which gives a contradiction.

Consider now the case when (2.74) holds and $\Re(A)$ is positive definite on $D$. Taking the difference between (2.81) and (2.80) yields

$$\int_D (I - A)\nabla v_\ell \cdot \nabla \overline{v}_\ell dx + \int_{B_R} A\nabla w_\ell \cdot \nabla \overline{w}_\ell dx$$
$$+ \int_D ((I - A)\nabla w_\ell \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) - (I - A)\nabla \overline{w}_\ell \cdot \nabla(v_\ell + w_\ell))\, dx \to 0$$

and taking the real part implies

$$\int_D (I - \Re(A))\nabla v_\ell \cdot \nabla \overline{v}_\ell dx + \int_{B_R} \Re(A)\nabla w_\ell \cdot \nabla \overline{w}_\ell dx$$
$$- i \int_D \left(\Im(A)\nabla w_\ell \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) - \Im(A)\nabla \overline{w}_\ell \cdot \nabla(v_\ell + w_\ell)\right) dx \to 0.$$

Taking the imaginary part of (2.82) implies that

$$- \int_D \Im(A)\nabla(v_\ell + w_\ell) \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) dx \to 0.$$

Now let $\lambda$ be a positive parameter that will be fixed later. The last two identities give

$$\int_D (I - \Re(A))\nabla v_\ell \cdot \nabla \overline{v}_\ell dx + \int_D \Re(A)\nabla w_\ell \cdot \nabla \overline{w}_\ell dx - \lambda \int_D \Im(A)\nabla(v_\ell + w_\ell) \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) dx$$
$$- i \int_D \left(\Im(A)\nabla w_\ell \cdot \nabla(\overline{v}_\ell + \overline{w}_\ell) - \Im(A)\nabla \overline{w}_\ell \cdot \nabla(v_\ell + w_\ell)\right) dx \to 0.$$

Let us denote by $M(\nabla v_\ell, \nabla w_\ell)$ the term under the integral over $D$ in this identity. We observe that

$$M(X, Y) = (I - \Re(A))X \cdot \overline{X} + (1 - \eta)\Re(A)Y \cdot \overline{Y} - \lambda\Im(A)(X + Y) \cdot (\overline{X} + \overline{Y})$$
$$+ |(\eta\Re(A))^{1/2} Y + i(\eta\Re(A))^{-1/2}\Im(A)(X + Y)|^2 - |(\eta\Re(A))^{-1/2}\Im(A)(X + Y)|^2.$$

Choosing

$$\lambda > \alpha + \sup_{x \in D} \|\Im(A)(x)\|/(\eta\|\Re(A)(x)\|)$$

we obtain from assumption (2.74) that

$$M(X, Y) \geq \theta|X|^2.$$

This implies that $\|\nabla v_\ell\|_{L^2(D)} \to 0$ and therefore yields a contradiction as in the first case. □

In view of Theorems 2.40 and 2.39 we now can state the following application of Corollary 2.8 and Theorem 2.14.

**Theorem 2.41.** *Assume that Assumptions* 2.5 *and* 2.6 *hold. Then the results of Theorems* 2.4, 2.18*, and* 2.19 *hold true in the present case.*

For the factorization method, a splitting of the real part of the operator $T$ into a coercive real operator and a compact operator is needed.

Let $B_R$ be a ball of radius $R$ containing $D$. With the notation of the proof of Theorem 2.40, if $w \in H^1_{loc}(\mathbb{R}^3)$ (respectively, $w' \in H^1_{loc}(\mathbb{R}^3)$) is the solution of 2.64 with

$(\boldsymbol{\varphi}, \psi) = \mathcal{I}(v)$ (respectively, $(\boldsymbol{\varphi}, \psi) = \mathcal{I}(v'))$ and $v \in H^1_{\mathrm{inc}}(D)$ (respectively, $v' \in H^1_{\mathrm{inc}}(D)$), then

$$(T\,\mathcal{I}(v), \mathcal{I}(v')) = -\frac{1}{4\pi}\int\limits_D ((A-I)\nabla(v+w)\cdot\nabla\overline{v}' + k^2(1-n)(v+w)\overline{v}')\,dx \quad (2.83)$$

and from the variational formulation of (2.64) (see, for instance, (1.55)) with test function equal to $w'$,

$$\int\limits_D ((A-I)\nabla(v+w)\cdot\nabla\overline{w}' + k^2(1-n)(v+w)\overline{w}')\,dx \\ + \int\limits_{B_R}(\nabla w\cdot\nabla\overline{w}' - k^2 w\overline{w}')\,dx - \int\limits_{|x|=R}\frac{\partial w}{\partial r}\overline{w}'\,ds = 0. \quad (2.84)$$

Consequently, adding (2.84) to $-4\pi$ times (2.83) gives

$$-4\pi(T\,\mathcal{I}(v), \mathcal{I}(v')) = \int\limits_D (A-I)\nabla(v+w)\cdot(\nabla\overline{v}' + \nabla\overline{w}')dx + \int\limits_{B_R}\nabla w\cdot\nabla\overline{w}'dx \\ + \int\limits_D k^2(1-n)(v+w)(\overline{v}'+\overline{w}')\,dx - \int\limits_{B_R} k^2 w\overline{w}'dx - \int\limits_{|x|=R}\frac{\partial w}{\partial r}\overline{w}'\,ds. \quad (2.85)$$

Adding (2.84) to $4\pi$ times (2.83) implies

$$4\pi(T\,\mathcal{I}(v), \mathcal{I}(v')) = -\int\limits_D (A-I)\nabla(v+w)\cdot(\nabla\overline{v}' - \nabla\overline{w}')dx - \int\limits_{B_R}\nabla w\cdot\nabla\overline{w}'dx \\ - \int\limits_D k^2(1-n)(v+w)(\overline{v}'-\overline{w}')\,dx + \int\limits_{B_R} k^2 w\overline{w}'dx + \int\limits_{|x|=R}\frac{\partial w}{\partial r}\overline{w}'\,ds$$

and rearranging the terms on the right-hand side we get

$$4\pi(T\,\mathcal{I}(v), \mathcal{I}(v')) = \int\limits_D (I-A)\nabla v\cdot\nabla\overline{v}'dx + \int\limits_{B_R} A\nabla w\cdot\nabla\overline{w}'dx \\ + \int\limits_D ((I-A)\nabla w\cdot\nabla\overline{v}' - (I-A)\nabla w'\cdot\nabla\overline{v})\,dx \\ - \int\limits_D k^2(1-n)(v+w)(\overline{v}'-\overline{w}')\,dx + \int\limits_{B_R} k^2 w\overline{w}'dx + \int\limits_{|x|=R}\frac{\partial w}{\partial r}\overline{w}'\,ds. \quad (2.86)$$

Let us introduce the operators $T_0^{\pm} : L^2(D)^3 \times L^2(D) \to L^2(D)^3 \times L^2(D)$ such that

$$-4\pi(T_0^-\,\mathcal{I}(v), \mathcal{I}(v')) = \int\limits_D (A-I)\nabla(v+w)\cdot(\nabla\overline{v}'+\nabla\overline{w}')dx + \int\limits_{B_R}\nabla w\cdot\nabla\overline{w}'dx + \int\limits_D v\overline{v}'dx \quad (2.87)$$

and

$$4\pi(T_0^+ \mathcal{I}(v), \mathcal{I}(v')) = \int_D (I - A)\nabla v \cdot \nabla \overline{v}' dx + \int_{B_R} A\nabla w \cdot \nabla \overline{w}' dx$$
$$+ \int_D ((I - A)\nabla w \cdot \nabla \overline{v}' - (I - A)\nabla \overline{w}' \cdot \nabla v)\, dx + \int_D v\overline{v}' dx. \quad (2.88)$$

Then, from the fact that $w, w' \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{D})$ and the Rellich compact embedding theorems, one easily concludes that

$$T - T_0^{\pm} : H^1_{\mathrm{inc}}(D) \to L^2(D)^3 \times L^2(D)$$

is compact. We already see from the expression of $T_0^+$ that the case of $I - A$ positive definite on $D$ is more delicate to analyze since $T_0^+$ is not self-adjoint, nor can it be written as the sum of self-adjoint and compact operators. For instance, one cannot apply the $(F^*F)^{1/4}$ method in this case. However, in the case when $A$ is real and $A - I$ is positive definite on $D$ we can state the following.

**Theorem 2.42.** *Assume that $A$ and $n$ are real valued, $A - I$ is positive definite on $D$, and $k$ is not a transmission eigenvalue. Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of $(F^*F)^{1/4}$.*

***Proof.*** We recall that in this case the operator $F$ is normal (Theorem 1.43). One easily sees from (2.87) that $T_0^-$ is self-adjoint and coercive on $H^1_{\mathrm{inc}}(D)$. Moreover, since $k$ is not a transmission eigenvalue, we have that $F$ is injective with dense range, and from the first part of the proof of Theorem 2.40 we get that $\Im(T)$ is positive. We then conclude the result using Theorem 2.24 and Lemma 2.43 □

**Lemma 2.43.** *For $z \in \mathbb{R}^3$ we have that $z \in D$ if and only if $\phi_z$ is in the range of $\mathcal{H}^*$.*

***Proof.*** This lemma is a simple consequence of Lemma 2.21 since $\mathcal{H}^*(0, \cdot)$ coincides with the operator $\mathcal{H}^*$ in Lemma 2.21. □

We now consider the $F_\sharp$ method. Once again, the case $A - I$ nonnegative can be treated in a similar way as in the case $A = I$. With the notation of Section 2.4.4 we have the following theorem.

**Theorem 2.44.** *Assume that there exists $\theta \in [-\pi/2, 0]$ such that $\Re(e^{i\theta}(A - I))$ is positive definite in $D$. Assume in addition that $k$ is not a transmission eigenvalue. Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of $(F_\sharp^\theta)^{1/2}$.*

***Proof.*** The case $\theta = -\pi/2$ is the case where $\Im(A)$ is positive definite in $D$. Then using (2.35) one gets that $T^\theta = \Im(T)$ satisfies Assumption 2.4 with $Y = Y^* = L^2(D)^4$. For the case $\theta \neq -\pi/2$, we get from (2.87) that the operator $\Re(e^{i\theta} T_0^-)$ is coercive on $H^1_{\mathrm{inc}}(D)$. As in the proof of Lemma 2.25, $\Im(T^\theta)$ is injective on the range $\overline{\mathcal{R}(\mathcal{H})}$ since $k$ is not a transmission eigenvalue. Assumption 2.4 is then verified. □

In the case $A - I$ nonpositive we content ourselves with the following result, assuming that the imaginary part is not too large.

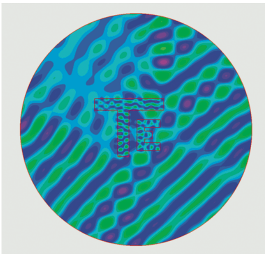
**Theorem 2.45.** *Assume that $k$ is not a transmission eigenvalue, $\Re(I-A)\xi\cdot\bar{\xi}\geq\alpha|\xi|^2$ and $\Re(A)\xi\cdot\bar{\xi}\geq\gamma|\xi|^2$ for all $\xi\in\mathbb{C}^3$ in $D$. Assume in addition that $\|\Im(A)\|_{L^\infty(D)}<\sqrt{\alpha\gamma}$. Then $z\in D$ if and only if $\Phi_\infty(\cdot,z)$ is in the range of $(F_\sharp)^{1/2}$.*

***Proof.*** We observe from (2.88) that

$$4\pi\Re(T_0^+\mathcal{I}(v),\mathcal{I}(v))=\int\limits_D\Re(I-A)\nabla v\cdot\nabla\overline{v}dx+\int\limits_{B_R}\Re(A)\nabla w\cdot\nabla\overline{w}dx$$
$$-i\int\limits_D\left(\Im(A)\nabla w\cdot\nabla\overline{v}-\Im(A)\nabla\overline{w}\cdot\nabla v\right)dx+\int\limits_D v\overline{v}dx.$$

The assumptions on $A$ then ensure that $\Re(T_0^+)$ is coercive on $H^1_{\mathrm{inc}}(D)$. We then conclude as in the proof of Theorem 2.44. □

The conditions of Theorem 2.44 can be weakened but at the expense of changing the expression for $F_\sharp$ (adding a sufficiently large imaginary part). This is left as an exercise to the reader.



# Chapter 3
# The Interior Transmission Problem

The interior transmission problem, as already mentioned in Chapter 2, plays an essential role in inverse scattering theory for inhomogeneous media. It is a boundary value problem for a coupled pair of partial differential equations in a bounded domain which corresponds to the support of the scatterer. This boundary value problem is not elliptic in the sense of Agmon–Douglas–Nirenberg and hence its study calls for new techniques. The homogeneous form of the interior transmission problem is referred to as the transmission eigenvalue problem and the corresponding eigenvalues as transmission eigenvalues. Typical concerns associated with these problems are (1) the *Fredholm property and solvability* of the interior transmission problems, (2) the *discreteness* of the transmission eigenvalues, (3) the *existence* of transmission eigenvalues, and (4) the *determination* of transmission eigenvalues from scattering data and the relationship between them and the material properties of the inhomogeneous medium. All these questions are at the core of inverse scattering theory. This chapter is concerned with the Fredholm property and solvability of the interior transmission problem corresponding to different kinds of inhomogeneous media.

We discuss in Section 3.1 the isotropic problem and more specifically the simple case where the contrast $n-1$ does not change sign in $D$. In this case a formulation of the problem as a fourth order partial differential equation can be obtained and then studied variationally. This approach that was first employed in [154] is also very convenient for the study of the existence of transmission eigenvalues, which is the subject of the next chapter. We then discuss in Section 3.1.2 the more delicate case where $n-1$ can vanish in a region strictly included in $D$. In this case, one can still derive a variational formulation similar to the previous case by including the equations in the region $n=1$ as a constraint in the variational space. This section can be skipped in a first reading. A more general problem is discussed in Section 3.1.3 where the contrast may change sign in a domain strictly contained in $D$. This case was first investigated in [159] (see also the approach in [124] for smooth coefficients). Our discussion follows the approach due to Kirsch in [111] where the same results as in [159] are obtained for a real valued refractive index by using a variational approach. Contrary to the case with voids, this approach cannot fit into the analytical framework developed in the next chapter to study existence of transmission eigenvalues. We introduce in Section 3.1.4 an alternative approach to study the interior transmission problem (3.1) based on boundary integral equations. Although the boundary integral method recovers the same type of solvability results discussed in Section 3.1.3 we believe that it merits discussion in this monograph for its mathematical and computational interest. Our presentation closely follows [79]. This section can also be skipped in a first reading.

The anisotropic problem is considered in Section 3.2. When a contrast is present in the main operator, the functional framework for the interior transmission problem becomes different, and hence a different approach is used to treat this case. As for the isotropic problem, we first consider the simpler case where a contrast sign is the same in all of $D$ as presented in [31] and [45]. This configuration is treated in Section 3.2.1 first in the case $n = 1$ and second in the case $n \neq 1$, where the functional framework is different. The case where the anisotropic contrast changes sign inside $D$ is treated using the T-coercivity approach as in [24] and [60]. We also refer the reader to [125] for methods based on elliptic theory for partial differential equations.

The differences in the treatment of the isotropic and anisotropic cases clearly indicate that the study of the problem where both configurations are mixed on the boundary is more difficult and would require new approaches.

## 3.1 ▪ Solvability of the Interior Transmission Problem for Isotropic Media

Let $D \subset \mathbb{R}^3$ be the support of an isotropic inhomogeneous media with refractive index $n \in L^\infty(D)$ such that $\Re(n) \geq n_0 > 0$ and $\Im(n) \geq 0$. Throughout this chapter, we assume that $\partial D$ is Lipschitz unless otherwise indicated. The interior transmission problem corresponding to the scattering problem for this isotropic inhomogeneous medium was already introduced in (2.6). Here we recall it for the reader's convenience: Given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ find $w \in L^2(D)$, $v \in L^2(D)$ with $w - v \in H^2(D)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w - v = f & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \partial D, \end{cases} \tag{3.1}$$

where the equations for $w$ and $v$ are understood in the distributional sense and the boundary conditions are well defined for the difference $w - v$.

**Definition 3.1.** Values of $k \in \mathbb{C}$ for which the homogeneous interior transmission problem

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \partial D \end{cases} \tag{3.2}$$

has nontrivial solutions $w \in L^2(D)$ and $v \in L^2(D)$, such that $w - v \in H_0^2(D)$, are called transmission eigenvalues.

At first glance it seems unclear why we are not formulating the problem in the usual energy space $H^1(D)$. However, there is a simple observation which indicates that the interior transmission problem does not fit into the standard framework of partial differential equations of the second order. For simplicity assume that $f = 0$. Then we multiply the first equation by a test function $\varphi$ and the second equation by a test function $\psi$ such that $\varphi = \psi$ on $\partial D$, integrate by parts, and use the boundary condition to obtain

$$-\int\limits_D \nabla w \cdot \nabla\overline{\varphi}\, dx + \int\limits_D \nabla v \cdot \nabla\overline{\psi}\, dx + k^2 \int\limits_D (nw\overline{\varphi} - v\overline{\psi})\, dx = -\int\limits_{\partial D} h\overline{\varphi}. \tag{3.3}$$

Obviously, this cannot be a compact perturbation of a coercive antilinear form due to the fact that the norm of the gradient of $w$ and $v$ appear with different signs. Hence in the isotropic case the standard variational approach for elliptic equations does not apply to the above variational equation in the energy space $H^1(D)$. We remark that if there is contrast in the main operator (i.e., in the anisotropic case that will be discussed in Section 3.2), the corresponding $H^1(D)$ variational formulation leads to a compact perturbation of a coercive problem under some kind of sign control on the contrast. Furthermore, it is easy to find a function in $L^2(D)$ with its gradient not in $L^2(D)$, and this function satisfies both equations in (3.1) with right-hand sides and zero boundary data, meaning that in general solutions to (3.1) can simply be in $L^2(D)$. As will become clear as we proceed with our discussion, the interior transmission problem (3.1) essentially depends on the contrast $n-1$ of the medium and different analytical techniques are needed to study it depending on the assumptions on $n-1$.

Given the structure of the boundary conditions in (3.1), it makes sense to introduce the difference $u := w - v$ as a new unknown and try to obtain an equation for $u$. Indeed, subtracting the second equation from the first, we have that

$$\Delta u + k^2 nu = -k^2(n-1)v \qquad \text{in } D, \tag{3.4}$$

which should be considered together with

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{3.5}$$

and the boundary conditions

$$u = f \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = h \qquad \text{on } \partial D. \tag{3.6}$$

To eliminate $v$ we should be able to divide by $n-1$ and then apply the Helmholtz operator. This motivates us to consider in the following the case when the division by $n-1$ is possible, i.e., $n-1$ is bounded away from zero.

## 3.1.1 ▪ The Case of One Sign Contrast

We start by assuming that the real part of the contrast $n-1$ does not change sign in $D$, more specifically, either $\Re(n(x)) - 1 \geq \alpha > 0$ or $1 - \Re(n(x)) \geq \alpha > 0$ for almost all $x \in D$ and some $\alpha > 0$. Letting

$$n_* = \inf_D \Re(n) \qquad \text{and} \qquad n^* = \sup_D \Re(n), \tag{3.7}$$

the above assumption means that either $n_* > 1$ or $0 < n^* < 1$. Under this assumption it is now possible to write (3.1) as a boundary value problem for the fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \quad \text{in } D, \tag{3.8}$$

$$u = f \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = h \qquad \text{on } \partial D, \tag{3.9}$$

where it is assumed that $u := w - v \in H^2(D)$. The functions $v$ and $w$ are related to $u$ through

$$v = -\frac{1}{k^2(n-1)}(\Delta u + k^2 nu) \quad \text{and} \quad w = -\frac{1}{k^2(n-1)}(\Delta u + k^2 u). \tag{3.10}$$

This fourth order formulation of the interior transmission problem was first introduced in [154] and later used in [44], [46], and [144] (see also [47]). For given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ let $\theta \in H^2(D)$ be a lifting function [131] such that $\theta = f$ and $\partial\theta/\partial\nu = h$ on $\partial D$ and $\|\theta\|_{H^2(D)} \le c\big(\|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)}\big)$ for some $c > 0$. Then letting $u_0 := u - \theta \in H_0^2(D)$, we can write (3.8)–(3.9) as an equivalent variational problem for $u_0$: Find a function $u_0 \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u_0 + k^2 u_0)(\Delta\overline{\psi} + k^2 n\overline{\psi})\, dx = \int_D \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta\overline{\psi} + k^2 n\overline{\psi})\, dx \tag{3.11}$$

for all $\psi \in H_0^2(D)$. Obviously,

$$F : \psi \mapsto \int_D \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta\overline{\psi} + k^2 n\overline{\psi})\, dx$$

is a bounded antilinear functional on $H_0^2(D)$. Let $\ell \in H_0^2(D)$ be such that $F(\psi) = (\ell, \psi)_{H^2(D)}$ for all $\psi \in H_0^2(D)$, which is uniquely provided by the Riesz representation theorem and satisfies

$$\|\ell\|_{H^2(D)} \le c_1\|\theta\|_{H^2(D)} \le c_2\left(\|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)}\right). \tag{3.12}$$

Problem (3.11), and hence the original interior transmission problem (3.1), is equivalent to the following operator equation in $H_0^2(D)$ for $u_0$,

$$\mathbb{T}u_0 - k^2\mathbb{T}_1 u_0 + k^4\mathbb{T}_2 u_0 = \ell, \tag{3.13}$$

where $\mathbb{T} : H_0^2(D) \to H_0^2(D)$, $\mathbb{T}_1 : H_0^2(D) \to H_0^2(D)$, and $\mathbb{T}_2 : H_0^2(D) \to H_0^2(D)$ are the bounded linear operators defined by the mean of the Riesz representation theorem as

$$(\mathbb{T}u, \psi)_{H^2(D)} = \int_D \frac{1}{n-1}\Delta u\, \Delta\overline{\psi}\, \mathrm{d}x \qquad \text{for all } u, \psi \in H_0^2(D), \tag{3.14}$$

$$\begin{aligned}(\mathbb{T}_1 u, \psi)_{H^2(D)} &= -\int_D \frac{1}{n-1} u\, \Delta\overline{\psi}\, dx - \int_D \frac{n}{n-1}\Delta u\, \overline{\psi}\, dx \\ &= -\int_D \frac{1}{n-1}\left(\Delta u\, \overline{\psi} + u\, \Delta\overline{\psi}\right) dx + \int_D \nabla u \cdot \nabla\overline{\psi}\, dx \qquad \text{for all } u, \psi \in H_0^2(D),\end{aligned} \tag{3.15}$$

$$(\mathbb{T}_2 u, \psi)_{H^2(D)} = \int_D \frac{n}{n-1} u\, \overline{\psi}\, dx \qquad \text{for all } u, \psi \in H_0^2(D). \tag{3.16}$$

The operator $\mathbb{T}$ in the case of $n_* > 1$ (or $-\mathbb{T}$ in the case of $0 < n^* < 1$) is coercive since when $1 < n_* \le \Re(n) \le n^*$

$$\Re\,(\mathbb{T}u, u)_{H^2(D)} \ge \frac{1}{n^* - 1}(\Delta u, \Delta u)_{L^2(D)} \ge c\|u\|_{H^2(D)}$$

(with a similar calculation when $0 < n^* < 1$), where we have used that, for $u \in H_0^2(D)$, $\|u\|_{H^2(D)}$ is equivalent to $\|\Delta u\|_{L^2(D)}$ [131]. Furthermore, the bounded linear operators $\mathbb{T}_1$ and $\mathbb{T}_2$ are compact, which is a consequence of the compact embedding of $H_0^2(D)$ in $L^2(D)$. For the reader's convenience we prove the compactness of $\mathbb{T}_1$. Indeed for the part $\mathbb{T}_1^{(1)}$ of the operator $\mathbb{T}_1$ given by the first integral in (3.15) we have

$$\|\mathbb{T}_1^{(1)} u\|_{H^2} = \sup_{0 \neq \psi \in H^2} \frac{1}{\|\psi\|_{H^2}} \left| \int_D \frac{1}{n-1} u \, \Delta \overline{\psi} \, dx \right| \leq C \|u\|_{L^2},$$

and hence for a sequence $\{u_n\}$ bounded in $H^2(D)$, thanks to the compact embedding of $H_0^2(D)$ in $L^2(D)$, we obtain that a subsequence of $\left\{\mathbb{T}_1^{(1)} u_n\right\}$ converges strongly in $H^2(D)$. The second integral in (3.15) yields the same result (consider the adjoint). Hence we can conclude that $\mathbb{T}_1$ is compact. Exactly the same reasoning holds for $\mathbb{T}_2$. Thus we can conclude that the Fredholm alternative can be applied to (3.13); in particular, uniqueness implies the existence of a unique solution. The homogeneous equation

$$(\mathbb{T} - k^2 \mathbb{T}_1 + k^4 \mathbb{T}_2) u = 0 \tag{3.17}$$

is equivalent to the transmission eigenvalue problem (see Definition 3.1).

We have now proven the following theorem concerning the solvability of the interior transmission problem (3.1) in the case when $n \in L^\infty(D)$, such that $\Re(n) \geq n_0 > 0$ and $\Im(n) \geq 0$, and either $n_* > 1$ or $n^* < 1$, where $n_*$ and $n^*$ are given by (3.7).

**Theorem 3.2.** *Assume that $k \in \mathbb{C}$ is not a transmission eigenvalue. Then for any given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ there exists a unique solution of the interior transmission problem* (3.1) *such that $w \in L^2(D)$, $v \in L^2(D)$, $u := w - v \in H^2(D)$, and*

$$\|w\|_{L^2(D)} + \|v\|_{L^2(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right)$$

*for some positive constant $C > 0$, with a similar estimate for $\|u\|_{H^2(D)}$.*

**Theorem 3.3.** *If $n \in L^\infty(D)$ is such that $\Im(n) > 0$ almost everywhere in region $D_0 \subset D$ with positive measure, then there are no* real *transmission eigenvalues.*

***Proof.*** Assume that $w$ and $v$ solve the transmission eigenvalue problem corresponding to a real transmission eigenvalue $k$, i.e., $u := w - v \in H_0^2(D)$ solves (3.11) with $\theta = 0$. Taking $\psi = u$ in (3.11) and regrouping the terms yields

$$\int_D \frac{1}{n-1} |\Delta u + k^2 u|^2 \, dx + k^4 \int_D |u|^2 \, dx - k^2 \int_D |\nabla u|^2 \, dx = 0.$$

Since $\Im(1/(n-1)) < 0$ in $D_0$ and all the terms in the above equation are real except for the first one, by taking the imaginary part we obtain that $\Delta u + k^2 u = 0$ in $D_0$ and hence, from (3.10), $w = 0$ in $D_0$. By Weyl's lemma $w \in H^2_{loc}(D)$. The unique continuation principle implies that $w = 0$ in $D$. Therefore since $w - v = u$ we have that the Cauchy datum of $v = -u$ is zero on $\partial D$, which finally implies that also $v = 0$ in $D$. Hence $k$ real is not a transmission eigenvalue. ☐

**Theorem 3.4.** *Assume that $n \in L^\infty(D)$ such that $\Re(n) \geq n_0 > 0$, $\Im(n) \geq 0$, and either $n_* > 1$ or $n^* < 1$, where $n_*$ and $n^*$ are given by* (3.7). *Then the set of transmission* *eigenvalues $k \in \mathbb{C}$ is discrete (possibly empty) with $+\infty$ as the only possible accumulation point. The multiplicity of the eigenvalues is finite with finite-dimensional generalized eigenspaces.*

***Proof.*** As discussed above, $k \in \mathbb{C}$ is a transmission eigenvalue if and only if

$$\mathbb{T}u - \tau\mathbb{T}_1 u + \tau^2\mathbb{T}_2 u = 0 \tag{3.18}$$

has nonzero solution $u \in H_0^2(D)$, where $\mathbb{T}$, $\mathbb{T}_1$, and $\mathbb{T}_2$ are defined by (3.14), (3.15), and (3.16), respectively, where we set $\tau := k^2$.

Assume first that $\Im(n) > 0$ almost everywhere in a region $D_0 \subset D$ with positive measure. We have that $\mathbb{T}$ is coercive, and $\tau\mathbb{T}_1 - \tau^2\mathbb{T}_2$ is compact and depends analytically on $\tau \in \mathbb{C}$. From Theorem 3.3 we have that the kernel of $\mathbb{T} - \tau\mathbb{T}_1 + \tau^2\mathbb{T}_2$ is trivial for real $\tau > 0$. Now the Analytic Fredholm Theorem 1.12 implies that the kernel of $\mathbb{T} - \tau\mathbb{T}_1 + \tau^2\mathbb{T}_2$ is trivial for all $\tau \in \mathbb{C}$ except for a discrete set of $\tau \in \mathbb{C}$ with infinity as the only possible accumulation point, which proves the statement of the theorem in this case.

Next if $\Im(n) = 0$ in $D$, then the coercive operator $\mathbb{T}$, compact operator $\mathbb{T}_1$, and nonnegative compact operator $\mathbb{T}_2$ are all self-adjoint. Therefore $\mathbb{T}^{\frac{1}{2}}$ is positive and $\mathbb{T}^{-\frac{1}{2}}$ exists. Hence we have that (3.18) becomes

$$u - \tau\mathbb{K}_1 u + \tau^2\mathbb{K}_2 u = 0,$$

where the compact operators $\mathbb{K}_1 : H_0^2(D) \to H_0^2(D)$ and $\mathbb{K}_2 : H_0^2(D) \to H_0^2(D)$ are given by $\mathbb{K}_1 = \mathbb{T}^{-1/2}\mathbb{T}_1\mathbb{T}^{-1/2}$ and $\mathbb{K}_2 = \mathbb{T}^{-1/2}\mathbb{T}_2\mathbb{T}^{-1/2}$. Now noting that $\mathbb{K}_2$ is nonnegative, we set $U := \left(u, \tau\mathbb{K}_2^{1/2}u\right)$ to obtain

$$\left(\mathbf{K} - \frac{1}{\tau}\mathbf{I}\right)U = 0, \qquad U \in H_0^2(D) \times H_0^2(D)$$

with the compact operator $\mathbf{K} : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbf{K} := \begin{pmatrix} \mathbb{K}_1 & -\mathbb{K}_2^{1/2} \\ \mathbb{K}_2^{1/2} & 0 \end{pmatrix}.$$

Then the theorem follows from the spectral properties of compact operators in Hilbert spaces. $\square$

Theorems 3.2 and 3.4 state that the interior transmission problem is well-posed for all $k \in \mathbb{C}$ except for at most a countable discrete set of wave numbers $k$ with infinity as the only possible accumulation point.

### 3.1.2 ▪ Variational Approach for Media with Voids

The above analysis can be extended to inhomogeneous media with voids, i.e., the inhomogeneity $D \subset \mathbb{R}^3$ contains regions $D_0 \subset D$ which can possibly be multiply connected such that $D \setminus \overline{D}_0$ is connected, for which $n(x) = 1$. For the purpose of discussion in this section we still assume that the real part of $n(x) - 1$ is bounded away from zero and keeps the same sign in $D \setminus \overline{D}_0$ and for technical reasons here we will assume that both $\partial D$ and $\partial D_0$ are $C^2$-smooth surfaces with $\nu$ the unit normal vector directed outwards to $D$ and $D_0$ (see Figure 3.1). We will denote by $n_*$ and $n^*$ the essential infimum and supremum of $n \in L^\infty(D \setminus \overline{D}_0)$, i.e., given by (3.7), where $D$ is replaced by $D \setminus \overline{D}_0$. Here we will

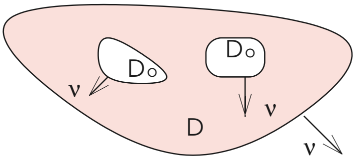



**Figure 3.1.** *Configuration of the media with voids.*

present an approach introduced in [33] (see [78] for Maxwell's equations). In the next section we present a more general approach to study the interior transmission problem for media with changing sign contrast which includes the case of interior voids. The analytical framework developed in this section will be used in the next chapter to prove the existence of real transmission eigenvalues as well as estimates for them. Similarly to Section 3.1.1, since $1/(n-1)$ is bounded in $D \setminus \overline{D}_0$, we obtain for $u := w - v$

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \qquad \text{in } D \setminus \overline{D}_0, \tag{3.19}$$

together with

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial D. \tag{3.20}$$

Inside $D_0$ one has

$$\left(\Delta + k^2\right) u = 0 \qquad \text{in } D_0 \tag{3.21}$$

with the continuity of the Cauchy data across $\partial D_0$

$$u^+ = u^- \qquad \text{and} \qquad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \tag{3.22}$$

where, for a generic function $\phi$,

$$\phi^\pm(x) = \lim_{h \to 0^+} \phi(x \pm h\nu_x) \quad \text{and} \quad \frac{\partial \phi^\pm(x)}{\partial \nu_x} = \lim_{h \to 0^+} \nu_x \cdot \nabla \phi(x \pm h\nu_x) \tag{3.23}$$

for $x \in \partial D_0$. The latter equations for $u$ are not sufficient to define $w$ and $v$ inside $\partial D_0$, and therefore one needs to add an additional unknown inside $D_0$, for instance, the function $w$ that satisfies

$$\left(\Delta + k^2\right) w = 0 \qquad \text{in } D_0 \tag{3.24}$$

with the continuity of the Cauchy data across $\partial D_0$ that can be written as

$$\begin{aligned} \left(\frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u\right)^+ &= w^- \qquad \text{and} \\ \frac{\partial}{\partial \nu} \left(\frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u\right)^+ &= \frac{\partial w^-}{\partial \nu}. \end{aligned} \tag{3.25}$$

We note that (3.25) is interpreted as equalities between functions in $H^{-\frac{1}{2}}(\partial D_0)$ and $H^{-\frac{3}{2}}(\partial D_0)$, respectively.

It is easily verified that the solutions $u \in H^2(D)$ and $w \in L^2(D_0)$ to (3.19)–(3.25) equivalently define a weak solution $w$ and $v$ to (3.1) by

$$w := \frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u \ \text{ in } D \setminus \overline{D}_0 \quad \text{and} \quad v := w - u \ \text{ in } D. \tag{3.26}$$

We establish existence and uniqueness results for the solution of the above interior transmission problem using a variational approach. The main difficulty in obtaining the variational formulation is to properly choose the function space that correctly handles the transmission conditions (3.22) and (3.25). More precisely, classical variational formulations of equations (3.19), (3.21), and (3.24) would require $u \in H^2(D \setminus \overline{D}_0) \cap H^1(D)$ and $v \in H^1(D_0)$ but this regularity is not sufficient to handle all boundary terms in (3.22) and (3.25). The proposed approach in the following treats equation (3.19) variationally and includes (3.21)–(3.22) in the variational space. More precisely we define

$$V(D, D_0, k) := \{u \in H^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}, \tag{3.27}$$

which is a Hilbert space equipped with the $H^2(D)$ scalar product and look for the solution $u$ in $V(D, D_0, k)$. We also consider the closed subspace

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}. \tag{3.28}$$

Let $u \in V(D, D_0, k)$ and consider a test function $\psi \in V_0(D, D_0, k)$. For the sake of presentation we assume that $u$ and $\psi$ are regular enough to justify the various integration by parts and then use a denseness argument. Multiplying (3.19) by $\psi$ and integrating by parts we obtain

$$\begin{aligned} 0 &= \int_{D\setminus\overline{D}_0} \left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u \, \bar{\psi} \, dx \qquad (3.29) \\ &= \int_{D\setminus\overline{D}_0} \left( \left(\Delta + k^2\right) \frac{1}{n-1} \left(\Delta + k^2\right) u + k^2 (\Delta + k^2) u \right) \bar{\psi} \, dx \\ &= \int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \left(\Delta + k^2\right) \bar{\psi} \, dx + k^4 \int_{D\setminus\overline{D}_0} u \, \bar{\psi} \, dx + k^2 \int_{D\setminus\overline{D}_0} \Delta u \, \bar{\psi} \, dx \\ &\quad + \int_{\partial D_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \frac{\partial \bar{\psi}}{\partial \nu} \, ds - \int_{\partial D_0} \frac{\partial}{\partial \nu} \left( \frac{1}{n-1} \left(\Delta + k^2\right) u \right) \bar{\psi} \, ds. \end{aligned}$$

Using the fact that $\bar{\psi} \in V_0(D, D_0, k)$, the boundary conditions (3.25), and equation (3.24) we obtain that

$$\int_{\partial D_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \frac{\partial \bar{\psi}}{\partial \nu} \, ds - \int_{\partial D_0} \frac{\partial}{\partial \nu} \left( \frac{1}{n-1} \left(\Delta + k^2\right) u \right) \bar{\psi} \, ds = 0. \tag{3.30}$$

Therefore we finally have that

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \left(\Delta + k^2\right) \bar{\psi} \, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u + k^2 u) \, \bar{\psi} \, dx = 0, \tag{3.31}$$

which is required to be valid for all $\psi \in V_0(D, D_0, k)$. For given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ let $\theta \in H^2(D)$ be the lifting function such that $\theta = f$ and $\partial\theta/\partial\nu = h$ on $\partial D$ as discussed in Section 3.1. Using a cutoff function we can guarantee that $\theta = 0$ in $D_\theta$ such that $D_0 \subset D_\theta \subset D$. The variational formulation amounts to finding $u_0 = u - \theta \in$

$V_0(D, D_0, k)$ such that

$$\int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta+k^2\right)u_0\,\left(\Delta+k^2\right)\bar{\psi}\,dx + k^2\int\limits_{D\setminus\overline{D}_0}\left(\Delta u_0+k^2u_0\right)\bar{\psi}\,dx$$
$$= \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta+k^2\right)\theta\,\left(\Delta+k^2\right)\bar{\psi}\,dx + k^2\int\limits_{D\setminus\overline{D}_0}\left(\Delta\theta+k^2\theta\right)\bar{\psi}\,dx \tag{3.32}$$

for all $\psi \in V_0(D, D_0, k)$. As one can see, the above variational formulation involves only $u$ (in particular, it does not involve $w$). The following lemma shows that the existence of $w$ is implicitly contained in the variational formulation.

**Lemma 3.5.** *Assume that $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$, and let $(\beta,\alpha) \in H^{-\frac{1}{2}}(\partial D_0)\times H^{-\frac{3}{2}}(\partial D_0)$ such that*

$$\langle \beta,\,\partial\psi/\partial\nu\rangle_{H^{-\frac{1}{2}}(\partial D_0),H^{\frac{1}{2}}(\partial D_0)} - \langle \alpha,\,\psi\rangle_{H^{-\frac{3}{2}}(\partial D_0),H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.33}$$

*for all $\psi \in V_0(D, D_0, k)$. Then there exists a unique $w \in L^2(D_0)$ such that $\Delta w + k^2 w = 0$ in $D_0$ and $(w, \partial w/\partial\nu) = (\beta,\alpha)$ on $\partial D_0$.*

***Proof.*** Assume that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D_0$. Let $w \in L^2(D_0)$ be a weak solution of $\Delta w + k^2 w = 0$ in $D_0$ and $w = \beta$ on $\partial D_0$ (see remark below on how one can construct this solution from $H^1(D_0)$ solutions by using a classical duality argument, i.e., the traces of $w$ and $\partial w/\partial\nu$ can be defined in this case by a duality argument; see also [131]). Then applying Green's formula to $w$ and a test function $\psi \in V_0(D, D_0, k)$ we get

$$\langle w,\,\partial\psi/\partial\nu\rangle_{H^{-\frac{1}{2}}(\partial D_0),H^{\frac{1}{2}}(\partial D_0)} - \langle \partial w/\partial\nu,\,\psi\rangle_{H^{-\frac{3}{2}}(\partial D_0),H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.34}$$

and therefore

$$\langle \partial w/\partial\nu - \alpha,\,\psi\rangle_{H^{-\frac{3}{2}}(\partial D_0),H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.35}$$

for all $\psi \in V_0(D, D_0, k)$. We know that the traces of Herglotz wave functions are dense in $H^{\frac{3}{2}}(\partial D_0)$ (see [169, Theorem 4]) provided that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D_0$ and, since $V_0(D, D_0, k)$ contains the set of Herglotz wave functions, we can conclude that the traces on $\partial D_0$ of functions in $V_0(D, D_0, k)$ are dense in $H^{\frac{3}{2}}(\partial D_0)$. Hence $\partial w/\partial\nu = \alpha$ and the result follows. The case when $k^2$ is not a Neumann eigenvalue can be treated by choosing $w \in L^2(D_0)$ to be a weak solution of $\Delta w + k^2 w = 0$ in $D_0$ such that $\partial w/\partial\nu = \alpha$ on $\partial D_0$ and using the denseness of normal traces on $\partial D_0$ of functions in $V_0(D, D_0, k)$ in $H^{\frac{1}{2}}(\partial D_0)$ (the denseness result follows from [169, Theorem 3]). The uniqueness of $w$ is obvious. □

**Remark 3.6.** We briefly recall the construction of $L^2$ solutions for the Helmholtz equation in $D_0$. Assume that $k^2$ is not a Dirichlet eigenvalue and let $g \in H^{\frac{1}{2}}(\partial D_0)$ and $u \in H^1(D_0)$ satisfy $\Delta u + k^2 u = 0$ in $D_0$ and $u = g$ on $\partial D_0$. Let $v \in H^1(D_0)$ be a solution of $\Delta v + k^2 v = u$ such that $v = 0$ on $\partial D_0$. Then standard regularity results imply that $v \in H^2(D_0)$, and there exists a constant $c$ independent of $v$ and $u$ such that $\|v\|_{H^2(D_0)} \le c\|u\|_{L^2(D_0)}$. Using Green's formula one easily obtains

$$\|u\|^2_{L^2(D_0)} = \left|\int\limits_{\partial D_0} g\,\partial v/\partial\nu\,ds\right| \le \|g\|_{H^{-\frac{1}{2}}(\partial D_0)}\|\partial v/\partial\nu\|_{H^{\frac{1}{2}}(\partial D_0)}$$
$$\le C\|g\|_{H^{-\frac{1}{2}}(\partial D_0)}\|u\|_{L^2(D_0)}, \tag{3.36}$$

and therefore the solution operator $g \to u$ is continuous from $H^{-\frac{1}{2}}(\partial D_0)$ into $L^2(D_0)$. Similar arguments also show that if $k^2$ is not an eigenvalue for the Neumann problem, then the solution operator $g \to u$ where $u \in H^1(D_0)$ satisfies $\Delta u + k^2 u = 0$ in $D_0$ and $\partial u/\partial\nu = g$ is continuous from $H^{-\frac{3}{2}}(\partial D_0)$ into $L^2(D_0)$.

**Remark 3.7.** If the solution of the variational problem (3.32) is in $H^4(D \setminus \overline{D}_0)$, then one can use the Calderòn projection [133] operator to construct $w$ in $D_0$ and thus avoid the assumption on $k^2$ in Lemma 3.5.

We now can state the equivalence between solutions to interior transmission problem (3.1) and solutions to the variational formulation (3.32).

**Theorem 3.8.** *Assume that $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$ and either $n_* > 1$ or $0 < n^* < 1$. Then the existence and uniqueness of a solution $w \in L^2(D)$ and $v \in L^2(D)$, $u := w - v \in H_0^2(D)$, to the interior transmission problem* (3.1) *is equivalent to the existence and uniqueness of a solution $u_0 \in V_0(D, D_0, k)$ of the variational problem* (3.32)*.*

***Proof.*** It remains only to verify that any solution to (3.32) defines a weak solution $w$ and $v$ to the interior transmission problem (3.1). Taking a test function $\psi$ to be a $C^\infty$ function with compact support in $D \setminus \overline{D}_0$, one can easily verify from (3.31) that $u$ satisfies (3.19). In particular, the function

$$w^+ := \left( -\frac{1}{k^2(n-1)}(\Delta + k^2)u \right)_{|D\setminus\overline{D}_0}$$

satisfies $w^+ \in L^2(D \setminus \overline{D}_0)$ and $(\Delta + k^2 n)w^+ = 0$ in $D \setminus \overline{D}_0$. For an arbitrary test function $\psi \in C^\infty(D \setminus \overline{D}_0)$ we can apply Green's formula and (3.31) to obtain

$$\left\langle w^+, \partial\psi/\partial\nu \right\rangle_{H^{-\frac{1}{2}}(\partial D_0), H^{\frac{1}{2}}(\partial D_0)} - \left\langle \partial w^+/\partial\nu, \psi \right\rangle_{H^{-\frac{3}{2}}(\partial D_0), H^{\frac{3}{2}}(\partial D_0)} = 0. \tag{3.37}$$

Finally, applying Lemma 3.5, we now obtain the existence of $w^- \in L^2(D_0)$ satisfying (3.24) and (3.25). □

We now proceed with the proof of existence of a solution to (3.32).

**Theorem 3.9.** *Let $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ and assume that $n \in L^\infty(D)$ is such that $n = 1$ in $D_0$, $\Re(n) \geq c > 0$, and $\Im(n) \geq 0$ almost everywhere in $D \setminus \overline{D}_0$. Assume further that either $n_* > 1$ or $0 < n^* < 1$. Then* (3.32) *satisfies the Fredholm alternative. In particular, if the homogeneous variational problem (i.e.,* (3.32) *with $\theta = 0$) has only the trivial solution $u_0 = 0$, then* (3.32) *has a unique solution which depends continuously on the data $f$ and $h$.*

***Proof.*** We define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\begin{aligned} \mathcal{A}(u_0, \psi) = & \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1} \left( \Delta u_0 \, \Delta\bar{\psi} + \nabla u_0 \cdot \nabla\bar{\psi} + u_0 \, \bar{\psi} \right) \, dx \\ & \pm \int\limits_{D_0} \left( \nabla u_0 \cdot \nabla\bar{\psi} + u_0 \, \bar{\psi} \right) \, dx \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\mathcal{B}_k(u_0,\psi) &= k^2 \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(u_0(\Delta\bar{\psi}+k^2\bar{\psi}) + (\Delta u_0 + k^2 n u_0)\bar{\psi}\right) dx \\ &- \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\nabla u_0\cdot\nabla\bar{\psi} + u_0\,\bar{\psi}\right)\, dx \mp \int\limits_{D_0}\left(\nabla u_0\cdot\nabla\bar{\psi} + u_0\,\bar{\psi}\right)\, dx,\end{aligned} \tag{3.39}$$

where the upper sign corresponds to the case when $n_* > 1$, whereas the lower sign corresponds to the case when $n^* < 1$. In terms of these forms the variational equation (3.32) for $u_0 \in V_0(D, D_0, k)$ becomes

$$\mathcal{A}(u_0,\psi) + \mathcal{B}_k(u_0,\psi) = \mathcal{A}(\theta,\psi) + \mathcal{B}_k(\theta,\psi) \quad \text{for all } \psi \in V_0(D,D_0,k). \tag{3.40}$$

It is clear that if the real part of $1/(n-1)$ is positive definite or negative definite, then there exists a positive constant $\gamma$, which only depends on $n$, such that

$$|\mathcal{A}(u_0,u_0)| \geq \gamma(\|\Delta u_0\|^2_{L^2(D\setminus\overline{D}_0)} + \|u_0\|^2_{H^1(D)}). \tag{3.41}$$

Let $\epsilon = 1/(1+k^4)$, so that $0<\epsilon<1$ and $\epsilon k^4 < 1$. Since $\Delta u_0 = -k^2 u_0$ in $D_0$ one also has that

$$\begin{aligned}|\mathcal{A}(u_0,u_0)| &\geq \gamma\epsilon\|\Delta u_0\|^2_{L^2(D)} + \gamma(1-\epsilon k^4)\|u_0\|^2_{H^1(D)} \\ &= (\gamma/(1+k^4))(\|\Delta u_0\|^2_{L^2(D)} + \|u_0\|^2_{H^1(D)}).\end{aligned} \tag{3.42}$$

From standard elliptic regularity results we deduce that

$$|\mathcal{A}(u_0,u_0)| \geq (\tilde{\gamma}/(1+k^4))\|u_0\|^2_{H^2(D)}, \tag{3.43}$$

where $\tilde{\gamma}$ only depends on $D$ and $n$. Therefore $\mathcal{A}$ defines a continuous and coercive sesquilinear form on $V_0(D,D_0,k)\times V_0(D,D_0,k)$. Moreover if $|1/(n-1)|$ and $n$ are bounded, then the compact embedding of $H^2_0(D)$ into $H^1(D)$ (Rellich's theorem) implies that $\mathcal{B}_k$ defines a compact perturbation of $\mathcal{A}$, while the right-hand side of (3.40) defines a continuous antilinear form on $V_0(D,D_0,k)$. The result of our theorem now follows from an application of the Fredholm alternative. □

We can prove a similar result as in Theorem 3.3 concerning uniqueness of the variational equation (3.32).

**Theorem 3.10.** *If* $n \in L^\infty(D)$ *is such that* $\mathrm{Im}(n) > 0$ *almost everywhere in* $D\setminus\overline{D}_0$*, then there are no* real *transmission eigenvalues.*

***Proof.*** Assume that the homogeneous problem

$$\mathcal{A}(u_0,\psi) + \mathcal{B}_k(u_0,\psi) = 0 \text{ for all } \psi \in V_0(D,D_0,k) \tag{3.44}$$

has a nontrivial solution $u_0 \in V_0(D,D_0,k)$. First taking $\psi = u_0$ in (3.44) we obtain

$$\begin{aligned}0 = &\int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u_0 + k^2 u_0|^2\, dx + k^4 \int\limits_{D\setminus\overline{D}_0} |u_0|^2\, dx \\ &- k^2 \int\limits_{D\setminus\overline{D}_0} |\nabla u_0|^2\, dx - k^2 \int\limits_{\partial D_0} \bar{u}_0^+ \frac{\partial u_0^+}{\partial\nu}\, ds.\end{aligned} \tag{3.45}$$

Using Green's first identity for $u_0$ in $D_0$ and the continuity of the Cauchy data of $u_0$ across $\partial D_0$ we can rewrite (3.45) as

$$0 = \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u_0 + k^2 u_0|^2\,dx + k^4 \int\limits_{D\setminus\overline{D}_0} |u_0|^2\,dx - k^2 \int\limits_{D\setminus\overline{D}_0} |\nabla u_0|^2\,dx$$
$$+ k^4 \int\limits_{D_0} |u_0|^2\,dx - k^2 \int\limits_{D_0} |\nabla u_0|^2\,dx. \tag{3.46}$$

Since $\Im(1/(n-1)) < 0$ in $D \setminus \overline{D}_0$ and all the terms in the above equation are real except for the first one, by taking the imaginary part we obtain that $\Delta u_0 + k^2 u_0 = 0$ in $D \setminus \overline{D}_0$, and since $u_0$ has zero Cauchy data on $\partial D$ we obtain that $u_0 = 0$ in $D \setminus \overline{D}_0$ and therefore $k$ is not a transmission eigenvalue. Note that the proof requires that $\Im(n) > 0$ almost everywhere in all of $D \setminus \overline{D}_0$. $\square$

**Remark 3.11.** Note that, by Theorem 3.8, if $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$, then the uniqueness of (3.32) is equivalent to $k \in \mathbb{C}$ not being a transmission eigenvalue (see also Remark 3.7). Furthermore, under the additional assumptions of Theorem 3.9, the interior transmission problem (3.1) has a unique solution depending continuously on the data, provided that $k \in \mathbb{C}$ is not a transmission eigenvalue.

It is possible to use the analytical framework developed here to prove that (3.32) and hence (3.1) fails to have a unique solution for at most a discrete set of values of $k$ with $+\infty$ as the only possible accumulation point. However, in the next section we will prove discreteness of transmission eigenvalues for a larger class of refractive indices which establishes this result as a special case since the set of Dirichlet and Neumann eigenvalues for $-\Delta$ in $D_0$ consists of a discrete set of real $k^2$ accumulating at $+\infty$. We refer interested readers to Section 4.2.1 in [33] for the proof of this discreteness result using the variational approach of this section.

**Remark 3.12.** The approach described in this section provides a general analytical framework to analyze the interior transmission problem for inhomogeneous media containing different types of inclusions $D_0$. We refer the reader to [43] to see how the approach can be modified to the case when $D_0$ is a nonpenetrable inclusion with Dirichlet boundary condition.

### 3.1.3 ▪ The Case of Sign Changing Contrast

In this section we investigate the solvability of the interior transmission problem (3.1) under less restrictive assumptions on the real part of the contrast. More specifically, we assume that there is a neighborhood of the boundary $\mathcal{N}$ (that is an open subdomain $\mathcal{N} \subset D$ with $\partial D \subset \overline{\mathcal{N}}$) where we impose conditions on the contrast $n-1$ (to become precise later on), and in $D \setminus \mathcal{N}$ the contrast $n-1$ can be anything (of course under the physical assumptions on the refractive index $n$ stated at the beginning of this chapter). The Fredholm property of the interior transmission problem and the discreteness of transmission eigenvalues for this general case were first investigated in [159]. The approach in [159] was revisited in [111] for real valued refractive index where the same results were obtained by using a variational approach. Our discussion follows the approach due to Kirsch in [111].

We recall the interior transmission problem formulated for $u := \frac{1}{k^2}(w - v)$ and $v$: Given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$, find $u \in H^2(D)$ and $v \in L^2(D)$ such that

$$\begin{cases} \Delta u + k^2 n u = -(n-1)v & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = f \quad \text{and} \quad \dfrac{\partial u}{\partial \nu} = h & \text{on } \partial D. \end{cases} \tag{3.47}$$

With the help of a lifting function $\theta \in H^2(D)$ such that $\theta = f$ and $\partial\theta/\partial\nu = h$ on $\partial D$ introduced in Section 3.1, it is possible to transform (3.47) into the following problem: Given $F \in L^2(D)$, find $u \in H_0^2(D)$ and $v \in L^2(D)$ such that

$$\begin{cases} \Delta u + k^2 n u = -(n-1)v + F & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \tag{3.48}$$

The above equations are assumed to be satisfied in the following weak sense:

$$\int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx = 0,$$

$$\int_D (\Delta u + k^2 n u + (n-1)v)\overline{\varphi}\,dx = \int_D F\,\overline{\varphi}\,dx$$

for all $\psi \in H_0^2(D)$ and $\varphi \in L^2(D)$. Let us denote $X(D) := H_0^2(D) \times L^2(D)$ equipped with the norm $\|(u,v)\|_{X(D)} = \|u\|_{H^2(D)} + \|v\|_{L^2(D)}$, and the corresponding inner product $\langle\cdot,\cdot\rangle_{X(D)}$. Then (3.48) can be written in the following equivalent variational form: Find $(u,v) \in X(D)$ such that for all $(\psi,\varphi) \in X(D)$

$$\int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx + \int_D (\Delta u + k^2 n u)\overline{\varphi} + (n-1)v\overline{\varphi}\,dx = \int_D F\,\overline{\varphi}\,dx. \tag{3.49}$$

For any $k \in \mathbb{C}$ we define the sesquilinear form $\mathcal{A}_k : X(D) \times X(D) \to \mathbb{C}$ by

$$\mathcal{A}_k(u,v;\psi,\varphi) := \int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx + \int_D (\Delta u + k^2 n u)\overline{\varphi} + (n-1)v\overline{\varphi}\,dx \tag{3.50}$$

for all $(u,v) \in X(D)$ and $(\psi,\varphi) \in X(D)$. For later use we also define the following auxiliary sesquilinear form $\hat{\mathcal{A}}_k : X(D) \times X(D) \to \mathbb{C}$:

$$\hat{\mathcal{A}}_k(u,v;\psi,\varphi) := \int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx + \int_D ((\Delta u + k^2 u)\overline{\varphi} + (n-1)v\overline{\varphi})\,dx \tag{3.51}$$

for all $(u,v) \in X(D)$ and $(\psi,\varphi) \in X(D)$. The Riesz representation theorem yields the existence of bounded linear operators $A_k, \hat{A}_k : X(D) \to X(D)$ such that

$$\mathcal{A}_k(u,v;\psi,\varphi) = \langle A_k(u,v), (\psi,\varphi)\rangle_{X(D)} \quad \text{for all } (u,v), (\psi,\varphi) \in X(D) \tag{3.52}$$

with an analogous expression for $\hat{A}_k$. Hence the interior transmission problem is equivalent to the following operator equation:

$$A_k(u,v) = \ell, \qquad (u,v) \in X(D), \tag{3.53}$$

where $\ell \in X(D)$ is the Riesz representative of the antilinear functional $\varphi \mapsto \int_D F\,\overline{\varphi}\,dx$.

**Theorem 3.13.** *For any two $k_1, k_2 \in \mathbb{C}$ the differences $A_{k_1} - \hat{A}_{k_2}$ and $A_{k_1} - A_{k_2}$ are compact.*

***Proof.*** Let $(u_j, v_j) \in X(D)$ converge weakly to zero in $X(D)$, and let $(\psi, \varphi) \in X(D)$. Then we have

$$\left(\mathcal{A}_{k_1} - \hat{\mathcal{A}}_{k_2}\right)(u_j, v_j; \psi, \varphi) = (k_1^2 - k_2^2)\int\limits_D \overline{\psi} v_j\,dx + \int\limits_D \left(k_1^2 n - k_2^2\right) u_j \overline{\varphi}\,dx.$$

Since $u_j \rightharpoonup 0$ in $H_0^2(D)$, Rellich's compact embedding theorem implies that $u_j \to 0$ in $L^2(D)$. Furthermore,

$$\left|\int\limits_D \left(k_1^2 n - k_2^2\right) u_j \overline{\varphi}\,dx\right| \le \|k_1^2 n - k_1^2\|_{L^\infty(D)} \|u_j\|_{L^2(D)} \|\varphi\|_{L^2(D)}. \tag{3.54}$$

Next let $z_j \in H^1(D)$ with $\Delta z_j = v_j$ in $D$ and $z_j = 0$ on $\partial D$. Since $z_j \rightharpoonup 0$ in $H^1(D)$, then $z_j \to 0$ in $L^2(D)$ and thus we have

$$\left|\int\limits_D \overline{\psi} v_j\,dx\right| = \left|\int\limits_D \overline{\psi}\Delta z_j\,dx\right| = \left|\int\limits_D \Delta\overline{\psi} z_j\,dx\right| \le \|z_j\|_{L^2(D)} \|\psi\|_{H^2(D)}. \tag{3.55}$$

Thus (3.54) and (3.55) imply

$$\begin{aligned}
\|(A_{k_1} - \hat{A}_{k_2})(u_j, v_j)\|_{X(D)} &= \sup_{0 \ne (\psi,\varphi) \in X(D)} \left|\left(\mathcal{A}_{k_1} - \hat{\mathcal{A}}_{k_2}\right)(u_j, v_j; \psi, \varphi)\right| \\
&\le C\left(\|u_j\|_{L^2(D)} + \|z_j\|_{L^2(D)}\right),
\end{aligned}$$

whence $(A_{k_1} - \hat{A}_{k_2})(u_j, v_j)$ converges strongly to zero in $X(D)$. This prove compactness of $A_{k_1} - \hat{A}_{k_2}$. The proof for $A_{k_1} - A_{k_2}$ follows along the same lines. $\square$

Theorem 3.13 suggests that we need to show the invertibility of $\hat{A}_k$ for some $k \in C$. At this point we need to assume that $\Re(n(x)) - 1 \ge \alpha > 0$ or $1 - \Re(n(x)) \ge \alpha > 0$ for almost all $x \in \mathcal{N}$ and some $\alpha > 0$. Denoting

$$n_\star = \inf_{\mathcal{N}} \Re(n) \qquad \text{and} \qquad n^\star = \sup_{\mathcal{N}} \Re(n) \tag{3.56}$$

(notice that here the inf and sup are taken over the boundary neighborhood $\mathcal{N}$ as opposed to the entire $D$ as in (3.7)), the latter assumption means that either $n_\star > 1$ or $0 < n^\star < 1$.

**Lemma 3.14.** *Assume that $n \in L^\infty(D)$ is such that either $n_\star > 1$ or $0 < n_\star < n^\star < 1$. Then there exist constants $c > 0$ and $d > 0$ such that for all $k = i\kappa$, $\kappa > 0$, the following estimate holds:*

$$\int\limits_{D\setminus\mathcal{N}} |v|^2\,dx \le c e^{-2d\kappa} \int\limits_{\mathcal{N}} |\Re(n) - 1||v|^2\,dx \tag{3.57}$$

*for all solutions $v \in L^2(D)$ of $\Delta v - \kappa^2 v = 0$ in $D$.*

***Proof.*** We choose a neighborhood $\mathcal{N}'$ of the boundary $\partial D$ such that $d = \mathrm{dist}(D \setminus \mathcal{N}, \mathcal{N}') > 0$ and a function $\rho \in C^\infty(D)$ with compact support in $D$ such that $\rho = 1$ in $D \setminus \mathcal{N}'$. Applying Green's formula (1.10) to $\rho v$ and noting that $\rho v \equiv v$ in $D \setminus \mathcal{N}'$, that is, $\Delta \rho v - \kappa^2 \rho v = 0$ in $D \setminus \mathcal{N}'$, yields

$$\begin{aligned}\rho(x)v(x) &= -\int_D \left[\Delta(\rho v)(y) - \kappa^2(\rho v)(y)\right] \frac{e^{-\kappa|x-y|}}{|x-y|}\, dy \\ &= -\int_{\mathcal{N}'} \left[2\nabla\rho(y)\cdot\nabla v(y) + v(y)\Delta\rho(y)\right] \frac{e^{-\kappa|x-y|}}{|x-y|}\, dy \\ &= \int_{\mathcal{N}'} \left[2\nabla\cdot\nabla\rho(y)\frac{e^{-\kappa|x-y|}}{|x-y|} - \Delta\rho(y)\frac{e^{-\kappa|x-y|}}{|x-y|}\right] v(y)\, dy.\end{aligned}$$

For $x \in D \setminus \mathcal{N}$ we can conclude that

$$|v(x)| \le c_1 e^{-d\kappa} \int_{\mathcal{N}'} |v(y)|\, dy$$

for some constant $c_1 > 0$ that depends only on $D$, $\mathcal{N}$, $\mathcal{N}'$, and $\rho$. Thus, from the above, using the Cauchy–Schwarz inequality we now obtain

$$|v(x)|^2 \le c_1^2 e^{-2d\kappa} |\mathcal{N}| \int_{\mathcal{N}} |v(y)|^2\, dx \le \frac{c_1^2|\mathcal{N}|}{\delta} e^{-2d\kappa} \int_{\mathcal{N}} |\Re(n)(y) - 1||v(y)|^2\, dy,$$

where $\delta = n_\star - 1$ if $n_\star > 1$ or $\delta = 1 - n^\star$ if $n^\star < 1$. Integrating with respect to $x$ over $D \setminus \mathcal{N}$ implies the result. □

**Theorem 3.15.** *There exists a $\kappa_0 > 0$ and a positive constant $c > 0$ such that for all $\kappa \ge \kappa_0$*

$$\sup_{(\psi,\varphi)\neq 0} \frac{\left|\hat{\mathcal{A}}_{i\kappa}(u,v;\psi,\varphi)\right|}{\|(\psi,\varphi)\|_{X(D)}} \ge c\|(u,v)\|_{X(D)} \qquad \textit{for all } (u,v) \in X(D). \tag{3.58}$$

***Proof.*** Thanks to Lemma 3.14 we can find a $\kappa_0 > 0$ such that

$$\int_{D\setminus\mathcal{N}} |\Re(n) - 1||v|^2\, dx \le \|(n-1)\|_{L^\infty(D)} \int_{D\setminus\mathcal{N}} |v|^2\, dx \le \frac{1}{2}\int_{\mathcal{N}} |\Re(n) - 1||v|^2\, dx \tag{3.59}$$

for all solutions of $\Delta v - \kappa^2 v = 0$ in $D$ and all $\kappa \ge \kappa_0$. Let us assume by contradiction that a constant $c > 0$ such that (3.58) holds does not exist, in which case we can find a sequence $\{(u_j, v_j)\} \in X(D)$ with $\|(u_j, v_j)\|_{X(D)} = 1$ and

$$\sup_{(\psi,\varphi)\neq 0} \frac{\left|\hat{\mathcal{A}}_{i\kappa}(u_j,v_j;\psi,\varphi)\right|}{\|(\psi,\varphi)\|_{X(D)}} \to 0, \qquad j \to \infty. \tag{3.60}$$

There is a weakly convergent subsequence (still denoted by $\{(u_j, v_j)\}$) such that $u_j \rightharpoonup u$ in $H_0^2(D)$ and $v_j \rightharpoonup v$ in $L^2(D)$ for some $(u, v) \in X(D)$. From (3.60) we see that $(u, v)$ satisfy $\Delta v - \kappa^2 v = 0$ and $\Delta u - \kappa^2 u = -(n-1)v$ in $D$.

As a first step, we show that the weak limits are zero, i.e., $u = v = 0$ in $D$. To this end, we notice that

$$\Re\left(\hat{\mathcal{A}}_{i\kappa}(u, v; -u, v)\right) = \int_D \Re(n-1)|v|^2\,dx = 0. \tag{3.61}$$

Now using (3.59), (3.61), and the fact that $\Re(n) - 1$ has one sign in $\mathcal{N}$, we have

$$\begin{aligned}\int_{\mathcal{N}} |\Re(n)-1||v|^2\,dx &= \left|\int_{\mathcal{N}} \Re(n-1)|v|^2\,dx\right| = \left|\int_{D\setminus\mathcal{N}} \Re(n-1)|v|^2\,dx\right| \\ &\le \int_{D\setminus\mathcal{N}} |\Re(n-1)||v|^2\,dx \le \frac{1}{2}\int_{\mathcal{N}} |\Re(n)-1||v|^2\,dx,\end{aligned}$$

and thus $v = 0$ in $\mathcal{N}$. Unique continuation yields that $v = 0$ in $D$ and hence also $u = 0$ in $D$ since $0 = -\hat{\mathcal{A}}_{i\kappa}(u, 0; 0, u) = \int_D \left(|\nabla u|^2 + \kappa^2|u|^2\right)\,dx$.

We now arrive at a contradiction. We choose a neighborhood $\mathcal{N}'$ of $\partial D$ such that $\overline{\mathcal{N}'} \subset \mathcal{N} \cup \partial D$ and a nonnegative function $\eta \in C^\infty(D)$ such that $\eta = 0$ in $D \setminus \mathcal{N}$ and $\eta = 1$ in $\mathcal{N}'$. Set $\psi = \eta u_j$ and $\varphi = -\eta v_j$ in (3.60). Since $\{(\eta u_j, -\eta v_j)\}$ is bounded in $X(D)$ we have that

$$\int_{\mathcal{N}} (\Delta\eta\overline{u_j} - \kappa^2\eta\overline{u_j})v_j\,dx - \int_{\mathcal{N}} [(\Delta u_j - \kappa^2 u_j)\eta\overline{v_j} + (n-1)\eta|v_j|^2]\,dx \to 0, \qquad j \to \infty,$$

and hence

$$\Re\left(\int_{\mathcal{N}} \left[2v_j\nabla\eta\cdot\nabla\overline{u_j} + \overline{u_j}v_j\Delta\eta - (n-1)\eta|v_j|^2\right]\,dx\right) \to 0, \qquad j \to \infty. \tag{3.62}$$

Since $u_j \rightharpoonup u$ in $H_0^2(D)$ then $\|u_j\|_{H^1(D)} \to 0$ due to the compact embedding of $H^2(D)$ in $H^1(D)$. Hence the first two terms of (3.62) go to zero as $j \to \infty$ and we are left with

$$\int_{\mathcal{N}} (\Re(n)-1)\eta|v_j|^2\,dx \to 0, \qquad j \to \infty.$$

Since $\Re(n) - 1$ has one sign in $\mathcal{N}$ and $|\Re(n) - 1|\eta > \alpha > 0$ in $\mathcal{N}'$ ($\alpha = n_\star$ if $n_\star > 1$ and $\alpha = n^\star$ if $n^\star < 1$ in $\mathcal{N}$), we can conclude that $v_j \to 0$ in $L^2(\mathcal{N}')$.

Now we choose a third neighborhood $\mathcal{N}''$ of $\partial D$ such that $\overline{\mathcal{N}''} \subset \mathcal{N}' \cup \partial D$ and a nonnegative function $\tilde{\eta} \in C^\infty(D)$ such that $\tilde{\eta} = 0$ in $\mathcal{N}''$ and $\eta = 1$ in $D \setminus \mathcal{N}'$. Let $z_j \in H^2(D)$ be the solution of $\Delta z_j - \kappa^2 z_j = v_j$ in $D$ and $z_j = 0$ on $\partial D$. Taking $\psi = \tilde{\eta}z_j$ and $\varphi = 0$ in (3.60) and noting that $\{\tilde{\eta}z_j\}$ is bounded in $H^2(D)$ yields

$$\int_{D\setminus\mathcal{N}''} \left[\Delta(\tilde{\eta}\overline{z_j}) - \kappa^2\tilde{\eta}\overline{z_j}\right] v_j\,dx \to 0, \qquad j \to \infty,$$

that is,

$$\int_{D\setminus\mathcal{N}''} \left[\tilde{\eta}|v_j|^2 + 2(\nabla\tilde{\eta}\cdot\nabla\overline{z_j})v_j + \overline{z_j}\Delta\tilde{\eta}v_j\right]\,dx \to 0, \qquad j \to \infty. \tag{3.63}$$

Since $v_j \rightharpoonup 0$ in $L^2(D)$ we conclude that $z_j \rightharpoonup 0$ in $H^2(D)$ and hence $z_j \to 0$ in $H^1(D)$. Noting that $\tilde{\eta} = 1$ in $D \setminus \mathcal{N}'$ and is nonnegative in $D \setminus \mathcal{N}''$ we have that in addition $v_j \to 0$ in $L^2(D \setminus \mathcal{N}')$. Altogether we have shown that $v_j \to 0$ in $L^2(D)$. Finally, let $\psi = 0$ and $\varphi = \Delta u_j - \kappa^2 u_j$ in (3.60), which yields

$$\frac{1}{\|\Delta u_j - \kappa^2 u_j\|_{L^2(D)}} \int_D [|\Delta u_j - \kappa^2 u_j|^2 + (n-1)v_j(\Delta \overline{u_j} - \kappa^2 \overline{u_j})]\, dx \to 0, \qquad j \to \infty,$$

that is,

$$\|\Delta u_j - \kappa^2 u_j\|_{L^2(D)} + \int_D (n-1)v_j \frac{\Delta \overline{u_j} - \kappa^2 \overline{u_j}}{\|\Delta u_j - \kappa^2 u_j\|_{L^2(D)}}\, dx \to 0, \qquad j \to \infty,$$

which since $v_j \to 0$ in $L^2(D)$ implies that $\Delta u_j - \kappa^2 u_j \to 0$ in $L^2(D)$. Therefore $\Delta u_j$ converges strongly to zero in $L^2(D)$ (note that $u_j \rightharpoonup 0$ in $H_0^2(D)$ and hence $u_j \to 0$ in $L^2(D)$). Since $\|\Delta u_j\|_{L^2(D)}$ is equivalent to $\|u_j\|_{H_0^2(D)}$ we have shown that $u_j \to 0$ in $H^2(D_0)$.

Concluding, we have shown that $(u_j, v_j) \to 0$ in $X(D)$, which is a contradiction. This proves the theorem. □

Appealing to the inf-sup condition in Theorem 3.15 implies the following invertibility property for $\hat{A}_k$.

**Corollary 3.16.** *Let $\kappa > 0$ be such that the inf-sup condition* (3.58) *is valid. Then the operator $\hat{A}_{i\kappa} : X \to X$ is invertible with bounded inverse.*

Combining Theorem 3.13 and Corollary 3.16, we have the following theorem concerning the solvability of the interior transmission problem.

**Theorem 3.17.** *Assume that $n \in L^\infty(D)$ with $\Re(n) > n_0 > 0$ and $\Im(n) \geq 0$ almost everywhere in D and either $\inf_{\mathcal{N}} \Re(n) > 1$ or $\sup_{\mathcal{N}} \Re(n) < 1$ for some neighborhood $\mathcal{N}$ of the boundary $\partial D$. Furthermore, assume that $k \in \mathbb{C}$ is not a transmission eigenvalue. Then for any given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$, the interior transmission problem* (3.1) *has a unique solution $w \in L^2(D)$ and $v \in L^2(D)$ with $w - v \in H^2(D)$ and the following a priori estimates hold:*

$$\|w\|_{L^2(D)} + \|v\|_{L^2(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right),$$

$$\|u\|_{H^2(D)} \leq C \left( \|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right)$$

*with some positive constant $C > 0$.*

Next we derive sufficient conditions under which the set of transmission eigenvalues in $\mathbb{C}$ is discrete (possibly empty) with $+\infty$ as the only accumulation point. To this end we first show that there exists a wave number $k$ that is not a transmission eigenvalue.

**Theorem 3.18.** *Assume that $n \in L^\infty(D)$ with $\Re(n) > n_0 > 0$, $\Im(n) \geq 0$ almost everywhere in D, and $\sup_{\mathcal{N}} \Re(n) < 1$ for some neighborhood $\mathcal{N}$ of the boundary $\partial D$. Then, for some $\kappa > 0$, the operator $A_{i\kappa} : X(D) \to X(D)$ is invertible with bounded inverse.*

***Proof.*** It suffices to prove that $A_{i\kappa} : X(D) \to X(D)$ is injective for some $\kappa$ since $\hat{A}_{i\kappa} : X(D) \to X(D)$ is invertible and $\hat{A}_{i\kappa} - A_{i\kappa} : X(D) \to X(D)$ is compact. We prove it by contradiction, i.e., we assume that for every $\kappa > 0$ there exist functions $(u, v) \in X(D)$ with $\|(u, v)\|_{X(D)} = 1$ and $A_{i\kappa}(u, v) = 0$. Therefore, $u \in H_0^2(D)$ and $v \in L^2(D)$ satisfy

$$\Delta u - \kappa^2 nu = -(n-1)v \qquad \text{and} \qquad \Delta v - \kappa^2 v = 0 \text{ in } D. \tag{3.64}$$

We let $\delta = \|n - 1\|_{L^\infty(D)} ce^{-2d\kappa}$, where $c > 0$ is the constant appearing in Lemma 3.14. Multiplying the first equation in (3.64) by $\overline{v}$, integrating over $D$, and using Green's second identity and the second equation in (3.64) yields

$$\int_D \kappa^2(n-1)u\overline{v}\,dx = \int_D (n-1)|v|^2\,dx. \tag{3.65}$$

Multiplying the first equation in (3.64) by $\overline{u}$, integrating over $D$, and using Green's first identity together with (3.65) yields

$$\int_D \left[|\nabla u|^2 + \kappa^2 n|u|^2\right]\,dx = \int_D (n-1)v\overline{u}dx = \frac{1}{\kappa^2}\int_D (\overline{n}-1)|v|^2\,dx. \tag{3.66}$$

Since $\Re(n) > n_0$ in $D$ and $\|u\| > 0$, on one hand we see from (3.66) that $\int_D (\Re(n) - 1)|v_j|^2\,dx > 0$. On the other hand, recalling that $\sup_{\mathcal{N}} \Re(n) < 1$, from Lemma 3.14 it follows that

$$\begin{aligned}\int_D (\Re(n)-1)|v|^2\,dx &= \int_{\mathcal{N}} (\Re(n)-1)|v|^2\,dx + \int_{D\setminus\mathcal{N}} (\Re(n)-1)|v|^2\,dx \\ &= \int_{\mathcal{N}} (\Re(n)-1)|v|^2\,dx + O(e^{-2d\kappa}) < 0\end{aligned}$$

for $\kappa > 0$ sufficiently large, which is a contradiction. $\square$

**Remark 3.19.** The proof of Theorem 3.18 in fact shows that there are possibly at most finitely many transmission eigenvalues in the imaginary axis.

**Remark 3.20.** It is possible to prove the result in Theorem 3.18 also for the case when $\inf_{\mathcal{N}} \Re(n) > 1$. We do not present the proof here since it is more technical. Instead we refer the reader to the proof of Theorem 2.7 in [111] for the proof of this result when $\Im(n) = 0$ almost everywhere in $D$ and $\inf_{\mathcal{N}} n > 1$. The proof there can be easily adapted to the case of $\Im(n) > 0$ and $\inf_{\mathcal{N}} \Re(n) > 1$.

We are ready now to state the result concerning the discreteness of transmission eigenvalues.

**Theorem 3.21.** *Assume that $n \in L^\infty(D)$ with $\Re(n) > n_0 > 0$, $\Im(n) \geq 0$ almost everywhere in $D$, and either $\sup_{\mathcal{N}} \Re(n) < 1$ or $\inf_{\mathcal{N}} \Re(n) > 1$ for some neighborhood $\mathcal{N}$ of the boundary $\partial D$. Then the set of transmission eigenvalues is at most discrete with $+\infty$ as the only accumulation point.*

***Proof.*** As discussed above, transmission eigenvalues are the values of $k \in \mathbb{C}$ for which the kernel of $A_k$ is nontrivial. Thanks to Theorem 3.18 and Remark 3.20 we choose $\kappa_0 > 0$

such that $A_{i\kappa_0}$ is invertible and write the equation $A_k(u,v) = 0$ in the form

$$(u,v) + A_{i\kappa_0}^{-1}(A_k - A_{i\kappa_0})(u,v) = 0.$$

Now the fact that $A_k - A_{i\kappa_0} : X(D) \to X(D)$ is compact, due to Theorem 3.13, allows us to prove the result of the theorem by appealing to the Analytic Fredholm Theorem 1.12. $\square$

**Remark 3.22.** It is possible to relax the coercivity assumption on the contrast $n-1$ in the case when $n$ is complex valued. More specifically, in [159] it is shown that transmission eigenvalues form at most a discrete set if $\inf_{\mathcal{N}} \Re(e^{i\theta}(n-1)) > 0$ in some neighborhood $\mathcal{N}$ of the boundary $\partial D$ for some $\theta \in (-\pi/2, \pi/2)$. These assumptions are not optimal. Nevertheless it seems that some type of sign condition on the contrast $n-1$ near the boundary is necessary for the interior transmission problem to be of Fredholm type [21].

### 3.1.4 ▪ Boundary Integral Equation Method

In this section we introduce an alternative approach to studying the interior transmission problem (3.1) based on boundary integral equations. Although the boundary integral method recovers the same type of solvability results discussed in the previous sections of this chapter, we believe that it merits discussion in this monograph for its mathematical and computational interest. Our presentation follows closely [79].

We start by assuming that the refractive index $0 < n \neq 1$ is a positive constant different from one and that $\partial D$ is a smooth surface of class $C^2$ (the smoothness of the boundary is needed for certain mapping properties of boundary integral operators although this assumption is not necessary in the analysis of the interior transmission problem). Introducing the notation $k_n := \sqrt{n}k$, the interior transmission problem for this particular case reads as follows: Given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ find $w \in L^2(D)$, $v \in L^2(D)$, such that $w - v \in H^2(D)$ satisfies

$$\begin{cases} \Delta w + k_n^2 w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w - v = f & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \partial D. \end{cases} \tag{3.67}$$

We recall the fundamental solution to the Helmholtz equation introduced in (1.8),

$$\Phi_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y, \tag{3.68}$$

where here we indicate the dependence on $k$. A formal application of Green's representation formula to the solution $v$ and $w$ of (3.67) gives that for $x \in D$

$$v(x) = \int_{\partial D} \left( \frac{\partial v(y)}{\partial \nu_y}\Phi_k(x,y) - v(y)\frac{\partial}{\partial \nu_y}\Phi_k(x,y) \right) ds_y, \tag{3.69}$$

$$w(x) = \int_{\partial D} \left( w(y)\frac{\partial}{\partial \nu_y}\Phi_{k_n}(x,y) - \frac{\partial w(y)}{\partial \nu_y}\Phi_{k_n}(x,y) \right) ds_y. \tag{3.70}$$

Now for a generic function $u$ defined in $\mathbb{R}^3 \setminus \partial D$ we denote

$$u^{\pm}(x) = \lim_{h \to 0^+} \nu \cdot u(x \pm h\nu), \qquad x \in \partial D,$$

$$\frac{\partial u(x)}{\partial \nu}^{\pm} = \lim_{h \to 0^+} \nu \cdot \nabla u(x \pm h\nu), \qquad x \in \partial D,$$

where we recall that $\nu$ is the unit outward normal vector to $\partial D$. We denote by $\mathcal{S}_k$ and $\mathcal{D}_k$ the single and double layer boundary potentials defined by

$$\left(\mathcal{S}_k\psi\right)(x) := \int_{\partial D} \psi(y)\Phi_k(x,y)\,dy, \qquad x \in \mathbb{R}^3 \setminus \partial D, \tag{3.71}$$

$$\left(\mathcal{D}_k\psi\right)(x) := \int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi_k(x,y)\,dy, \qquad x \in \mathbb{R}^3 \setminus \partial D, \tag{3.72}$$

with similar expressions for $\mathcal{S}_{k_n}$ and $\mathcal{D}_{k_n}$. It can be shown [99], [119], [133] that for $-1 \le s \le 1$, the mapping $\mathcal{S}_k : H^{s-\frac{1}{2}}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^3)$ is continuous and the mappings $\mathcal{D}_k : H^{s+\frac{1}{2}}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^3 \setminus \overline{D})$ and $\mathcal{D}_k : H^{s+\frac{1}{2}}(\partial D) \to H^{s+1}(D)$ are continuous. We define the restriction of $\mathcal{S}_k$ and $\mathcal{D}_k$ to the boundary $\partial D$ by

$$(S_k\psi)(x) := \int_{\partial D} \psi(y)\Phi(x,y)ds_y, \qquad x \in \partial D, \tag{3.73}$$

$$(K_k\psi)(x) := \int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y, \qquad x \in \partial D, \tag{3.74}$$

and the restriction of the normal derivative of $\mathcal{S}_k$ and $\mathcal{D}_k$ to the boundary $\partial D$ by

$$(K'_k\psi)(x) := \frac{\partial}{\partial \nu_x}\int_{\partial D} \psi(y)\Phi(x,y)ds_y, \qquad x \in \partial D, \tag{3.75}$$

$$(T_k\psi)(x) := \frac{\partial}{\partial \nu_x}\int_{\partial D} \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y, \qquad x \in \partial D. \tag{3.76}$$

It is known that [99], [133]

$$S_k : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{\frac{1}{2}+s}(\partial D), \qquad K_k : H^{\frac{1}{2}+s}(\partial D) \longrightarrow H^{\frac{1}{2}+s}(\partial D), \tag{3.77}$$

$$K'_k : H^{-\frac{1}{2}+s}(\partial D) \longrightarrow H^{-\frac{1}{2}+s}(\partial D), \qquad T_k : H^{\frac{1}{2}+s}(\partial D) \longrightarrow H^{-\frac{1}{2}+s}(\partial D), \tag{3.78}$$

are continuous for $-1 \le s \le 1$. It can be shown [119] that for smooth densities the single layer potential and the normal derivative of the double layer potential are continuous across $\partial D$, i.e.,

$$(\mathcal{S}_k\psi)^+ = (\mathcal{S}_k\psi)^- = S_k\psi \qquad \text{on } \partial D, \tag{3.79}$$

$$\frac{\partial(\mathcal{D}_k\psi)^+}{\partial \nu} = \frac{\partial(\mathcal{D}_k\psi)^-}{\partial \nu} = T_k\psi \qquad \text{on } \partial D, \tag{3.80}$$

while the normal derivative of the single layer potential and the double layer potential are discontinuous across $\partial D$ and satisfy the following jump relations:

$$\frac{\partial(\mathcal{S}_k\psi)^{\pm}}{\partial \nu} = K'_k\psi \mp \frac{1}{2}\psi \qquad \text{on } \partial D, \tag{3.81}$$

$$(\mathcal{D}_k\psi)^{\pm} = K_k\psi \pm \frac{1}{2}\psi \qquad \text{on } \partial D. \tag{3.82}$$

As the reader has already seen, the solutions $v$ and $w$ of the interior transmission problem (3.1) are simply $L^2_\Delta(D)$ functions, where

$$L^2_\Delta(D) := \left\{u \in L^2(D) \quad \text{such that } \Delta u \in L^2(D)\right\}$$

with a similar definition for $L^2_\Delta(\mathbb{R}^3 \setminus \overline{D})$. Therefore (cf. Remark 3.6) their trace and their normal derivative on the boundary live in $H^{-\frac{1}{2}}(\partial D)$ and $H^{-\frac{3}{2}}(\partial D)$, respectively. Hence the representation formulas (3.69) and (3.70) suggest that we must work with single layer potentials $\mathcal{S}_k$ with density in $H^{-\frac{3}{2}}(\partial D)$ and double layer potentials $\mathcal{D}_k$ with density in $H^{-\frac{1}{2}}(\partial D)$ (i.e., for $s = -1$ in the above). Both obviously satisfy the Helmholtz equation in the distributional sense, and hence we can conclude that $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$, $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^3 \setminus \overline{D})$, and $\mathcal{D}_k : H^{-\frac{1}{2}}(\partial D) \to L^2_\Delta(D)$, $L^2_\Delta(\mathbb{R}^3 \setminus \overline{D})$ are continuous. More importantly, by a duality argument, it is possible to extend the jump relations (3.79), (3.80), (3.81), and (3.82) to the case of potentials with weaker densities. More specifically, the following lemma is proven in Theorem 3.1 in [79] (see also [133]).

**Lemma 3.23.** *The single layer potential* $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$*,* $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(\mathbb{R}^3 \setminus \overline{D})$ *and the double layer potential* $\mathcal{D}_k : H^{-\frac{1}{2}}(\partial D) \to L^2_\Delta(D)$*,* $L^2_\Delta(\mathbb{R}^3 \setminus \overline{D})$ *satisfy the jump relations on* $\partial D$

$$(\mathcal{S}_k\psi)^+ = (\mathcal{S}_k\psi)^- = S_k\psi \quad \text{and} \quad \frac{\partial(\mathcal{S}_k\psi)^\pm}{\partial\nu} = K'_k\psi \mp \frac{1}{2}\psi \qquad \text{in } H^{-\frac{1}{2}}(\partial D),$$
$$(\mathcal{D}_k\psi)^\pm = K_k\psi \pm \frac{1}{2}\psi \quad \text{and} \quad \frac{\partial(\mathcal{D}_k\psi)^+}{\partial\nu} = \frac{\partial(\mathcal{D}_k\psi)^-}{\partial\nu} = T_k\psi \quad \text{in } H^{-\frac{3}{2}}(\partial D),$$

*where the bounded linear operators*

$$S_k : H^{-\frac{3}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D), \qquad K_k : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{1}{2}}(\partial D),$$
$$K'_k : H^{-\frac{3}{2}}(\partial D) \longrightarrow H^{-\frac{3}{2}}(\partial D), \qquad T_k : H^{-\frac{1}{2}}(\partial D) \longrightarrow H^{-\frac{3}{2}}(\partial D),$$

*are given by* (3.73)*,* (3.74)*,* (3.75)*, and* (3.76)*, respectively.*

To arrive at a system of boundary integral equations equivalent to the interior transmission problem (3.1) for $v \in L^2(D)$ and $w \in L^2(D)$ we introduce two unknowns

$$\alpha := \left.\frac{\partial v}{\partial \nu}\right|_{\partial D} \in H^{-\frac{3}{2}}(\partial D) \qquad \text{and} \qquad \beta := v|_{\partial D} \in H^{-\frac{1}{2}}(\partial D) \tag{3.83}$$

and use the ansatz (3.69) and (3.70) along with the boundary conditions in (3.1) to write

$$v = \mathcal{S}_k\alpha - \mathcal{D}_k\beta \qquad \text{and} \qquad w = \mathcal{S}_{k_n}\alpha - \mathcal{D}_{k_n}\beta + \mathcal{S}_{k_n}h - \mathcal{D}_{k_n}f, \tag{3.84}$$

where we note

$$\left.\frac{\partial w}{\partial \nu}\right|_{\partial D} = \alpha + f \qquad \text{and} \qquad w|_{\partial D} = \beta + h. \tag{3.85}$$

Using the jump relations in Lemma 3.23 and once again the boundary conditions in (3.1) we arrive at the following system of integral equations:

$$Z_n(k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = F_n(k)\begin{pmatrix} h \\ f \end{pmatrix}, \tag{3.86}$$

where

$$Z_n(k) := \begin{pmatrix} S_{k_n} - S_k & -K_{k_n} + K_k \\ -K'_{k_n} + K'_k & T_{k_n} - T_k \end{pmatrix} \tag{3.87}$$

and

$$F_n(k) := \begin{pmatrix} -S_{k_n} & \frac{1}{2}I + K_{k_n} \\ -\frac{1}{2} + K'_{k_n} & -T_{k_n} \end{pmatrix}.$$

Since $h \in H^{\frac{1}{2}}(\partial D)$ and $f \in H^{\frac{3}{2}}(\partial D)$, the mapping properties (3.77) and (3.78) for $s = 1$ imply that $F_n(k)\binom{h}{f} \in H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$.

To understand the mapping properties of the operator $Z_n(k)$ we must recall some regularity results concerning single and double layer potentials and consequently the associated boundary integral operators. Notice that the components in (3.87) are more regular that each of the operators involved since the singular part, which is independent of $k$, cancels.

**Lemma 3.24.** *Assume that $k, k_n \in \mathbb{C}$ have nonzero real part. Then the operators $\mathcal{S}_k - \mathcal{S}_{k_n} : H^{-\frac{3}{2}}(\partial D) \to H^3(D)$ and $\mathcal{D}_k - \mathcal{D}_{k_n} : H^{-\frac{1}{2}}(\partial D) \to H^3(D)$ are continuous.*

***Proof.*** We sketch here the proof following the proof of Theorem 3.2 in [79]. First we notice that $\mathcal{V}_k - \mathcal{V}_{k_n}$, where $\mathcal{V}_k$ is the volume potential defined by

$$(\mathcal{V}_k\psi)(x) := \int_D \psi(y)\Phi_k(x,y)\,dy,$$

is a pseudodifferential operator of order $-4$. This follows from applying Theorem 7.1.1 in [99] on integral operators with pseudohomogeneous kernels to the operator $\mathcal{V}_k - \mathcal{V}_{k_n}$ whose kernel takes the form $a(x, x-y)$, where

$$\begin{aligned} a(x,z) &:= \frac{e^{ik|z|} - e^{ik_n|z|}}{4\pi|z|} \\ &= \frac{i}{4\pi}(k - k_n) - \frac{1}{4\pi}\sum_{j=0}^{\infty} \frac{i^j}{(j+2)!}(k^{j+2} - k_n^{j+2})|z|^{j+1}. \end{aligned}$$

Now using Theorem 8.5.8 in [99] it is possible to deduce from this the regularity result for the difference of the single layer potentials $\mathcal{S}_k - \mathcal{S}_{k_n}$. Finally, the fact that

$$(\mathcal{D}_k - \mathcal{D}_{k_n})\psi = -\nabla \cdot (\mathcal{S}_k - \mathcal{S}_{k_n})(\nu\psi) \tag{3.88}$$

implies that the regularity result for $\mathcal{D}_k - \mathcal{D}_{k_n}$ can also be deduced from the regularity property of the difference of the single layer potentials. $\square$

Later on in our analysis we would like to decompose the operator $Z_n(k)$ into an invertible operator and a compact operator. Hence we will need to find more regular operators, and the way to achieve this is to eliminate the principal part in the asymptotic expansion of the kernel of the operator $\mathcal{V}_k - \mathcal{V}_{k_n}$. To this end we consider the operator

$$(\mathcal{V}_k - \mathcal{V}_{k_n}) + \gamma(k)(\mathcal{V}_{i|k|} - \mathcal{V}_{i|k_n|}),$$

where

$$\gamma(k) := \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}$$

and which has the kernel $\tilde{a}(x, x-y)$, where

$$\begin{aligned}\tilde{a}(x,z) &:= \frac{e^{ik|z|} - e^{ik_n|z|}}{4\pi|z|} + \frac{e^{-|kz|} - e^{-|k_n z|}}{4\pi|z|} \\ &= \frac{1}{4\pi}\left[i(k-k_n) - \frac{k^2-k_n^2}{|k|+|k_n|}\right] - \sum_{j=0}^{\infty} \tilde{a}_{j+2}(x,z)\end{aligned}$$

with

$$\tilde{a}_{j+2}(x,z) := \frac{1}{4\pi(j+3)!}\left[i^{j+1}(k^{j+3} - k_n^{j+3}) + (-1)^j\left(|k|^{j+3} - |k_n|^{j+3}\right)\gamma(k)\right]|z|^{j+2}$$

for all $j \geq 0$, which satisfies

$$\tilde{a}_p(x, tz) = t^p \tilde{a}_p(x,z).$$

From [99, Theorem 7.1.1], we deduce that

$$\left((\mathcal{V}_k - \mathcal{V}_{k_n}) + \gamma(k)(\mathcal{V}_{i|k|} - \mathcal{V}_{i|k_n|})\right)\varphi(x) = \int_D \tilde{a}(x, x-y)\varphi(y)dy$$

is a pseudodifferential operator of order $-5$ since $\tilde{a}$ is a pseudohomogeneous kernel of degree 2. Then, applying Theorem 8.5.8 in [99] and (3.88), we can immediately prove the following regularity result for the operators $(\mathcal{S}_k - \mathcal{S}_{k_n}) + \gamma(k)(\mathcal{S}_{i|k|} - \mathcal{S}_{i|k_n|})$ and $(\mathcal{D}_k - \mathcal{D}_{k_n}) + \gamma(k)(\mathcal{D}_{i|k|} - \mathcal{D}_{i|k_n|})$.

**Lemma 3.25.** *Assume that $k, k_n \in \mathbb{C}$ have nonzero real part. Then the operators*

$$(\mathcal{S}_k - \mathcal{S}_{k_n}) + \gamma(k)(\mathcal{S}_{i|k|} - \mathcal{S}_{i|k_n|}) : H^{-\frac{3}{2}}(\partial D) \to H^3(D)$$

*and*

$$(\mathcal{D}_k - \mathcal{D}_{k_n}) + \gamma(k)(\mathcal{D}_{i|k|} - \mathcal{D}_{i|k_n|}) : H^{-\frac{1}{2}}(\partial D) \to H^3(D)$$

*are continuous.*

We now return to our main system of integral equation (3.86) which, if it is uniquely solvable, is equivalent to the interior transmission problem (3.1) via Green's representation formulas (3.69) and (3.70) using (3.83) and (3.85). Lemma 3.24 implies that $Z_n(k) : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is continuous.

In the next step we want to show that $Z_n(k)$ is a Fredholm operator of index zero. To this end we decompose $Z_n(k)$ as

$$Z_n(k) = -\gamma(k) Z_n(i|k|) + \left(Z_n(k) + \gamma(k) Z_n(i|k|)\right).$$

From Lemma 3.25 and the classic trace theorems we know that $Z_n(k) + \gamma(k)Z(i|k|) : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is compact. Hence it suffices to show that $Z_n(i|k|) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is invertible.

**Lemma 3.26.** $Z_n(i|k|) : H^{-3/2}(\partial D) \times H^{-1/2}(\partial D) \to H^{3/2}(\partial D) \times H^{1/2}(\partial D)$ *is coercive, i.e.,*

$$\left|\left\langle Z_n(i|k|)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\rangle\right| \geq C\left(\|\alpha\|^2_{H^{\frac{3}{2}}(\partial D)} + (\|\beta\|^2_{H^{\frac{1}{2}}(\partial D)}\right),$$

*where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$.*

***Proof.*** For simplicity we set $\kappa := |k|$ and $\kappa_n := |k_n|$. Let $\alpha$ be in $H^{-3/2}(\partial D)$ and $\beta \in H^{-1/2}(\partial D)$ and consider the following problem:

$$\begin{cases} (\Delta - \kappa^2)(\Delta - \kappa_n^2)u = 0 & \text{in } \mathbb{R}^3 \backslash \partial D, \\ [\Delta u]_{\partial D} = \beta(\kappa_n^2 - \kappa^2) & \text{on } \partial D, \\ \left[\dfrac{\partial(\Delta u)}{\partial \nu}\right]_{\partial D} = \alpha(\kappa_n^2 - \kappa^2) & \text{on } \partial D, \end{cases} \tag{3.89}$$

where for a generic function $u$, $[u] := u^+ - u^-$ denotes the jump of $u$ across the boundary $\partial D$. Multiplying the first equation by $\varphi \in H^2(\mathbb{R}^3)$, integrating by parts on both sides of $\partial D$, and using the jump conditions on $\partial D$, we can reformulate (3.89) as the following variational problem: Find $u \in H^2(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3\backslash\partial D} (\Delta u - \kappa^2 u)(\Delta\overline{\varphi} - \kappa_n^2\overline{\varphi})dx = -\int_{\partial D} (\kappa_n^2 - \kappa^2)\left(\alpha\overline{\varphi} - \beta\frac{\partial\overline{\varphi}}{\partial\nu}\right) ds \tag{3.90}$$

for all $\varphi \in H^2(\mathbb{R}^3)$. We remark that $u = (\mathcal{S}_{i\kappa_n} - \mathcal{S}_{i\kappa})\alpha - (\mathcal{D}_{i\kappa_n} - \mathcal{D}_{i\kappa})\beta$ obviously solves (3.90). Using the Lax–Milgram theorem, the existence and uniqueness of a solution $u \in H^2(\mathbb{R}^3)$ to (3.90) can be established. Thus the only solution to (3.90) is $u = (\mathcal{S}_{i\kappa_n} - \mathcal{S}_{i\kappa})\alpha - (\mathcal{D}_{i\kappa_n} - \mathcal{D}_{i\kappa})\beta$. In particular,

$$u|_{\partial D} = (S_{i\kappa_n} - S_{i\kappa})\alpha - (K_{i\kappa_n} - K_{i\kappa})\beta \quad \text{and} \quad \left.\frac{\partial u}{\partial\nu}\right|_{\partial D} = (K'_{i\kappa_n} - K'_{i\kappa})\alpha - (T_{i\kappa_n} - T_{i\kappa})\beta.$$

Taking $\varphi = u$ in (3.90) we obtain

$$\int_{\mathbb{R}^3\backslash\partial D} (\Delta u - \kappa^2 u)(\Delta\overline{u} - \kappa_n^2\overline{u})dx = -\int_{\partial D} (\kappa_n^2 - \kappa^2)\left(\alpha\overline{u} - \beta\frac{\partial\overline{u}}{\partial\nu}\right) ds. \tag{3.91}$$

The inequality

$$\int_{\mathbb{R}^3\backslash\partial D} (\Delta u - \kappa^2 u)(\Delta\overline{u} - \kappa_n^2\overline{u})dx \geq C\|u\|^2_{H^2(\mathbb{R}^3)}$$

along with (3.91) implies that

$$\left| \langle \alpha, u\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} - \left\langle \beta, \frac{\partial u}{\partial\nu}\right\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)} \right| \geq C'\|u\|^2_{H^2(\mathbb{R}^3)}. \tag{3.92}$$

Next we want to show that there exists $C_1 > 0$ such that $\|\alpha\|_{H^{-\frac{3}{2}}(\partial D)} \leq C_1\|u\|_{H^2(\mathbb{R}^3)}$. To this end, we take $\varphi \in H^{3/2}(\partial D)$ such that $\|\varphi\|_{H^{3/2}(\partial D)} = 1$. Then there exists $\tilde{\varphi} \in H^2(\mathbb{R}^3)$ such that $\tilde{\varphi}|_{\partial D} = \varphi$ and $\frac{\partial\tilde{\varphi}}{\partial\nu}|_{\partial D} = 0$. From (3.90) we have that

$$\begin{aligned} \left|\langle\alpha, \varphi\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)}\right| &= \frac{1}{|\kappa_n^2 - \kappa^2|}\left|\int_{\mathbb{R}^3\backslash\partial D} (\Delta u - \kappa^2 u)(\Delta\overline{\tilde{\varphi}} - \kappa_n^2\overline{\tilde{\varphi}})dx\right| \\ &\leq C\|u\|_{H^2(\mathbb{R}^3)}\|\tilde{\varphi}\|_{H^2(\mathbb{R}^3)} \leq C_1\|u\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

because $\|\tilde{\varphi}\|_{H^2(\mathbb{R}^3)} \leq \|\varphi\|_{H^{3/2}(\partial D)} = 1$. Hence $\|\alpha\|_{H^{-3/2}(\partial D)} \leq C_1\|u\|_{H^2(\mathbb{R}^3)}$.

Similarly we show that $\|\beta\|_{H^{-1/2}(\partial D)} \leq C_2\|u\|_{H^2(\mathbb{R}^3)}$ for some constant $C_2 > 0$. Indeed, take $\psi \in H^{1/2}(\partial D)$ such that $\|\psi\|_{H^{1/2}(\partial D)} = 1$ and choose $\tilde{\psi} \in H^2(\mathbb{R}^3)$ such that $\tilde{\psi}|_{\partial D} = 0$ and $\frac{\partial \tilde{\psi}}{\partial \nu}|_{\partial D} = \psi$. Then

$$\left|\langle \beta, \psi \rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)}\right| = \frac{1}{|\kappa_n^2 - \kappa^2|} \left| \int\limits_{\mathbb{R}^3 \setminus \partial D} (\Delta u - \kappa^2 u)(\Delta \overline{\tilde{\psi}} - \kappa_n^2 \overline{\tilde{\psi}}) dx \right|$$
$$\leq C \|u\|_{H^2(\mathbb{R}^3)} \|\tilde{\psi}\|_{H^2(\mathbb{R}^3)} \leq C_2 \|u\|_{H^2(\mathbb{R}^3)}$$

since $\|\tilde{\psi}\|_{H^2(\mathbb{R}^3)} \leq \|\psi\|_{H^{1/2}(\partial D)} = 1$, whence $\|\beta\|_{H^{-1/2}(\partial D)} \leq C_2 \|u\|_{H^2(\mathbb{R}^3)}$.

We have now all the ingredients to show the coercivity property for $Z(i|k|)$. Thus,

$$\left| \left\langle Z_n(i|k|) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle \right| = \left| \langle (S_{i\kappa_n} - S_{i\kappa})\alpha - (K_{i\kappa_n} - K_{i\kappa})\beta, \alpha \rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} \right.$$
$$\left. + \left\langle -(K'_{i\kappa_n} - K'_{i\kappa})\alpha + (T_{i\kappa_n} - T_{i\kappa})\beta, , \beta \right\rangle_{H^{\frac{1}{2}}(\partial D), H^{-\frac{1}{2}}(\partial D)} \right|$$
$$\geq \left| \langle u|_{\partial D}, \alpha \rangle_{H^{\frac{3}{2}}(\partial D), H^{-\frac{3}{2}}(\partial D)} + \left\langle -\frac{\partial u}{\partial \nu}|_{\partial D}, \beta \right\rangle_{H^{\frac{1}{2}}(\partial D), H^{-\frac{1}{2}}(\partial D)} \right|$$
$$\geq C' \|u\|^2_{H^2(\mathbb{R}^3)} \geq \frac{C'}{C_1} \|\alpha\|^2_{H^{-3/2}(\partial D)} + \frac{C'}{C_2} \|\beta\|^2_{H^{-1/2}(\partial D)},$$

which proves the result. □

Summarizing the above analysis, we can now state the following result.

**Theorem 3.27.** *The operator $Z_n(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is Fredholm with index zero and is analytic on $k \in \mathbb{C} \setminus \mathbb{R}^-$. The kernel of $Z_n(k)$ is trivial for all $k \in \mathbb{C} \setminus \mathbb{R}^-$ except for at most a discrete set with $+\infty$ as the only possible accumulation point.*

***Proof.*** Thanks to Lemma 3.25 along with the classic trace theorems and Lemma 3.26, the operator $Z_n(k)$ is the sum of the compact operator $Z_n(k) + \gamma(k)Z(i|k|)$ and the coercive operator $-\gamma(k)Z(i|k|)$. Hence it is Fredholm of index zero. The analyticity of $Z_n(k)$ on $k$ is a direct consequence of the fact that the kernels of the boundary integral operators that compose $Z_n(k)$ are analytic functions of $k \in \mathbb{C} \setminus \mathbb{R}^-$. Finally, since $Z(i\kappa)$ for $\kappa > 0$ is invertible, an application of the Analytic Fredholm Theorem 1.12 implies that the kernel of $Z_n(k)$ is trivial for all $k \in \mathbb{C} \setminus \mathbb{R}^-$ except for at most a discrete set with $+\infty$ as the only possible accumulation point. □

We remark that the set of values of $k \in \mathbb{C}$ for which the kernel of $Z_n(k)$ fails to be trivial is larger than the set of transmission eigenvalues. In addition to transmission eigenvalues, it also contains the so-called exterior transmission eigenvalues (see [38] and [72] for the relevance of the exterior transmission eigenvalues to the scattering theory of inhomogeneous media). The next theorem shows the relation between transmission eigenvalues and the kernel of $Z_n(k)$ using the fact that in the Green's representation (3.69) and (3.70) a solution to (3.1) corresponds to nonradiating fields. To this end, let

$$P^\infty(\alpha, \beta)(\hat{x}) = \frac{1}{4\pi} \int\limits_{\partial D} \left( \beta(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \alpha(y) e^{-ik\hat{x}\cdot y} \right) ds(y),$$
$$P_n^\infty(\alpha, \beta)(\hat{x}) = \frac{1}{4\pi} \int\limits_{\partial D} \left( \beta(y) \frac{\partial e^{-ik_n\hat{x}\cdot y}}{\partial \nu(y)} - \alpha(y) e^{-ik_n\hat{x}\cdot y} \right) ds(y),$$

which are the far field patterns of $v$ and $w$ defined by (3.69) and (3.70), respectively.

**Theorem 3.28.** *The following statements are equivalent:*

(i) *There exist a nontrivial solution* $v, w \in L^2(D)$ *to* (3.1) *such that* $w - v \in H^2(D)$.

(ii) *There exist* $\alpha \neq 0$ *in* $H^{-3/2}(\partial D)$ *and* $\beta \neq 0$ *in* $H^{-1/2}(\partial D)$ *such that*

$$Z_n(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P^\infty(\alpha,\beta) = 0.$$

(iii) *There exist* $\alpha \neq 0$ *in* $H^{-3/2}(\partial D)$ *and* $\beta \neq 0$ *in* $H^{-1/2}(\partial D)$ *such that*

$$Z_n(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \quad \textit{and} \quad P_n^\infty(\alpha,\beta) = 0.$$

***Proof.*** From the construction of the operator $Z_n$, it remains to show that (ii) implies (i) and that (iii) implies (i). Assume that there exist $\alpha \in H^{-1/2}(\partial D)$ and $\beta \in H^{1/2}(\partial D)$ satisfying $Z_n(k)\binom{\alpha}{\beta} = 0$. We define $v = \mathcal{S}_k\alpha - \mathcal{D}_k\beta$ and $w = \mathcal{S}_{k_n}\alpha - \mathcal{D}_{k_n}\beta$ in $\mathbb{R}^3 \setminus \partial D$. The mapping properties of single and double layer potentials show that $v$ and $w$ are in $L^2(D)$ and $w - v \in H^2(D)$ and they satisfy $\Delta v + k^2 v = 0$ and $\Delta w + k^2 n w = 0$ in $D$. Now assume that $P^\infty(\alpha,\beta) = 0$. We want to show that $v \neq 0$. From Rellich's Lemma we deduce that $v = 0$ in $\mathbb{R}^d \backslash D$. Assume that $v = 0$ also in $D$. We have in particular that $[v]_{\partial D} = \left[\frac{\partial v}{\partial \nu}\right]_{\partial D} = 0$, and from the jump properties of the single and double layer potentials we also have that $[v]_{\partial D} = -\beta$ and $\left[\frac{\partial v}{\partial \nu}\right]_{\partial D} = -\alpha$. This contradicts the fact that $(\alpha,\beta) \neq (0,0)$. Then $v \neq 0$ in $D$. In a similar way we can show that if $P_n^\infty(\alpha,\beta) = 0$, then $w \neq 0$. $\square$

We can now use the integral equation framework to study the solvability of the interior transmission problem and show the discreteness of transmission eigenvalues for media with contrasts. To present the idea we first consider piecewise homogeneous media where we assume that $D = \overline{D}_1 \cup \overline{D}_2$ such that $D_1 \subset D$ and $D_2 := D \setminus \overline{D}_1$ and consider the simple case when $n := n_1$ in $D_1$ and $n := n_2$ in $D_2$, where $n_1 > 0$, $n_2 > 0$ are two positive constants such that $(n_1 - 1)(n_2 - 1) < 0$. We denote $\Sigma = \partial D_1$, which is assumed to be a $C^2$ smooth surface with $\nu$ the unit normal vector to either $\partial D$ or $\Sigma$ outward to $D$ and $D_1$, respectively (see Figure 3.2). We let $k_1 = k\sqrt{n_1}$ and $k_2 = k\sqrt{n_2}$. In the following we use the notation $\mathcal{S}_k^{\partial D}$, $\mathcal{D}_k^{\partial D}$, and $\mathcal{S}_k^{\Sigma}$, $\mathcal{D}_k^{\Sigma}$ in order to differentiate between the potentials with densities defined on $\partial D$ or $\Sigma$. We also use the notation

$$\left(S_k^{\partial D}\psi\right)(x) = \int_{\partial D} \psi(y)\Phi(x,y)ds_y, \qquad x \in \partial D,$$

$$\left(S_k^{\Sigma}\psi\right)(x) = \int_{\Sigma} \psi(y)\Phi(x,y)ds_y, \qquad x \in \Sigma,$$

$$\left(S_k^{\partial D,\Sigma}\psi\right)(x) = \int_{\partial D} \psi(y)\Phi(x,y)ds_y, \qquad x \in \Sigma,$$

$$\left(S_k^{\Sigma,\partial D}\psi\right)(x) = \int_{\Sigma} \psi(y)\Phi(x,y)ds_y, \qquad x \in \partial D,$$

with the respective notation for the other operators $K_k$, $K_k^{'}$, and $T_k$. Letting

$$\alpha := \left.\frac{\partial v}{\partial \nu}\right|_{\partial D} = \left.\frac{\partial w}{\partial \nu}\right|_{\partial D} - f \in H^{-\frac{3}{2}}(\partial D) \quad \text{and} \quad \beta := v|_{\partial D} = w|_{\partial D} - h \in H^{-\frac{1}{2}}(\partial D)$$

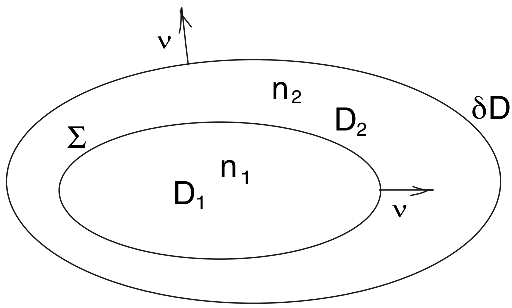



**Figure 3.2.** *Configuration of the geometry for two homogeneous media.*

and

$$\tilde{\alpha} := \left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} \in H^{-\frac{1}{2}}(\Sigma) \qquad \text{and} \qquad \tilde{\beta} := w|_{\Sigma} \in H^{\frac{1}{2}}(\Sigma),$$

the solution to (3.1) can be written as

$$v = \mathcal{S}_k^{\partial D}\alpha - \mathcal{D}_k^{\partial D}\beta \qquad \text{in } D \tag{3.93}$$

and

$$w = \begin{cases} \mathcal{S}_{k_2}^{\partial D}\alpha - \mathcal{D}_{k_2}^{\partial D}\beta + \mathcal{S}_{k_2}^{\partial D}h - \mathcal{D}_{k_2}^{\partial D}f - \mathcal{S}_{k_2}^{\Sigma}\tilde{\alpha} + \mathcal{D}_{k_2}^{\Sigma}\tilde{\beta} & \text{in } D_2, \\ \mathcal{S}_{k_1}^{\Sigma}\tilde{\alpha} - \mathcal{D}_{k_1}^{\Sigma}\tilde{\beta} & \text{in } D_1. \end{cases} \tag{3.94}$$

Note that the interior regularity for the solutions to the Helmholtz equation implies that $v$ and $w$ are at least in $H^1_{loc}(D)$ . Using the boundary conditions on $\partial D$ and continuity of the Cauchy data of $w$ across $\Sigma$, we arrive at the following system of integral equations:

$$\underbrace{\begin{pmatrix} S_{k_2}^{\partial D} - S_k^{\partial D} & -K_{k_2}^{\partial D} + K_k^{\partial D} \\ -K_{k_2}^{'\partial D} + K_k^{'\partial D} & T_{k_2}^{\partial D} - T_k^{\partial D} \end{pmatrix}}_{=Z_{n_2}^{\partial D}(k)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \underbrace{\begin{pmatrix} S_{k_2}^{\Sigma,\partial D} & -K_{k_2}^{\Sigma,\partial D} \\ -K_{k_2}^{'\Sigma,\partial D} & T_{k_2}^{\Sigma,\partial D} \end{pmatrix}}_{=Z^{\Sigma,\partial D}(k)} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} -S_{k_2}^{\partial D} & \frac{1}{2}I + K_{k_2}^{\partial D} \\ -\frac{1}{2} + K_{k_2}^{'\partial D} & -T_{k_2}^{\partial D} \end{pmatrix}}_{=F_{n_2}(k)} \begin{pmatrix} h \\ f \end{pmatrix}$$

and

$$\underbrace{\begin{pmatrix} S_{k_2}^{\Sigma} + S_{k_1}^{\Sigma} & -K_{k_2}^{\Sigma} - K_{k_1}^{\Sigma} \\ -K_{k_2}^{'\Sigma} - K_{k_1}^{'\Sigma} & T_{k_2}^{\Sigma} - T_{k_1}^{\Sigma} \end{pmatrix}}_{=\tilde{Z}_{n_1,n_2}^{\Sigma}(k)} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} -S_{k_2}^{\partial D,\Sigma} & K_{k_2}^{\partial D,\Sigma} \\ K_{k_2}^{'\partial D,\Sigma} & -T_{k_2}^{\partial D,\Sigma} \end{pmatrix}}_{=Z^{\partial D,\Sigma}(k)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The operator $\tilde{Z}^{\Sigma}_{n_1,n_2}(k) : H^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$ is invertible since it corresponds to the following transmission problem: For given $\phi \in H^{\frac{1}{2}}(\Sigma)$ and $\psi \in H^{-\frac{1}{2}}(\Sigma)$ find $v \in H^1(\mathbb{R}^3 \setminus \overline{D_1})$ and $w \in H^1(D_1)$ such that

$$\begin{cases} \Delta w + k^2 n_1 w = 0 \quad \text{in } D_1, \\ \Delta \omega + k^2 n_2 \omega = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D_1}, \\ w - \omega = \phi \quad \text{on } \Sigma, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial \omega}{\partial \nu} = \psi \quad \text{on } \Sigma, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial \omega}{\partial r} - ik\sqrt{n_2}\omega\right) = 0, \end{cases} \tag{3.95}$$

which is well known to be uniquely solvable [29] (see also Chapter 1 of this book). Indeed, using Green's representation formula for the solution $\omega$ and $w$ of (3.95) it is easy to see that (3.95) is equivalent to the integral equation

$$\tilde{Z}^{\Sigma}_{n_1,n_2}(k) \begin{pmatrix} \left.\dfrac{\partial \omega}{\partial \nu}\right|_{\Sigma} \\ \omega|_{\Sigma} \end{pmatrix} = \begin{pmatrix} -S^{\Sigma}_{k_1} & \frac{1}{2}I + K^{\Sigma}_{k_1} \\ -\frac{1}{2} + K'^{\Sigma}_{k_1} & -T^{\Sigma}_{k_1} \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}.$$

The interior transmission problem can clearly be written as

$$\mathcal{Z}(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = F_{n_2}(k) \begin{pmatrix} h \\ f \end{pmatrix}, \tag{3.96}$$

where $\mathcal{Z}(k) : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is given by

$$\mathcal{Z}(k) = Z^{\partial D}_{n_2}(k) + Z^{\Sigma,\partial D}(k) \left(\tilde{Z}^{\Sigma}_{n_1,n_2}(k)\right)^{-1} Z^{\partial D,\Sigma}(k).$$

Now the operator $Z^{\partial D}_{n_2}(k)$ corresponds to the interior transmission problem with $n := n_2$ which is studied above and, thanks to Theorem 3.27, is a Fredholm operator of index zero. Furthermore, the operator $Z^{\Sigma,\partial D}(k)\big(\tilde{Z}^{\Sigma}_{n_1,n_2}(k)\big)^{-1} Z^{\partial D,\Sigma}(k)$ is compact as a product of compact operators and bounded operators. All operators involved in the expression of $\mathcal{Z}(k)$ are analytic $k \in \mathbb{C} \setminus \mathbb{R}^-$. Thus we have shown the following result.

**Theorem 3.29.** *The operator $\mathcal{Z}(k) : H^{-\frac{3}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \times H^{\frac{1}{2}}(\partial D)$ is Fredholm with index zero and is analytic on $k \in \mathbb{C} \setminus \mathbb{R}^-$.*

The idea presented above for the case of a piecewise homogeneous medium can be generalized to a more general case when the medium inside $D_1$ is not necessarily homogeneous. More specifically, in the more general case where the refractive index $n(x)$ in $D_1$ is such that $n \in L^\infty(D_1)$, $\Re(n) \geq \alpha > 0$, $\Im(n) \geq 0$, and $n \neq 1$ is a positive constant in $D_2$, we can use exactly the same approach as above to prove the result in Theorem 3.27 by replacing the fundamental solution $\Phi_{k_1}(\cdot, y)$ with the free space fundamental solution $\mathbb{G}(\cdot, y)$ of

$$\Delta \mathbb{G}(\cdot, y) + k^2 n(x) \mathbb{G}(\cdot, y) = -\delta_y \qquad \text{in } \mathbb{R}^3$$

in the distributional sense together with the Sommerfeld radiation condition, where $n(x)$ is extended by its constant value in $D_2$ to the whole space $\mathbb{R}^3$. Because $\Phi_{k_2}(\cdot, y) - \mathbb{G}(\cdot, y)$

solves the Helmholtz equation with wave number $k_2$ in the neighborhood of $\Gamma$ the mapping properties of the integral operators do not change. We refer the reader to Section 4.2 of [79] for more details.

In fact the above idea can be applied even in a more general case, provided that $n$ is a positive constant not equal to one in a neighborhood of $\partial D$. More precisely, consider a neighborhood $D_2$ of $\partial D$ in $D$ with a $C^2$ smooth boundary (e.g., one can take $D_2$ to be the region in $D$ bounded by $\partial D$ and $\Sigma := \{x - \epsilon\nu(x),\ x \in \partial D\}$ for some $\epsilon > 0$ where $\nu$ is the outward unit normal vector to $\partial D$). Assume that the refractive index in $D_2$ is a positive constant $n \neq 1$, whereas in $D_1 := D \setminus \overline{D_2}$ the refractive index is such that the transmission problem

$$\begin{cases} \Delta w + k^2 n(x) w = 0 \quad \text{in } D_1, \\ \Delta \omega + k^2 n \omega = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D_1}, \\ w - \omega = \phi \quad \text{on } \Sigma, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial \omega}{\partial \nu} = \psi \quad \text{on } \Sigma, \\ \lim\limits_{r \to \infty} r \left( \dfrac{\partial \omega}{\partial r} - ik\sqrt{n_2}\omega \right) = 0 \end{cases} \tag{3.97}$$

is well-posed. Then a result similar to that in Theorem 3.27 holds true in this case. Indeed, without going into details, in $D_2$ we can express $v$ and $w$ by (3.93) and (3.94), respectively, and in $D_1$ we leave the expressions for $v$ and $w$ in the form of a partial differential equation with Cauchy data connected to $w$ in $D_2$. Hence it is possible to obtain an equation of the form (3.96), where the operator $\mathcal{Z}(k)$ is written as

$$\mathcal{Z}(k) = Z_n^{\partial D}(k) + Z^{\Sigma,\partial D}(k) \left(A(k)\right)^{-1} Z^{\partial D,\Sigma}(k),$$

where now $\mathbf{A}(k)$ is the invertible solution operator corresponding to the well-posed transmission problem (3.97).

The above discussion implies that the Fredholm alternative can be applied to the interior transmission problem (3.1), provided that the refractive index is a positive constant different from one in a neighborhood of the boundary $\partial D$ and otherwise satisfies the assumptions for which the direct scattering problem is well-posed. Note that this analysis includes the case when inside $D$ there are obstacles with different types of boundary conditions. The solvability of the interior transmission problem (3.1) for almost all $k \in \mathbb{C}$ amounts to proving that there exists a wave number $k$ which is not a transmission eigenvalue. Assumptions on $n$ under which the latter is true are discussed in Section 3.1.3 and in [159]. It is possible to derive different boundary integral equations equivalent to the interior transmission problem. In [50] the transmission eigenvalue problem is analyzed as one single boundary integral equation in terms of the Dirichlet-to-Neumann or Robin-to-Neumann operators. In particular, when this formulation is used to compute transmission eigenvalues, it results in a noticeable reduction of computational costs.

## 3.2 ▪ Solvability of the Interior Transmission Problem for Anisotropic Media

We turn our attention to the interior transmission problem corresponding to the scattering problem for the anisotropic inhomogeneous media introduced in Section 1.4.2, which reads as follows: Given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$, $\ell_1 \in H^{-1}(D)$, and $\ell_2 \in L^2(D)$, find

$w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = \ell_1 & \text{in } D, \\ \Delta v + k^2 v = \ell_2 & \text{in } D, \\ w - v = f & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \partial D, \end{cases} \tag{3.98}$$

where

$$\frac{\partial u}{\partial \nu_A} := \nu \cdot A\nabla u.$$

**Definition 3.30.** Values of $k \in \mathbb{C}$ for which the homogeneous interior transmission problem

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu} & \text{on } \partial D \end{cases} \tag{3.99}$$

has nontrivial solutions $w \in H^1(D)$ and $v \in H^1(D)$ are called *transmission eigenvalues*.

As in the case of isotropic media we are concerned with whether the interior transmission problem, or a compact perturbation of it, has a unique solution that depends continuously on the data. In many applications discussed in Chapter 2, (3.98) appears with $\ell_1 = \ell_2 = 0$. However, in our presentation here we include possibly nonzero $\ell_1$ and $\ell_2$; this case is needed, for instance, in the proof of the uniqueness theorem in Section 1.4.2. In general we will assume that the support $D \subset \mathbb{R}^3$ of the anisotropic inhomogeneous media has Lipschitz boundary $\partial D$, unless mentioned otherwise, and $\nu$ is the unit normal vector directed outwards to $D$. The assumptions on the constitutive material properties are those introduced in Section 2.5, which we recall here for the sake of the reader's convenience: $A$ is a $3 \times 3$ symmetric matrix with $L^\infty(D)$-entries such that

$$\overline{\xi} \cdot \Re(A)\xi \geq \gamma\,|\xi|^2 \quad \text{and} \quad \overline{\xi} \cdot \Im(A)\xi \leq 0$$

for all $\xi \in \mathbb{C}^3$, almost everywhere for $x \in \overline{D}$ and some constant $\gamma > 0$, whereas $n \in L^\infty(D)$ is a complex valued scalar function such that $\Re(n) > 0$ and $\Im(n) \geq 0$. For the purpose of this section and for later use we make the following notation:

$$\begin{aligned} a_* &:= \inf_D \inf_{|\xi|=1} \xi \cdot \Re(A)\xi > 0, \\ a^* &:= \sup_D \sup_{|\xi|=1} \xi \cdot \Re(A)\xi < \infty, \\ n_* &:= \inf_D \Re(n) > 0 \quad \text{and} \quad n^* := \sup_D \Re(n) < \infty. \end{aligned} \tag{3.100}$$

Various techniques are used to analyze the interior transmission problem depending on the assumptions on the constitutive material parameters $A$ and $n$.

### 3.2.1 ▪ The Case of One Sign Contrast in $A$

In this section we consider the case when the contrast $A - I$ does not change sign in $D$; more specifically, we assume that either $a_* > 1$ or $0 < a^* < 1$. To present our ideas we start the discussion with the case when $a_* > 1$ following [31] and [45] (see also [29]). We first study an intermediate problem called the *modified interior transmission problem*, which turns out to be a compact perturbation of our original transmission problem. The following is the modified interior transmission problem: Given $f \in H^{\frac{1}{2}}(\partial D)$, $h \in H^{-\frac{1}{2}}(\partial D)$, a real valued function $\gamma \in C(\bar{D})$, and two functions $\ell_1 \in L^2(D)$ and $\ell_2 \in L^2(D)$, find $w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\begin{cases} \nabla \cdot A\nabla w - \gamma w = \ell_1 & \text{in } D, \\ \Delta v - v = \ell_2 & \text{in } D, \\ w - v = f & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \partial D. \end{cases} \tag{3.101}$$

This is exactly the problem whose well-posedness is needed in the proof of the uniqueness theorem in Section 1.4.2. We now reformulate (3.101) as an equivalent variational problem. To this end, we define the Hilbert space

$$W(D) := \left\{ \mathbf{v} \in \left(L^2(D)\right)^3 : \nabla \cdot \mathbf{v} \in L^2(D) \quad \text{and} \quad \nabla \times \mathbf{v} = 0 \right\}$$

equipped with the norm $\|\mathbf{v}\|^2_W = \|\mathbf{v}\|^2_{L^2(D)} + \|\nabla \cdot \mathbf{v}\|^2_{L^2(D)}$. We denote by $\langle \cdot,\, \cdot \rangle$ the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. The duality pairing

$$\langle \varphi,\, \boldsymbol{\psi} \cdot \nu \rangle = \int_D \varphi \, \nabla \cdot \boldsymbol{\psi} \, dx + \int_D \nabla \varphi \cdot \boldsymbol{\psi} \, dx \tag{3.102}$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$ will be of particular interest in what follows.

We next introduce the sesquilinear form $\mathcal{A}$ defined on $\{H^1(D) \times W(D)\}^2$ by

$$\begin{aligned} \mathcal{A}(U, V) = &\int_D A\nabla w \cdot \nabla \bar{\varphi} \, dx + \int_D m \, w \, \bar{\varphi} \, dx + \int_D \nabla \cdot \mathbf{v} \, \nabla \cdot \bar{\boldsymbol{\psi}} \, dx + \int_D \mathbf{v} \cdot \bar{\boldsymbol{\psi}} \, dx \\ &- \left\langle w,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle - \langle \bar{\varphi},\, \mathbf{v} \cdot \nu \rangle , \end{aligned} \tag{3.103}$$

where $U := (w, \mathbf{v})$ and $V := (\varphi, \boldsymbol{\psi})$ are in $H^1(D) \times W(D)$. We denote by $L : H^1(D) \times W(D) \to \mathbb{C}$ the bounded antilinear functional given by

$$L(V) = \int_D (\ell_1 \, \bar{\varphi} + \ell_2 \, \nabla \cdot \bar{\boldsymbol{\psi}}) \, dx + \langle \bar{\varphi},\, h \rangle - \left\langle f,\, \bar{\boldsymbol{\psi}} \cdot \nu \right\rangle . \tag{3.104}$$

Then the variational formulation of problem (3.101) is to find $U = (w, \mathbf{v}) \in H^1(D) \times W(D)$ such that

$$\mathcal{A}(U, V) = L(V) \qquad \text{for all} \quad V \in H^1(D) \times W(D). \tag{3.105}$$

The following theorem states the equivalence between problems (3.101) and (3.105); for the proof see Theorem 6.5 of [29].

**Theorem 3.31.** *The problem* (3.101) *has a unique solution* $(w, v) \in H^1(D) \times H^1(D)$ *if and only if the problem* (3.105) *has a unique solution* $U = (w, \mathbf{v}) \in H^1(D) \times W(D)$.

*Moreover if $(w, v)$ is the unique solution to* (3.101)*, then $U = (w, \nabla v)$ is the unique solution to* (3.105)*. Conversely, if $U = (w, \mathbf{v})$ is the unique solution to* (3.105)*, then the unique solution $(w, v)$ to* (3.101) *is such that $\mathbf{v} = \nabla v$.*

We now investigate the modified interior transmission problem in the variational formulation (3.105).

**Lemma 3.32.** *Assume that $a_* > 1$ and $\gamma(x) \geq a_*$. Then problem* (3.105) *has a unique solution $U = (w, \mathbf{v}) \in H^1(D) \times W(D)$. This solution satisfies the a priori estimate*

$$\begin{aligned} \|w\|_{H^1(D)} + \|\mathbf{v}\|_W \leq 2C\frac{a_* + 1}{a_* - 1} \Big( &\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} \Big), \end{aligned} \tag{3.106}$$

*where the constant $C > 0$ is independent of $\ell_1$, $\ell_2$, $f$, $h$, and $a_*$.*

***Proof.*** The trace theorems and Schwarz's inequality ensure the continuity of the antilinear functional $L$ on $H^1(D) \times W(D)$ and the existence of a constant $C$ independent of $\rho_1$, $\rho_2$, $f$, and $h$ such that

$$\|L\| \leq C \left( \|\ell_1\|_{L^2} + \|\ell_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}} \right). \tag{3.107}$$

On the other hand, if $U = (w, \mathbf{v}) \in H^1(D) \times W(D)$, the assumptions that $a_* > 1$ and $\gamma(x) \geq a_*$ imply

$$|\mathcal{A}(U, U)| \geq a_* \|w\|^2_{H^1} + \|\mathbf{v}\|^2_W - 2\,\mathrm{Re}\left(\langle \bar{w},\, \mathbf{v}\rangle\right). \tag{3.108}$$

According to the duality identity (3.102), one has by Schwarz's inequality that

$$|\,\langle \bar{w},\, \mathbf{v}\rangle\,| \leq \|w\|_{H^1} \|\mathbf{v}\|_W$$

and therefore

$$|\mathcal{A}(U, U)| \geq a_* \|w\|^2_{H^1} + \|\mathbf{v}\|^2_W - 2\, \|w\|_{H^1} \|\mathbf{v}\|_W\,.$$

Using the identity $\alpha x^2 + y^2 - 2xy = \frac{\alpha+1}{2}\left(x - \frac{2}{\alpha+1}\, y\right)^2 + \frac{\alpha-1}{2}x^2 + \frac{\alpha-1}{\alpha+1}y^2$ with $\alpha = a_*$, we conclude that

$$|\mathcal{A}(U, U)| \geq \frac{a_* - 1}{a_* + 1} \left( \|\mathbf{v}\|^2_W + \|w\|^2_{H^1} \right),$$

whence $\mathcal{A}$ is coercive. The continuity of $\mathcal{A}$ follows easily from Schwarz's inequality and the classic trace theorems. Lemma 3.32 is now a direct consequence of the Lax–Milgram lemma applied to (3.105). ∎

Combining Theorem 3.31 and Lemma 3.32 gives the following result concerning the well-posedness of the modified interior transmission problem.

**Corollary 3.33.** *Assume that $a_* > 1$ and $\gamma(x) \geq a_*$. Then the modified interior transmission problem* (3.101) *has a unique solution $(w, v)$ that satisfies*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leq c \left( \|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} + \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} \right)$$

*with $c > 0$ independent of $\ell_1$, $\ell_2$, $f$, $h$.*

It is possible to perform the same analysis for the case when $0 < a^* < 1$ and prove a statement similar to that in Corollary 3.33 for $\gamma$ chosen such that $a^* < \gamma < 1$. This is done by arriving at a similar variational formulation where the roles of $w$ and $v$ are interchanged, i.e., making the substitution $\nabla w = \mathbf{w}$ (see [45] for the details).

Summarizing the above analysis, we can state the following result concerning the solvability of interior transmission problem (3.98).

**Theorem 3.34.** *Assume that either $a_* > 1$ or $0 < a^* < 1$. Then the Fredholm alternative can be applied to* (3.98). *In particular if $k$ is not a transmission eigenvalue, then* (3.98) *has a unique solution $(w, v) \in H^1(D) \times H^1(D)$ that satisfies the estimate*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leq c\left(\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} + \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right)$$

*with $c > 0$ independent of $\ell_1$, $\ell_2$, $f$, $h$.*

***Proof.*** Let us consider $a^* > 1$ (the other case can be handled in exactly the same way). Set

$$\mathcal{X}(D) = \left\{(w, v) \in H^1(D) \times H^1(D) : \nabla\cdot A\nabla v \in L^2(D) \text{ and } \Delta w \in L^2(D)\right\} \quad (3.109)$$

and consider the operator $\mathcal{G}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{G}(w, v) = \left(\nabla\cdot A\nabla w - \gamma w, \Delta v - v, (w - v)_{|\partial D}, \left(\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu}\right)_{|\partial D}\right)$$

with a constant $\gamma > 1$. Obviously $\mathcal{G}$ is continuous, and from Corollary 3.33 we know that the inverse of $\mathcal{G}$ exists and is continuous. Now consider the operator $\mathcal{T}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}(w, v) = \left((k^2\, n + \gamma)w,\ (k^2\ + 1)v,\ 0,\ 0\right).$$

From the compact embedding of $H^1(D)$ into $L^2(D)$, the operator $\mathcal{T}$ is compact. Hence the injectivity of $\mathcal{G} + \mathcal{T}$, which is equivalent to $k$ not being a transmission eigenvalue, implies $(\mathcal{G} + \mathcal{T})^{-1}$ exists (i.e., the existence of a unique solution to (3.98)) and is bounded (i.e., this solution satisfies the a priori estimate stated in the formulation of Theorem 3.34). □

In general we cannot conclude the solvability of the interior transmission problem as $k$ may be a transmission eigenvalue (see Definition 3.30). Similarly to the case of isotropic media, it is of great interest to know what assumptions on $A$ and $n$ guarantee that transmission eigenvalues either do not exist or form a countable set. The following theorem concerning the nonexistence of transmission eigenvalues holds under no assumptions on the contrasts $A - I$ and $n - 1$.

**Theorem 3.35.** *Assume that $A \in \left(C^1(D)\right)^{3\times 3}$ and $n \in C(D)$. If either $\Im(n) > 0$ or $\Im\left(\bar{\xi} \cdot A\,\xi\right) < 0$ at a point $x_0 \in D$, then the interior transmission problem* (3.98) *has at most one solution for $k$ real (i.e., there are no transmission eigenvalues).*

***Proof.*** Let $w$ and $v$ be a solution of the homogeneous interior transmission problem (3.99). Applying the divergence theorem to $\overline{w}$ and $A\nabla w$, using the boundary condition, and applying Green's first identity to $\overline{v}$ and $v$, we obtain

$$\int_D \nabla\overline{w}\cdot A\nabla w\,dy - \int_D k^2 n|w|^2\,dy = \int_{\partial D} \overline{w}\cdot\frac{\partial w}{\partial \nu_A}\,dy = \int_D |\nabla v|^2\,dy - \int_D k^2|v|^2\,dy.$$

Hence

$$\Im\left(\int_D \nabla\overline{w}\cdot A\nabla w\,dy\right) = 0 \quad\text{and}\quad \Im\left(\int_D n|v|^2\,dy\right) = 0. \tag{3.110}$$

If $\Im(n) > 0$ at a point $x_0 \in D$, and hence by continuity in a small disk $\Omega_\epsilon(x_0)$, then the second equality of (3.110) and the unique continuation principle (Theorem 17.2.6 in [98]) imply that $v \equiv 0$ in $D$. From the boundary conditions in (3.99), and the integral representation formula, $w$ also vanishes in $D$. In the case when $\Im\left(\bar{\xi}\cdot A\,\xi\right) < 0$ at a point $x_0 \in D$ for all $\xi \in \mathbb{C}^3$, and hence by continuity in a small ball $\Omega_\epsilon(x_0)$, from the first equality of (3.110) we obtain that $\nabla w \equiv 0$ in $\Omega_\epsilon(x_0)$ and from the equation $w \equiv 0$ in $\Omega_\epsilon(x_0)$, whence again from the unique continuation principle $w \equiv 0$ in $D$. Similarly as above, this implies that $v = 0$ also, which ends the proof. $\square$

**Remark 3.36.** The result of Theorem 3.35 holds true for $A \in (L^\infty(D))^{3\times3}$ and $n \in L^\infty(D)$, but in this case one has to assume that either $\Im(n) > 0$ or $\Im\left(\bar{\xi}\cdot A\,\xi\right) < 0$ almost everywhere in $D$

In view of Theorem 3.35 and Remark 3.36 we now assume that both $A$ and $n$ are real valued, and show that under appropriate assumptions the transmission eigenvalues $k \in \mathbb{C}$ form at most a discrete set with $+\infty$ as the only accumulation point. To this end, it suffices to show that there exists a $\kappa \in \mathbb{C}$ which is not a transmission eigenvalue. Indeed, let us define the operator $\mathcal{L}_k$ from $\mathcal{X}(D)$ into $L^2(D)\times L^2(D)\times H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)$ by

$$\mathcal{L}_k(w,v) = \left(\nabla\cdot A\nabla w + k^2 nw, \Delta v + k^2 v, (w-v)_{|\partial D}, \left(\frac{\partial w}{\partial\nu} - \frac{\partial v}{\partial \nu}\right)_{|\partial D}\right),$$

where $\mathcal{X}(D)$ is defined by (3.109). Obviously the family of operators $\mathcal{L}_k$ depends analytically on $k \in \mathbb{C}$. If we can show that $\mathcal{L}_\kappa$ is injective for some $\kappa \in \mathbb{C}$ (i.e., this $\kappa$ is not a transmission eigenvalue), then, thanks to Theorem 3.34, $\mathcal{L}_\kappa^{-1}$ exists and is bounded. Then, writing

$$\mathcal{L}_k = \mathcal{L}_\kappa\left(I - \mathcal{L}_\kappa^{-1}(\mathcal{L}_\kappa - \mathcal{L}_k)\right),$$

the discreteness of transmission eigenvalues follows form the fact that

$$\mathcal{L}_\kappa - \mathcal{L}_k = \left((\kappa^2-k^2)nw,\ (\kappa^2-k^2)v,\ 0,\ 0\right)$$

is compact. The approach to showing that $\mathcal{L}_\kappa$ is injective for some $\kappa \in \mathbb{C}$ depends fundamentally on whether $n \equiv 1$ or $n \not\equiv 1$. Hence in the following we distinguish between these two cases.

**Discreteness of Transmission Eigenvalues for $n \equiv 1$.** Here we assume that $\Im(A) = 0$ and either $a_* > 1$ or $0 < a^* < 1$. The transmission eigenvalue problem for $n \equiv 1$ reads

$$\begin{cases} \nabla\cdot A\nabla w + k^2 w = 0 & \text{in } D,\\ \Delta v + k^2 v = 0 & \text{in } D,\\ w = v & \text{on } \partial D,\\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial\nu} & \text{on } \partial D \end{cases} \tag{3.111}$$

with $v \in H^1(D)$ and $w \in H^1(D)$. The structure of this problem resembles the problem studied in Section 3.1.1. The main idea is to make an appropriate substitution and rewrite it as a transmission eigenvalue problem with contrast in the lower order terms and hence use a fourth order formulation as in Section 3.1.1. This approach was first introduced in [34] and later used in [30] and [46]. To this end, let $w \in H^1(D)$ and $v \in H^1(D)$ satisfy (3.111) and make the substitution

$$\mathbf{w} = A\nabla w \in L^2(D)^3 \quad \text{and} \quad \mathbf{v} = \nabla v \in L^2(D)^3.$$

Since from (3.100) $A^{-1}$ exists and is bounded, we have that

$$\nabla w = A^{-1}\mathbf{w}.$$

Taking the gradient of the first two equations in (3.111), we obtain that $\mathbf{w}$ and $\mathbf{v}$ satisfy

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 A^{-1}\mathbf{w} = 0 \tag{3.112}$$

and

$$\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0, \tag{3.113}$$

respectively, in $D$. Obviously the second boundary condition in (3.111) implies that

$$\nu \cdot \mathbf{v} = \nu \cdot \mathbf{w} \qquad \text{on } \partial D, \tag{3.114}$$

whereas the equations in (3.111) yield

$$-k^2 w = \nabla \cdot \mathbf{w} \qquad \text{and} \qquad -k^2 v = \nabla \cdot \mathbf{v}$$

which together with the first boundary condition in (3.111) give

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v} \qquad \text{on } \partial D. \tag{3.115}$$

We can now formulate the interior transmission eigenvalue problem (3.111) in terms of $\mathbf{w}$ and $\mathbf{v}$. In addition to the usual energy spaces $H^1(D)$ and $H^1_0(D)$, we introduce the Sobolev spaces

$$H(\text{div}, D) := \left\{\mathbf{u} \in L^2(D)^3 : \ \nabla \cdot \mathbf{u} \in L^2(D)\right\},$$
$$H_0(\text{div}, D) := \{\mathbf{u} \in H(\text{div}, D) : \ \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\}$$

and

$$\mathcal{H}(D) := \left\{\mathbf{u} \in H(\text{div}, D) : \ \nabla \cdot \mathbf{u} \in H^1(D)\right\},$$
$$\mathcal{H}_0(D) := \left\{\mathbf{u} \in H_0(\text{div}, D) : \ \nabla \cdot \mathbf{u} \in H^1_0(D)\right\} \tag{3.116}$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u}, \mathbf{v})_{L^2(D)} + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{H^1(D)}.$$

Letting $N := A^{-1}$, in terms of new vector valued functions $\mathbf{w}$ and $\mathbf{v}$ the transmission eigenvalue problem (3.111) can be written as the equivalent problem

$$\begin{cases} \nabla(\nabla\cdot\mathbf{w}) + k^2 N\mathbf{w} = 0 & \text{in } D, \\ \nabla(\nabla\cdot\mathbf{v}) + k^2\mathbf{v} = 0 & \text{in } D, \\ \nu\cdot\mathbf{w} = \nu\cdot\mathbf{v} & \text{on } \partial D, \\ \nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{v} & \text{on } \partial D \end{cases} \tag{3.117}$$

with $\mathbf{w}\in(L^2(D))^3$, $\mathbf{v}\in(L^2(D))^3$ such that $\mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$.

Following Section 3.1.1, we can now write (3.117) as an equivalent eigenvalue problem for $\mathbf{u} := \mathbf{w}-\mathbf{v}\in\mathcal{H}_0(D)$ satisfying the forth order equation

$$\left(\nabla\nabla\cdot + k^2 N\right)(N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right) = 0 \quad \text{in} \quad D, \tag{3.118}$$

which in the variational form reads as follows: Find $\mathbf{u}\in\mathcal{H}_0(D)$ such that

$$\int_D (N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right)\cdot\left(\nabla\nabla\cdot\overline{\mathbf{u}'} + k^2 N\overline{\mathbf{u}'}\right)\,dx = 0 \tag{3.119}$$

for all $\mathbf{u}'\in\mathcal{H}_0(D)$. The variational equation (3.119) can in turn be written as an operator equation

$$\mathbb{A}_k\mathbf{u} - k^2\mathbb{B}\mathbf{u} = 0 \quad \text{or} \quad \tilde{\mathbb{A}}_k\mathbf{u} - k^2\mathbb{B}\mathbf{u} = 0 \qquad \text{for} \quad \mathbf{u}\in\mathcal{H}_0(D), \tag{3.120}$$

where the bounded linear operators $\mathbb{A}_k : \mathcal{H}_0(D)\to\mathcal{H}_0(D)$, $\tilde{\mathbb{A}}_k : \mathcal{H}_0(D)\to\mathcal{H}_0(D)$, and $\mathbb{B} : \mathcal{H}_0(D)\to\mathcal{H}_0(D)$ are defined by means of the Riesz representation theorem

$$(\mathbb{A}_k\mathbf{u},\mathbf{u}')_{\mathcal{H}_0(D)} = \mathcal{A}_k(\mathbf{u},\mathbf{u}') \qquad \text{and} \qquad (\tilde{\mathbb{A}}_k\mathbf{u},\mathbf{u}')_{\mathcal{H}_0(D)} = \tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}') \tag{3.121}$$

and

$$(\mathbb{B}\mathbf{u},\mathbf{u}')_{\mathcal{H}_0(D)} = \mathcal{B}(\mathbf{u},\mathbf{u}') \tag{3.122}$$

with the sesquilinear forms $\mathcal{A}_k$, $\tilde{\mathcal{A}}_k$, and $\mathcal{B}$ given by

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}') := \left((N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right), \left(\nabla\nabla\cdot\mathbf{u}' + k^2\mathbf{u}'\right)\right)_D + k^4\left(\mathbf{u},\mathbf{u}'\right)_D,$$

$$\begin{aligned}\tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{v}) := {} & \left(N(I-N)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right), \left(\nabla\nabla\cdot\mathbf{u}' + k^2\mathbf{u}'\right)\right)_D \\ & + \left(\nabla\nabla\cdot\mathbf{u}, \nabla\nabla\cdot\mathbf{v}\right)_D,\end{aligned}$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}) := \left(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}\right)_D,$$

respectively, where $(\cdot,\,\cdot)_D$ denotes the $L^2(D)$-inner product.

In our discussion we must distinguish between the two cases $a_* > 1$ and $0 < a^* < 1$ (note that $a_*$ and $a^*$ are the infimum in $D$ of the smallest eigenvalue of $A$ and the supremum in $D$ of the largest eigenvalue of $A$, respectively). The assumption that $0 < a_* \le a^* < 1$ implies that $\xi\cdot(N-I)^{-1}\xi \ge \alpha|\xi|^2$ for all $\xi\in\mathbb{R}^3$ almost everywhere in $D$ and some constant $\alpha > 0$ since

$$\inf_{\substack{\xi\,\in\,\mathbb{C}^3\\ \|\xi\|\,=\,1}} \bar{\xi}\cdot(A^{-1}-I)^{-1}\,\xi = \frac{1}{\sup_\xi \bar{\xi}\cdot A^{-1}\,\xi - 1} \ge \frac{1}{1/a^* - 1} = \alpha,$$

where

$$\alpha := \frac{a^*}{1-a^*} > 0. \tag{3.123}$$

On the other hand, the assumption that $1 < a_* \le a^* < \infty$ implies that $\xi \cdot N(I-N)^{-1}\xi \ge \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^3$ almost everywhere in $D$ and some constant $\alpha > 0$. Indeed, noting that $A^{-1}(I-A^{-1})^{-1} = (I-A^{-1})^{-1} - I$ we have

$$\inf_{\substack{\xi \in \mathbb{C}^3 \\ \|\xi\|=1}} \bar{\xi} \cdot A^{-1}(I-A^{-1})^{-1}\,\xi = \inf_{\xi} \bar{\xi} \cdot (I-A^{-1})^{-1}\,\xi - 1$$

$$= \frac{1}{1-\sup_{\xi} \bar{\xi} \cdot A^{-1}\,\xi} - 1 \ge \frac{1}{1-1/a_*} - 1 = \alpha,$$

where

$$\alpha := \frac{1}{a_*-1} > 0. \tag{3.124}$$

**Theorem 3.37.** *Let $\lambda_1(D)$ be the first eigenvalue of $-\Delta$ on $D$. Then*

1. *for $0 < a^* < 1$, real wave numbers $k > 0$ such that $k^2 < a^*\lambda_1(D)$ are not transmission eigenvalues;*
2. *for $a_* > 1$, real wave numbers $k > 0$ such that $k^2 < \lambda_1(D)$ are not transmission eigenvalues.*

***Proof.*** First we recall that for $\nabla \cdot u \in H_0^1(D)$, using the Poincaré inequality, we have that

$$\|\nabla \cdot \mathbf{u}\|^2_{L^2(D)} \le \frac{1}{\lambda_1(D)}\|\nabla\nabla \cdot \mathbf{u}\|^2_{L^2(D)}, \tag{3.125}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on $D$.

Now assume that $a^* < 1$ which from the above implies $\xi \cdot (N(x)-I)^{-1}\xi \ge \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and almost every $x \in D$ with $\alpha$ given by (3.123). Then we have that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \ge \alpha\|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} + k^4\|\mathbf{u}\|^2_{L^2(D)}.$$

Setting $X = \|\nabla\nabla \cdot \mathbf{u}\|_{L^2(D)}$ and $Y = k^2\|\mathbf{u}\|_{L^2(D)}$ we have that

$$\|\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\|^2_{L^2(D)} \ge X^2 - 2XY + Y^2$$

and therefore

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \ge \alpha X^2 - 2\alpha XY + (\alpha+1)Y^2. \tag{3.126}$$

From the identity,

$$\alpha X^2 - 2\alpha XY + (\alpha+1)Y^2 = \epsilon\left(Y - \frac{\alpha}{\epsilon}X\right)^2 + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)X^2 + (1+\alpha-\epsilon)Y^2$$

for $\alpha < \epsilon < \alpha+1$ and (3.125) we have that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) \ge \left(\alpha - \frac{\alpha^2}{\epsilon}\right)\|\nabla\nabla \cdot \mathbf{u}\|^2_{L^2(D)} + (1+\alpha-\epsilon)k^2\|\mathbf{u}\|^2_{L^2(D)}$$

$$- k^2\frac{1}{\lambda_1(D)}\|\nabla\nabla \cdot \mathbf{u}\|^2_{L^2(D)}.$$

Therefore, if $k^2 < \left(\alpha - \alpha^2/\epsilon\right)\lambda_1(D)$ for every $\alpha < \epsilon < \alpha+1$, then $\mathcal{A}_k(\cdot,\cdot) - k^2\mathcal{B}(\cdot,\cdot)$ is coercive and hence $\mathbb{A}_k - k^2\mathbb{B}$ is invertible. In particular taking $\epsilon$ arbitrarily close to $\alpha+1$ we have that if $k^2 < \frac{\alpha}{1+\alpha}\lambda_1(D) = a^*\lambda_1(D)$, then $k$ is not a transmission eigenvalue, which proves the first part.

Next, let $a_* > 0$, which from the above implies $\xi \cdot N(x)(I - N(x))^{-1}\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^3$ almost everywhere for $x \in D$ with $\alpha$ given by (3.124). Then in exactly the same way as for the first part we obtain

$$\tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) \geq (1+\alpha-\epsilon)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + \left(\alpha - \frac{\alpha^2}{\epsilon}\right)k^2\|\mathbf{u}\|^2_{L^2(D)}$$
$$- k^2\frac{1}{\lambda_1(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}.$$

In particular, $\tilde{\mathcal{A}}_k(\cdot,\cdot) - k^2\mathcal{B}(\cdot,\cdot)$ is coercive as long as $k^2 < (1+\alpha-\epsilon)\lambda_1(D)$. Hence by taking $\epsilon > 0$ arbitrarily close to $\alpha$ we have that, for $k^2 < \lambda_1(D)$, $\tilde{\mathbb{A}}_k - k^2\mathbb{B}$ is invertible, which proves the second part. $\square$

Combining Theorem 3.37 with the discussion right below Remark 3.36, we can state the following result.

**Theorem 3.38.** *Assume that $n \equiv 1$, $\Im(A) = 0$, and either $a_* > 1$ or $0 < a^* < 1$. Then the transmission eigenvalues form a discrete (possibly empty) set in $\mathbb{C}$ with $+\infty$ as the only possible accumulation point.*

**Discreteness of Transmission Eigenvalues for $n \not\equiv 1$.** Again from Theorem 3.35 and Remark 3.36 we can assume that $\Im(A) = 0$ and $\Im(n) = 0$, and either $a_* > 1$ or $0 < a^* < 1$, and we consider the transmission eigenvalue problem (3.99). While we have assumed that the contrast $A - I$ does not change sign in $D$, our goal here is to prove the discreteness of transmission eigenvalues under less restrictive assumptions on $n-1$, more specifically, allowing $n-1$ to change sign in $D$. To this end, we see that a natural variational formulation equivalent to the transmission eigenvalue problem is as follows: Find $(w,v) \in \mathcal{H}(D)$ such that

$$\int_D A\nabla w \cdot \nabla\overline{w}'\,dx - \int_D \nabla v \cdot \nabla\overline{v}'\,dx - k^2\int_D nw\,\overline{w}'\,dx + k^2\int_D v\,\overline{v}'\,dx = 0 \qquad (3.127)$$

for all $(w',v') \in \mathcal{H}(D)$, where $\mathcal{H}(D)$ denotes the Sobolev space

$$\mathcal{H}(D) := \left\{(w,v) \in H^1(D) \times H^1(D) : \, w - v \in H^1_0(D)\right\}, \qquad (3.128)$$

equipped with the $H^1(D)$ Cartesian product norm. To this end, taking $w' = v' = 1$ in (3.127), we first notice that the solution $(w,v)$ of (3.99) satisfies

$$k^2\int_D (nw - v)dx = 0.$$

This suggests considering (3.127) in a subspace of $\mathcal{H}(D)$ defined by

$$\mathcal{Y}(D) := \left\{(w,v) \in \mathcal{H}(D) \text{ such that } \int_D (nw-v)dx = 0\right\}.$$

Now suppose $\int_D (n-1)dx \neq 0$. Arguing by contradiction, one can in a standard manner prove the existence of a Poincaré constant $C_P > 0$ (which depends only on $D$ and $n$) such that

$$\|w\|_D^2 + \|v\|_D^2 \leq C_P(\|\nabla w\|_D^2 + \|\nabla v\|_D^2) \quad \text{for all } (w,v) \in \mathcal{Y}(D). \tag{3.129}$$

We observe that $k \neq 0$ is a transmission eigenvalue if and only if there exists a nontrivial element $(v,w) \in \mathcal{Y}(D)$ such that

$$a_k((w,v),(w',v')) = 0 \quad \text{for all } (w',v') \in \mathcal{Y}(D),$$

where the sesquilinear from $a_k(\cdot,\cdot) : \mathcal{Y}(D) \times \mathcal{Y}(D) \to \mathbb{C}$ is defined by

$$a_k((w,v),(w',v')) := \int\limits_D A\nabla w \cdot \nabla \overline{w}'\, dx - \int\limits_D \nabla v \cdot \nabla \overline{v}'\, dx - k^2 \int\limits_D nw\,\overline{w}'\, dx + k^2 \int\limits_D v\,\overline{v}'\, dx.$$

If $A_k : \mathcal{Y}(D) \to \mathcal{Y}(D)$ is the bounded linear operator defined by means of the Riesz representation theorem by

$$(A_k(w,v),(w',v'))_{\mathcal{Y}(D)} := a_k((w,v),(w',v')),$$

our goal is to find a $k \in \mathbb{C}$ for which the operator $A_k$ is invertible. To this end, we observe that $a_k(\cdot,\cdot)$ is not coercive for any $k \in \mathbb{C}$ due to the different signs in front of the gradient terms, but employing the arguments in [24] and [60], we show in the following that $a_k(\cdot,\cdot)$ is so-called $T$-coercive for some particular values of $k$, and this suffices to show that $A_k$ is invertible for those $k$. The $T$-coercivity property can be interpreted as a form of the Babuška–Brezzi inf-sup conditions. More specifically, the idea behind it is to replace $a_k(\cdot,\cdot)$ by $a_k^T(\cdot,\cdot)$ defined by

$$a_k^T((w,v),(w',v')) := a_k((w,v),\mathbf{T}(w',v')) \tag{3.130}$$

for all $((w,v),(w',v')) \in \mathcal{Y}(D) \times \mathcal{Y}(D)$ with the operator $\mathbf{T} : \mathcal{Y}(D) \to \mathcal{Y}(D)$ being an isomorphism. If we can choose the isomorphism $\mathbf{T}$ such that $a^T(\cdot,\cdot)$ is coercive, then, using the Lax–Milgram theorem and the fact that $\mathbf{T}$ is an isomorphism, we can deduce that the operator $\mathbf{A}_k : \mathcal{Y}(D) \to \mathcal{Y}(D)$ is invertible.

To present the idea of how to apply the $T$-coercivity approach, we focus on the case when $0 < a^* < 1$. Letting

$$\lambda(v) := 2\frac{\int_D (n-1)v\, dx}{\int_D (n-1)\, dx},$$

we consider the mapping $\mathbf{T} : \mathcal{Y}(D) \to \mathcal{Y}(D)$ defined by

$$\mathbf{T} : (w,v) \mapsto (w - 2v + \lambda(v), -v + \lambda(v)).$$

Note that $\lambda(\lambda(v)) = 2\lambda(v)$, which implies that $\mathbf{T}^2 = I$, and hence $\mathbf{T}$ is an isomorphism in $\mathcal{Y}(D)$. Then for all $(w,v) \in \mathcal{Y}(D)$ we have that

$$\begin{aligned}
&\left|a_k^T((w,v),(w,v))\right| \\
&\quad = |(A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2(A\nabla w, \nabla v)_D \\
&\qquad - k^2\left((nw,w)_D + (v,v)_D - 2(nw,v)_D\right)| \\
&\quad \geq (A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2\left|(A\nabla w, \nabla v)_D\right| \\
&\qquad - |k|^2\left((nw,w)_D + (v,v)_D + 2\left|(nw,v)_D\right|\right) \\
&\quad \geq (1 - \sqrt{a^*})\left((A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D\right) \\
&\qquad - |k|^2\Big(1 + \sqrt{n^*})((nw,w)_D + (v,v)_D\Big),
\end{aligned} \tag{3.131}$$

where in the first inequality we use the fact that $\int_D (nw - v)dx = 0$ and hence $\lambda(v)$ cancels. Note that in fact we introduce $\lambda(v)$ to ensure that the operator $\mathbf{T}$ has values in $\mathcal{Y}(D)$.

Now, if we choose $k \in \mathbb{C}$ such that

$$|k|^2 < \frac{a_*(1-\sqrt{a^*})}{C_P \, \max(n^*, 1)\,(1+\sqrt{n^*})}, \tag{3.132}$$

then $a_k^T$ and hence $A_k$ are invertible in $\mathcal{Y}(D)$; in other words all $k \in \mathbb{C}$ satisfying (3.132) are not transmission eigenvalues.

The case $a_* > 1$ can be handled in a similar way by using the isomorphism $\mathbf{T} : \mathcal{Y}(D) \to \mathcal{Y}(D)$ defined by

$$\mathbf{T} : (w, v) \mapsto (w - \lambda(w), -v + 2w - \lambda(w)).$$

In particular in this case all $k \in \mathbb{C}$ such that

$$|k|^2 < \frac{(1-1/\sqrt{a_*}))}{C_P \, \max(n^*, 1)\,(1+1/\sqrt{n_*})} \tag{3.133}$$

are not transmission eigenvalues.

Combining the above analysis with the discussion right below Remark 3.36, we can prove the following result.

**Theorem 3.39.** *Assume that either* $0 < a^* < 1$ *or* $a_* > 1$*, and* $\int_D (n-1)dx \neq 0$*. Then the transmission eigenvalues form a discrete (possibly empty) set in* $\mathbb{C}$ *with* $+\infty$ *as the only possible accumulation point.*

Summarizing, in the case when $\Re(A) - I$ is bounded away from zero and does not change sign in $D$, and either $\Im(A) < 0$ or $\Im(n) > 0$ in a subset of $D$, then the interior transmission problem (3.98) has a unique solution which depends continuously on the data. Furthermore, if $\Im(A) = 0$ and $\Im(n) = 0$ in $D$, and $A - I$ is bounded away from zero and does not change sign in $D$ and $\int_D (n-1)dx \neq 0$, then the interior transmission problem (3.98) has a unique solution depending continuously on the data except for a possibly discrete set of wave numbers $k \in \mathbb{C}$ with $+\infty$ the only possible accumulation point, referred to as transmission eigenvalues.

### 3.2.2 ▪ The Case of Sign Changing Contrast in $A$

We return to the solvability question of (3.98), but here we allow for $\Re(A) - I$ to change sign inside $D$. The $T$-coercivity approach used to prove Theorem 3.39 can be applied to study the interior transmission problem in this case. To this end, without loss of generality, we can take $f = 0$ in (3.98). Otherwise from the trace theorem it is possible to find $v_0 \in H^1(D)$ supported in $\overline{D}$ such that $v_0|_{\partial D} = f$ with $\|f\|_{H^{\frac{1}{2}}(\partial D)} \leq C\|v_0\|_{H^1(D)}$ with $C$ a positive constant, and then $w$ and $v - v_0$ satisfy the interior transmission problem with $f := 0$, $h := h + \partial v_0/\partial\nu$, and $\ell := \ell_2 + \Delta v_0 + k^2 v_0$. Similarly to (3.127), the interior transmission problem (3.98) is equivalently formulated as follows: Find $(w, v) \in \mathcal{H}(D)$ such that

$$\int_D A\nabla w \cdot \nabla \overline{w}'\, dx - \int_D \nabla v \cdot \nabla \overline{v}'\, dx - k^2 \int_D nw\, \overline{w}'\, dx + k^2 \int_D v\, \overline{v}'\, dx$$

$$= \int_{\partial D} h\overline{w'}\, ds - \int_D \ell_1 \overline{w'}\, dx - \int_D \ell_2 \overline{v'}\, dx \quad \text{for all} \quad (w', v') \in \mathcal{H}(D), \tag{3.134}$$

where $\mathcal{H}(D)$ is defined by (3.128). Let us define the bounded sesquilinear forms $a_k(\cdot,\cdot)$, $a(\cdot,\cdot), b(\cdot,\cdot) : \mathcal{H}(D) \times \mathcal{H}(D) \to \mathbb{C}$ by

$$a_k((w,v),(w',v')) := \int_D A\nabla w\cdot\nabla\overline{w}'\,dx - \int_D \nabla v\cdot\nabla\overline{v}'\,dx - k^2\int_D n w\,\overline{w}'\,dx + k^2\int_D v\,\overline{v}'\,dx,$$

$$a((w,v),(w',v')) := \int_D A\nabla w\cdot\nabla\overline{w}'\,dx - \int_D \nabla v\cdot\nabla\overline{v}'\,dx + \kappa^2\int_D \gamma w\,\overline{w}'\,dx - \kappa^2\int_D v\,\overline{v}'\,dx$$

for some constants $\kappa > 0$ and $\gamma > 0$ (to become precise later) and

$$b((w,v),(w',v')) := -(\kappa^2+k^2)\int_D (\gamma - n)w\,\overline{w}'\,dx + (\kappa^2+k^2)\int_D v\,\overline{v}'\,dx,$$

and the bounded antilinear functional $L : \mathcal{H}(D) \to \mathbb{C}$ by

$$L(w',v') := \int_{\partial D} h\overline{w'}\,ds - \int_D \ell_1\overline{w'}\,dx - \int_D \ell_2\overline{v'}\,dx.$$

Letting $\mathbf{A} : \mathcal{H}(D) \to \mathcal{H}(D)$ and $\mathbf{B} : \mathcal{H}(D) \to \mathcal{H}(D)$ be the bounded linear operators defined by means of the Riesz representation theorem

$$(\mathbf{A}_k(w,v),(w',v'))_{\mathcal{H}(D)} = a_k((w,v),(w',v')), \tag{3.135}$$

$$(\mathbf{A}(w,v),(w',v'))_{\mathcal{H}(D)} = a((w,v),(w',v')), \tag{3.136}$$

$$(\mathbf{B}(w,v),(w',v'))_{\mathcal{H}(D)} = b((w,v),(w',v')), \tag{3.137}$$

respectively, and $\ell \in \mathcal{H}(D)$ the Riesz representative of $L$ defined by

$$(\ell,(w',v'))_{\mathcal{H}(D)} = L(w',v'),$$

then the interior transmission problem entails finding $(w,v) \in \mathcal{H}(D)$ satisfying

$$\mathbf{A}_k(w,v) := (\mathbf{A}+\mathbf{B})(w,v) = \ell.$$

Thanks to the compact embedding of $H^1(D)$ in $L^2(D)$, $\mathbf{B}$ is a compact operator since obviously $\|\mathbf{B}(w,v)\|_{\mathcal{H}(D)}$ is bounded by $C\|(w,v)\|_{L^2(D)\times L^2(D)}$, where $C$ is a positive constant. Hence it suffices to show that $\mathbf{A}$ is invertible for some $\kappa > 0$ and $\gamma > 0$ in order to conclude that $\mathbf{A}+\mathbf{B}$ is a Fredholm operator of index zero, in which case the interior transmission problem (3.98) has a unique solution, provided $k$ is not a transmission eigenvalue (see Definition 3.30). To prove the invertibility of $\mathbf{A}$ we employ the $T$-coercivity argument as discussed above in Theorem 3.39.

At this point we need to assume that there exists a $\delta$-neighborhood $\mathcal{N}$ of the boundary $\partial D$ in $D$, i.e.,

$$\mathcal{N} := \{x \in D : \operatorname{dist}(x,\partial D) < \delta\}$$

such that $\Im(A) = 0$ in $\mathcal{N}$ and either $0 < a^\star < 1$ or $a_\star > 1$, where

$$\begin{aligned} a_\star &:= \inf_{x\in\mathcal{N}} \inf_{\substack{\xi\,\in\,\mathbb{R}^3\\ |\xi|\,=\,1}} \xi\cdot A(x)\xi > 0, \\ a^\star &:= \sup_{x\in\mathcal{N}} \sup_{\substack{\xi\,\in\,\mathbb{R}^3\\ |\xi|\,=\,1}} \xi\cdot A(x)\xi < \infty. \end{aligned} \tag{3.138}$$

Note that the above requirements hold only in the boundary neighborhood $\mathcal{N}$, whereas in $D \setminus \overline{\mathcal{N}}$ there are no assumptions on the contrast $A - I$ and $\Im(A)$ besides the physical assumptions stated at the beginning of Sections 3.2.

Let us start with the case when $0 < a^* < 1$ and choose $0 < \gamma < 1$. We introduce $\chi \in \mathcal{C}^\infty(\overline{D})$, a cutoff function such that $0 \leq \chi \leq 1$ is supported in $\overline{\mathcal{N}}$ and equals one in a neighborhood of the boundary, and define the isomorphism $\mathbf{T} : \mathcal{H}(D) \to \mathcal{H}(D)$ by

$$\mathbf{T} : (w, v) \mapsto (w - 2\chi v, -v).$$

(Note again that $\mathbf{T}$ is an isomorphism since $\mathbf{T}^2 = I$.) We then have that for all $(w, v) \in \mathcal{H}(D)$

$$\begin{aligned} \left|a^T((w,v),(w,v))\right| = &\left|(A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2(A\nabla w, \nabla(\chi v))_D \right.\\ &\left. + \kappa^2 \left(\gamma(w,w)_D + (v,v)_D - 2\gamma(w,\chi v)_D\right)\right|. \end{aligned} \tag{3.139}$$

Using Young's inequality, we can write

$$\begin{aligned} 2\left|(A\nabla w, \nabla(\chi v))_D\right| &\leq 2\left|(\chi A\nabla w, \nabla v)_{\mathcal{N}}\right| + 2\left|(A\nabla w, \nabla(\chi)v)_{\mathcal{N}}\right| \\ &\leq \eta(A\nabla w, \nabla w)_{\mathcal{N}} + \eta^{-1}(A\nabla v, \nabla v)_{\mathcal{N}} \\ &\quad + \alpha(A\nabla w, \nabla w)_{\mathcal{N}} + \alpha^{-1}(A\nabla(\chi)v, \nabla(\chi)v)_{\mathcal{N}} \end{aligned} \tag{3.140}$$

and

$$2\left|(\gamma w, \chi v)_D\right| \leq \beta(\gamma w, w)_{\mathcal{N}} + \beta^{-1}(\gamma v, v)_{\mathcal{N}} \tag{3.141}$$

for arbitrary constants $\alpha > 0$, $\beta > 0$, and $\eta > 0$. Substituting (3.140) and (3.141) into (3.139), we now obtain

$$\begin{aligned} \left|a^T((w,v),(w,v))\right| \geq\ &(A\nabla w, \nabla w)_{D\setminus\overline{\mathcal{N}}} + (\nabla v, \nabla v)_{D\setminus\overline{\mathcal{N}}} \\ &+ \kappa^2\left(\gamma(w,w)_{D\setminus\overline{\mathcal{N}}} + (v,v)_{D\setminus\overline{\mathcal{N}}}\right) \\ &+ ((1-\eta-\alpha)A\nabla w, \nabla w)_{\mathcal{N}} + ((I - \eta^{-1}A)\nabla v, \nabla v)_{\mathcal{N}} \\ &+ \kappa^2((1-\beta)\gamma w, w)_{\mathcal{N}} + ((\kappa^2(1-\beta^{-1}\gamma) - \sup_{\mathcal{N}}|\nabla\chi|^2 a^\star\alpha^{-1})v, v)_{\mathcal{N}}. \end{aligned} \tag{3.142}$$

Taking $\eta$, $\alpha$, and $\beta$ such that $a^\star < \eta < 1$, $\gamma < \beta < 1$, and $0 < \alpha < 1 - \eta$, and $\kappa > 0$ large enough we obtain the coercivity of $a^T(\cdot,\cdot)$, which implies that $\mathbf{A}$ is invertible.

Exactly in the same way we can treat the case when $a^\star > 1$. More specifically we chose $\gamma > 1$, define the isomorphism $\mathbf{T} : \mathcal{H}(D) \to \mathcal{H}(D)$ by

$$\mathbf{T} : (w, v) \mapsto (w, -v + 2\chi w),$$

and do exactly the same calculations as for the case of $0 < a^\star < 1$ to obtain the $T$-coercivity of $a(\cdot,\cdot)$ and consequently the invertibility of $\mathbf{A}$.

Thus we have proven the following result.

**Theorem 3.40.** *Assume that there exists a neighborhood $\mathcal{N}$ of the boundary $\partial D$ where $\Im(A) = 0$ and either $0 < a^\star < 1$ or $a_\star > 1$ (see (3.138)). Then the interior transmission problem (3.98) satisfies the Fredholm alternative, i.e., there exists a unique solution depending continuously on the data, provided $k$ is not a transmission eigenvalue.*

**Remark 3.41.** In view of the result of Theorem 3.36, the above theorem implies the well-posedness of the interior transmission problem (3.98), provided that either $\Im(A) < 0$ in a subregion of $D \setminus \overline{\mathcal{N}}$ or $\Im(n) > 0$ is a subregion of $D$.

**Remark 3.42.** The assumption that $A$ is real in some neighborhood of $\partial D$ in Theorem 3.41 can be relaxed. In particular, by taking the real part in (3.142) the estimates can be carried through if either $\sup\limits_{\mathcal{N}} \xi \cdot (-\Im(A))\xi < \inf\limits_{\mathcal{N}} \xi \cdot \Re(A)\xi$ or $0 < \sup\limits_{\mathcal{N}} \xi \cdot \Re(A)\xi < 1$, for some neighborhood $\mathcal{N}$ of the boundary $\partial D$.

We conclude this section by proving a discreteness result concerning transmission eigenvalues in the case when $\Im(A) = \Im(n) \equiv 0$. To this end let us introduce

$$n_\star := \inf_{x \in \mathcal{N}} n(x) > 0 \quad \text{and} \quad n^\star := \sup_{x \in \mathcal{N}} n(x) < \infty. \tag{3.143}$$

**Theorem 3.43.** *Assume that either* $0 < a^\star < 1$ *and* $0 < n^\star < 1$ *or* $a_\star > 1$ *and* $n_\star > 1$. *Then the set of transmission eigenvalues* $k \in \mathbb{C}$ *is discrete with* $+\infty$ *as the only possible accumulation point.*

***Proof.*** First we notice that $\mathbb{A}_{i\kappa}$ for $\kappa > 0$ is invertible. Indeed, $\mathbb{A}_{i\kappa}$ defined by (3.136) coincides with $\mathbb{A}$ defined by (3.137), where $\gamma$ is replaced by $n(x)$, and hence the proof of $T$-coercivity occurs in the same way as in (3.142) thanks to the assumptions on $n(x)$. Then the result of the theorem follows from the fact that $\mathbb{A}_k - \mathbb{A}_{i\kappa}$ is compact and by an application of the Analytic Fredholm Theorem 1.12. Note that the mapping $k \mapsto \mathbb{A}_k$ is analytic in $k \in \mathbb{C}$. $\square$

**Remark 3.44.** The state-of-the-art sufficient conditions on the real valued coefficients $A$ and $n$ to ensure solvability of the interior transmission problem for all wave numbers $k \in \mathbb{C}$ except for an infinite discrete set of transmission eigenvalues that accumulate only to infinity can be found in [140], [141]. More specifically, these assumptions are that the matrix valued function $A$ and the scalar function $n$ are real valued and continuous in $\overline{D}$ with $C^2$ boundary $\partial D$ such that

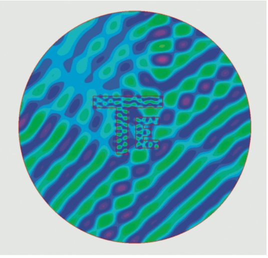
$$(A(x)\nu, \nu)\,(A(x)\xi, \xi) - (A(x)\nu, \xi) \neq 1 \quad \text{for all } x \in \partial D$$

and for every unit vector $\xi \in \mathbb{R}^3 \setminus \{0\}$ orthogonal to the outward unit normal vector $\nu$ at any $x \in \partial D$ and

$$(A(x)\nu, \nu)\, n(x) \neq 1 \qquad \text{for all } x \in \partial D,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^3$. The first condition is known as the complementing condition (see [2]). The analyses in these papers employ techniques that are essentially different from the variational approach systematically developed throughout this monograph. The interested reader can consult these papers for more details.

We end our discussion in this section by mentioning that, as indicated earlier in the isotropic media case, some sign condition on the contrast $A - I$ is needed to prove the Fredholm property of the interior transmission problem as well as the discreteness of the set of transmission eigenvalues. This is also the case in a series of papers [122], [123], [124], [125], where an alternative approach to investigating the transmission eigenvalue problem for anisotropic media was introduced, and a study of the counting function for transmission eigenvalues was initiated. Although it is not yet understood whether the assumption on the contrast $A - I$ not changing sign in a neighborhood of the boundary is optimal, there is an indication that it cannot be relaxed too much. More specifically, in [21] it is shown that if the contrast $A - I$ changes sign up to the boundary, then the interior transmission problem may lose its Fredholm property. The extension to Maxwell's equations of all the techniques discussed in this chapter can be found in [23], [48], [54], [78].



# Chapter 4
# The Existence of Transmission Eigenvalues

In the previous chapter we have only considered the solvability of the interior transmission problem and have provided sufficient conditions on the material properties that guarantee that the transmission eigenvalues form at most a discrete set. The study of these questions was mainly motivated by the application of sampling methods introduced in Chapter 2. In particular, knowing that the transmission eigenvalues form at most a discrete set was deemed to be sufficient since the transmission eigenvalues were something to be avoided in the context of these reconstruction techniques. Our attention from now on will be to obtain qualitative information on the material properties of the scattering media using real transmission eigenvalues since, as we show in Section 5.1, they can be determined from the far field data. Thus the existence of transmission eigenvalues as well as the derivation of inequalities connecting transmission eigenvalues and the constitutive material properties become central questions, and this chapter is dedicated to their investigation. We remind the reader that the transmission eigenvalue problem is non-self-adjoint and nonlinear. Hence questions related to the existence of transmission eigenvalues or the structure of associated eigenvectors appeal for nonstandard approaches.

Our discussion in this chapter will be mainly limited to the approach introduced in [144] and refined in [44] which, under appropriate assumptions on the contrast in the medium, transforms the transmission eigenvalue problem into a parametric eigenvalue problem for an auxiliary self-adjoint operator, and this provides a structure to obtain Faber–Krahn-type inequalities and monotonicity properties for the real transmission eigenvalues. The abstract framework is presented in Section 4.1.

We proceed in Section 4.2 with the application of this theory to prove the existence of real transmission eigenvalues for isotropic media under a fixed sign for the contrast. We rely on the variational framework introduced in the previous chapter.

We show in Section 4.2.1 how the analysis can also be adapted to include the case of media with voids discussed in Section 3.1.2. The main difficulty here is how to cope with dependence of the variational space on $k$. The reader can skip this section in a first reading.

One of the interesting points of the analytical framework of Section 4.1 is that it allows the derivation of inequalities on real transmission eigenvalues that may be exploited in the inverse medium problem. We present these inequalities in Section 4.2.2 and complement our discussion with some results from the literature on free zones for complex transmission eigenvalues.

When the index of refraction changes sign inside $D$, our analytical framework no longer applies. As an opening for possible other strategies to prove the existence of transmission eigenvalues, we outline at the end of Section 4.2.2 the approach proposed in [152] that allows us to obtain information on the spectrum in the complex plane.

The study of the transmission eigenvalue problem in the case of absorbing media and background was initiated in [35] (see also [80]), and we present some of these results in Section 4.2.3

We address in Section 4.3.2 the general case of anisotropic media. The existence of transmission eigenvalues for this case is more delicate since the nonlinear eigenvalue problem is no longer quadratic. We follow here the approach in [49] for fixed contrast sign. Similarly to the case of isotropic media, alternative approaches have been introduced to investigate the spectral properties of the anisotropic transmission eigenvalue problem under the assumptions that the contrast has one sign only in a neighborhood of the boundary (see, for instance, [122] and [125]). These techniques are not presented here.

## 4.1 ▪ Analytical Tools

In this section we develop the general analytical framework that will be the theoretical foundation of our method to prove the existence of real transmission eigenvalues.

Let $X$ be an infinite-dimensional separable Hilbert space with scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$, and let $\mathbb{A}$ be a bounded, positive definite, and self-adjoint operator on $X$. Under these assumptions $\mathbb{A}^{\pm 1/2}$ are well defined (cf. [151]). In particular, $\mathbb{A}^{\pm 1/2}$ are also bounded, positive definite, and self-adjoint operators, $\mathbb{A}^{-1/2}\mathbb{A}^{1/2} = I$ and $\mathbb{A}^{1/2}\mathbb{A}^{1/2} = \mathbb{A}$. We shall consider the spectral decomposition of the operator $\mathbb{A}$ with respect to self-adjoint nonnegative compact operators. The next two theorems [46] indicate the main properties of such a decomposition.

**Definition 4.1.** *A bounded linear operator $\mathbb{A}$ on a Hilbert space $X$ is said to be nonnegative if $(\mathbb{A}u, u) \geq 0$ for every $u \in X$. $\mathbb{A}$ is said to be coercive (or positive definite) if $(\mathbb{A}u, u) \geq \beta\|u\|^2$ for some positive constant $\beta$.*

In the following $N(\mathbb{B})$ denotes the null space of the operator $\mathbb{B}$.

**Theorem 4.2.** *Let $\mathbb{A}$ be a bounded, self-adjoint, and coercive operator on a Hilbert space, and let $\mathbb{B}$ be a nonnegative, self-adjoint, and compact linear operator with null space $N(\mathbb{B})$. There exist an increasing sequence of positive real numbers $(\lambda_j)_{j\geq 1}$ and a sequence $(u_j)_{j\geq 1}$ of elements of $X$ satisfying*

$$\mathbb{A}u_j = \lambda_j \mathbb{B}u_j$$

*and*

$$(\mathbb{B}u_j, u_\ell) = \delta_{j\ell}$$

*such that each $u \in [\mathbb{A}(N(\mathbb{B}))]^\perp$ can be expanded in a series*

$$u = \sum_{j=1}^{\infty} \gamma_j u_j.$$

*If $N(\mathbb{B})^\perp$ has infinite dimension, then $\lambda_j \to +\infty$ as $j \to \infty$.*

***Proof.*** This theorem is a direct consequence of the Hilbert–Schmidt theorem applied to the nonnegative self-adjoint compact operator $\tilde{\mathbb{B}} = \mathbb{A}^{-1/2}\mathbb{B}\mathbb{A}^{-1/2}$. Let $(\mu_j)_{j\geq 1}$ be the decreasing sequence of positive eigenvalues and $(v_j)_{j\geq 1}$ the corresponding eigenfunctions associated with $\tilde{\mathbb{B}}$ that form an orthonormal basis for $N(\tilde{\mathbb{B}})^\perp$. Note that zero is the only possible accumulation point for the sequence $(\mu_j)$. Straightforward calculations show that

$$\lambda_j = 1/\mu_j \qquad \text{and} \qquad u_j = \sqrt{\lambda_k}\,\mathbb{A}^{-1/2}v_j$$

satisfy

$$\mathbb{A}u_j = \lambda_j \mathbb{B}u_j.$$

Obviously if $w \in \mathbb{A}\,(N(\mathbb{B}))$, then $w = \mathbb{A}z$ for some $z \in N(\mathbb{B})$ and hence

$$(u_j, w) = \lambda_j(\mathbb{A}^{-1}\mathbb{B}u_j, w) = \lambda_j(\mathbb{A}^{-1}\mathbb{B}u_j, \mathbb{A}z) = \lambda_j(\mathbb{B}u_j, z) = 0,$$

which means that $u_j \in [\mathbb{A}\,(N(\mathbb{B}))]^\perp$. Furthermore, any $u \in [\mathbb{A}\,(N(\mathbb{B}))]^\perp$ can be written as $u = \sum_j \gamma_j u_j = \sum_j \gamma_j \sqrt{\lambda_j}\mathbb{A}^{-1/2}v_j$ since $\mathbb{A}^{1/2}u \in \left[N(\mathbb{A}^{-1/2}\mathbb{B}\mathbb{A}^{-1/2})\right]^\perp$. This ends the proof of the theorem. $\square$

**Theorem 4.3.** *Let $\mathbb{A}$, $\mathbb{B}$, and $(\lambda_j)_{j\geq 1}$ be as in Theorem 4.2 and define the Rayleigh quotient as*

$$R(u) = \frac{(\mathbb{A}u, u)}{(\mathbb{B}u, u)}$$

*for $u \notin N(\mathbb{B})$, where $(\cdot\,,\,\cdot)$ is the inner product on $X$. Then the following min-max principles hold:*

$$\lambda_j = \min_{W\in\mathcal{U}_j^{\mathbb{A}}}\left(\max_{u\in W\setminus\{0\}} R(u)\right) = \max_{W\in\mathcal{U}_{j-1}^{\mathbb{A}}}\left(\min_{u\in(\mathbb{A}(W+N(\mathbb{B})))^\perp\setminus\{0\}} R(u)\right),$$

*where $\mathcal{U}_j^{\mathbb{A}}$ denotes the set of all $j$-dimensional subspaces of $[\mathbb{A}\,(N(\mathbb{B}))]^\perp$.*

***Proof.*** The proof follows the classical proof of the Courant–Fischer min-max principle [127] and is given here for the reader's convenience. It is based on the fact that if $u \in [\mathbb{A}(N(B))]^\perp$, then from Theorem 4.2 we can write $u = \sum_j \gamma_j u_j$ for some coefficients $\gamma_j$, where the $u_j$ are defined in Theorem 4.2 (note that the $u_j$ are orthogonal with respect to the inner product induced by the self-adjoint invertible operator $\mathbb{A}$). Then using the facts that $(\mathbb{B}u_j, u_\ell) = \delta_{j\ell}$ and $\mathbb{A}u_j = \lambda_j\mathbb{B}u_j$ it is easy to see that

$$R(u) = \frac{1}{\sum_j |\gamma_j|^2}\sum_j \lambda_j|\gamma_j|^2.$$

Therefore, if $W_j \in \mathcal{U}_j^{\mathbb{A}}$ denotes the space generated by $\{u_1, \ldots, u_j\}$, we have that

$$\lambda_j = \max_{u\in W_j\setminus\{0\}} R(u) = \min_{u\in[\mathbb{A}(W_{j-1}+N(\mathbb{B}))]^\perp\setminus\{0\}} R(u).$$

Next let $W$ be any element of $\mathcal{U}_j^{\mathbb{A}}$. Since $W$ has dimension $j$ and $W \subset [\mathbb{A}(N(\mathbb{B}))]^\perp$, then $W \cap [\mathbb{A}W_{j-1} + \mathbb{A}(N(\mathbb{B}))]^\perp \neq \{0\}$. Therefore

$$\begin{aligned}\max_{u\in W\setminus\{0\}} R(u) &\geq \min_{u\in W\cap[\mathbb{A}(W_{j-1}+N(\mathbb{B}))]^\perp\setminus\{0\}} R(u)\\ &\geq \min_{u\in[\mathbb{A}(W_{j-1}+N(\mathbb{B}))]^\perp\setminus\{0\}} R(u) = \lambda_j,\end{aligned}$$

which proves the first equality of the theorem. Similarly, if $W$ has dimension $j-1$ and $W \subset [\mathbb{A}(N(\mathbb{B}))]^\perp$, then $W_j \cap (\mathbb{A}W)^\perp \neq \{0\}$. Therefore

$$\min_{u \in [\mathbb{A}(W+N(\mathbb{B}))]^\perp \setminus \{0\}} R(u) \leq \max_{u \in W_j \cap (\mathbb{A}W)^\perp \setminus \{0\}} R(u) \leq \max_{u \in W_j \setminus \{0\}} R(u) = \lambda_j,$$

which proves the second equality of the theorem. $\square$

The following corollary shows that it is possible to remove the dependence on $\mathbb{A}$ in the choice of the subspaces in the min-max principle for the eigenvalues $\lambda_j$.

**Corollary 4.4.** *Let $\mathbb{A}$, $\mathbb{B}$, $(\lambda_j)_{j\geq 1}$, and $R$ be as in Theorem* 4.3. *Then*

$$\lambda_j = \min_{W \subset \mathcal{U}_j} \left( \max_{u \in W \setminus \{0\}} R(u) \right), \tag{4.1}$$

*where $\mathcal{U}_j$ denotes the set of all $j$-dimensional subspaces $W$ of $X$ such that $W \cap N(\mathbb{B}) = \{0\}$.*

***Proof.*** From Theorem 4.3 and the fact that $\mathcal{U}_j^{\mathbb{A}} \subset \mathcal{U}_j$ it suffices to prove that

$$\lambda_j \leq \min_{W \subset \mathcal{U}_j} \left( \max_{u \in W \setminus \{0\}} R(u) \right).$$

Let $W \in \mathcal{U}_j$ and let $v_1, v_2, \ldots, v_k$ be a basis for $W$. Each vector $v_j$ can be decomposed into a sum $v_j^0 + \tilde{v}_j$, where $\tilde{v}_j \in [\mathbb{A}(N(\mathbb{B}))]^\perp$ and $v_j^0 \in N(\mathbb{B})$ (which is the orthogonal decomposition with respect to the inner product induced by $\mathbb{A}$). Since $W \cap N(B) = \{0\}$, the space $\tilde{W}$ generated by $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j$ has dimension $j$. Moreover, $\tilde{W} \subset [\mathbb{A}(N(\mathbb{B}))]^\perp$. Now let $\tilde{u} \in \tilde{W}$. Obviously $\tilde{u} = u - u^0$ for some $u \in W$ and $u^0 \in N(\mathbb{B})$. Since $\mathbb{B}u^0 = 0$ and $(\mathbb{A}u^0, \tilde{u}) = 0$ we have that

$$R(u) = \frac{(\mathbb{A}\tilde{u}, \tilde{u}) + (\mathbb{A}u^0, u^0)}{(\mathbb{B}\tilde{u}, \tilde{u})} = R(\tilde{u}) + \frac{(\mathbb{A}u^0, u^0)}{(\mathbb{B}\tilde{u}, \tilde{u})}.$$

Consequently, since $\mathbb{A}$ is positive definite and $\mathbb{B}$ is nonnegative, we obtain

$$R(\tilde{u}) \leq R(u) \leq \max_{u \in W \setminus \{0\}} R(u).$$

Finally, taking the maximum with respect to $\tilde{u} \in \tilde{W} \subset [\mathbb{A}(N(\mathbb{B}))]^\perp$ in the above inequality, we obtain from Theorem 4.3 that

$$\lambda_j \leq \max_{u \in W \setminus \{0\}} R(u),$$

which completes the proof after taking the minimum over all $W \subset \mathcal{U}_j$. $\square$

The following theorem provides the theoretical basis of our analysis of the existence of transmission eigenvalues. This theorem is a simple consequence of Theorem 4.3 and Corollary 4.4.

**Theorem 4.5.** *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0, \infty[$ to the set of bounded, self-adjoint, and coercive operators on the Hilbert space $X$, and let $\mathbb{B}$ be a self-adjoint and nonnegative compact bounded linear operator on $X$. We assume that there exist two* *positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{A}_{\tau_0} - \tau_0\mathbb{B}$ is positive on $X$;*

2. *$\mathbb{A}_{\tau_1} - \tau_1\mathbb{B}$ is nonpositive on an $\ell$-dimensional subspace $W_j$ of $X$.*

*Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, \ell$ has at least one solution in $[\tau_0, \tau_1]$, where $\lambda_j(\tau)$ is the $j$th eigenvalue (counting multiplicity) of $\mathbb{A}_\tau$ with respect to $\mathbb{B}$, i.e., $N(\mathbb{A}_\tau - \lambda_j(\tau)\mathbb{B}) \neq \{0\}$.*

***Proof.*** First we can deduce from (4.1) that for all $j \geq 1$, $\lambda_j(\tau)$ is a continuous function of $\tau$. Assumption 1 shows that $\lambda_j(\tau_0) > \tau_0$ for all $j \geq 1$. Assumption 2 implies in particular that $W_j \cap N(\mathbb{B}) = \{0\}$. Hence, another application of (4.1) implies that $\lambda_j(\tau_1) \leq \tau_1$ for $1 \leq j \leq \ell$. The desired result is now obtained by applying the intermediate value theorem. □

We now explicitly state a particular case of Theorem 4.5, which is the version used in [144] and is needed here to analyze the transmission eigenvalue problem for anisotropic media. Let $X$ be an infinite-dimensional separable Hilbert space, and let $\mathbb{T}_k : X \to X$ be a family of compact symmetric bounded linear operators. Furthermore, assume that the mapping $k \longmapsto T_k$ from $]0, +\infty[$ to the space of compact symmetric bounded linear operators is continuous. The Hilbert–Schmidt theorem [151] ensures the existence of a sequence of real eigenvalues $(\mu_j(k))_{j\geq 1}$ of the operator $\mathbb{T}_k$ for any fixed $k > 0$, accumulating to 0 where positive eigenvalues are ordered in the decreasing order and negative eigenvalues ordered in the increasing order. The Courant–Fischer max-min principle (see [127, page 319])

$$\mu_j(k) = \min_{W\in\mathcal{U}_j^{\mathbb{A}}} \max_{u\in W\setminus\{0\}} \frac{(\mathbb{T}_k u, u)_X}{\|u\|_X} = \max_{W\in\mathcal{U}_{j-1}^{\mathbb{A}}} \min_{u\in W^\perp\setminus\{0\}} \frac{(\mathbb{T}_k u, u)_X}{\|u\|_X} \tag{4.2}$$

for positive eigenvalues (with a similar expression for negative eigenvalues since max-min applied to $-\mathbb{T}$ gives min-max) implies that $\mu_j(k)$ are continuous function of $k$. The question of interest is to find $k > 0$ for which the kernel of $\mathbb{I} + \mathbb{T}_k$ is nontrivial, where $\mathbb{I}$ is the identity operator, in other words to find the zeros of

$$\mu_j(k) + 1 = 0, \qquad j \geq 1.$$

**Theorem 4.6.** *Assume that*

1. *there is a $\kappa_0$ such that $\mathbb{I} + \mathbb{T}_{\kappa_0}$ is positive on $X$;*

2. *there is a $\kappa_1 > \kappa_0$ such that $\mathbb{I} + \mathbb{T}_{\kappa_1}$ is nonpositive on a $p$-dimensional subspace $W_k$ of $X$.*

*Then the equation $\mu_j(k) + 1 = 0$ has $p$ solutions in $[\kappa_0, \kappa_1]$ counting their multiplicity.*

***Proof.*** If $\mathbb{I}+\mathbb{T}_{\kappa_0}$ is positive, then from (4.2) $\mu_j(\kappa_0)+1 > 0$. Now assumption 2 and another application of (4.2) imply that $\mu_j(\kappa_1) + 1 \leq 0$ for $j = 1, \ldots, p$, counting the multiplicity. Since $\mu_j(k) + 1$ is a continuous function of $k$, the mean value theorem implies that for each $j$, $1 \leq j \leq p$, there is a $k \in [\kappa_0, \kappa_1]$ such that $\mu_j(k) + 1 = 0$. □

## 4.2 ▪ Existence of Transmission Eigenvalues for Isotropic Media

In this section we are concerned with proving the existence of real transmission eigenvalues, i.e., the values of $k > 0$ for which

$$\Delta w + k^2 n(x) w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D,$$
$$w = v \quad \text{and} \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D,$$

have nontrivial solutions $w \in L^2(D)$ and $v \in L^2(D)$, such that $w - v \in H^2(D)$, which are referred to as the corresponding eigenfunctions.

As already mentioned the transmission eigenvalue problem is non-self-adjoint, and in Chapter 6 it is shown that for special cases of spherically stratified media there exist complex eigenvalues (see also [71]). For general media, we limit ourselves to proving the existence of real eigenvalues for two reasons: first, our approach based on auxiliary self-adjoint operators works only for real eigenvalues, and second, the real eigenvalues are of particular interest in the application to the inverse scattering problem since only they can be measured from scattering data. Therefore in view of Theorem 3.3 we now assume that $n \in L^2(D)$ is a real valued function (i.e., $\Im(n) \equiv 0$) such that

$$n_* = \inf_{x \in D} n(x) > 0 \qquad \text{and} \qquad n^* = \sup_{x \in D} n(x) < +\infty. \tag{4.3}$$

For historical reasons we mention that the first result on the existence of real transmission eigenvalues was obtained for spherically stratified media when $D := B_R$, where $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ is a ball of radius $R$ centered at the origin and $n := n(r)$ is a radial function [69], which we include here for the sake of completeness.

**Theorem 4.7.** *Assume that $n \in C^2[0, R]$, $\Im(n(r)) = 0$, and either $n(R) \neq 1$ or $n(R) = 1$, and $\frac{1}{R}\int_0^R \sqrt{n(\rho)}d\rho \neq 1$. Then there exists an infinite discrete set of transmission eigenvalues with spherically symmetric eigenfunctions.*

***Proof.*** To show existence, we restrict ourselves to spherically symmetric solutions to (3.30) and look for solutions of the form

$$v(r) = a_0 j_0(kr) \qquad \text{and} \qquad w(r) = b_0 \frac{y(r)}{r},$$

where

$$y'' + k^2 n(r) y = 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

where $j_0(r)$ is the spherical Bessel function of order zero. Using the Liouville transformation

$$z(\xi) \;:=\; [n(r)]^{\frac{1}{4}}\, y(r), \quad \text{where} \quad \xi(r) \;:=\; \int_0^r [n(\rho)]^{\frac{1}{2}}\, d\rho,$$

we arrive at the following initial value problem for $z(\xi)$:

$$z'' + [k^2 - p(\xi)] z = 0\,, \quad z(0) = 0\,, \quad z'(0) = \big[n(0)\big]^{-\frac{1}{4}}, \tag{4.4}$$

where

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} \;-\; \frac{5}{16}\frac{[n'(r)]^2}{[n(r)]^3}.$$

Now exactly in the same way as in [69], [75], by writing (4.4) as a Volterra integral equation and using the methods of successive approximations, for $k > 0$ we obtain the following asymptotic behavior for $y$:

$$y(r) = \frac{1}{k\left[n(0)\,n(r)\right]^{1/4}} \sin\left(k \int_0^r \left[n(\rho)\right]^{1/2} d\rho\right) + O\left(\frac{1}{k^2}\right), \tag{4.5}$$

$$y'(r) = \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos\left(k \int_0^r [n(\rho)]^{1/2}\, d\rho\right) + O\left(\frac{1}{k}\right) \tag{4.6}$$

uniformly on $[0, R]$. Applying the boundary conditions on $r = R$, we see that a nontrivial solution to (3.30) exists if and only if

$$d_0(k) = \det\begin{pmatrix} \dfrac{y(R)}{R} & j_0(kR) \\ \dfrac{d}{dr}\left(\dfrac{y(r)}{r}\right)_{r=R} & k\, j_0'(kR) \end{pmatrix} = 0\,.$$

Since $j_0(kr) = \sin kr/kr$, from the above asymptotic behavior of $y(r)$ we have that

$$d_0(k) = \frac{1}{kR^2}\left[A\sin(k\delta R)\, \cos(kR) - B\cos(k\delta R)\, \sin(kR)\right] + O\left(\frac{1}{k^2}\right), \tag{4.7}$$

where

$$\delta = \frac{1}{R}\int_0^R \sqrt{n(\rho)}d\rho, \qquad A = \frac{1}{[n(0)n(R)]^{1/4}}\,, \qquad B = \left[\frac{n(R)}{n(0)}\right]^{1/4}.$$

If $n(R) = 1$, since $\delta \neq 1$ the first term in (4.7) is a periodic function if $\delta$ is rational and almost-periodic (see [75]) if $\delta$ is irrational and in either case takes both positive and negative values. This means that $d_0(k)$ has infinitely many real zeros, which proves the existence of infinitely many real transmission eigenvalues. Now if $n(R) \neq 1$, then $A \neq B$ and the above argument holds independently of the value of $\delta$. $\square$

We refer the reader to Chapter 6 for more results on the spectral properties of the transmission eigenvalue problem for spherically stratified media.

The following result is an important tool in our proofs of the existence of real eigenvalues for general media and can be obtained by separating variables in the transmission eigenvalue problem (3.2).

**Corollary 4.8.** *Let $D := B_R$, and let $n > 0$ be a positive constant such that $n \neq 1$. The infinitely many real zeros of*

$$d_\ell(k) = \det\begin{pmatrix} j_\ell(ka) & j_\ell(k\sqrt{n}a) \\ j_\ell'(ka) & \sqrt{n}j_\ell'(k\sqrt{n}a) \end{pmatrix} = 0$$

*are transmission eigenvalues for the media $B_R, n$, where $j_\ell(r)$, $\ell \geq 0$, are spherical Bessel function of order $n$.*

We denote by $k_{a,n}$ the smallest real eigenvalue (which is not necessarily the smallest real zero of $d_0(k)$)

We now turn our attention to general inhomogeneous media. Setting $\tau := k^2$, in Section 3.1 it is shown that the transmission eigenvalue problem is equivalent to

$$\int_D \frac{1}{n-1}(\Delta u + \tau u)(\Delta\overline{\psi} + \tau n\overline{\psi})\, dx = 0 \qquad \text{for all } \psi \in H_0(D) \tag{4.8}$$

or

$$\mathbb{T}u - \tau\mathbb{T}_1 u + \tau^2\mathbb{T}_2 u = 0, \tag{4.9}$$

where the coercive operator $\mathbb{T}$, compact operator $\mathbb{T}_1$, and nonnegative compact operator $\mathbb{T}_2$ are defined by (3.14), (3.15), and (3.16), respectively. Note that for real valued refractive indices these operators are self-adjoint. However, the quadratic pencil of operators (4.9) after linearization does not correspond to an eigenvalue problem for a self-adjoint compact operator. Indeed, since $\mathbb{T}$ is coercive, $\mathbb{T}^{\frac{1}{2}}$ is positive and $\mathbb{T}^{-\frac{1}{2}}$ exists. Hence we have that

$$u - \tau\mathbb{K}_1 u + \tau^2\mathbb{K}_2 u = 0, \tag{4.10}$$

where the self-adjoint compact operators $\mathbb{K}_1 : H_0^2(D) \to H_0^2(D)$ and $\mathbb{K}_2 : H_0^2(D) \to H_0^2(D)$ are given by $\mathbb{K}_1 = \mathbb{T}^{-1/2}\mathbb{T}_1\mathbb{T}^{-1/2}$ and $\mathbb{K}_2 = \mathbb{T}^{-1/2}\mathbb{T}_2\mathbb{T}^{-1/2}$. Now noting that $\mathbb{K}_2$ is nonnegative, we set $U := (u, \tau\mathbb{K}_2^{1/2}u)$ to obtain

$$\left(\mathbf{K} - \frac{1}{\tau}\mathbf{I}\right) U = 0, \qquad U \in H_0^2(D) \times H_0^2(D),$$

for the compact (non-self-adjoint) operator $\mathbf{K} : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbf{K} := \begin{pmatrix} \mathbb{K}_1 & -\mathbb{K}_2^{1/2} \\ \mathbb{K}_2^{1/2} & 0 \end{pmatrix}.$$

Obviously although each of the entries in $\mathbf{K}$ are self-adjoint, $\mathbf{K}$ itself is not self-adjoint.

To proceed further, following [44] we define the following bounded sesquilinear forms on $H_0^2(D) \times H_0^2(D)$,

$$\mathcal{A}_\tau(u,\psi) = \left(\frac{1}{n-1}(\Delta u + \tau u),\, (\Delta\psi + \tau\psi)\right)_D + \tau^2\,(u,\,\psi)_D\,, \tag{4.11}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,\psi) &= \left(\frac{1}{1-n}(\Delta u + \tau n u),\, (\Delta\psi + \tau n\psi)\right)_D + \tau^2\,(nu,\,\psi)_D \\ &= \left(\frac{n}{1-n}(\Delta u + \tau u),\, (\Delta\psi + \tau\psi)\right)_D + (\Delta u,\, \Delta\psi)_D\,,\end{aligned} \tag{4.12}$$

$$\mathcal{B}(u,\psi) = (\nabla u,\, \nabla\psi)_D\,, \tag{4.13}$$

where $(\cdot\,,\,\cdot)_D$ denotes the $L^2(D)$ inner product. Using the Riesz representation theorem we now define the bounded linear operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$, and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ by

$$(\mathbb{A}_\tau u, \psi)_{H^2(D)} = \mathcal{A}_\tau(u,\psi), \qquad \left(\tilde{\mathbb{A}}_\tau u, \psi\right)_{H^2(D)} = \tilde{\mathcal{A}}_\tau(u,\psi),$$

$$(\mathbb{B}u, \psi)_{H^2(D)} = \mathcal{B}(u,\psi).$$

In terms of these operators we can rewrite (4.8) as

$$(\mathbb{A}_\tau u - \tau\mathbb{B}u, \psi)_{H^2(D)} = 0 \quad \text{or} \quad \left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u, \psi\right)_{H^2(D)} = 0 \tag{4.14}$$

for all $\psi \in H_0^2(D)$, which means that $k$ is a transmission eigenvalue if and only if $\tau := k^2$ is such that the kernel of the operator $\mathbb{A}_\tau u - \tau\mathbb{B}$ or the operator $\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}$ is not trivial.

In order to analyze (4.14), we recall the following results from [46] about the properties of the above operators. To this end, let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in $D$ and assume that either $n^* < 1$ or $n_* > 1$.

**Lemma 4.9.** *The operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$, $\tilde{\mathbb{A}}_\tau : H_0^2(D) \to H_0^2(D)$, $\tau > 0$, and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ are self-adjoint. If $n_* > 1$, then $\mathbb{A}_\tau$ is positive definite, whereas if $0 < n_* < n^* < 1$, then $\tilde{\mathbb{A}}_\tau$ is positive definite. In addition, $\mathbb{B}$ is positive and compact.*

***Proof.*** Obviously $\mathbb{A}_\tau$, $\tilde{\mathbb{A}}_\tau$, and $\mathbb{B}$ are self-adjoint since $n$ and $\tau$ are real. Now assume that $n_* > 1$. Then since $\frac{1}{n(x)-1} > \frac{1}{n^*-1} = \gamma > 0$ almost everywhere in $D$, we have

$$\begin{aligned}
(\mathbb{A}_\tau u, u)_{H^2(D)} &\geq \gamma\|\Delta u + \tau u\|_{L^2}^2 + \tau^2\|u\|_{L^2}^2 \\
&\geq \gamma\|\Delta u\|_{L^2}^2 - 2\gamma\tau\|\Delta u\|_{L^2}\|u\|_{L^2} + (\gamma+1)\tau^2\|u\|_{L^2}^2 \\
&= \epsilon\left(\tau\|u\|_{L^2} - \frac{\gamma}{\epsilon}\|\Delta u\|_{L^2(D)}\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|_{L^2(D)}^2 \\
&\quad + (1+\gamma-\epsilon)\tau^2\|u\|_{L^2}^2 \\
&\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|_{L^2(D)}^2 + (1+\gamma-\epsilon)\tau^2\|u\|_{L^2}^2
\end{aligned} \tag{4.15}$$

for some $\gamma < \epsilon < \gamma+1$. Furthermore, since $\nabla u \in H_0^1(D)^2$, using the Poincaré inequality we have that

$$\|\nabla u\|_{L^2(D)}^2 \leq \frac{1}{\lambda_1(D)}\|\Delta u\|_{L^2(D)}^2.$$

Hence we can conclude that

$$(\mathbb{A}_\tau u, u)_{H^2(D)} \geq C_\tau\|u\|_{H^2(D)}^2$$

for some positive constant $C_\tau$. Consequently $\mathbb{A}_\tau$ is positive definite and hence invertible. Exactly in the same way, one can prove that if $0 < n^* < 1$, then

$$\left(\tilde{\mathbb{A}}_\tau u, u\right)_{H^2(D)} \geq C_\tau\|u\|_{H^2(D)}^2$$

for some positive constant $C_\tau$ since in this case $\frac{n(x)}{1-n(x)} > \frac{n_*}{1-n_*} = \gamma > 0$ almost everywhere in $D$.

We now consider the operator $\mathbb{B}$. By definition $\mathbb{B}$ is nonnegative, and furthermore the compact embedding of $H^2(D)$ into $H^1(D)$ and the fact that $\nabla u \in H_0^1(D)$ imply that $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ is compact since $\|\mathbb{B}u\|_{H^2(D)} \leq c\|u\|_{H^1(D)}$. $\square$

**Lemma 4.10.**

1. *If $n_* > 1$, then*

$$(\mathbb{A}_\tau u - \tau\mathbb{B}u, u)_{H^2} \geq \alpha\|u\|_{H^2}^2 \quad \textit{for all} \quad 0 < \tau < \frac{\lambda_1(D)}{n^*}.$$

2. *If* $n^* < 1$, *then*

$$\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u, u\right)_{H^2} \geq \alpha \|u\|^2_{H^2} \quad \textit{for all} \quad 0 < \tau < \lambda_1(D).$$

***Proof.*** Assume that $n_* > 1$. Then $\frac{1}{n(x)-1} > \frac{1}{n^*-1} = \gamma > 0$ almost everywhere in $D$. We have

$$\begin{aligned}(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H_0^2} &= \mathcal{A}_\tau(u,u) - \tau \|\nabla u\|^2_{L^2} \\ &\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right) \|\Delta u\|^2_{L^2} + (1+\gamma-\epsilon)\|u\|^2_{L^2} - \tau\|\nabla u\|^2_{L^2}\end{aligned} \tag{4.16}$$

for $\gamma < \epsilon < \gamma + 1$. Since $\nabla u \in H_0^1(D)$, using the Poincaré inequality we have that

$$\|\nabla u\|^2_{L^2(D)} \leq \frac{1}{\lambda_1(D)} \|\Delta u\|^2_{L^2(D)},$$

and hence we obtain

$$(\mathbb{A}_\tau u - \tau \mathbb{B} u,\, u)_{H_0^2} \geq \left(\gamma - \frac{\gamma^2}{\epsilon} - \frac{\tau}{\lambda_1(D)}\right) \|\Delta u\|^2_{L^2} + \tau(1+\gamma-\epsilon)\|u\|^2_{L^2}.$$

Thus $\mathbb{A}_\tau - \tau\mathbb{B}$ is positive as long as $\tau < (\gamma - \frac{\gamma^2}{\epsilon})\lambda_1(D)$. In particular, choosing $\gamma = \frac{1}{n^*-1}$, and taking $\epsilon$ arbitrarily close to $\gamma + 1$, the latter becomes $\tau < \frac{\gamma}{1+\gamma}\lambda_1(D) = \frac{\lambda_1(D)}{n^*}$.

Next assume that $0 < n^* < 1$. Then $\frac{n(x)}{1-n(x)} > \frac{n_*}{1-n_*} = \gamma > 0$. Hence

$$\begin{aligned}\left(\tilde{\mathbb{A}}_\tau u - \tau \mathbb{B} u,\, u\right)_{H_0^2} &= \tilde{\mathcal{A}}_\tau(u,u) - \tau\|\nabla u\|^2_{L^2} \\ &\geq (1+\gamma-\epsilon-\tau\frac{1}{\lambda_1(D)})\|\Delta u\|^2_{L^2} + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|u\|^2_{L^2}\end{aligned} \tag{4.17}$$

for $\gamma < \epsilon < \gamma + 1$. Thus $\tilde{\mathbb{A}}_\tau - \tau\mathbb{B}$ is positive as long as $\tau < (1+\gamma-\epsilon)\lambda_1(D)$. In particular, taking $\epsilon$ arbitrarily close to $\gamma$, the latter becomes $\tau < \lambda_1(D)$. ☐

Obviously $\mathbb{A}_\tau$ and $\tilde{\mathbb{A}}_\tau$ depend continuously on $\tau \in (0, +\infty)$. From the above discussion, $k > 0$ is a transmission eigenvalue if for $\tau = k^2$ the kernel of the operator $\mathbb{A}_\tau - \tau\mathbb{B}$ if $n_* > 1$, or the kernel of the operator $\tilde{\mathbb{A}}_\tau - \tau\mathbb{B}$ if $n^* < 1$, is nontrivial. In order to analyze the kernel of these operators, we consider the auxiliary eigenvalue problems

$$\mathbb{A}_\tau u - \lambda(\tau)\mathbb{B} u = 0, \qquad u \in H_0^2(D), \qquad \text{if} \quad n_* > 1, \tag{4.18}$$

and

$$\tilde{\mathbb{A}}_\tau u - \lambda(\tau)\mathbb{B} u = 0, \qquad u \in H_0^2(D), \qquad \text{if} \quad n^* < 1. \tag{4.19}$$

Thus a transmission eigenvalue $k > 0$ is such that $\tau := k^2$ solves $\lambda(\tau) - \tau = 0$, where $\lambda(\tau)$ is an eigenvalue corresponding to (4.18) or (4.19) in the respective cases. Our goal is now to apply Theorem 4.5 to (4.18) or (4.19) to prove the existence of an infinite set of transmission eigenvalues.

**Remark 4.11.** The multiplicity of transmission eigenvalues is finite since if $k_0$ is a transmission eigenvalue, then, letting $\tau_0 := k_0^2$, the kernel of $\mathbb{I} - \tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ if $n_* > 1$, or $\mathbb{I} - \tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ if $n^* < 1$, is finite since the operators $\tau_0\mathbb{A}_{\tau_0}^{-1/2}\mathbb{B}\mathbb{A}_{\tau_0}^{-1/2}$ and $\tau_0\tilde{\mathbb{A}}_{\tau_0}^{-1/2}\mathbb{B}\tilde{\mathbb{A}}_{\tau_0}^{-1/2}$ are compact. (See also Theorem 3.4.)

We are now ready to prove the main theorem of this section.

**Theorem 4.12.** *Let $n \in L^\infty(D)$ satisfy either one of the following assumptions:*

1. $1 < n_* \le n(x) \le n^* < \infty$.

2. $0 < n_* \le n(x) \le n^* < 1$.

*Then there exists an infinite set of real transmission eigenvalues with $+\infty$ as the only accumulation point.*

***Proof.*** Assume that assumption 1 holds, which also implies that

$$0 < \frac{1}{n^*-1} \le \frac{1}{n(x)-1} \le \frac{1}{n_*-1} < \infty.$$

Therefore, from Lemma 4.9, $\mathbb{A}_\tau$ and $\mathbb{B}$ satisfy the requirement of Theorem 4.5 with $X = H_0^2(D)$, and from Lemma 4.10 they also satisfy assumption 1 of Theorem 4.5 with $\tau_0 \le \lambda_1(D)/n^*$.

Next let $k_{1,n_*}$ be the first transmission eigenvalue for the ball $B$ of radius $R = 1$ and let $n := n_*$. By a scaling argument, it is obvious that $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\epsilon > 0$ with index of refraction $n_*$. Now take $\epsilon > 0$ small enough such that $D$ contains $m := m(\epsilon) \ge 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2, \dots, B_\epsilon^m$ of radius $\epsilon$, that is, $\overline{B_\epsilon^j} \subset D$, $j = 1, \dots, m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \ne i$. Then $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction $n_*$, and let $u^{B_\epsilon^j,n_*} \in H_0^2(B_\epsilon^j)$, $j = 1, \dots, m$, be the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j,n_*}$ to the whole of $D$ is obviously in $H_0^2(D)$ due to the boundary conditions on $\partial B_{\epsilon,n_*}^j$. Furthermore, the vectors $\{\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^m\}$ are linearly independent and orthogonal in $H_0^2(D)$ since they have disjoint supports. From (4.8) we have that

$$\begin{aligned} 0 &= \int_D \frac{1}{n_*-1}(\Delta\tilde{u}^j + k_{\epsilon,n_*}^2\tilde{u}^j)(\Delta\overline{\tilde{u}}^j + k_{\epsilon,n_*}^2 n_*\overline{\tilde{u}}^j)\,dx \qquad (4.20)\\ &= \int_D \frac{1}{n_*-1}|\Delta\tilde{u}^j + k_{\epsilon,n_*}^2\tilde{u}^j|^2\,dx + k_{\epsilon,n_*}^4\int_D |\tilde{u}^j|^2\,dx - k_{\epsilon,n_*}^2\int_D |\nabla\tilde{u}^j|^2\,dx \end{aligned}$$

for $j = 1, \dots, m$. Let us denote by $\mathcal{U}$ the $m$-dimensional subspace of $H_0^2(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, \dots, m$, satisfies (4.20) and they have disjoint supports, we have that for $\tau_1 := k_{\epsilon,n_*}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$\begin{aligned} (\mathbb{A}_{\tau_1}\tilde{u} - \tau_1\mathbb{B}\tilde{u},\, \tilde{u})_{H_0^2(D)} &= \int_D \frac{1}{n-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int_D |\tilde{u}|^2\,dx - \tau_1\int_D |\nabla\tilde{u}|^2\,dx \\ &\le \int_D \frac{1}{n_*-1}|\Delta\tilde{u} + \tau_1\tilde{u}|^2\,dx + \tau_1^2\int_D |\tilde{u}|^2\,dx - \tau_1\int_D |\nabla\tilde{u}|^2\,dx = 0. \qquad (4.21) \end{aligned}$$

This means that assumption 2 of Theorem 4.5 is also satisfied, and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0,\, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and $k_{\epsilon,n_*}$ both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite, we have shown, by letting $\epsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at $\infty$.

If the index of refraction is such that assumption 2 holds, then we have that

$$0 < \frac{n_*}{1-n_*} \le \frac{n(x)}{1-n(x)} \le \frac{n^*}{1-n^*} < \infty,$$

and therefore according to Lemmas 4.9 and 4.10, $\tilde{\mathbb{A}}_\tau$ and $\mathbb{B}$, $\tau > 0$, satisfy the requirements and assumption 1 of Theorem 4.5 with $X = H_0^2(D)$ for $\tau_0 \le \lambda_1(D)$. In this case we can estimate

$$\begin{aligned}\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u,\, u\right)_{H_0^2(D)} &= \int_D \frac{n}{1-n}|\Delta u + \tau u|^2\,dx + \int_D |\Delta u|^2\,dx - \tau\int_D |\nabla u|^2\,dx \\ &\le \int_D \frac{n^*}{1-n^*}|\Delta u + \tau u|^2\,dx + \int_D |\Delta u|^2\,dx - \tau\int_D |\nabla u|^2\,dx. \end{aligned} \tag{4.22}$$

The rest of the proof for checking the validity of assumption 2 of Theorem 4.5 goes exactly in the same way as for the previous case if one replaces $n_*$ by $n^*$. This proves the result. □

### 4.2.1 ▪ Media with Voids

The above analysis can be adapted to include the case of media with voids discussed in Section 3.1.2. In this case the transmission eigenvalue problem is formulated in variational form as finding $u \in V_0(D, D_0, k)$ such that

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta + k^2\right)u\,\left(\Delta + k^2\right)\bar{\psi}\,dx + k^2\int_{D\setminus\overline{D}_0}\left(\Delta u + k^2 u\right)\bar{\psi}\,dx = 0 \tag{4.23}$$

for all $\psi \in V_0(D, D_0, k)$, where the Hilbert space $V_0(D, D_0, k)$ is defined by (3.27). As shown in Section 3.1.2, this variational formulation is equivalent to the transmission eigenvalue problem, provided that $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$. With this understanding, our goal is to show the existence of $k > 0$ such that the homogeneous problem

$$\mathcal{A}(u,\psi) + \mathcal{B}_k(u,\psi) = 0 \text{ for all } \psi \in V_0(D, D_0, k) \tag{4.24}$$

has a nonzero solution $u \in V_0(D, D_0, k)$, where the sesquilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ on $V_0(D, D_0, k) \times V_0(D, D_0, k)$ are defined by (3.38) and (3.39), respectively. Let $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ and $B_k$ be the self-adjoint operators associated with $\mathcal{A}$ and $\mathcal{B}_k$, respectively, by using the Riesz representation theorem (note that $A_k$ depends on $k$ since the space of definition depends on $k$). In the proof of Theorem 3.9 it is shown that the operator $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is positive definite, i.e., $A_k^{-1} : V_0(D, D_0, k) \to V_0(D, D_0, k)$ exists, and the operator $B_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is compact. Hence we can define the operator $A_k^{-1/2}$ [151], in particular $A_k^{-1/2}$ is also bounded, self-adjoint, and positive definite. Thus we have that (4.24) is equivalent to finding $u \in V_0(D, D_0, k)$ such that

$$u + A_k^{-1/2}B_k A_k^{-1/2}u = 0. \tag{4.25}$$

In particular, if $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$, $k$ is a transmission eigenvalue if and only if the operator

$$I_k + A_k^{-1/2} B_k A_k^{-1/2} : V_0(D, D_0, k) \to V_0(D, D_0, k) \tag{4.26}$$

has a nontrivial kernel, where $I_k$ is the identity operator on $V_0(D, D_0, k)$. To avoid dealing with function spaces depending on $k$, we introduce the orthogonal projection operator $P_k$ from $H_0^2(D)$ onto $V_0(D, D_0, k)$ and the corresponding injection $R_k : V_0(D, D_0, k) \to H_0^2(D)$. Then one easily sees that $A_k^{-1/2} B_k A_k^{-1/2}$ is injective on $V_0(D, D_0, k)$ if and only if

$$\mathbb{I} + \mathbb{T}_k : H_0^2(D) \to H_0^2(D) \tag{4.27}$$

is injective, where

$$\mathbb{T}_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D)$$

and $\mathbb{I}$ is the identity operator on $H_0(D)$. Indeed, if $u + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u = 0$, then by taking the inner product of the latter with the component $w = u - P_k u$ which is orthogonal to $P_k u$, we have that

$$\begin{aligned} 0 &= (u,\, w)_{H^2} + \left( R_k A_k^{-1/2} B_k A_k^{-1/2} P_k u,\, w \right)_{H^2} \\ &= (w,\, w)_{H^2} + \left( A_k^{-1/2} B_k A_k^{-1/2} P_k u,\, P_k w \right)_{H^2} = \|w\|_{H^2}^2, \end{aligned} \tag{4.28}$$

and hence $w = 0$. The injectivity of $A_k^{-1/2} B_k A_k^{-1/2}$ now implies the injectivity of (4.27) since the component $P_k u$ is in $V_0(D, D_0, k)$. The converse is obvious. Furthermore, compactness of $B_k$ implies that $\mathbb{T}_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D)$ is also compact. Therefore we have that $k > 0$ is a transmission eigenvalue, provided that the kernel of $\mathbb{I} + \mathbb{T}_k$ is nontrivial.

**Lemma 4.13.** *The mapping $k \to \mathbb{T}_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is continuous from $]0, +\infty[$ to the space of bounded linear compact self-adjoint operators in $H_0^2(D)$*

***Proof.*** The proof is straightforward but technical, and we refer the reader to Theorem 4.5 and Corollary 4.6 of [33]. □

Now we can apply Theorem 4.6 to $\mathbb{T}_k$ to prove the existence of real transmission eigenvalues. To this end we recall the notation

$$n_* := \inf_{D \setminus \overline{D}_0} (n) \quad \text{and} \quad n^* := \sup_{D \setminus \overline{D}_0} (n).$$

**Theorem 4.14.** *Let $n \in L^\infty(D)$, $n = 1$ in $D_0$, satisfy either one of the following assumptions: for $x \in D \setminus \overline{D}_0$,*

1. $1 < n_* \le n(x) \le n^* < \infty$,
2. $0 < n_* \le n(x) \le n^* < 1$.

*Then there exists an infinite countable set of transmission eigenvalues with $+\infty$ as the only accumulation point.*

***Proof.*** First we assume that assumption 1 holds, in which case we have

$$0 < \frac{1}{n^* - 1} \le \frac{1}{n(x) - 1} \le \frac{1}{n_* - 1} < \infty \qquad \text{in} \qquad D \setminus \overline{D}_0.$$

First we note that $\mathbb{I} + \mathbb{T}_k$, where $\mathbb{T}_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$, is positive on $H_0^2(D)$ if and only if $A_k + B_k$ is positive on $V_0(D, D_0, k)$.

Next, combining the terms in (4.23) in a different way, we have that for $u \in V_0(D, D_0, k)$

$$\begin{aligned}(A_k u + B_k u, u)_{H_0^2(D)} &= \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u + k^2 n u|^2\,dx - k^4 \int\limits_{D\setminus\overline{D}_0} n|u|^2\,dx \\ &\quad + k^2 \int\limits_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^4 \int\limits_{D_0} |u|^2\,dx + k^2 \int\limits_{D_0} |\nabla u|^2\,dx. \end{aligned} \tag{4.29}$$

For $n^* = \sup_{D\setminus\overline{D}_0} n > 1$, if the sum of the last four terms in (4.29) is nonnegative, then we have that $A_k + B_k$ is positive. Hence we have

$$\begin{aligned} -k^2 \int\limits_{D\setminus\overline{D}_0} n|u|^2\,dx + \int\limits_{D\setminus\overline{D}_0} |\nabla u|^2\,dx - k^2 \int\limits_{D_0} |u|^2\,dx + \int\limits_{D_0} |\nabla u|^2\,dx \qquad (4.30)\\ \ge \int\limits_{D} |\nabla u|^2\,dx - k^2 n^* \int\limits_{D} |u|^2\,dx \ge (\lambda_1(D) - k^2 n^*)\|u\|^2_{L^2(D)}. \end{aligned}$$

Therefore for all $\kappa_0 > 0$ such that $\kappa_0^2 \le \frac{\lambda_1(D)}{n^*}$ we have that $A_k + B_k$ is positive in $V_0(D, D_0, k)$ and hence $\mathbb{I} + \mathbb{T}_k$ satisfies assumption 1 of Theorem 4.6.

Next we proceed in the same way as in the proof of Theorem 4.12. To this end, take $\epsilon > 0$ small enough such that $D \setminus \overline{D}_0$ contains $m := m(\epsilon) \ge 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2, \ldots, B_\epsilon^m$ of radius $\epsilon$. With $k_{1,n_*}$ being the first transmission eigenvalue for the ball $B$ of radius $R = 1$ and $n := n_*$, we take $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ as the first transmission eigenvalue for each of these balls with index of refraction $n_*$, and $u^{B_\epsilon^j, n_*} \in H_0^2(B_\epsilon^j)$, $j = 1, \ldots, m$, the corresponding eigenfunction. The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j, n_*}$ to the whole of $D$ is obviously in $V_0(D, D_0, k)$, and the vectors $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$ are linearly independent and orthogonal since they have disjoint supports in $D \setminus \overline{D}_0$. Let us denote by $\mathcal{U}$ the $m$-dimensional subspace of $V_0(D, D_0, k)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, \ldots, m$, satisfies (4.20) and they have disjoint supports, we have that for $\kappa_1 := k_{\epsilon,n_*}$ and for every $\tilde{u}^j \in \mathcal{U}$ (note that $\tilde{u}^j = 0$ in a neighborhood of $D_0$)

$$\begin{aligned} &(A_{\kappa_1}\tilde{u} + B_{\kappa_1}\tilde{u}, \tilde{u})_{H_0^2(D)} \\ &\quad= \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta\tilde{u} + \kappa_1\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \kappa_1^2 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx \\ &\quad\le \int\limits_{D\setminus\overline{D}_0} \frac{1}{n_*-1}|\Delta\tilde{u} + \kappa_1^2\tilde{u}|^2\,dx + \kappa_1^4 \int\limits_{D\setminus\overline{D}_0} |\tilde{u}|^2\,dx - \kappa_1^2 \int\limits_{D\setminus\overline{D}_0} |\nabla\tilde{u}|^2\,dx = 0. \end{aligned} \tag{4.31}$$

This means that $\mathbb{I} + \mathbb{T}_k$ satisfies assumption 2 of Theorem 4.6, and therefore there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\kappa_0,\, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$

and $k_{\epsilon,n_*}$ both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown that there exists an infinite countable set of transmission eigenvalues that accumulate at $+\infty$.

Now consider the case $0 < n_* \le n(x) \le n^* < 1$. Similarly to the previous case, from the definitions (3.38) and (3.39) of $A_k$ and $B_k$, we have that

$$(A_k u + B_k u, u)_{H_0^2(D)} = \int\limits_{D\setminus\overline{D}_0} \frac{1}{1-n}|\Delta u + k^2 u|^2\, dx - k^4 \int\limits_{D\setminus\overline{D}_0} |u|^2\, dx + k^2 \int\limits_{D\setminus\overline{D}_0} |\nabla u|^2\, dx - k^4 \int\limits_{D_0} |u|^2\, dx + k^2 \int\limits_{D_0} |\nabla u|^2\, dx. \quad (4.32)$$

Hence we have that $A_k + B_k$ is positive as long as

$$-k^2 \int\limits_{D\setminus\overline{D}_0} n|u|^2\, dx + \int\limits_{D\setminus\overline{D}_0} |\nabla u|^2\, dx - k^2 \int\limits_{D_0} |u|^2\, dx + \int\limits_{D_0} |\nabla u|^2\, dx \quad (4.33)$$
$$\ge \int\limits_D |\nabla u|^2\, dx - k^2 \int\limits_D |u|^2\, dx \ge (\lambda_1(D) - k^2)\|u\|^2_{L^2(D)} \ge 0.$$

Therefore, for all $\kappa_0 > 0$ such that $\kappa_0^2 \le \lambda_1(D)$, $\mathbb{I} + \mathbb{T}_k$ satisfies assumption 1 of Theorem 4.6. The rest of the proof can be done in exactly the same way as for the first part, where $n_*$ is replaced by $n^*$. $\square$

## 4.2.2 ▪ Inequalities for Transmission Eigenvalues

The proofs of Theorems 4.12 and 4.14 provide as byproduct inequalities on real transmission eigenvalues that can be used in the inverse medium problem to obtain information about the material properties of the scatterer. We start by stating Faber–Krahn-type inequalities which are merely a consequence of Lemma 4.10 for media without voids, and (4.29)–(4.30) and (4.32)–(4.33) for media with voids.

**Theorem 4.15.** *Let $n \in L^\infty(D)$ and $n = 1$ in $D_0$ ($D_0$ is possibly empty) and denote $0 < n_* := \inf_{D\setminus\overline{D}_0}(n)$ and $n^* := \sup_{D\setminus\overline{D}_0}(n) \le \infty$. Then all real transmission eigenvalues $k > 0$ satisfy*

1. $k^2 \ge \frac{\lambda_1(D)}{n^*}$ *if* $1 < n_*$ *or*

2. $k^2 \ge \lambda_1(D)$ *if* $n^* < 1$,

*where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in $D$.*

The above inequalities are not isoperimetric. The proof of Theorem 4.12 implies the following monotonicity results for a sequence of eigenvalues which can be seen as a type of "isoperimetric" inequality for transmission eigenvalues in terms of the refractive index for fixed $D$. Let $k_j := k_j(n(x), D) > 0$ for $j \in \mathbb{N}$ be the increasing sequence of the transmission eigenvalues for the media with support $D$ and refractive index $n(x)$ such that $t_j = k_j^2$ is the smallest zero of $\lambda_j(\tau, D, n(x)) = \tau$, where $\lambda_j(\tau, D, n(x))$, $j \ge 1$, are the eigenvalues of the auxiliary problem (see the proof of Theorem 4.12) given by

$$\lambda_j(\tau, D, n(x)) = \min_{W \in \mathcal{U}_j} \max_{\substack{u \in W \\ \|\nabla u\|_{L^2} = 1}} \int_D \frac{1}{n(x)-1} |\Delta u + \tau u|^2 \, dx + \tau^2 \|u\|^2_{L^2(D)}, \tag{4.34}$$

where $\mathcal{U}_j$ denotes the set of all $j$-dimensional subspaces $W$ of $H_0^2(D)$. Then the following monotonicity property for transmission eigenvalues is true.

**Theorem 4.16.** *Let* $n \in L^\infty(D)$ *and* $0 < n_* = \inf_D(n)$, $n^* := \sup_D(n) \leq +\infty$. *Assume that* $B_1$ *and* $B_2$ *are two balls of radius* $r_1$ *and* $r_2$*, respectively, such that* $B_1 \subset D \subset B_2$. *Then*

1. *if* $1 < n_*$, *then*

$$k_j(n^*, B_2) \leq k_j(n^*, D) \leq k_j(n(x), D) \leq k_j(n_*, D) \leq k_j(n_*, B_1);$$

2. *if* $n^* < 1$, *then*

$$k_j(n_*, B_2) \leq k_j(n_*, D) \leq k_j(n(x), D) \leq k_j(n^*, D) \leq k_j(n^*, B_1).$$

*In particular, these inequalities hold true for the smallest transmission eigenvalue* $k_1(n(x), D)$.

***Proof.*** For simplicity of presentation we prove the theorem for the smallest transmission eigenvalue. Take $1 < n_*$. Then for any $u \in H_0^2(D)$ such that $\|\nabla u\|_{L^2(D)} = 1$ we have

$$\begin{aligned} \frac{1}{n^*-1} \|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)} &\leq \int_D \frac{1}{n(x)-1} |\Delta u + \tau u|^2 \, dx + \tau^2 \|u\|^2_{L^2(D)} \\ &\leq \frac{1}{n_*-1} \|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)}. \end{aligned} \tag{4.35}$$

Therefore from (4.34) we have that for an arbitrary $\tau > 0$

$$\begin{aligned} \lambda_1(\tau, B_2, n^*) &\leq \lambda_1(\tau, D, n^*) \leq \lambda_1(\tau, D, n(x)) \\ &\leq \lambda_1(\tau, D, n_*) \leq \lambda_1(\tau, B_1, n_*). \end{aligned}$$

Now for $\tau_1 := k_{1,n_*}/r_1$, $B_1 \subset D$, from the proof of Theorem 4.12 we have that $\lambda_1(\tau_1, D, n(x)) - \tau_1 \leq 0$. On the other hand, for $\tau_0 := k_{1,n^*}/r_2$, $D \subset B_2$, we have $\lambda_1(\tau_0, B_2, n^*) - \tau_0 = 0$ and hence $\lambda_1(\tau_0, D, n(x)) - \tau_0 \geq 0$. Therefore the first eigenvalue $k_{1,D,n(x)}$ corresponding to $D$ and $n(x)$ is between $k_{1,n^*}/r_2$ and $k_{1,n_*}/r_1$. Note that there is no transmission eigenvalue for $D$ and $n(x)$ that is less than $k_{1,n^*}/r_2$. Indeed, if there is a transmission eigenvalue strictly less than $k_{1,n^*}/r_2$, then by the monotonicity of the eigenvalues of the auxiliary problem with respect to the domain and the fact that for $\tau$ small enough there are no transmission eigenvalues, we would have found an eigenvalue of the ball $B_2$ and $n^*$ that is strictly smaller than the first eigenvalue. The case of $n^* < 1$ can be proven in the same way if $n_*$ is replaced by $n^*$.

Now it is clear how to modify the same argument for the smallest zero of $\lambda_j(\tau, D, n(x)) = \tau$. ∎

**Remark 4.17.** We remark that obviously the balls $B_1$ and $B_2$ in Theorem 4.16 can be replaced by any two domains such that $D_1 \subset D \subset D_2$. Also for fixed $D$ and two media with the same support $D$ and refractive indices $n_1(x)$ and $n_2(x)$ both in $L^\infty(D)$ the proof of Theorem 4.16 can be adapted in an obvious way to prove the following:

1. If $1 < \alpha \le n_1(x) \le n_2(x)$ for almost all $x \in D$, then
$$k_j(n_2(x), D) \le k_j(n_1(x), D).$$
2. If $0 < \alpha \le n_1(x) \le n_2(x) \le \beta < 1$ for almost all $x \in D$, then
$$k_j(n_1(x), D) \le k_j(n_2(x), D).$$

Theorem 4.16 shows in particular that for a constant index of refraction the first transmission eigenvalue $k_1(n, D)$ as a function of $n$ for $D$ fixed is monotonically increasing if $n > 1$ and is monotonically decreasing if $0 < n < 1$. In fact in [30] it is shown that this monotonicity is strict which leads to the following uniqueness result for a constant index of refraction in terms of the first transmission eigenvalue, which is the only known inverse spectral result for general media (see Chapter 6 for results on inverse spectral problems for spherically stratified media).

**Theorem 4.18.** *The constant index of refraction $n$ is uniquely determined from a knowledge of the corresponding smallest transmission eigenvalue $k_1(n, D) > 0$, provided that it is known a priori that either $n > 1$ or $0 < n < 1$.*

***Proof.*** Here we show the proof for the case of $n > 1$ (see [30] for the case of $0 < n < 1$). Consider two homogeneous media with constant indices of refraction $n_1$ and $n_2$ such that $1 < n_1 < n_2$, and let $u_1 := w_1 - v_1$, where $w_1, v_1$ is the nonzero solution of (3.2) with $n(x) := n_1$ corresponding to the first transmission eigenvalue $k_1(n_1, D)$. Now, setting $\tau_1 = k_1^2(n_1, D)$ and after normalizing $u_1$ such that $\|\nabla u_1\|_{L^2(D)} = 1$, we have
$$\frac{1}{n_1 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} = \tau_1 = \lambda_1(\tau_1, D, n_1). \tag{4.36}$$
Furthermore, we have
$$\frac{1}{n_2 - 1}\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)} < \frac{1}{n_1 - 1}\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)}$$
for all $u \in H_0^2(D)$ such that $\|\nabla u\|_{L^2(D)} = 1$ and all $\tau > 0$. In particular for $u = u_1$ and $\tau = \tau_1$
$$\frac{1}{n_2 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} < \frac{1}{n_1 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)}.$$
But using (4.36) we have
$$\lambda(\tau_1, D, n_2) \le \frac{1}{n_2 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} < \lambda_1(\tau_1, D, n_1),$$
and hence for this $\tau_1$ we have a strict inequality, i.e.,
$$\lambda_1(\tau_1, D, n_2) < \lambda_1(\tau_1, D, n_1). \tag{4.37}$$
Obviously (4.37) implies the first zero $\tau_2$ of $\lambda_1(\tau, D, n_2) - \tau = 0$ is such that $\tau_2 < \tau_1$, and therefore we have that $k_1(n_2, D) < k_1(n_1, D)$ for the first transmission eigenvalues $k_1(n_1, D)$ and $k_1(n_2, D)$ corresponding to $n_1$ and $n_2$, respectively. Hence we have shown that if $n_1 > 1$ and $n_2 > 1$ are such that $n_1 \ne n_2$, then $k_1(n_1, D) \ne k_1(n_2, D)$, which proves uniqueness. $\square$

We finally present a monotonicity result for the first transmission eigenvalue corresponding to media with voids. For a fixed $D$, denote by $k_1(D_0, n)$ the first transmission eigenvalue corresponding to the void $D_0$ and the index of refraction $n$.

**Theorem 4.19.** *If $D_0 \subseteq \tilde{D}_0$ and $n, \tilde{n} \in L^2(D)$ such that $n(x) \leq \tilde{n}(x)$ for almost every $x \in D$, then*

1. $k_1(D_0, \tilde{n}) \leq k_1(\tilde{D}_0, n)$ *if* $1 < \alpha \leq n(x) \leq \tilde{n}(x)$;
2. $k_1(D_0, n) \leq k_1(\tilde{D}_0, \tilde{n})$ *if* $0 < \alpha \leq n(x) \leq \tilde{n}(x) \leq \beta < 1$.

***Proof.*** Consider the first case. Repeating the proof of Theorem 4.14 with $\kappa_0 > 0$ such that $\kappa_0^2 = \frac{\lambda_1(D)}{\sup_D(\tilde{n})}$ and $\kappa_1 = k_1(D_0, n)$, one deduces that $k_1(D_0, \tilde{n}) \leq k_1(D_0, n)$. It remains to show that for fixed $n$, $k_1(D_0, n) \leq k_1(\tilde{D}_0, n)$. To this end, again from the proof of Theorem 4.14, $A_{\kappa_0} + B_{\kappa_0}$ is positive for $\kappa_0 > 0$ such that $\kappa_0^2 = \frac{\lambda_1(D)}{\sup_D(n)}$. Next let $\kappa_1 = k_1(\tilde{D}_0, n)$ and let $v \in V_0(D, \tilde{D}_0, \kappa_1)$ be its corresponding eigenvector. Then

$$\begin{aligned}
(A_{\kappa_1} v + B_{\kappa_1} v, v)_{H_0^2(D)} &= \int\limits_{D \setminus \overline{D}_0} \frac{1}{n-1} |\Delta v + \kappa_1^2 n v|^2 \, dx - \kappa_1^4 \int\limits_D n |v|^2 \, dx \\
&\quad + \kappa_1^2 \int\limits_D |\nabla u|^2 \, dx \\
&= \int\limits_{D \setminus \overline{\tilde{D}}_0} \frac{1}{n-1} |\Delta v + \kappa_1^2 n v|^2 \, dx - \kappa_1^4 \int\limits_D n |v|^2 \, dx \\
&\quad + \kappa_1^2 \int\limits_D |\nabla u|^2 \, dx = 0,
\end{aligned}$$

which implies from Theorem 4.6 that there exists a transmission eigenvalue in $[\kappa_0, \kappa_1]$ for media with void $D_0$ and refractive index $n$. The same type of argument shows that this indeed is the first eigenvalue. Hence we have that $k_1(D_0, n) \leq k_1(\tilde{D}_0, n)$, which proves the estimates in the first case. The second case can be handled similarly, and we leave it to the reader as an exercise. $\square$

Note that although the transmission eigenvalue problem (3.2) has the structure of a quadratic pencil of operators (4.10), it appears that available results on quadratic pencils [132] are not applicable to the transmission eigenvalue problem due to the incorrect signs of the involved operators. The crucial assumption in our analysis in this chapter is that the contrast does not change sign inside $D$, i.e., $n-1$ is either positive or negative and bounded away from zero in $D$, except that we allow that $n = 1$ in a subregion of $D$. By using weighted Sobolev spaces it is also possible in a similar way to this chapter to consider the case when $n - 1$ goes smoothly to zero at the boundary $\partial D$ [67], [74], [96], [156]. However, the real interest is in investigating the case when $n - 1$ is allowed to change sign in $D$. The question of discreteness of transmission eigenvalues in the latter case has been related to the uniqueness of the sound speed for the wave equation with an arbitrary source, which is a question that arises in thermoacoustic imagining [86]. In the general case $n \geq c > 0$ with no assumptions on the sign of $n - 1$, the study of the transmission eigenvalue problem is completely open. As the reader has seen in Chapter 3, the discreteness of transmission eigenvalues is obtained under the assumption that $n - 1$ has a fixed sign in a neighborhood of the boundary. In the case when both the domain $D$ and refractive index $n(x)$ are $C^\infty$-smooth, with the additional assumption that $n \neq 1$ on the boundary $\partial D$, a complete characterization of the spectrum of the transmission eigenvalue problem is presented in [152]. This study is done in the framework of semiclassical analysis [59], relating the transmission eigenvalue problem to the spectrum of a Hilbert–Schmidt operator whose resolvent exhibits the desired growth properties following the approach of Agmon in [1]. For the sake of completeness, we sketch here the main points of this approach.

Let $n \in C^\infty(\overline{D})$, where $D \subset \mathbb{R}^3$ such that $\partial D$ is of class $C^\infty$. Furthermore, we assume that $n(x) \geq n_0 > 0$ for $x \in D$ and $n \neq 1$ on $\partial D$ (note that by continuity the latter means that $n \neq 1$ in a neighborhood of $\partial D$). As the reader has already seen, the transmission eigenvalue problem can be written in terms of $u := \frac{1}{k^2}(w-v) \in H_0^2(D)$ and $v \in L^2(D)$ as

$$\begin{aligned} \frac{1}{n}\Delta u + \frac{n-1}{n} v + k^2 u = 0 \qquad & \text{in } D, \\ \Delta v + k^2 v = 0 \qquad & \text{in } D. \end{aligned} \tag{4.38}$$

For $z \in \mathbb{C}$ define the operator $B_z : H_0^2(D) \times \{L^2(D),\ \Delta u \in L^2(D)\} \to L^2(D) \times L^2(D)$ by

$$(u, v) \mapsto (f, g),$$

where

$$\frac{1}{n}\Delta u + \frac{n-1}{n} v - zu = f \qquad \text{in } D, \tag{4.39}$$

$$\Delta v - zv = g \qquad \text{in } D. \tag{4.40}$$

We already know from Section 3.1.3 that there is a fixed $z \in \mathbb{C}$ such that $R_z := B_z^{-1}$ is bounded. The spectral properties of the transmission eigenvalue problem can be deduced from the spectral analysis of $B_z$ or, more precisely, its inverse $R_z$. Indeed if $\eta$ is an eigenvalue of $B_z$, then $k \in \mathbb{C}$ such that $k^2 = -z - \eta$ is a transmission eigenvalue with the same eigenfunction. To this end, a key tool is the following lemma, which is a direct consequence of Proposition 4.2 and the proof of Theorem 5 in [152]. Indeed the statement of the lemma is a slight modification of the celebrated result of Agmon stated in Theorem 16.4 in [1].

**Theorem 4.20.** *Let $H$ be a Hilbert space, and let $S : H \to H$ be a bounded linear operator. If $\lambda^{-1}$ is in the resolvent of $S$, define*

$$S_\lambda = S(I - \lambda S)^{-1}. \tag{4.41}$$

*Assume $S^p : H \to H$ is a Hilbert–Schmidt operator for some integer $p \geq 2$. For the operator $S$, assume there exist $0 \leq \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ such that $\theta_k - \theta_{k-1} < \frac{\pi}{2p}$ for $k = 2, \dots, N$ and $2\pi - \theta_N + \theta_1 < \frac{\pi}{2p}$ satisfying the condition that there exists $r_0 > 0$, $c > 0$, such that $\sup_{r \geq r_0} \|(S)_{re^{i\theta_k}}\|_{H \to H} \leq c$ for $k = 1, \dots, N$. Then eigenvalues of $S$ exist and the space spanned by the nonzero generalized eigenfunctions is dense in the closure of the range of $S^p$.*

One can now apply Theorem 4.20 to the operator $S := R_z$ for fixed $z$ and $H := L^2(D) \times L^2(D)$ to derive the desired spectral decomposition for $R_z$, noting that $(R_z)_\lambda = R_{z+\lambda}$, where $(R_z)_\lambda$ is defined by (4.41) with $S$ replaced by $R_z$. To this end one needs to prove the following:

1. A regularity result for the solution of (4.38). This part is quite technical, and the approach involves results from pseudodifferential calculus. For the details we refer the reader to [152]. In particular, it is possible to prove that $R_z$ is two orders smoothing, i.e., the mapping $R_z : H^2(D) \times L^2(D) \to H^4(D) \times H^2(D)$ is bounded, which first proves that $R_z$ on $H^2(D) \times L^2(D)$ is compact, and then applying Theorem 13.5 in [1] proves that $R_z^2$ on $H^2(D) \times L^2(D)$ is Hilbert–Schmidt.

2. Then using the theory of pseudodifferential operators for symbols with a parameter, it is possible to prove a growth condition for $R_z$ along the rays such as stated in Theorem 4.20 for $p = 2$. This step is also technical, and more details can be found in Section 3.1 in [152].

The final result of the above effort is stated in the following theorem.

**Theorem 4.21.** *Assume that $n \in C^\infty(\overline{D})$, where $D \subset \mathbb{R}^3$ is such that $\partial D$ is of class $C^\infty$ and $n(x) \geq n_0 > 0$ for $x \in D$ and $n \neq 1$ on $\partial D$. Then there exist an infinite number of transmission eigenvalues $k \in \mathbb{C}$ and the space spanned by the generalized eigenfunctions is dense in $H_0^2(D) \times \left\{L^2(D),\ \Delta u \in L^2(D)\right\}$.*

We note that although in [152] the refractive index is allowed to be complex valued, the analysis there does not imply any result on transmission eigenvalues for absorbing media, i.e., when the refractive index depends on the wave number.

Another important question is the location of the transmission eigenvalues in the complex plane $\mathbb{C}$. A first attempt to localize transmission eigenvalues in the complex plane is done in [30], followed by [97], where it is shown that almost all transmission eigenvalues $k^2$ are confined to a parabolic neighborhood of the positive real axis. However, it is desirable to know if there exists a half-plane in $\mathbb{C}$ free of transmission eigenvalues. This is an important question for analyzing the time-domain interior transmission problem, which is the main building block for the time-domain linear sampling method for inhomogeneous media [52]. This question was first answered in [164]. The apparatus of this paper relies heavily on microlocal analysis to construct a parametrix for the involved operators, and it is impossible to develop in this monograph the needed rigorous mathematical framework. We only state the main results here for the reader's convenience and refer to [164], [165], [166] for the proofs and a more complete picture. To this end, we first remark that the transmission eigenvalue problem can be recast in terms of the difference of two Dirichlet-to-Neumann operators. More precisely, let us define $\Lambda_q(k) : \varphi \mapsto \frac{\partial u}{\partial \nu}$, where $u$ solves

$$\begin{cases} \Delta u + k^2 q u = 0 & \text{in } D, \\ u = \varphi & \text{on } \partial D \end{cases}$$

(provided $k^2$ is not a Dirichlet eigenvalue). Then the transmission eigenvalue problem can be viewed as finding $k \in \mathbb{C}$ for which there exists a nontrivial $u$ such that

$$\mathcal{T}(k)u := \Lambda_n(k)u - \Lambda_1(k)u = 0.$$

The operator $\mathcal{T}(k) : H^{-1/2+s}(\partial D) \to H^{1/2+s}(\partial D)$, $0 \leq s \leq 1$, is one order smoothing and is Fredholm with index zero. The eigenvalue free zone in $\mathbb{C}$ corresponds to $k \in \mathbb{C}$ for which $\mathcal{T}(k)^{-1}$ exists. In [164] it is proven that all transmission eigenvalues $k$ lie in a horizontal strip about the real axis. In addition this paper provides $k$-explicit bounds for the norm of the inverse of $\mathcal{T}(k)$ as well as Weyl's asymptotic estimates for the transmission eigenvalues. The main tool in obtaining these results is the derivation of refined high frequency estimates for the Dirichlet-to-Neumann operator in the framework of semiclassical analysis. Therefore these results require $C^\infty$ regularity for both $D$ and $n$ and are summarized in the following theorem.

**Theorem 4.22.** *Assume that $n \in C^\infty(\overline{D})$, $\partial D$ is of class $C^\infty$, $n(x) \geq n_0 > 0$ for $x \in D$, and $n \neq 1$ on $\partial D$. The following hold:*

(i) *There are no transmission eigenvalues in the region $\{k \in \mathbb{C} : \ |\Im(k)| > \gamma\}$ for some constant $\gamma > 0$. In this region, $\mathcal{T}(k)^{-1} : H^1(\partial D) \to L^2(\partial D)$ is bounded, and if in addition $\Re(k) > 1$,*

$$\|\mathcal{T}(k)^{-1}\| \leq c|k|^{-1} \quad \textit{for some } c > 0.$$

(ii) *Let $N(r) := \#\{k$ transmission eigenvalues $|k| \leq r\}$; then*

$$N(r) = \frac{r^3}{6\pi^2} \int_D \left(1 + n(x)^{3/2}\right) dx + O_\epsilon\left(r^{2+\epsilon}\right)$$

*for all $0 < \epsilon \ll 1$, where the order term depends on $\epsilon$.*

The completeness results in [152] and Weyl's estimates in [164] are recovered in [87] for less regular $\partial D$ and $n$, including the case when the Laplace operator $\Delta$ in both equations in the transmission eigenvalue problem is replaced by $\nabla \cdot A\nabla$ with the same matrix valued coefficient $A$ of class $C^2(\overline{D})$ (i.e., the contrast is only due to the lower order term). More specifically, these results are obtained for $\partial D$ of class $C^3$ and $n \in C^1(\overline{D})$, $n \neq 1$, on $\partial D$. The analysis is based on the theory of Hilbert–Schmidt operators, but well-posedness estimates are obtained in the $L^p$-framework, avoiding the use of microlocal analysis and allowing for less regularity. For the detailed proofs we refer the reader to [87]

More results on transmission eigenvalues for isotropic media, including Weyl-type asymptotic estimates for the counting function for transmission eigenvalues, can be found in [124], [126], [140], [145], [146], [147]. We conclude this section by listing a few important open questions. Although for spherically symmetric media it is proven that complex transmission eigenvalues exist, for general media $(D, n)$ it is not known if this is the case. Also in the case when the contrast $n - 1$ is of one sign in a neighborhood of $\partial D$ but otherwise it changes sign inside $D$, it is not known if *real* transmission eigenvalues exist, a question which is important in the use of transmission eigenvalues to obtain information on the refractive index $n$. Nothing is known about the spectral properties of the transmission eigenvalue problem in the case when there is a point $P$ on the boundary $\partial D$ for which $n - 1$ changes sign in every neighborhood of $P$ inside $D$.

### 4.2.3 ▪ Remarks on Absorbing Media

The refractive index $n(x)$ for an absorbing media depends on the wave number $k$; more precisely, for a large range of frequencies it assumes the form

$$n(x) = \epsilon(x) + i\frac{\gamma(x)}{k}$$

for real valued functions $\epsilon$ and $\gamma$. The reader can view the complex part in the refractive index as arising from the Fourier transform of the damping which involves the time derivative of the field. In our analysis in the previous chapters we have considered the complex valued refractive index where we have ignored the dependence on $k$ of the imaginary part. This is fine as long as we are considering a fixed frequency, and this is the case in our discussion of the direct scattering problem, the reconstruction techniques, and the solvability of the interior transmission problem. However, in order to correctly investigate the spectral properties of the transmission eigenvalue problem for absorbing media, it is necessary to take into consideration the $k$-dependence of the refractive index since $k$ is the eigenvalue parameter. At this time, very little is known about the spectral properties of the transmission eigenvalue problem in this case, and in many recent studies (e.g., [152]) the $k$-dependence on the refractive index is dropped.

The study of the transmission eigenvalue problem in the general case of absorbing media and background has been initiated in [35] (see also [80]), and we now present these results. In particular we prove that the set of transmission eigenvalues in the open right complex half-plane is at most discrete, provided that the contrast in the real part of the index of refraction does not change sign in $D$. Furthermore, using perturbation theory, we show that if the absorption in the inhomogeneous media and (possibly) in the background is small enough, then there exist (at least) a finite number of complex transmission eigenvalues each near a real transmission eigenvalue associated with the corresponding nonabsorbing media and background.

Before we start with our presentation, we alert the reader that up to now we have considered for simplicity a homogeneous nonabsorbing background with refractive index scaled to one. On the other hand, as the reader has by now seen, the interior transmission problem depends on the refractive index of the scattering inhomogeneity and the refractive index of the background in the region $D$ occupied by this inhomogeneity. The difference of the refractive index of the inhomogeneity and background, referred to as the contrast in the media, fundamentally characterize the properties of the interior transmission problem. In order to introduce the reader to the interior transmission problem arising from scattering due to an inhomogeneity embedded in a complex background, in this section we consider an inhomogeneous (possibly absorbing) background to the scattering inhomogeneity.

The interior transmission eigenvalue problem for an inhomogeneous absorbing media of support $D$ occupying a part of an inhomogeneous absorbing background is formulated as

$$\Delta w + k^2 \left( \epsilon_1(x) + i\frac{\gamma_1(x)}{k} \right) w = 0 \qquad \text{in } D, \tag{4.42}$$

$$\Delta v + k^2 \left( \epsilon_0(x) + i\frac{\gamma_0(x)}{k} \right) v = 0 \qquad \text{in } D, \tag{4.43}$$

$$v = w \qquad \text{on } \partial D, \tag{4.44}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D, \tag{4.45}$$

where $w \in L^2(D)$ and $v \in L^2(D)$ such that $w - v \in H_0^2(D)$. Here we assume that $\epsilon_1 \in L^\infty(D)$ and $\gamma_1 \in L^\infty(D)$ such that $\epsilon_1(x) \geq \eta_1 > 0$, $\gamma_1(x) \geq 0$ almost everywhere in $D$, and similarly $\epsilon_0 \in L^\infty(D)$ and $\gamma_0 \in L^\infty(D)$ such that $\epsilon_0(x) \geq \eta_0 > 0$, $\gamma_0(x) \geq 0$. Similarly to Section 3.1.1, it is possible to write (4.42)–(4.45) as an eigenvalue problem for the fourth order differential equation

$$\left(\Delta + k^2\epsilon_1(x) + ik\gamma_1(x)\right) \frac{1}{k\epsilon_c(x) + i\gamma_c(x)} \left(\Delta + k^2\epsilon_0(x) + ik\gamma_0(x)\right) u = 0 \tag{4.46}$$

for $u \in H_0^2(D)$, where we denote by $\epsilon_c := (\epsilon_1 - \epsilon_0)$ and $\gamma_c := (\gamma_1 - \gamma_0)$ the respective contrasts. Obviously if $u \in H_0^2(D)$ satisfies (4.46), then

$$w := \frac{-1}{k^2\epsilon_c + ik\gamma_c}(\Delta + k^2\epsilon_0 + ik\gamma_0)u \in L^2(D)$$

and $v = w - u \in L^2(D)$ satisfies (4.42)–(4.45).

In variational form (4.46) is formulated as the problem of finding $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{k\epsilon_c + i\gamma_c}\left[\Delta u + (k^2\epsilon_0 + ik\gamma_0)u\right]\left[\Delta \overline{v} + (k^2\epsilon_1 + ik\gamma_1)\overline{v}\right]\, dx = 0 \qquad (4.47)$$

for all $v \in H_0^2(D)$. It is easy to see that the interior transmission problem (4.42)–(4.45) does not have purely imaginary eigenvalues $k = i\tau$ as long as $\tau > 0$ is such that $\tau\epsilon_c + \gamma_c > 0$. Indeed, after integrating by parts and using the zero boundary conditions, we have that

$$\begin{aligned}
0 &= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}\left[\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right]\left[\Delta \overline{u} - (\tau^2\epsilon_1 + \tau\gamma_1)\overline{u}\right]\, dx \\
&= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}\left|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right|^2 dx - \tau\int_D \left[\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right]\overline{u}\, dx \\
&= \int_D \frac{1}{\tau\epsilon_c + \gamma_c}\left|\Delta u - (\tau^2\epsilon_0 + \tau\gamma_0)u\right|^2 dx + \tau\int_D |\nabla u|^2\, dx + \tau^2\int_D (\tau\epsilon_0 + \gamma_0)|u|^2\, dx,
\end{aligned}$$

which implies that $u = 0$ in $D$. In a similar way, by exchanging subindices $_1$ and $_0$ one can show the same result for $\tau\epsilon_c + \gamma_c < 0$. The situation is not clear for $k = i\tau$ for which $\tau\epsilon_c + \gamma_c$ changes sign. For example, if $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\gamma_0 > 0$, and $\gamma_1 > 0$ are all positive constants, then $k = i\tau_0$, where $\tau_0 = \frac{\gamma_1 - \gamma_0}{\epsilon_1 - \epsilon_0}$ is an eigenvalue and the corresponding eigenspace is infinite-dimensional since for any solution $v$ to the Helmholtz equation $\Delta v - \tau_0(\tau_0\epsilon_0 + i\gamma_0)v = 0$, $v$ and $w = v$ are eigenfunctions.

**Remark 4.23.**

1. If $\epsilon_c(x) \geq \theta > 0$ and $\gamma_c(x) \geq 0$ almost everywhere in $D$, then $k = i\tau$, where $\tau$ is such that $\tau \geq -\frac{\sup_D \gamma_c}{\inf_D \epsilon_c}$ or $\tau \leq -\frac{\inf_D \gamma_c}{\sup_D \epsilon_c}$ is not a transmission eigenvalue.

2. If $\epsilon_c(x) \geq \theta > 0$ and $|\gamma_c(x)| < M$ almost everywhere in $D$, then $k = i\tau$, where $\tau > 0$ is large enough such that $\tau \geq \frac{M}{\inf_D \epsilon_c}$ is not a transmission eigenvalue.

In the following we assume that the real part of $k \in \mathbb{C}$ is positive. Furthermore, we assume that the contrast $\epsilon_c$ is bounded and does not change sign; more specifically, due to the symmetric role of $\epsilon_1$ and $\epsilon_0$, we require that $0 < \theta \leq \epsilon_c(x) < N$ almost everywhere in $D$, whereas the contrast $\gamma_c$ is only bounded, i.e., $|\gamma_c(x)| < M$ almost everywhere in $D$.

**Lemma 4.24.** *Assume that $0 < \theta \leq \epsilon_c(x) < N$ and $|\gamma_c(x)| < M$ almost everywhere in $D$. Then the set of transmission eigenvalues in the region $G_\sigma := \{k = \kappa + i\tau : \kappa \geq \sigma > 0$ and $\tau \leq 2M/\theta\} \cup \{k = \kappa + i\tau : \kappa \in \mathbb{R}$ and $\tau \geq 2M/\theta\}$ is discrete.*

***Proof.*** Let us define the following sesquilinear forms on $H_0^2(D)$:

$$\mathcal{A}_k(u,v) = \int\limits_D \frac{1}{k\epsilon_c + i\gamma_c} \Delta u \, \Delta \overline{v} \, dx,$$

$$\mathcal{B}_k(u,v) = \int\limits_D \left[ k \frac{k\epsilon_1 + i\gamma_1}{k\epsilon_c + i\gamma_c} \Delta u \, \overline{v} + k \frac{k\epsilon_0 + i\gamma_0}{k\epsilon_c + i\gamma_c} u \, \Delta \overline{v} + k^2 \frac{(k\epsilon_0 + i\gamma_0)(k\epsilon_1 + i\gamma_1)}{k\epsilon_c + i\gamma_c} u \, \overline{v} \right] dx.$$

From our assumption we have that $|k\epsilon_c + i\gamma_c| \geq \beta > 0$ almost everywhere in $D$, and therefore the above bilinear forms define bounded linear operators $\mathbf{A}_k : H_0^2(D) \to H_0^2(D)$ and $\mathbf{B}_k : H_0^2(D) \to H_0^2(D)$ by means of the Riesz representation theorem. In terms of these operators the transmission eigenvalue problem takes the form

$$(\mathbf{A}_k + \mathbf{B}_k)\, u = 0, \qquad u \in H_0^2(D). \tag{4.48}$$

In particular, $k$ is a transmission eigenvalue if and only if the kernel of the operator $\mathbf{A}_k + \mathbf{B}_k$ is nontrivial. In the same way as is Section 3.1.1 one can prove that $\mathbf{A}_k$ is invertible for fixed $k \in G_\sigma \subset \mathbb{C}$ and $\mathbf{B}_k$ is compact. Since (4.48) becomes $\left(\mathbf{I} + \mathbf{A}_k^{-1}\mathbf{B}_k\right) u = 0$, if $k$ is a transmission eigenvalue $-1$ is an eigenvalue of the compact (non-self-adjoint) operator $\mathbf{A}_k^{-1}\mathbf{B}_k$ and hence transmission eigenvalues have finite multiplicity. Note that the eigenfunctions of $\mathbf{A}_k^{-1}\mathbf{B}_k$ are elements of the kernel of $\mathbf{A}_k + \mathbf{B}_k$, and vice versa.

Next we show that the set of transmission eigenvalues is discrete, and to this end we apply the Analytic Fredholm Theorem. Obviously the bilinear forms $\mathcal{A}_k(\cdot,\cdot)$ and $\mathcal{B}_k(\cdot,\cdot)$ depend analytically on $k \in G_\sigma \subset \mathbb{C}$, and thus the mappings $k \mapsto \mathbf{A}_k$ and $k \mapsto \mathbf{B}_k$ are weakly analytic in this region and hence strongly analytic [69]. Therefore, $k \mapsto \mathbf{A}_k^{-1}$ is also strongly analytic and so is $k \mapsto \mathbf{A}_k^{-1}\mathbf{B}_k$. Furthermore, from Remark 4.23, $k_0 = i\tau$ for some $\tau > 2M/\theta$ is not a transmission eigenvalue, i.e., the kernel of $\mathbf{A}_{k_0} + \mathbf{B}_{k_0}$, and hence of $\mathbf{I} + \mathbf{A}_{k_0}^{-1}\mathbf{B}_{k_0}$, is nontrivial. Hence from the Analytic Fredholm Theorem 1.12 we can conclude that the set of transmission eigenvalues in the region $G_\sigma \subset \mathbb{C}$ of the complex plane is discrete (possibly empty) with $\infty$ as the only possible accumulation point. ∎

Now since the region $k \in \mathbb{C}$ such that $\Re(k) > 0$ is included in $\bigcup_{n=1}^\infty G_{1/n}$ we have proven the following theorem.

**Theorem 4.25.** *Assume that* $0 < \theta \leq \epsilon_c(x) < N$ *and* $|\gamma_c(x)| < M$ *almost everywhere in* $D$. *Then the set of transmission eigenvalues* $k \in \mathbb{C}$, $\Re(k) > 0$, *is discrete (possibly empty).*

The existence of transmission eigenvalues for absorbing media is in general an open problem. However, for small enough conductivities $\gamma_0$ and $\gamma_1$, using perturbation theory [106] it is possible to show the existence of transmission eigenvalues near the real axis. The following theorem is just a reformulation of Theorem 4.12.

**Theorem 4.26.** *Assume that both* $\gamma_0 = 0$ *and* $\gamma_1 = 0$ *almost everywhere in* $D$ *and* $\epsilon_0 \in L^\infty(D)$ *and* $\epsilon_1 \in L^\infty(D)$ *are such that* $\epsilon_0(x) \geq \theta_0 > 0$, $\epsilon_1(x) \geq \theta_1 > 0$, *and* $\epsilon_c := \epsilon_1 - \epsilon \geq \theta > 0$ *almost everywhere in* $D$. *Then there exists an infinite set of positive real transmission eigenvalues that accumulate only at* $+\infty$. *Furthermore, the smallest real transmission eigenvalue* $k_1 > 0$ *satisfies* $k_1 > \frac{\lambda_1(D)}{\sup_D \epsilon_c}$, *where* $\lambda_1(D) > 0$ *is the first Dirichlet eigenvalue for* $-\Delta$ *in* $D$.

Our aim is to now use the upper semicontinuity of the spectrum of linear operators. To this end we rewrite the eigenvalue problem (4.42)–(4.45) in a different equivalent form.

Note that we already know by Theorem 4.25 that in the right half-plane (4.42)–(4.45) has a discrete point spectrum. Obviously in terms of $v$ and $u := w - v$, (4.42)–(4.45) can be written as

$$\Delta u + \left(k^2\epsilon_1 + ik\gamma_1\right) u + \left(k^2\epsilon_c + ik\gamma_c\right) v = 0 \qquad \text{in } D, \tag{4.49}$$

$$\Delta v + \left(k^2\epsilon_0 + ik\gamma_0\right) v = 0 \qquad \text{in } D, \tag{4.50}$$

together with the boundary conditions

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{4.51}$$

These equations make sense for $u = H_0^2(D)$ and $v \in L^2(D)$ such that $\Delta v \in L^2(D)$. Setting $X(D) := H_0^2(D) \times \left\{v \in L^2(D) :\ \Delta v \in L^2(D)\right\}$, we can define the linear operators $\mathbb{A}, \mathbb{B}_\gamma, \mathbb{D}_\epsilon : L^2(D) \times L^2(D) \to L^2(D) \times L^2(D)$ by

$$\mathbb{A} = \begin{pmatrix} \Delta_{00} & 0 \\ 0 & \Delta \end{pmatrix}, \qquad \mathbb{B}_\gamma = \begin{pmatrix} i\gamma_1 & i\gamma_c \\ 0 & i\gamma_0 \end{pmatrix}, \qquad \mathbb{D}_\epsilon = \begin{pmatrix} \epsilon_1 & \epsilon_c \\ 0 & \epsilon_0 \end{pmatrix},$$

where $\Delta_{00}$ indicates that the Laplacian acts on a function in $H_0^2(D)$, i.e., one with zero Cauchy data on $\partial D$. Let $\mathbf{p} := \binom{u}{v}$ and note that the domain of definition of $\mathbb{A}$ is $X(D)$ and $\mathbb{A}$ is an unbounded densely defined operator in $L^2(D) \times L^2(D)$. Furthermore, $\mathbb{A}$ is a closed operator, i.e., for any sequence $\{\mathbf{p}_n\} \in X(D)$ such that $\mathbf{p}_n \to \mathbf{p}$ in $L^2(D) \times L^2(D)$ and $\mathbb{A}\mathbf{p}_n \to \mathbf{q}$, we have that $\mathbf{p} \in X(D)$ and $\mathbb{A}\mathbf{p} = \mathbf{q}$. Indeed, since $\|\Delta_{00} u\|_{L^2(D)}$ defines an equivalent norm in $H_0^2(D)$, if $u_n \to u$ in $L^2(D)$ and $\Delta_{00} u_n \to q_1$ in $L^2(D)$, then $u \in H_0^2(D)$ and $q_1 = \Delta_{00} u$. Similarly, if $v_n \to v$ in $L^2(D)$ and $\Delta v_n \to q_2$ in $L^2(D)$, then $\Delta v = q_2$. The operators $\mathbb{B}_\gamma$ and $\mathbb{D}_\epsilon$ are bounded in $L^2(D) \times L^2(D)$ and $\mathbb{D}_\epsilon^{-1}$ exists in $L^2(D) \times L^2(D)$ and is given by

$$\mathbb{D}_\epsilon^{-1} = \frac{1}{\epsilon_0 \epsilon_1} \begin{pmatrix} \epsilon_0 & -\epsilon_c \\ 0 & \epsilon_1 \end{pmatrix}.$$

Thus the transmission eigenvalue problem is equivalent to the following quadratic eigenvalue problem:

$$\mathbb{A}\mathbf{p} + k\mathbb{B}_\gamma \mathbf{p} + k^2 \mathbb{D}_\epsilon \mathbf{p} = \mathbf{0}, \qquad \mathbf{p} \in L^2(D) \times L^2(D). \tag{4.52}$$

Introducing $\mathbf{U} = \binom{\mathbf{p}}{k\mathbb{D}_\epsilon \mathbf{p}}$, the eigenvalue problem (4.52) becomes

$$(\mathbb{K}\mathbf{U} - k\mathbb{I}_{\epsilon,\gamma})\mathbf{U} = \mathbf{0}, \qquad \mathbf{U} \in (L^2(D) \times L^2(D))^2, \tag{4.53}$$

where the $4 \times 4$ matrix operators $\mathbb{K}$ and $\mathbb{I}_{\gamma,\epsilon}$ are given by

$$\mathbb{K} := \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \mathbb{I}_{\epsilon,\gamma} := \begin{pmatrix} -\mathbb{B}_\gamma & -\mathbb{I} \\ \mathbb{D}_\epsilon & 0 \end{pmatrix},$$

where $\mathbb{I}$ is the identity operator in $L^2(D) \times L^2(D)$. By straightforward calculation we obtain $\mathbb{I}_{\epsilon,\gamma}^{-1} := \mathbb{D}_\epsilon^{-1} \left(\begin{smallmatrix} 0 & \mathbb{I} \\ -\mathbb{D}_\epsilon & -\mathbb{B}_\gamma \end{smallmatrix}\right)$, which is a bounded operator in $L^2(D) \times L^2(D)$. Thus we have that the original transmission eigenvalue problem (4.42)–(4.45) is equivalent to an eigenvalue problem for the closed (unbounded) operator $\mathbb{T}_{\epsilon,\gamma} := \mathbb{I}_{\epsilon,\gamma}^{-1}\mathbb{K}$ (note that $\mathbb{T}_{\epsilon,\gamma}$ is closed because it is the product of a closed operator with a bounded operator in $(L^2(D) \times L^2(D))^2$). Let us denote by $\mathbb{T}_{\epsilon,\gamma=0}$ the operator defined as above corresponding to the nonabsorbing case, i.e., $\gamma_0 = 0$ and $\gamma_1 = 0$ almost everywhere in $D$ ($\mathbb{B}_{\gamma=0}$ becomes the zero operator). Let $\Sigma(\mathbb{T}_{\epsilon,\gamma})$ be the spectrum of $\mathbb{T}_{\epsilon,\gamma}$ and $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma})$ the resolvent of $\mathbb{T}_{\epsilon,\gamma}$. We have proven in Theorem 4.25 that $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma}) = (\mathbb{T}_{\epsilon,\gamma} - k\mathbb{I})^{-1}$ is well defined for all $k \in \mathbb{C}$ such that $\Re(k) > 0$ except for a discrete set of $k$ without any finite accumulation point (possibly empty). Furthermore, from Theorem 4.26 we already know that $\Sigma(\mathbb{T}_{\epsilon,\gamma=0})$ contains infinitely many isolated points lying on the positive real axis, which indeed are real transmission eigenvalues. Our aim is to use the stability of eigenvalues for closed operators under small perturbations as described in [106, Chapter 4, Section 3]. To this end we need to define what a small perturbation means and prove that $\mathbb{T}_{\epsilon,\gamma}$ is a small perturbation of $\mathbb{T}_{\epsilon,\gamma=0}$ assuming that the absorptions $\gamma_0$ and $\gamma_1$ are small enough.

To do this we set $\mathbb{P} := \mathbb{T}_{\epsilon,\gamma} - \mathbb{T}_{\epsilon,\gamma=0}$ and by straightforward calculation we see that the perturbation $\mathbb{P}$ is a bounded operator in $(L^2(D) \times L^2(D))^2$ given by

$$\mathbb{P} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{D}_\epsilon^{-1}\mathbb{B}_\gamma \end{pmatrix}.$$

According to [106], the perturbation $\mathbb{P}$ is considered small if the so-called gap between the two closed operators $\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}$, denoted by $\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0})$ is small. For the sake of the reader's convenience we include here the definition of the gap $\hat{\delta}(T,S)$ between two closed operators $T$ and $S$ on a Banach space $X$. In particular

$$\hat{\delta}(T,S) = \max(\delta(T,S), \delta(S,T)), \quad \text{where} \quad \delta(T,S) = \sup_{u\in G(T), \|u\|=1} \operatorname{dist}(u, G(S)),$$

where $G(T)$ and $G(S)$ are the graphs of $T$ and $S$, respectively, which are closed subsets of $X \times X$. In particular, if $S = T + A$ with $A$ a bounded operator in $X$, then (see [106, Chapter 4, Theorem 2.14])

$$\hat{\delta}(T+A, T) \le \|A\|.$$

In our case it is now easy to show that

$$\begin{aligned} \hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) &\le \|\mathbb{P}\| \le \|\mathbb{D}_\epsilon^{-1}\mathbb{B}_\gamma\| && (4.54)\\ &\le 4\frac{\sup_D(\epsilon_0) + \sup_D(\epsilon_1)}{\inf_D(\epsilon_0)\inf_D(\epsilon_1)}\left(\sup_D(\gamma_0) + \sup_D(\gamma_1)\right). && (4.55) \end{aligned}$$

Now let $k^*$ be a real transmission eigenvalue corresponding to the operator $\mathbb{T}_{\epsilon,\gamma=0}$, and consider a neighborhood $\mathcal{N}_\sigma(k^*) \subset \mathbb{C}$ of $k^*$ of radius $\sigma > 0$. Then there is an $\eta_{k^*} > 0$ (of course depending on $\sigma$) such that this neighborhood contains at least one point in $\Sigma(\mathbb{T}_{\epsilon,\gamma})$ as long as $\hat{\delta}(\mathbb{T}_{\epsilon,\gamma}, \mathbb{T}_{\epsilon,\gamma=0}) < \eta_{k^*}$ since otherwise from [106, Theorem 3.1, Chapter 4] $\mathcal{N}_\sigma(k^*)$ must be included in both resolvents, $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma})$ and $\mathcal{R}(k;\mathbb{T}_{\epsilon,\gamma=0})$. Thus we have shown that for small absorption there is at least one transmission eigenvalue near $k^*$.

**Theorem 4.27.** *Let $\epsilon_0 \in L^\infty(D)$ and $\epsilon_1 \in L^\infty(D)$ satisfy $\epsilon_0(x) \ge \theta_0 > 0$, $\epsilon_1(x) \ge \theta_1 > 0$, and $\epsilon_c := \epsilon_1 - \epsilon \ge \theta > 0$, and let $k_j > 0$, $j = 1, \dots, \ell$, be the first $\ell$ real transmission eigenvalues (multiple eigenvalues are counted once) corresponding to* (4.42)–(4.45) *for nonabsorbing media, i.e., for $\gamma_0 = \gamma_1 = 0$. Then for every $\sigma > 0$ there is an $\tilde{\eta} > 0$ (depending on $\sigma$) such that if the absorption in the media is such that $\sup_D \gamma_0 + \sup_D \gamma_1 < \tilde{\eta}$, there exist at least $\ell$ transmission eigenvalues corresponding to* (4.42)–(4.45) *in a $\sigma$-neighborhood of $k_j$, $j = 1, \dots, \ell$.*

***Proof.*** To prove this theorem, it suffices to choose $\tilde{\eta} = \max(\tilde{\eta}_{k_1}, \tilde{\eta}_{k_2}, \dots, \tilde{\eta}_{k_\ell})$ thanks to (4.54), where

$$\tilde{\eta}_{k_j} < \eta_{k_i}\frac{\inf_D(\epsilon_0)\inf_D(\epsilon_1)}{4\sup_D(\epsilon_0) + 4\sup_D(\epsilon_1)}$$

and $\eta_{k_j}$ is the size of the perturbation corresponding to $k_j$, $j = 1, \ldots, \ell$, as discussed above. $\Box$

**Remark 4.28.** The approach developed in this section can be seen as a development of the continuity property for the resolvent of the transmission eigenvalue problem. In particular, for a real valued refractive index the same analysis can be done to show that if the real valued refractive index in the media is slightly perturbed, then so are the transmission eigenvalues.

We finish by noticing that the discussion in this section is the only result known up to date on the existence of transmission eigenvalues for absorbing and dispersive media, i.e., media with $k$-dependent complex valued refractive indices. The difficulty in analyzing the spectrum of the transmission eigenvalue problem in this case lies in the fact that the problem can no longer be viewed as an eigenvalue problem with $k^2$ as the eigenvalue parameter. It would also be interesting for the applications to know if real transmission eigenvalues exist if both the inhomogeneity and the part of the background occupied by the inhomogeneity are absorbing. Note that, as already seen here, real transmission eigenvalues do not exist if only the inhomogeneity is absorbing.

## 4.3 ▪ Existence of Transmission Eigenvalues for Anisotropic Media

We now return our attention to the transmission eigenvalue problem for anisotropic media (3.99) and prove the existence of real transmission eigenvalues under a sign restriction on the contrast. As the reader has already learned from Section 3.2, the transmission eigenvalue problem for anisotropic media assumes a different structure, provided whether $n \equiv 1$ or $n \not\equiv 1$.

Let us recall the transmission eigenvalue problem for anisotropic media:

$$\begin{cases} \nabla \cdot A \nabla w + k^2 n w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu_A} = \dfrac{\partial v}{\partial \nu} & \text{on } \partial D \end{cases} \tag{4.56}$$

with $w \in H^1(D)$ and $v \in H^1(D)$, where in view of Theorem 3.35 we assume that $\Im(A) = 0$ and $\Im(n) = 0$ and remind the reader of the notation

$$\begin{aligned} &a_* := \inf_D \inf_{|\xi|=1} \xi \cdot A\xi > 0 \quad \text{and} \quad a^* := \sup_D \sup_{|\xi|=1} \xi \cdot A\xi < \infty, \\ &n_* := \inf_D n > 0 \quad \text{and} \quad n^* := \sup_D n < \infty. \end{aligned} \tag{4.57}$$

### 4.3.1 ▪ The Case $n \equiv 1$

We start by assuming that $n(x) \equiv 1$ for almost all $x \in D$ and in addition $\Im(A) = 0$ and either $a_* > 1$ or $0 < a^* < 1$. Under these assumptions, in Section 3.2 (right below Remark 3.36), it is shown that real transmission eigenvalues, i.e., the values of $k > 0$ for which there exist nonzero solutions $v \in H^1(D)$ and $w \in H^1(D)$ of

$$\begin{aligned} \nabla \cdot A\nabla w + k^2 w = 0 \quad &\text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D, \\ w = v \quad &\text{and} \quad \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D, \end{aligned}$$

are the values of $\tau := k^2$ for which the kernel of the operators

$$\mathbb{A}_\tau - \tau\mathbb{B} \quad \text{or} \quad \tilde{\mathbb{A}}_\tau - \tau\mathbb{B}, \qquad \text{defined in } \mathcal{H}_0(D), \tag{4.58}$$

is nontrivial. Here we recall

$$\begin{aligned} H_0(\text{div}, D) &:= \left\{\mathbf{u} \in L^2(D)^2,\ \nabla \cdot \mathbf{u} \in L^2(D),\ \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\right\}, \\ \mathcal{H}_0(D) &:= \left\{\mathbf{u} \in H_0(\text{div}, D) :\ \nabla \cdot \mathbf{u} \in H_0^1(D)\right\}, \end{aligned}$$

and the bounded linear operators $\mathbb{A}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, $\tilde{\mathbb{A}}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ are defined via the Riesz representation theorem, respectively, applied to the forms

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{u}') := \left((N - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{u}' + \tau\mathbf{u}'\right)\right)_D + \tau^2 \left(\mathbf{u}, \mathbf{u}'\right)_D,$$

$$\begin{aligned} \tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) := &\left(N(I - N)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{u}' + \tau\mathbf{u}'\right)\right)_D \\ &+ \left(\nabla\nabla \cdot \mathbf{u}, \nabla\nabla \cdot \mathbf{v}\right)_D, \end{aligned}$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) := (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_D$$

with $N = A^{-1}$ and $(\cdot, \cdot)_D$ denoting the $L^2(D)$-inner product (see (3.121) and the equations following). Exactly in the same way as in Lemma 4.9 we can prove the following result.

**Lemma 4.29.** *The bounded self-adjoint operator $\mathbb{A}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is positive definite if $0 < a^* < 1$, whereas $\tilde{\mathbb{A}}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is positive definite if $a_* > 1$.*

**Lemma 4.30.** *The self-adjoint nonnegative linear operator $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is compact.*

***Proof.*** Let $\mathbf{u}_n$ be a bounded sequence in $\mathcal{H}_0(D)$. Hence there exists a subsequence, denoted again by $\mathbf{u}_n$, which converges weakly to $\mathbf{u}^0$ in $\mathcal{H}_0(D)$. Since $\nabla \cdot \mathbf{u}_n$ is also bounded in $H^1(D)$, from Rellich's compactness theorem we have that $\nabla \cdot \mathbf{u}_n$ converges strongly to $\nabla \cdot \mathbf{u}^0$ in $L^2(D)$. But

$$\|\mathbb{B}(\mathbf{u}_n - \mathbf{u}^0)\|_{\mathcal{H}_0(D)} \le \|\nabla \cdot (\mathbf{u}_n - \mathbf{u}^0)\|_{L^2(D)},$$

which proves that $\mathbb{B}\mathbf{u}_n$ converges strongly to $\mathbb{B}\mathbf{u}^0$. $\square$

The kernel of the operator $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is given by

$$\text{Kernel}(\mathbb{B}) = \{\mathbf{u} \in \mathcal{H}_0(D);\ \nabla \cdot \mathbf{u} = 0\},$$

which is obvious from the representation

$$(\mathbb{B}\mathbf{u}, \mathbf{v})_{\mathcal{H}_0(D)} = (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_D.$$

To carry over the approach of Section 4.2 to the eigenvalue problem for anisotropic media, we also need to consider the corresponding transmission eigenvalue problem for a ball $B_R$ of radius $R$ centered at the origin with a constant index of refraction $0 < n \neq 1$, which is formulated as

$$\Delta w + k^2 n w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{for} \quad |x| < R, \tag{4.59}$$

$$w = v \quad \text{and} \quad \frac{1}{n}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{for} \quad |x| = R. \tag{4.60}$$

By separation of variables we can prove the following lemma.

**Lemma 4.31.** *Let $D := B_R$, and let $n > 0$ be a positive constant such that $n \neq 1$. The infinitely many real zeros of*

$$d_n(k) = \det \begin{pmatrix} j_n(kR) & j_n(k\sqrt{n}R) \\ j_n'(kR) & \frac{1}{\sqrt{n}} j_n'(k\sqrt{n}R) \end{pmatrix} = 0$$

*are transmission eigenvalues for the anisotropic media with support $B_R$ and refractive index $A := \frac{1}{n} I$.*

We denote by $k_{R,n}$ the smallest real eigenvalue. An eigenfunction corresponding to $k_{R,n}$ is $\mathbf{u}^{B_R,n} = n\nabla w^{B_R,n} - \nabla v^{B_R,n} \in \mathcal{H}_0(B_R)$, where $w^{B_R,n}, v^{B_R,n}$ is a nonzero solution to (4.59)–(4.60). Furthermore, $\mathbf{u}^{B_R,n}$ satisfies

$$\int\limits_{B_R} \frac{1}{n-1}(\nabla\nabla\cdot \mathbf{u}^{B_R,n} + k_{R,n}^2 \mathbf{u}^{B_R,n}) \cdot (\nabla\nabla\cdot \overline{\mathbf{u}}^{B_R,n} + k_{R,n}^2 n \overline{\mathbf{u}}^{B_R,n})\, dx = 0. \tag{4.61}$$

By definition $\mathbf{u}^{B_R,n}$ is not in the kernel of $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$. Finally, if $\overline{B}_R \subset D$, then the extension by zero $\tilde{\mathbf{u}}$ of $\mathbf{u}^{B_R,n}$ to the whole $D$ is in $\mathcal{H}_0(D)$, respectively.

Now we have all the pieces to repeat word for word the proof of Theorem 4.12 to obtain the following theorem on the existence of real transmission eigenvalues for anisotropic media.

**Theorem 4.32.** *Assume $\Im(A) = 0$, $n \equiv 1$, and the matrix valued function $A$ satisfies either*

1. $1 < a_* \leq \xi \cdot A(x)\xi \leq a^* < \infty$ *or*
2. $0 < a_* \leq \xi \cdot A(x)\xi \leq a^* < 1$

*for almost all $x \in D$ and all $\xi \in \mathbb{R}^3$ with $\|\xi\| = 1$. Then there exists an infinite set of real transmission eigenvalues for the anisotropic media problem* (4.56) *with $+\infty$ as the only accumulation point.*

## 4.3.2 ▪ The Case $n \not\equiv 1$

We here discuss the existence of positive transmission eigenvalues in the general case of anisotropic media with $n \neq 1$. Unfortunately the existence of transmission eigenvalues for this case can only be shown under restrictive assumptions on $A - I$ and $n - 1$. The approach presented here follows along the lines of [49] where, motivated by the case of $n \equiv 1$, the transmission eigenvalue problem is formulated in terms of the difference $u := v - w$.

However, due to the lack of symmetry, the problem for $u$ is no longer a quadratic eigenvalue problem but takes the form of a more complicated nonlinear eigenvalue problem, as will become clear in the following.

**Example 4.33.** *The spherically symmetric case*: In the case when $D := B_R$ is a ball of radius $R$ centered at the origin and both constitutive material properties $A = a(r)I$ and $n = n(r)$ depend only on the radial variable, similarly to the isotropic media in Theorem 4.7 we can directly show that there exists an infinite set of transmission eigenvalues. We assume that both $a \in C^2[0, R]$ and $n \in C^2[0, R]$. Obviously if both $a \equiv 1$ and $n \equiv 1$, every $k > 0$ is a transmission eigenvalue (i.e., this corresponds to the case when there is no inhomogeneity and therefore no waves are scattered). To avoid such a situation we assume that either $a(R) \neq 1$ and $n(R) \neq 1$ or otherwise

$$\delta := \frac{1}{R} \int_0^R \left( \frac{n(r)}{a(r)} \right)^{\frac{1}{2}} dr \neq 1. \tag{4.62}$$

We restrict our attention to solutions of (4.56) that depend only on $r = |x|$, that is,

$$v(x) = a_0 j_0(kr),$$

where $j_0$ is the spherical Bessel function of order zero and $a_0$ is a constant. Next, making the substitution $w(x) = [a(r)]^{-1/2} W(x)$ we see that the first equation in (4.56) takes the form

$$\Delta W + \left( k^2 \frac{n(r)}{a(r)} - m(r) \right) W = 0,$$

where

$$m(r) = \frac{1}{\sqrt{a(r)}} \Delta \sqrt{a(r)}.$$

Hence, setting

$$w(x) = \frac{b_0}{[a(r)]^{\frac{1}{2}}} \frac{y(r)}{r},$$

where $b_0$ is a constant, straightforward calculations show that if $y$ is a solution of

$$y'' + \left( k^2 \frac{n(r)}{a(r)} - m(r) \right) y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

then $w$ satisfies the first equation in (4.56). Define $c(r)$ by

$$c(r) := \frac{n(r)}{a(r)}.$$

Again following [69], [75], in order to solve the above initial value problem for $y$ we use the Liouville transformation

$$z(\xi) \ := \ [c(r)]^{\frac{1}{4}} \, y(r), \quad \text{where} \quad \xi(r) \ := \int_0^r [c(\rho)]^{\frac{1}{2}} \, d\rho,$$

which yields the following initial value problem for $z(\xi)$:

$$z'' + [k^2 - p(\xi)]z = 0\,, \quad z(0) = 0\,, \quad z'(0) = \big[c(0)\big]^{-\frac{1}{4}}, \tag{4.63}$$

where

$$p(\xi) \;:=\; \frac{c''(r)}{4[c(r)]^2} \;-\; \frac{5}{16}\frac{[c'(r)]^2}{[c(r)]^3} \;+\; \frac{m(r)}{c(r)}\,.$$

Now exactly in the same way as in Theorem 4.7, (4.63) can then be rewritten as a Volterra integral equation and, for $k > 0$, using the method of successive approximations, we can obtain the asymptotic behavior for $y$ which is the same as (4.5) and (4.6) where $n(r)$ is replaced by $c(r)$. Applying the boundary conditions on $|x| = R$, the transmission eigenvalues are the zeros of

$$d_0(k) = \det \begin{pmatrix} \dfrac{1}{[a(R)]^{1/2}}\dfrac{y(R)}{R} & j_0(kR) \\ a(R)\dfrac{d}{dr}\left(\dfrac{1}{[a(r)]^{1/2}}\dfrac{y(r)}{r}\right)_{r=R} & k\,j_0'(kR) \end{pmatrix} = 0,$$

which has the same asymptotic expression as in (4.7), where

$$\delta := \frac{1}{R}\int_0^R \frac{n(r)}{a(r)}, \quad A = \frac{1}{[a(R)]^{1/2}}\,\frac{1}{[c(0)c(R)]^{1/4}}\,, \quad B = \big[a(R)\big]^{1/2}\left[\frac{c(R)}{c(0)}\right]^{1/4}.$$

Then, as in the proof of Theorem 4.7, we can conclude the existence of infinitely many eigenvalues, provided the above assumptions are met.

In the following we need to consider a particular case of the above spherically stratified media where $A = aI$ and $a \neq 1$ and $n \neq 1$ are both positive constants. Separation of variables leads to solutions of (4.56) of the form

$$v(r,\hat{x}) = a_\ell j_\ell(kr) Y_\ell^m(\hat{x})\,, \qquad w(r,\hat{x}) = b_\ell j_\ell\left(k\sqrt{\frac{n}{a}}\,r\right) Y_\ell^m(\hat{x}),$$

where $j_n$ are spherical Bessel functions of order $n$, $Y_n^m$ are the spherical harmonics, and $\hat{x} = x/r$. Then the corresponding transmission eigenvalues are zeros of the determinants

$$d_\ell(k) = \det \begin{pmatrix} j_\ell(kR) & j_\ell\left(k\sqrt{\dfrac{n}{a}}\,R\right) \\ k\,j_\ell'(kR) & k\sqrt{na}\,j_\ell'\left(k\sqrt{\dfrac{n}{a}}R\right) \end{pmatrix} = 0 \tag{4.64}$$

for $\ell \geq 0$. For later use we denote by $k_{a,n,R}$ the smallest transmission eigenvalue, which may not necessarily be the first zero of $d_0(k)$.

We now turn our attention to the general case (4.56). To simplify the expressions we set $\tau := k^2$ and observe that if $(v, w)$ satisfies (4.56), then, subtracting the equation for $v$ from the equation for $w$, we arrive at the equivalent formulation for $u := v - w \in H_0^1(D)$ and $v \in H^1(D)$:

$$\begin{aligned} \nabla\cdot A\nabla u + \tau n u \;&=\; \nabla\cdot(A-I)\nabla v + \tau(n-1)\,v \quad \text{in } D\,, \\ \nu\cdot A\nabla u \;&=\; \nu\cdot(A-I)\nabla v \quad \text{on } \partial D, \end{aligned} \tag{4.65}$$

along with

$$\Delta v + \tau v = 0 \quad \text{in } D. \tag{4.66}$$

The main idea of the proof of the existence of transmission eigenvalues consists in expressing $v$ in terms of $u$, using (4.65), and substituting the resulting expression into (4.66) in order to formulate the eigenvalue problem only in terms of $u$. In the case when $A = I$ this substitution is simple and leads to an explicit expression as a fourth order equation satisfied by $u$ as discussed in Section 3.1.1 (see also [110]). In the current case the substitution requires the inversion of the operator $\nabla \cdot \big[(A-I)\nabla \cdot\big] + \tau(n-1)$ with a Neumann boundary condition. It is then obvious that the case where $(A-I)$ and $(n-1)$ have the same sign is more problematic since in that case the operator may not be invertible for special values of $\tau$. This is why we only consider in detail the simpler case when $(A-I)$ and $(n-1)$ have the opposite sign almost everywhere in $D$. Thus we now assume that either $a^* < 1$ and $n_* > 1$, or $a_* > 1$ and $n^* < 1$.

Note that for given $u \in H_0^1(D)$, the problem (4.65) for $v \in H^1(D)$ is equivalent to the variational formulation

$$\int_D \big[(A-I)\nabla v \cdot \nabla\overline{\psi} - \tau\,(n-1)\,v\,\overline{\psi}\big]\,dx = \int_D \big[A\,\nabla u \cdot \nabla\overline{\psi} - \tau n\,u\,\overline{\psi}\big]\,dx \qquad (4.67)$$

for all $\psi \in H^1(D)$. The following result concerning the invertibility of the operator associated with (4.67) can be proven in a standard way using the Lax–Milgram lemma. We skip the proof here and refer the reader to [49].

**Lemma 4.34.** *Assume that either $a_* > 1$ and $0 < n^* < 1$, or $0 < a^* < 1$ and $n_* > 1$. Then for every $u \in H_0^1(D)$ and $\tau \geq 0$ there exists a unique solution $v := v_u \in H^1(D)$ of* (4.67). *The operator $A_\tau : H_0^1(D) \to H^1(D)$, defined by $u \mapsto v_u$, is bounded and depends continuously on $\tau \geq 0$.*

For fixed $u \in H_0^1(D)$, we now set $v_u := A_\tau u$ and denote by $\mathbb{L}_\tau u \in H_0^1(D)$ the unique Riesz representation of the bounded antilinear functional

$$\psi \mapsto \int_D \big[\nabla v_u \cdot \nabla\overline{\psi} - \tau\,v_u\,\overline{\psi}\big]\,dx \quad \text{for } \psi \in H_0^1(D)\,,$$

i.e.,

$$(\mathbb{L}_\tau u, \psi)_{H^1(D)} = \int_D \big[\nabla v_u \cdot \nabla\overline{\psi} - \tau\,v_u\,\overline{\psi}\big]\,dx \quad \text{for } \psi \in H_0^1(D)\,. \qquad (4.68)$$

Obviously $\mathbb{L}_\tau$ also depends continuously on $\tau$. Now we are able to connect a transmission eigenfunction, i.e., a nontrivial solution $(v, w)$ of (4.56), to the kernel of the operator $\mathbb{L}_\tau$.

**Theorem 4.35.** *The following statements are true:*

1. *Let $(w, v) \in H^1(D) \times H^1(D)$ be a transmission eigenfunction corresponding to some eigenvalue $\tau > 0$. Then $u = v - w \in H_0^1(D)$ satisfies $\mathbb{L}_\tau u = 0$.*

2. *Let $u \in H_0^1(D)$ satisfy $\mathbb{L}_\tau u = 0$ for some $\tau > 0$. Furthermore, let $v := v_u = A_\tau u \in H^1(D)$ be as in Lemma* 4.34, *i.e., the solution of* (4.67). *Then $\tau$ is a transmission eigenvalue with $(w, v) \in H^1(D) \times H^1(D)$ the corresponding transmission eigenfunction, where $w = v - u$.*

***Proof.*** Formula (4.68) implies that $(\mathbb{L}_\tau u, \psi)_{H^1(D)} = 0$ for all $\psi \in H_0^1(D)$, which means that $\mathbb{L}_\tau u = 0$.

The proof of the second part of the theorem is a simple consequence of the observation that (4.66) is equivalent to

$$\int_D \left[\nabla v \cdot \nabla \overline{\psi} - \tau\, v\, \overline{\psi}\right] dx \;=\; 0 \quad \text{for all } \psi \in H_0^1(D). \tag{4.69}$$

Hence $L_\tau u = 0$ implies that $v_u$ solves the Helmholtz equation in $D$. Since $w := v - u$ we have that the Cauchy data of $w$ and $v$ coincide. The equation for $w$ follows from (4.67). □

The operator $\mathbb{L}_\tau$ plays a role similar to that of the operator $\mathbb{A}_k - k^2\mathbb{B}$ for the case of $n \equiv 1$ discussed in the first part of this section.

**Theorem 4.36.** *The bounded linear operator* $\mathbb{L}_\tau : H_0^1(D) \to H_0^1(D)$ *satisfies*

1. $\mathbb{L}_\tau$ *is self-adjoint for all* $\tau > 0$*;*
2. $(\sigma \mathbb{L}_0 u, u)_{H^1(D)} \;\geq\; c\,\|u\|^2_{H^1(D)}$ *for all* $u \in H_0^1(D)$ *and* $c > 0$ *independent of* $u$*, where* $\sigma = 1$ *if* $a_* > 1$ *and* $0 < n^* < 1$*, and* $\sigma = -1$ *if* $0 < a^* < 1$ *and* $n_* > 1$*;*
3. $\mathbb{L}_\tau - \mathbb{L}_0$ *is compact.*

***Proof.*** 1. Let $u_1, u_2 \in H_0^1(D)$, and let $v_1 := v_{u_1}$ and $v_2 := v_{u_2}$ be the corresponding solution of (4.67). Then we have that

$$\begin{aligned}
(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} &= \int_D \left[\nabla v_1 \cdot \nabla \overline{u_2} - \tau\, v_1 \overline{u_2}\right] dx \\
&= \int_D \left[A\, \nabla v_1 \cdot \nabla \overline{u_2} - \tau n\, v_1\, \overline{u_2}\right] dx \\
&\quad - \int_D \left[(A - I)\nabla v_1 \cdot \nabla \overline{u_2} - \tau\,(n-1)\, v_1\, \overline{u_2}\right] dx\,.
\end{aligned}$$

Using (4.67) twice, first for $u = u_2$ and the corresponding $v = v_2$ and $\psi = v_1$ and then for $u = u_1$ and the corresponding $v = v_1$ and $\psi = v_2$, yields

$$\begin{aligned}
(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} &= \int_D \left[(A - I)\nabla v_1 \cdot \nabla \overline{v_2} - \tau\,(n-1)\, v_1\, \overline{v_2}\right] dx \\
&\quad - \int_D \left[A\, \nabla u_1 \cdot \nabla \overline{u_2} - \tau n\, u_1\, \overline{u_2}\right] dx,
\end{aligned} \tag{4.70}$$

which shows that $\mathbb{L}_\tau$ is self-adjoint.

2. In order to show that $\sigma\mathbb{L}_0 : H_0^1(D) \to H_0^1(D)$ is a coercive operator, we recall the definition (4.68) of $\mathbb{L}_0$ and use the fact that $v = v_u = u + w$ to obtain

$$(\mathbb{L}_0 u,\, u)_{H^1(D)} = \int_D \nabla v \cdot \nabla \overline{u}\, dx \;=\; \int_D |\nabla u|^2\, dx + \int_D \nabla w \cdot \nabla \overline{u}\, dx\,. \tag{4.71}$$

From (4.67) for $\tau = 0$ and $\psi = w$ we now have that

$$\int_D \nabla w \cdot \nabla \overline{u}\, dx = \int_D (A - I)\nabla w \cdot \nabla \overline{w}\, dx\,. \tag{4.72}$$

If $a_* > 0$, we have $\int_D (A - I)\nabla w \cdot \nabla \overline{w}\, dx \geq (a_* - 1)\|\nabla w\|^2_{L^2(D)} \geq 0$ and hence

$$(\mathbb{L}_0 u,\, u)_{H^1(D)} \geq \int_D |\nabla u|^2\, dx\,.$$

Since from Poincaré's inequality $\|\nabla u\|_{L^2(D)}$ is an equivalent norm on $H^1_0(D)$, this proves the strict coercivity of $\mathbb{L}_0$.

Now if $0 < a^* < 1$, from (4.70) with $u_1 = u_2 = u$ and $\tau = 0$ we have

$$\begin{aligned} -(\mathbb{L}_0 u,\, u)_{H^1(D)} &= -\int_D (A - I)\nabla w \cdot \nabla \overline{w}\, dx \;+\; \int_D A\,\nabla u \cdot \nabla \overline{u}\, dx \\ &\geq a_* \int_D |\nabla u|^2\, dx \end{aligned}$$

which proves the strict coercivity of $-\mathbb{L}_0$ since $a_* > 0$.

3. This follows from the compact embedding of $H^1_0(D)$ into $L^2(D)$. □

We are now in the position to establish the existence of infinitely many positive transmission eigenvalues, i.e., the existence of a sequence of $\tau_j > 0$, and corresponding $u_j \in H^1_0(D)$, such that $u_j \neq 0$ and $\mathbb{L}_{\tau_j} u_j = 0$. Obviously these $\tau > 0$ are such that the kernel of $\mathbb{I} + \mathbb{T}_\tau$ is not trivial, which corresponds to one being an eigenvalue of the compact self-adjoint operator $\mathbb{T}_\tau$, where $\mathbb{T}_\tau : H^1_0(D) \to H^1_0(D)$ is defined by

$$\mathbb{T}_\tau := (\sigma \mathbb{L}_0)^{-\frac{1}{2}} \left(\sigma(\mathbb{L}_\tau - \mathbb{L}_0)\right) (\sigma \mathbb{L}_0)^{-\frac{1}{2}}.$$

Thus we can conclude that real transmission eigenvalues have finite multiplicity. We can now use Theorem 4.6 to prove the main result of this section.

**Theorem 4.37.** *Assume that either $a_* > 1$ and $0 < n^* < 1$, or $0 < a^* < 1$ and $n_* > 1$. Then there exists an infinite sequence of positive transmission eigenvalues $k_j > 0$ ($\tau_j := k_j^2$) with $+\infty$ as the only accumulation point.*

***Proof.*** We sketch the proof only for the case of $a_* > 1$ and $0 < n^* < 1$ (i.e., take $\sigma = 1$ in Theorem 4.36). First, we recall that assumption 1 of Theorem 4.6 is satisfied with $\tau_0 = 0$ from Theorem 4.36, part 2. Next, from the definition of $\mathbb{L}_\tau$ and the fact that $v = w + u$, we have

$$\begin{aligned} &(\mathbb{L}_\tau u,\, u)_{H^1(D)} \\ &\quad= \int_D \left[\nabla v \cdot \nabla \overline{u} - \tau\, v\, \overline{u}\right] dx = \int_D \left[\nabla w \cdot \nabla \overline{u} - \tau\, w\, \overline{u} + |\nabla u|^2 - \tau\, |u|^2\right] dx. \end{aligned} \tag{4.73}$$

We also have that $w$ satisfies

$$\int_D \left[(A - I)\nabla w \cdot \nabla \overline{\psi} - \tau(n - 1)\, w\, \overline{\psi}\right] dx = \int_D \left[\nabla u \cdot \nabla \overline{\psi} - \tau\, u\, \overline{\psi}\right] dx \tag{4.74}$$

for all $\psi \in H^1(D)$. Now taking $\psi = w$ in (4.74) and substituting the result into (4.73) yields

$$\begin{aligned} &(\mathbb{L}_\tau u,\, u)_{H^1(D)} \\ &\quad= \int_D \left[(A - I)\nabla w \cdot \nabla \overline{w} - \tau\, (n - 1)\, |w|^2 + |\nabla u|^2 - \tau\, |u|^2\right] dx\,. \end{aligned} \tag{4.75}$$

Now let $\hat\tau$ be such that $\hat\tau := k^2_{n^*,a_*,R}$ (the first transmission eigenvalue corresponding to (4.59)–(4.60) for the disk $B_R$ with $a := a_*$ and $n := n^*$). We denote by $\hat v$, $\hat w$ the corresponding nonzero solutions and set $\hat u := \hat v - \hat w \in H_0^1(B_R)$. We denote the corresponding operator by $\hat{\mathbb{L}}_\tau$. Of course, by construction, we have that (4.75) still holds, i.e., since $\hat{\mathbb{L}}_{\hat\tau}\hat u = 0$,

$$
\begin{aligned}
0 &= \left(\hat{\mathbb{L}}_{\hat\tau}\hat u, \hat u\right)_{H^1(B_R)} \\
&= \int\limits_{B_R} \left[(a_*-1)|\nabla \hat v|^2 - \hat\tau\,(n^*-1)|\hat v|^2 + |\nabla \hat u|^2 - \hat\tau\,|\hat u|^2\right] dx\,.
\end{aligned}
\tag{4.76}
$$

Next we denote by $\tilde u \in H_0^1(D)$ the extension of $\hat u \in H_0^1(B_R)$ by zero to the whole of $D$, let $\tilde v := v_{\tilde u}$ be the corresponding solution to (4.67), and set $\tilde w := \tilde v - \tilde u$. In particular $\tilde w \in H^1(D)$ satisfies

$$
\begin{aligned}
&\int\limits_D \left[(A-I)\nabla\tilde w\cdot\nabla\overline{\psi} - \hat\tau\,(n-1)\,\tilde w\,\overline{\psi}\right] dx = \int\limits_D \left[\nabla\tilde u\cdot\nabla\overline{\psi} - \hat\tau\,\tilde u\,\overline{\psi}\right] dx \\
&= \int\limits_{B_R} \left[\nabla\hat u\cdot\nabla\overline{\psi} - \hat\tau\,\hat u\,\overline{\psi}\right] dx = \int\limits_{B_R} \left[(a_*-1)\nabla\hat w\cdot\nabla\overline{\psi} - \hat\tau\,(n^*-1)\,\hat w\,\overline{\psi}\right] dx
\end{aligned}
$$

for all $\psi \in H^1(D)$. Therefore, for $\psi = \tilde w$ we have

$$
\begin{aligned}
&\int\limits_D [(A-I)\nabla\tilde w\cdot\nabla\overline{\tilde w} - \hat\tau\,(n-1)\,|\tilde w|^2]\,dx \\
&\qquad = \int\limits_{B_R} [(a_*-1)\,\nabla\hat w\cdot\nabla\overline{\tilde w} + \hat\tau\,|n^*-1|\,\hat w\,\overline{\tilde w}]\,dx.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
&\int\limits_D ((A-I)\nabla\tilde w\cdot\nabla\overline{\tilde w} - \hat\tau\,(n-1)\,|\tilde w|^2)\,dx \\
&\le \left[\int\limits_{B_R} ((a_*-1)\,|\nabla\hat w|^2 + \hat\tau\,|n^*-1|\,|\hat w|^2)\,dx\right]^{\frac12} \left[\int\limits_{B_R} ((a_*-1)\,|\nabla\tilde w|^2 + \hat\tau\,|n^*-1|\,|\tilde w|^2)\,dx\right]^{\frac12} \\
&\le \left[\int\limits_{B_R} ((a_*-1)\,|\nabla\hat w|^2 - \hat\tau\,(n^*-1)\,|\hat w|^2)\,dx\right]^{\frac12} \left[\int\limits_D ((A-I)\nabla\tilde w\cdot\nabla\overline{\tilde w} - \hat\tau\,(n-1)\,|\tilde w|^2)\,dx\right]^{\frac12}
\end{aligned}
$$

since $|n-1| = 1-n \ge 1-n^* = |n^*-1|$. Hence we have

$$
\begin{aligned}
&\int\limits_D \left[(A-I)\nabla\tilde w\cdot\nabla\overline{\tilde w} - \hat\tau\,(n-1)\,|\tilde w|^2\right] dx \\
&\qquad \le \int\limits_{B_R} \left[(a_*-1)\,|\nabla\hat w|^2 - \hat\tau\,(n^*-1)\,|\hat w|^2\right] dx\,.
\end{aligned}
$$

Substituting this into (4.75) for $\tau = \hat{\tau}$ and $u = \tilde{u}$ yields

$$\left(\mathbb{L}_{\hat{\tau}}\tilde{u}, \tilde{u}\right)_{H^1(D)} = \int\limits_D \left[(A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{\tau}\,(n-1)\,|\tilde{v}|^2 + |\nabla\tilde{w}|^2 - \hat{\tau}\,|\tilde{w}|^2\right] dx$$
$$\leq \int\limits_{B_R} \left[(a_*-1)|\nabla\hat{w}|^2 - \hat{\tau}\,(n^*-1)\,|\hat{w}|^2 + |\nabla\hat{w}|^2 - \hat{\tau}\,|\hat{w}|^2\right] dx \;=\; 0$$

by (4.76). Hence from Theorem 4.6 we have that there is a transmission eigenvalue $k > 0$, such that $k^2 \in (0,\,\hat{\tau}]$. Finally, repeating this argument for disks of arbitrary small radius, we can show the existence of infinitely many transmission eigenvalues exactly in the same way as in the proof Theorem 4.12. In a similar way we can prove the same result for the case when $0 < a^* < 1$ and $n_* > 1$ where in the proof we consider the operator $-\mathbb{L}_\tau$ and the ball $B_R$ with $a := a^*$ and $n := n_*$. $\Box$

We end our discussion in this section by making a few comments on the case when $(A-I)$ and $(n-1)$ have the same sign. As indicated above, if we follow a similar procedure, then we are faced with the problem that (4.67) is not solvable for all $\tau$. For this reason it is only possible to prove the existence of a finite number of transmission eigenvalues under the restrictive assumption that $n^* - 1$ is small enough. To avoid repetition, we refer the reader to [49] for more details. We also mention that the approach of this section can be modified to include anisotropic media with small voids [95].

### 4.3.3 ▪ Inequalities for Transmission Eigenvalues

Similarly to the case of isotropic media, our proof of the existence of real transmission eigenvalues provides a framework for deriving inequalities between transmission eigenvalues and the matrix valued refractive index. In view of the fact that the matrix valued refractive index cannot be uniquely determined from scattering data, such inequalities become particularly important in the context of the inverse problem for anisotropic media since real transmission eigenvalues can be determined from far field data (see Section 5.1). In Section 5.1.1 we show that the inequalities and monotonicity properties of transmission eigenvalues can be used to obtain information about anisotropic media from scattering data.

Let us start with the case when $n \equiv 1$. Rephrasing Theorem 3.37 we have the following lower bounds for transmission eigenvalues.

**Theorem 4.38.** *Let $A \in (L^\infty(D))^{3\times 3}$, $\Im(A) = 0$, $n \equiv 1$ in $D$, and let $0 < a_*$ and $a^* \leq \infty$ be defined as in* (4.57). *Then all real transmission eigenvalues $k > 0$ satisfy*

1. $k^2 \geq a^*\lambda_1(D)$ *if* $0 < a^* < 1$ *or*
2. $k^2 \geq \lambda_1(D)$ *if* $1 < a_*$,

*where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in $D$.*

As the reader has already seen, the analytical structure of the transmission eigenvalue problem for anisotropic media with contrast only in $A$ resembles the one corresponding to isotropic media with $N := A^{-1}$. Hence, as in the proof of Theorem 4.16, we can prove a monotonicity property for transmission eigenvalues for anisotropic media. To this end let $k_j := k_j(A(x), D) > 0$ for $j \in \mathbb{N}$ be the increasing sequence of transmission eigenvalues for the media with support $D$ and refractive index $A$, such that $t_j = k_j^2$ is the smallest zero of $\lambda_j(\tau, D, A(x)) = \tau$, where $\lambda_j(\tau, D, A(x))$, $j \geq 1$, are the eigenvalues of the auxiliary problem (see Theorem 4.32) given by

$$\lambda_j(\tau, D, A) = \min_{W \in \mathcal{U}_j} \max_{\substack{\mathbf{u} \in W \\ \|\nabla \cdot \mathbf{u}\|_{L^2(D)} = 1}} \int_D (A^{-1} - I)^{-1} |\nabla \nabla \cdot \mathbf{u} + k^2 \mathbf{u}|^2 \, dx + k^4 \int_D |\mathbf{u}|^2 \, dx, \tag{4.77}$$

where $\mathcal{U}_j$ denotes the set of all $j$-dimensional subspaces $W$ of $\mathcal{H}_0(D)$. Then for this sequence of $k_j(A(x), D) > 0$ we have the following monotonicity property.

**Theorem 4.39.** *Let $A \in (L^\infty(D))^{3\times 3}$, $\Im(A) = 0$, $n \equiv 1$ in $D$, and let $0 < a_*$ and $a^* \leq \infty$ be defined as in* (4.57). *Assume that $B_1$ and $B_2$ are two balls such that $B_1 \subset D \subset B_2$. Then*

1. *if $a^* < 1$, then*

$$k_j(a_*, B_2) \leq k_j(a_*, D) \leq k_j(A(x), D) \leq k_j(a^*, D) \leq k_j(a^*, B_1);$$

2. *if $1 < a_*$, then*

$$k_j(a^*, B_2) \leq k_j(a^*, D) \leq k_j(A(x), D) \leq k_j(a_*, D) \leq k_j(a_{min}, B_1).$$

*In particular, these inequalities hold true for the smallest transmission eigenvalue $k_1(A(x), D)$.*

As a consequence of this theorem we have the following more general formulation of the monotonicity property for the sequence of transmission eigenvalues $k_j(A(x), D) > 0$ described above.

**Corollary 4.40.** *Let $D_1 \subset D \subset D_2$ and $A_1 < A < A_2$, where $A_1, A, A_2$ all satisfy the assumptions of Theorem* 4.39.

1. *If $A_1 < A < A_2 < I$, then*

$$k_j(A_1, D_2) \leq k_j(A_1, D) \leq k_j(A, D) \leq k_j(A_2, D) \leq k_j(A_2, D_1).$$

2. *If $I < A_1 < A < A_2$, then*

$$k_j(A_2, D_2) \leq k_j(A_2, D) \leq k_j(A, D) \leq k_j(A_1, D) \leq k_j(A_1, D_1).$$

*Here $I$ is a $3 \times 3$ identity matrix and for any two matrices $B < A$ means that the matrix $A - B$ is positive definite uniformly in $D$.*

Theorem 4.39 shows in particular that for $A = aI$, where $a \neq 1$ is a positive constant, the first transmission eigenvalue $k_1(a, D)$ as a function of $a$ for $D$ fixed is monotonically increasing if $a < 1$ and is monotonically decreasing if $a > 1$. As in Theorem 4.18, this leads to the following uniqueness result for the constant index of refraction in terms of the first transmission eigenvalue.

**Theorem 4.41.** *The constant index of refraction $A = aI$ is uniquely determined from knowledge of the corresponding smallest transmission eigenvalue $k_1(a, D) > 0$, provided that it is known a priori that either $a > 1$ or $0 < a < 1$.*

Next we consider the case when $n \neq 1$. Unfortunately, the proof of the existence of transmission eigenvalues in this case has a more complicated structure. Hence we can derive only an inequality for the first transmission eigenvalue.

**Theorem 4.42.** *Let $B_R \subset D$ be the largest disk contained in $D$ and $\lambda_1(D)$ the first Dirichlet eigenvalue of $-\Delta$ in $D$. Furthermore, let $k_1(A,n,D)$ be the first transmission eigenvalue corresponding to $D$, $A$, and $n$, and $0 < a_* \leq a^* < \infty$, $0 < n_* \leq n^* < \infty$ defined as in* (4.57).

1. *If $a_* > 1$ and $0 < n^* < 1$, then*
$$\lambda_1(D) \;\leq\; k_1^2(A,n,D) \leq k_1^2(a_*, n^*, B_R).$$

2. *If $0 < a^* < 1$ and $n_* > 1$, then*
$$\frac{a_*}{n^*}\,\lambda_1(D) \leq k_1^2(A,n,D) \;\leq\; k_1^2(a^*, n_*, B_R).$$

***Proof.*** The upper bounds in both cases are consequence of the proof of Theorem 4.37. We now prove a lower bound for the first transmission eigenvalue. To this end, let us assume that $a_* > 1$ and $0 < n^* < 1$ and consider (4.75), i.e.,

$$(\mathbb{L}_\tau u, u)_{H^1(D)} = \iint\limits_D \left[(A-I)\nabla w \cdot \nabla \overline{w} - \tau\,(n-1)\,|u|^2 + |\nabla u|^2 - \tau\,|u|^2\right] dx\,.$$

The first term is estimated by

$$\iint\limits_D \left[(A-I)\nabla w \cdot \nabla \overline{w} - \tau\,(n-1)\,|w|^2\right] dx \;\geq\; \min(a_*-1), \tau(1-n^*))\,\|w\|^2_{H^1(D)} \geq 0$$

and, since $u \in H^1_0(D)$, we have that $\|\nabla u\|^2_{L^2(D)} \geq \lambda_1(D)\,\|u\|^2_{L^2(D)}$, where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in $D$. Therefore, $(\mathbb{L}_\tau u, u)_{H^1(D)} > 0$ as long as $\tau < \lambda_1(D)$. Thus, we can conclude that all transmission eigenvalues $k$ are such that $k^2 \geq \lambda_1(D)$.

Next we consider $0 < a^* < 1$ and $n_* > 1$ and from (4.70) since $v = w + u$ we have that

$$-(\mathbb{L}_\tau u, u)_{H^1(D)} = \iint\limits_D \left[(I-A)(\nabla w + \nabla u)\cdot(\nabla \overline{w} + \nabla \overline{u}) + \tau(n-1)\,|w+u|^2\right] dx$$
$$+ \iint\limits_D \left[A\nabla u \cdot \nabla \overline{u} - \tau\, n\,|u|^2\right] dx\,.$$

In this case

$$\iint\limits_D \left[(I-A)|\nabla w + \nabla u|^2 + \tau(n-1)\,|w+u|^2\right] dx \geq C\,\|u+v\|^2_{H^1(D)} \geq 0,$$

where $C = \min((1-a^*), \tau(n_*-1))$, whereas

$$\iint\limits_D \left[A\nabla u \cdot \nabla \overline{u} - \tau\, n\,|u|^2\right] dx \geq \left[a_*\lambda_1(D) - \tau n^*\right] \|v\|^2_{L^2(D)}\,.$$

Hence if $0 < \tau < \frac{a_*}{n^*}\lambda_1(D)$, there are no transmission eigenvalues, which proves the lower bound in the second case. $\square$

We end this section by stating an estimate on transmission eigenvalues which is a consequence of the proof of Theorem 3.39.

**Theorem 4.43.** *Assume that either* $0 < a^* < 1$ *or* $a_* > 1$, *and* $\int_D (n-1)dx \neq 0$. *Then the nonzero eigenvalue* $k_1 \in \mathbb{C}$ *of the smallest modulus satisfies*

$$|k|^2 \geq \frac{a_*(1-\sqrt{a^*})}{C_P\ \max(n^*, 1)\,(1+\sqrt{n^*})}$$

*with* $C_P > 0$ *defined by*

$$\frac{1}{C_P} := \min_{\substack{(w,v)\,\in\,\mathcal{Y}(D)\\ (w,v)\,\neq\,(0,0)}} \frac{\|\nabla w\|^2_D + \|\nabla v\|^2_D}{\|w\|^2_D + \|v\|^2_D},$$

*where*

$$\mathcal{Y}(D) := \left\{ (w,v) \in H(D)\times H(D),\ w = v \,\text{on}\, \partial D,\ \int_D (nw - v)dx = 0 \right\}.$$

Note that under the weaker assumption on $n$ in Theorem 4.43 it is not known whether real transmission eigenvalues exist. In particular, the eigenvalue of the smallest modulus may not necessarily be real.

We end this section with the comment that, similarly to the case of isotropic media, alternative approaches have been introduced to investigate the spectral properties of the anisotropic transmission eigenvalue problem under the assumption of sign control in a neighborhood of the boundary. Under this assumption, in a series of papers [122], [123], [124], [125], [140], [141], [164] the existence of transmission eigenvalues is proven and a study of the counting function for transmission eigenvalues is initiated. In this regard, we explicitly state the results in [164] on the location of transmission eigenvalues in the complex plane and in [141] on the state-of-the-art assumptions on the coefficients that are needed to analyze the spectrum of the transmission eigenvalue problem. We state these result for a more general formulation of the transmission eigenvalue problem:

$$\begin{cases} \nabla\cdot A_1\nabla w + k^2 n_1 w = 0 & \text{in } D,\\ \nabla\cdot A_2\nabla v + k^2 n_2 v = 0 & \text{in } D,\\ w = v & \text{on } \partial D,\\ \dfrac{\partial w}{\partial \nu_{A_1}} = \dfrac{\partial v}{\partial \nu_{A_2}} & \text{on } \partial D, \end{cases} \tag{4.78}$$

where in general $A_1$ and $A_2$ are real symmetric matrix valued functions, positive definite, and bounded uniformly in $D$, and $n_1$ and $n_2$ are real scalar strictly positive functions bounded uniformly in $D$.

The following theorem is proven in [164].

**Theorem 4.44.** *Consider the transmission eigenvalue problem* (4.78) *and assume that* $\partial D$ *is* $C^\infty$, $A_{1,2}(x) := a_{1,2}(x)$ *are scalar functions in* $C^\infty(\overline{D})$ *and* $n \in C^\infty(\overline{D})$.

(1) *Assume that*

$$(a_1(x) - a_2(x))(a_1(x)n_1(x) - a_2(x)n_2(x)) < 0 \qquad \textit{for all } x \in \partial D. \tag{4.79}$$

*Then there exists a constant* $C > 0$ *such that there are no transmission eigenvalues in the region*

$$\{k \in \mathbb{C} : \Re(k) > 1, \, |\Im(k)| \geq C\}.$$

(2) *Assume that*

$$(a_1(x) - a_2(x))(a_1(x)n_1(x) - a_2(x)n_2(x)) > 0 \qquad \textit{for all } x \in \partial D. \tag{4.80}$$

*Then there exists a constant* $C > 0$ *such that there are no transmission eigenvalues in the region*

$$\left\{k \in \mathbb{C} : \Re(k) > 1, \, |\Im(k)| \geq C_\epsilon (\Re(k))^{1/2+\epsilon}\right\}.$$

*If in addition*

$$\frac{n_1(x)}{a_1(x)} \neq \frac{n_2(x)}{c_2(x)} \qquad \textit{for all } x \in \partial D,$$

*then there exists a constant* $C > 0$ *such that there are no transmission eigenvalues in the region*

$$\{k \in \mathbb{C} : \Re(k) > 1, \, |\Im(k)| \geq C \ln(\Re(k) + 1)\}.$$

(3) *Under the assumption of part* (1) *or part* (2)*, letting*

$$N(r) := \#\{k \text{ transmission eigenvalues } |k| \leq r\},$$

*it holds that*

$$N(r) = \frac{r^3}{6\pi^2} \int_D \left[ \left( \frac{n_1(x)}{a_1(x)} \right)^{3/2} + \left( \frac{n_2(x)}{a_2(x)} \right)^{3/2} \right] dx + O_\epsilon \left( r^{2+\epsilon} \right)$$

*for all* $0 < \epsilon \ll 1$*, where the order term depends on* $\epsilon$*.*

Note that Theorem 4.44(2) states that the imaginary part of transmission eigenvalues may blow up under assumption (4.80), although it is not known if these estimates are optimal. In this case, however, there are no transmission eigenvalues in a neighborhood of the imaginary axis (see [163]). On the other hand, under the assumption (4.79) in part (1) of the above theorem, transmission eigenvalues with real part greater than one lie in a strip around the real axis. Interestingly, it is clarified in Remark 4 in [163] that there exist infinitely many transmission eigenvalues near the imaginary axis in this case. Therefore the imaginary part of all transmission eigenvalues is not uniformly bounded (note that transmission eigenvalues form a discrete set in the complex plane with infinity as the only accumulation point). Moreover, there also exist infinitely many transmission eigenvalues around the real axis (see Remark 5 in [163]). Thus in this case it is not possible to find a half-space free of transmission eigenvalues. Summarizing, in the case of the transmission eigenvalue problem with contrast in the main operator, there is no known case of the existence of a half-plane free of transmission eigenvalues, which is important for sampling methods in inverse scattering with time domain data.

The proof of the next theorem can be found in [141].

**Theorem 4.45.** *Assume that $\partial D$ is in $C^2$, $A_1, A_2$ are matrix valued functions in $C(\overline{D})$, and $n_1, n_2$ are scalar functions in $C(\overline{D})$. Furthermore, assume that*

$$(A_2(x)\nu\cdot\nu)(A_2(x)\xi\cdot\xi)-(A_2(x)\nu\cdot\xi)^2 \neq (A_1(x)\nu\cdot\nu)(A_1(x)\xi\cdot\xi)-(A_1(x)\nu\cdot\xi)^2 \quad (4.81)$$

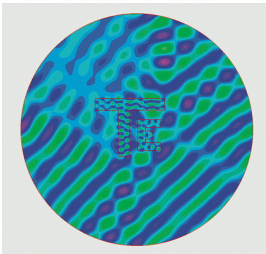
*for all $x \in \partial D$, with $\nu := \nu(x)$ the outward unit normal vector on $\partial D$, and for all unit vectors $\xi \in \mathbb{R}^3 \setminus \{0\}$ with $\xi \cdot \nu = 0$, and*

$$(A_2(x)\nu \cdot \nu)n_2(x) \neq (A_1(x)\nu \cdot \nu)n_1(x) \qquad \text{for all } x \in \partial D. \quad (4.82)$$

*Then the generalized eigenfunctions of* (4.78) *are complete in $L^2(D) \times L^2(D)$.*

Under the assumptions of Theorem 4.45 in [141] Weyl's estimate for the eigenvalue counting function of the same order as in part (3) of Theorem 4.44 is proven. The assumption (4.81) is equivalent to the Agmon–Douglis–Nirenberg complementing condition [2] and, together with the assumption (4.82), provides the most general up-to-date assumptions on the coefficients for which the transmission eigenvalue problem for anisotropic media is studied. These two assumptions can be viewed as a generalization of the sign conditions on the contrasts near the boundary that are needed in our discussion throughout this section.

We also mention that there is a considerable body of work connected with numerical computations of transmission eigenvalues [103], [104], [116]. We refer the reader to [54] and [51] for similar results on the transmission eigenvalue problem for Maxwell's equations.



## Chapter 5

# Determination of Transmission Eigenvalues from Far Field Data and Applications to Inverse Scattering

The goal of this chapter is to provide a glimpse of possible applications of transmission eigenvalues for providing solutions to some inverse problems. We first show how measurements of far field data for different frequencies lead to the identification of real transmission eigenvalues. We present three methods for showing this. The first and second methods are based on the LSM and GLSM formalisms and require that the shape of the scattering object be known. The third one is based on so-called inside-outside duality and the behavior of the phase of some eigenvalues of the far field operator. It does not need a priori knowledge of the domain $D$ but works only under very restrictive assumptions on the refractive index. The inside-outside duality also allows the construction of an approximation of the incident field associated with the transmission eigenfunction. In connection with this method, we explain how an appropriate change of the background allows for the construction of a different set of transmission eigenvalues under weaker assumptions on the refractive index. The use of an artificial background may have its own interest for solutions to the inverse problem. We outline some of the possible applications in the last section of this chapter.

## 5.1 ▪ The Determination of Transmission Eigenvalues from Far Field Data

We discuss in this section the determination of teal transmission eigenvalues from a knowledge of the far field data. This is important for applications as it shows that these quantities can be determined from measurements and therefore can be exploited in the solution of inverse problems. We restrict ourselves to the scattering problem for isotropic media defined by (1.27)–(1.29). We make the assumption that $\Im(n) = 0$, for which real transmission eigenvalues are proven to exist. The case of anisotropic media can be treated in a very similar way and is skipped here.

We present three approaches to determine transmission eigenvalues from the far field operator (2.1) as introduced in Chapter 2, namely, $F : L^2(S^2) \to L^2(S^2)$ defined by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \tag{5.1}$$

where $u_\infty(\hat{x}, d)$ denote the far field pattern.

The first and second approaches use the LSM and GLSM algorithms and require a priori knowledge of a nonempty open subset of $D$ (which is the support of $n-1$, where $n$ is the refractive index) [34]. For $z$ in this subregion we exploit the fact that an appropriate indicator function blows up if $k$ is a transmission eigenvalue, while it remains bounded if $k$ is not a transmission eigenvalue. The third approach uses a different philosophy [115]. It is based on an analysis of accumulation points of the normalized eigenvalues of the far field operator. Roughly speaking, transmission eigenvalues are detected when these normalized eigenvalues accumulate at two different points as the wave number approaches a transmission eigenvalue.

We remark that our presentation here is slightly different from the one in the indicated literature.

### 5.1.1 ▪ An Approach Based on LSM

The main assumption here is that the operator $F$ has dense range. This is indeed guaranteed if $k$ is not a nonscattering wave number, which is discussed in Chapter 7. Moreover we assume that a nonempty open subset of $D$ is known a priori and that $D$ is simply connected (see Remark 5.3 for a discussion of the case of a multiply connected domain $D$). We set $\phi_z(\hat{x}) := \frac{1}{4\pi}e^{-ik\hat{x}\cdot z}$ to be the far field pattern associated with the fundamental solution $\Phi(\cdot, z)$ of the Helmholtz equation. We let $g_z^\alpha \in L^2(S^2)$ be the solution to

$$(\alpha + F^*F)g_z^\alpha = F^*\phi_z.$$

Recall that $F = G\mathcal{H}$, where $\mathcal{H} : L^2(S^2) \to H_{\text{inc}}(D)$ is the Herglotz operator defined by (2.3) and $G : H_{\text{inc}}(D) \to L^2(S^2)$ is defined by (2.4). We prove the following result.

**Theorem 5.1.** *Assume that $n - 1 \geq \alpha > 0$ (respectively, $1 - n \geq \alpha > 0$) in $D$ for some constant $\alpha$ and that $k > 0$ is not a nonscattering wave number. Then for any ball $B \subset D$, $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ is bounded as $\alpha \to 0$ for almost every $z \in B$ if and only if $k$ is not a transmission eigenvalue.*

***Proof.*** If $k$ is not a transmission eigenvalue, then one can apply Theorems 2.26 and 2.33 to deduce that $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ is bounded as $\alpha \to 0$ for all $z$ in $D$. Now assume that $k$ is a transmission eigenvalue. Since $F$ has dense range (by the assumption that $k$ is not a nonscattering wave number, Theorem 1.17), then

$$Fg_z^\alpha \to \phi_z \quad \text{as } \alpha \to 0,$$

(cf. Theorem 1.31). Assume that there exists a ball $B \subset D$ such that for almost every $z \in B$, $\|\mathcal{H}g_z^\alpha\|_{L^2(D)} \leq M$ for some constant $M > 0$ as $\alpha \to 0$ (the constant $M$ may depend on $z$). Then (for fixed $z$) there exists a subsequence $v_n = \mathcal{H}g_z^{\alpha_n}$ that weakly converges to $v_z$ in $H_{\text{inc}}(D)$. Since $G$ is a compact operator, we deduce that $Gv_z = \phi_z$. Using Rellich's Lemma, one deduces the existence of a solution $(u_z, v_z) \in L^2(D) \times L^2(D)$ of the interior transmission problem

$$\begin{cases} \Delta u_z + k^2 n u_z = 0 \quad \text{in } D, \\ \Delta v_z + k^2 v_z = 0 \quad \text{in } D, \\ u_z - v_z = \Phi(\cdot, z) \quad \text{on } \partial D, \\ \partial(u_z - v_z)/\partial\nu = \partial\Phi(\cdot, z)/\partial\nu \quad \text{on } \partial D \end{cases} \tag{5.2}$$

such that the function $w_z = u_z - v_z \in H^2(D)$. As in Chapter 3, one verifies that $w_z$ satisfies

$$\int_D \frac{1}{n-1}(\Delta w_z + k^2 w_z)(\Delta\varphi + k^2 n\varphi)dx = 0 \quad \text{for all } \varphi \in H_0^2(D) \tag{5.3}$$

and

$$w_z = \Phi(\cdot, z) \quad \text{and} \quad \frac{\partial w_z}{\partial\nu} = \frac{\partial\Phi(\cdot, z)}{\partial\nu} \quad \text{on } \partial D.$$

Since $k$ is a transmission eigenvalue, according to the results of Chapter 3 there exists a nontrivial function $w_0 \in H_0^2(D)$ satisfying

$$(\Delta + k^2)\frac{1}{n-1}(\Delta w_0 + k^2 n w_0) = 0 \quad \text{in } D. \tag{5.4}$$

Taking $\varphi = w_0$ in (5.3) and applying Green's theorem twice yields, after using (5.4),

$$\int_{\partial D} \left(\frac{1}{n-1}(\Delta w_0 + k^2 n w_0)\right) \frac{\partial\Phi(\,\cdot\,, z)}{\partial\nu}\, ds - \int_{\partial D} \frac{\partial}{\partial\nu}\left(\frac{1}{n-1}(\Delta w_0 + k^2 n w_0)\right) \Phi(\,\cdot\,, z)\, ds = 0, \tag{5.5}$$

where these integrals have to be understood in the sense of $H^{\mp1/2}(\partial D)$ (respectively, $H^{\mp3/2}(\partial D)$) duality pairing. Defining $\psi(x) := \frac{1}{n-1}(\Delta + k^2 n(x))w_0(x)$ in $D$, we observe that

$$\Delta\psi + k^2\psi = 0 \quad \text{in } D.$$

Classical interior elliptic regularity results and the Green's representation theorem imply that

$$\psi(z) = \int_{\partial D} \left(\psi(x)\,\frac{\partial\Phi(x,z)}{\partial\nu} - \frac{\partial\psi(x)}{\partial\nu}\Phi(x,z)\right) ds_x \qquad \text{for } z \in D. \tag{5.6}$$

Equation (5.5) and the unique continuation principle now show that $\psi = 0$ in $D$. Therefore $(\Delta + k^2 n(x))w_0(x) = 0$ in $D$. Since $w_0 \in H_0^2(D)$ one deduces again from the unique continuation principle that $w_0 = 0$ in $D$, which is a contradiction. $\square$

There are two weak points of the characterization provided by Theorem 5.1. The first one is related to the assumption that $k$ should not be a nonscattering wave number. Necessary conditions for which a transmission eigenvalue can consist of nonscattering wave numbers is discussed in Section 7.2. In particular it is shown that if $D$ contains corners or edge singularities, then the set of nonscattering wave numbers is empty. The only known case for which the set of nonscattering wave numbers is not empty is the case of a spherically stratified index of refraction. We refer the reader to [34] for a way to get around this problem by exploiting the fact that the noisy operator has in general dense range.

The second weak point is indeed the fact that the characterization of transmission eigenvalues is given in terms of the behavior of $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$, which requires knowledge of

$D$. In practice, numerical experiments show that replacing $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ with $\|g_z^\alpha\|_{L^2(S^2)}$ provides satisfactory results [40], [32], [89].

## Numerical Examples

For the numerical experiments one needs to have access to points $z_i$ $i = 1, \dots, N$, inside the domain $D$. We then evaluate

$$k \mapsto \sum_{i=1}^{N} \|g_{z_i}^\alpha\|_{L^2(S^2)}$$

for some regularization parameter $\alpha$ that can be chosen using the Morozov discrepancy principle. This in turn assumes that one has access to the far field operator for a range of wave numbers that contain the sought transmission eigenvalues. We now give some numerical examples from [89] for a circular domain $D$ of radius $= 0.5$ with index of refraction $n = n_i$ in an inner circle and $n = n_e$ in the outer annulus (see Figure 5.1). More examples can be found in [42] and [95].

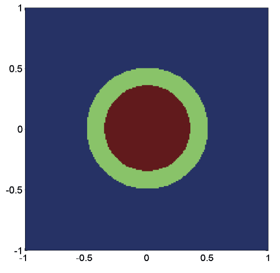



**Figure 5.1.** *Configuration of the refractive index in a circular domain $D$ of radius $0.5$. Reproduced from* [89] *with permission.*

In Figure 5.2 we indicate the behavior of $k \mapsto \|g_{z_i}^\alpha\|_{L^2(S^2)}$ for several choices of the refractive index $n_i$ and $n_e$ and for different choices of the points $z_i$. The parameter $\alpha$ is fixed using the Morozov discrepancy principle. Observe in particular that some peaks disappear (or are less sharp) for some choices of the points $z_i$. This confirms that several points are needed in order to obtain stable determination of the peaks that correspond to transmission eigenvalues.

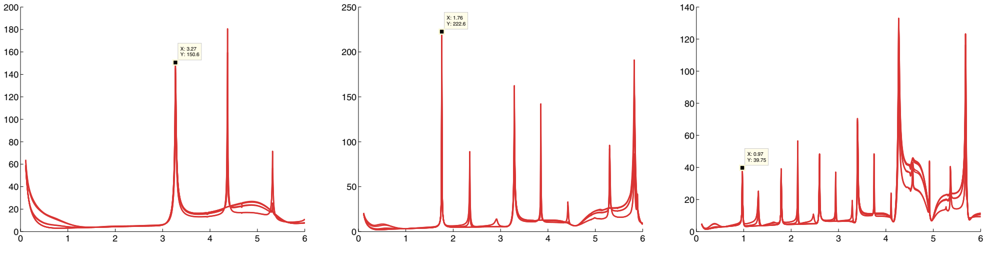


**Figure 5.2.** *From left to right, plots of $k \mapsto \|g_z^\alpha\|_{L^2(S^2)}$ for several choices of points $z$, respectively, for $(n_e, n_i) = (11, 5), (22, 19), (67, 61)$. Reproduced from* [89] *with permission.*

## 5.1.2 ▪ An Approach Based on GLSM

The if and only if statement in Theorem 5.1 relies on the link between the LSM and the factorization method (Theorem 2.33) and therefore requires a stronger assumption on the refractive index. We can avoid relying on this link and on the hypothesis that $k$ is not a non-scattering wave number using the following formulation, based on GLSM and assuming that $D$ is known. To this end, define

$$J_\alpha(\phi_z; g) := \alpha \|\mathcal{H}g\|^2_{L^2(D)} + \|Fg - \phi_z\|^2_{L^2(S^2)}$$

and set

$$j_\alpha(\phi_z) = \inf_{g \in L^2(S^2)} J_\alpha(\phi_z; g). \tag{5.7}$$

We then consider $g_z^\alpha$ to be the minimizing sequence satisfying

$$J_\alpha(\phi; g_z^\alpha) \leq j_\alpha(\phi) + p(\alpha) \tag{5.8}$$

with $0 < \frac{p(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$.

**Theorem 5.2.** *Assume that Assumption* 2.2 *holds. Then for any ball $B \subset D$, $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ is bounded and $\|Fg_z^\alpha - \phi_z\|_{L^2(S^2)} \to 0$ as $\alpha \to 0$ for almost every $z \in B$ if and only if $k$ is not a transmission eigenvalue.*

***Proof.*** The case when $k$ is not a transmission eigenvalue is a consequence of Theorem 2.9 with $B = \mathcal{H}^*\mathcal{H}$, the injectivity and dense range of $F$, and Theorem 2.3.

If $k$ is a transmission eigenvalue, $\|\mathcal{H}g_z^\alpha\|_{L^2(D)}$ is bounded, and $\|Fg_z^\alpha - \phi_z\|_{L^2(S^2)} \to 0$ as $\alpha \to 0$, one can obtain a contradiction in the same way as in the second part of the proof of Theorem 5.1. □

**Remark 5.3.** For Theorems 5.1 and 5.2, the assumption that $D$ is simply connected can be removed by assuming that the intersection of the set of points $z$ with each connected component of $D$ contains an open set with positive measure. Also in the case of a multiply connected domain $D$, if we restrict the set of points $z$ to a connected component of $D$, then one recovers the transmission eigenvalues related to that connected component. This has been observed and numerically tested in [89].

## 5.1.3 ▪ An Approach Based of the Eigenvalues of the Far Field Operator

We present in this section a different approach to identify transmission eigenvalues based on the behavior of the phase of the eigenvalues of the normal operator $F$. In many aspects, this approach can also be seen as the complement of the $(F^*F)^{1/4}$ method (Section 2.4.1) in determining transmission eigenvalues. It is referred to in the literature as the *inside-outside duality* method [115], [83]. We here adopt the notation of Section 2.4.2 and explicitly indicate the dependence on $k$ in our notation: For instance, the far field operator is denoted by $F_k$. We recall that

$$F_k = \mathcal{H}_k^* T_k \mathcal{H}_k$$

with $T_k : L^2(D) \to L^2(D)$ defined as in (2.17). We recall (Lemma 2.25) that if $n - 1 \geq \alpha > 0$ (respectively, $1 - n \geq \alpha > 0$) in $D$ for some constant $\alpha > 0$ and if $k > 0$ is not a transmission eigenvalue, then the operator $T_k : L^2(D) \to L^2(D)$ (respectively $-T_k$) satisfies Assumption 2.3 with $Y = Y^* = L^2(D)$. Moreover, the scattering operator

$$S_k = I + \frac{ik}{2\pi} F_k$$

is unitary, which is equivalent to $F_k$ being normal. If $k > 0$ is not a transmission eigenvalue, then $F_k$ is injective. We therefore can exhibit an orthonormal complete basis $(g_j(k))_{j=1,+\infty}$ of $L^2(S^2)$ such that

$$F_k g_j(k) = \lambda_j(k) g_j(k), \tag{5.9}$$

where $\lambda_j(k) \neq 0$ form a sequence of complex numbers that go to 0 as $j \to \infty$. Define

$$\hat{\lambda}_j(k) := \lambda_j(k)/|\lambda_j(k)|.$$

Since $\lambda_j(k)$ (for all $j$) lies on the circle of radius $2\pi/k$ and center $i2\pi/k$ and $\lambda_j \to 0$ as $j \to \infty$, the only possible accumulation points of the sequence $(\hat{\lambda}_j(k))$ are $-1$ and $+1$. From the proof of Theorem 2.24 we see that if $k$ is not a transmission eigenvalue and $n - 1 \geq \alpha > 0$, then $+1$ is the only accumulation point of $\hat{\lambda}_j(k)$. In the case when $1 - n \geq \alpha > 0$, applying Theorem 2.24 to $-F_k$ shows that $-1$ is the only accumulation point of $\hat{\lambda}_j(k)$. More precisely, writing

$$\lambda_j(k) = \frac{2\pi}{ik}(e^{i\delta_j(k)} - 1), \quad 0 < \delta_j(k) < 2\pi,$$

with $e^{i\delta_j(k)}$ being the eigenvalues of the scattering operator $S_k$, we can state the following result.

**Proposition 5.4.** *Assume that $k$ is not a transmission eigenvalue. If $n - 1 \geq \alpha > 0$ in $D$, then the sequence $(\delta_j(k))$ accumulates only at 0 (see Figure 5.3, right). If $1 - n \geq \alpha > 0$ in $D$, then the sequence $(\delta_j(k))$ accumulates only at $2\pi$.*

We deduce that if $k$ is not a transmission eigenvalue, we can order the phases $\delta_j(k)$ so that

$$\begin{aligned} &2\pi > \delta_1(k) \geq \delta_2(k) \geq \cdots \geq \delta_j(k) \geq \cdots > 0 \qquad &&\text{when } n - 1 \geq \alpha > 0,\\ &0 < \delta_1(k) \leq \delta_2(k) \leq \cdots \leq \delta_j(k) \leq \cdots < 2\pi \qquad &&\text{when } 1 - n \geq \alpha > 0. \end{aligned}$$

The following theorem provides a sufficient condition for the existence of a transmission eigenvalue.

**Theorem 5.5.** *Assume that $n - 1 \geq \alpha > 0$ (respectively, $1 - n \geq \alpha > 0$) in $D$ for some constant $\alpha$. Let $k_0 > 0$, and let $(k_\ell)$ be a sequence of positive numbers converging to $k_0$ as $\ell \to \infty$. Assume that the sequence $\delta_1(k_\ell) \to 2\pi$ or equivalently $\hat{\lambda}^\ell := \hat{\lambda}_1(k_\ell) \to -1$ (respectively, $\delta_1(k_\ell) \to 0$ or equivalently $\hat{\lambda}^\ell \to +1$) as $\ell \to \infty$. Then $k_0$ is a transmission eigenvalue.*

***Proof.*** The proof uses basically the same arguments as the proof of Theorem 2.24 and a continuity property with respect to $k$ of the operator $T_k$ formulated in Lemma 5.9 below. We shall consider only the case $n - 1 \geq \alpha > 0$ since the other case follows from the same arguments (replacing $F_k$ with $-F_k$). Define

$$\psi_\ell := \frac{1}{\sqrt{|\lambda_{j_\ell}|}} \mathcal{H}_{k_\ell} g_{j_\ell}.$$

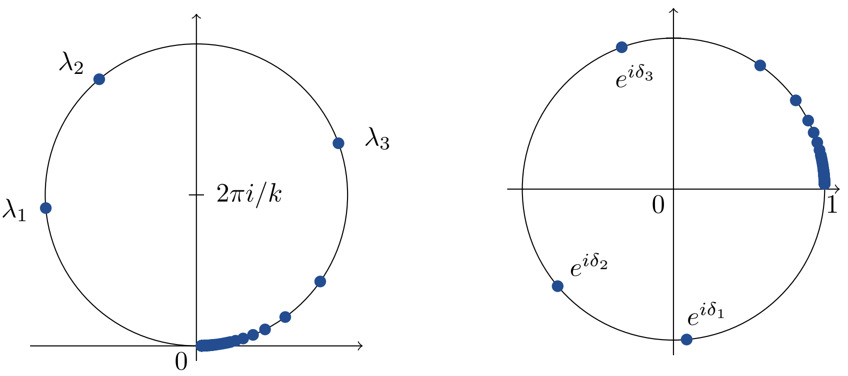



**Figure 5.3.** *Left: eigenvalues of $F_k$. Right: eigenvalues of $S_k$. Here the representation corresponds to a situation where $n-1 \geq \alpha > 0$ in $D$.*

From (2.43) we clearly have

$$(T_{k_\ell}\psi_\ell, \psi_\ell)_{L^2(D)} = \hat{\lambda}^\ell (g_{j_\ell}, g_{j_\ell})_{L^2(S^2)} = \hat{\lambda}^\ell \to -1. \tag{5.10}$$

Assume that $k_0$ is not a transmission eigenvalue. Then from Lemma 2.25 we deduce that $T_{k_0}$ is coercive. Using the continuity of $k \mapsto T_k$ we deduce that $T_{k_\ell}$ are uniformly coercive for $\ell$ sufficiently large since

$$|(T_{k_\ell}\psi, \psi)_{L^2(D)}| \geq |(T_{k_0}\psi, \psi)_{L^2(D)}| - \|T_{k_0} - T_{k_\ell}\| \|\psi\|^2_{L^2(D)}.$$

Choosing $\ell$ sufficiently large so that $\|T_{k_0} - T_{k_\ell}\| \leq \beta/2$, where $\beta$ is the coercivity constant for $T_{k_0}$, we get

$$|(T_{k_\ell}\psi, \psi)_{L^2(D)}| \geq \frac{\beta}{2} \|\psi\|^2_{L^2(D)}.$$

We then deduce from (5.10) that the sequence $(\psi_\ell)$ is bounded in $L^2(D)$. Therefore, up to a subsequence, one can assume that $(\psi_\ell)$ weakly converges to some $\psi_0$ in $L^2(D)$. Since

$$\Delta\psi_\ell + k_\ell^2 \psi_\ell = 0 \quad \text{in } D,$$

we deduce that

$$\Delta\psi_0 + k_0^2 \psi_0 = 0 \quad \text{in } D,$$

meaning that $\psi_0 \in \overline{\mathcal{R}(\mathcal{H}_{k_0})}$. Let us denote by $w_\ell \in H^2_{\text{loc}}(\mathbb{R}^3)$ the solution of (2.2) with $u_0 = \psi_\ell$ and $w_\ell^\infty$ the corresponding far field pattern. We recall from (2.35) that

$$4\pi \Im(T_{k_\ell}\psi_\ell, \psi_\ell) = k_\ell \int_{S^2} |w_\ell^\infty|^2 ds. \tag{5.11}$$

From Lemma 5.8, the Rellich compact embedding theorem, and the continuity of the mapping $w \to w^\infty$ from $L^2(D)$ into $L^2(S^2)$ we deduce that

$$\Im(T_{k_\ell}\psi_\ell, \psi_\ell) \to \Im(T_{k_0}\psi_0, \psi_0).$$

From (5.10) we then get $\Im(T_{k_0}\psi_0, \psi_0) = 0$ and therefore $\psi_0 = 0$ (since $k_0$ is not a transmission eigenvalue). We now note that

$$\frac{k_\ell^2}{4\pi}((n-1)\psi_\ell, \psi_\ell)_{L^2(D)} = (T_{k_\ell}\psi_\ell, \psi_\ell)_{L^2(D)} - \frac{k_\ell^2}{4\pi}((n-1)\psi_\ell, w_\ell)_{L^2(D)},$$

where $((n-1)\psi_\ell, w_\ell)_{L^2(D)} \to ((n-1)\psi_0, w_0)_{L^2(D)}$ by Lemma 5.8 and the Rellich compact embedding theorem. Consequently

$$0 \le \frac{k_\ell^2}{4\pi}((n-1)\psi_\ell, \psi_\ell)_{L^2(D)} \to -1$$

(see [138]), which is a contradiction. $\square$

We complement the result of Theorem 5.5 with the following convergence result that shows that one can reconstruct an approximation of the solution $v$ associated with a transmission eigenvalue [10].

**Theorem 5.6.** *Let $n$ and $k_\ell \to k_0$ be as in Theorem* 5.5*. Then the sequence*

$$v_\ell := \frac{1}{\|\mathcal{H}_{k_\ell} g_1(k_\ell)\|_{L^2(D)}} \mathcal{H}_{k_\ell} g_1(k_\ell),$$

*where $g_1$ is defined in* (5.9)*, has a strongly convergent subsequence to some $v$ in $L^2(D)$ satisfying $\Delta v + k_0^2 v = 0$ in $D$ and such that $v$ is associated with a transmission eigenpair.*

***Proof.*** Let $v$ be the weak limit in $L^2(D)$ of a subsequence of $v_\ell$ (this limit exists since $\|v_k\|_{L^2(D)} = 1$). To simplify the notation this subsequence is denoted the same as $v_\ell$.

We now have

$$(T_{k_\ell} v_\ell, v_\ell)_{L^2(D)} = \theta_\ell \, \hat{\lambda}^\ell \tag{5.12}$$

with $\theta_\ell := |\lambda_\ell| / \|\mathcal{H}_{k_\ell} g_{j_\ell}\|^2_{L^2(D)}$. From Lemma 5.9 one infers that $\theta_\ell$ is bounded and therefore, up to changing the subsequence, one can assume that $\theta_\ell \to \theta_0 \ge 0$. We then observe that

$$\Im(T_{k_\ell} v_\ell, v_\ell)_{L^2(D)} \to 0$$

and therefore $\Im(T_{k_0} v, v)_{L^2(D)} = 0$. Denote by $w \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ the solution of (2.2) with $u_0 = v$. We then get, using Rellich's Lemma as explained above, that $w \in H_0^2(D)$. We now prove that $v \ne 0$ and is the strong limit of the sequence $v_\ell$. We use the identity

$$((n-1)v_\ell, v_\ell)_{L^2(D)} = \frac{4\pi}{k_\ell^2}(T_{k_\ell} v_\ell, v_\ell)_{L^2(D)} - ((n-1)v_\ell, w_\ell)_{L^2(D)},$$

where $w_\ell \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ is the solution of (2.2) with $u_0 = v_\ell$, to deduce that

$$\limsup_{\ell\to\infty}((n-1)v_\ell, v_\ell)_{L^2(D)} \le -((n-1)v, w)_{L^2(D)},$$

where we used the strong convergence of $w_\ell$ to $w$ (using Lemma 5.8). One easily observes, using $w \in H_0^2(D)$, that

$$((n-1)v + w, v)_{L^2(D)} = 0$$

and get for the case $n-1 \ge \alpha > 0$,

$$\limsup_{\ell\to\infty}((n-1)v_\ell, v_\ell)_{L^2(D)} \le ((n-1)v, v)_{L^2(D)}.$$

We then obtain

$$\limsup_{\ell\to\infty}((n-1)v_\ell - v, v_\ell - v)_{L^2(D)} \le 0$$

and deduce the strong convergence of $v_\ell$ to $v$ in $L^2(D)$. The case $n-1 \le -\alpha < 0$ follows in the same way by replacing $(n-1)$ with $(1-n)$. This also proves that $v$ is not trivial and therefore is associated with a transmission eigenpair. □

**Remark 5.7.** We observe that the result of Theorem 5.6 also holds if we replace $g_1$ with $g_p$, $p > 1$ (defined in (5.9)), as long as $\delta_p(k_\ell)$ converges to the same value as $\delta_1(k_\ell)$. This may occur if the transmission eigenvalue has a multiplicity greater than one.

We now establish a continuity property with respect to $k$ of the operator $T_k$. We first show a uniform bound on the solution of (2.2) for wave numbers in a bounded interval.

**Lemma 5.8.** *Let $\psi \in L^2(D)$ and $w_k \in H^1_{\rm loc}(\mathbb{R}^3)$ be the solution of* (2.2) *for the wave number $k > 0$ and $u_0 = \psi$. Then, for all bounded intervals $I \subset \mathbb{R}_+$ and compact $K \in \mathbb{R}^3$, there exists a constant $C$ independent of $k$ and $\psi$ such that*

$$\|w_k\|_{H^1(K)} \le C\|\psi\|_{L^2(D)} \quad \textit{for all } k \in I.$$

***Proof.*** Using the Lippmann–Schwinger integral equation for $w_k$ (see Theorem (1.9)) we have

$$w_k + A_k w_k = -A_k\psi \quad \text{in } L^2(D), \tag{5.13}$$

where $A_k : L^2(D) \to L^2(D)$ is the compact operator defined by

$$A_k\varphi := k^2 \int_D \Phi_k(x,y)(1-n(y))\varphi(y)\,dy.$$

From the expression $\Phi_k(x,y) = \exp(ik|x-y|)/(4\pi|x-y|)$ one can easily verify that

$$\|A_k\varphi - A_{k'}\varphi\|_{L^2(D)} \le C|k-k'|\|\varphi\|_{L^2(D)} \tag{5.14}$$

and

$$\|A_k\varphi\|_{L^2(D)} \le C\|\varphi\|_{L^2(D)} \tag{5.15}$$

with a constant $C$ independent of $k, k' \in I$, and $\varphi$. Fix $\delta$ sufficiently small such that $2C\delta \le \inf_{k\in I}\|\mathcal{I} + A_k\|$. Let $k_0 \in I$. Writing $\mathcal{I} + A_k = (\mathcal{I} + A_{k_0}) + (A_k - A_{k_0})$, we then deduce from (5.14) that for $k \in (k_0 - \delta, k0 + \delta)$,

$$\|(\mathcal{I} + A_k)^{-1}\| \le 2\|(\mathcal{I} + A_{k_0})^{-1}\|.$$

Combined with (5.15) we observe that for $k \in (k_0 - \delta, k_0 + \delta)$

$$\|w_k\|_{L^2(D)} \le \tilde{C}\|\psi\|_{L^2(D)}$$

for some different constant $C$ independent of $k$. Since

$$w_k(x) = k^2 \int_D \Phi_k(x,y)(1-n(y))(\varphi(y) + w_k(y))dy, \quad x \in \mathbb{R}^3,$$

we then also get, with a different constant $C$, that

$$\|w_k\|_{H^1(K)} \le C\|\psi\|_{L^2(D)} \quad \text{for all } k \in (k_0 - \delta, k0 + \delta).$$

The result follows from considering a finite covering of $I$ with intervals of size $2\delta$. □

We now prove the following technical lemma needed in the proof of Theorem 5.5.

**Lemma 5.9.** *The mapping $k \mapsto T_k$ is continuous from $\mathbb{R}_+$ into the space of linear mappings from $L^2(D)$ into itself.*

***Proof.*** Consider two wave numbers $k > 0$ and $k' > 0$ in some given bounded interval $I$. Then

$$w_k - w_{k'} + A_k(w_k - w_{k'}) = -(A_k - Ak')(\psi + w_{k'}) \quad \text{in } L^2(D).$$

Therefore, using (5.13), (5.14), and Lemma 5.8 one deduces that

$$\|w_k\varphi - w_{k'}\|_{L^2(D)} \le C(k)|k-k'|\|\psi\|_{L^2(D)}$$

for some constant $C(k)$ independent of $k' \in I$ and $\psi$. The proof then directly follows from the expression of $T_k$. □

The criterion of Theorem 5.5 can be used as an indicator of transmission eigenvalues. However, the hard part is to prove that it occurs for every transmission eigenvalue. We refer the reader to [115] for a proof that this is the case for the first transmission eigenvalue if the contrast is a sufficiently large constant (or small perturbation of a constant). We show in the following how one can obtain the converse statement in the case of a specific modified background. This converse statement has been shown for Dirichlet obstacles in [83], where this method was first introduced. A discussion of numerical issues related to the use of this criterion can be found in [105].

## 5.2 ▪ Transmission Eigenvalues for Modified Background

Let $n_b \in L^\infty(\mathbb{R}^3)$ be an arbitrary given real valued function such that $n_b = 1$ in $\mathbb{R}^3 \setminus \overline{D_b}$, where $D_b \supset D$ is a bounded regular domain with connected complement. This parameter can be seen as the index of refraction of an artificial background. We denote by $u_b^s(\cdot, d) \in H^1_{loc}(\mathbb{R}^3)$ the solution of the scattering problem (1.27)–(1.29), where $n$ is replaced by $n_b$, and by $u_\infty(\cdot, d)$ the associated far field pattern. We recall that $u_b^s(\cdot, d)$ coincides with $u_b^s \in H^1_{loc}(\mathbb{R}^3)$ being the unique solution to

$$\begin{cases} \Delta u_b^s + k^2 n_b u_b^s = k^2(1-n_b)u^i \quad \text{in } \mathbb{R}^3, \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} |\partial u_b^s/\partial|x| - ik\, u_b^s|^2 \, ds = 0 \end{cases} \tag{5.16}$$

when $u^i = u^i(\cdot, d) := e^{ikd\cdot x}$. Similarly to (5.1) and (1.30), we introduce the far field operator for the background media, $F^b : L^2(S^2) \to L^2(S^2)$, defined by

$$(F^b g)(\hat{x}) := \int_{S^2} g(d)\, u^b_\infty(\hat{x}, d)\, ds(d),$$

and the corresponding scattering operator $S^b : L^2(S^2) \to L^2(S^2)$,

$$S^b := I + \frac{ik}{2\pi} F^b. \tag{5.17}$$

We then define the modified far field operator $F^m : L^2(S^2) \to L^2(S^2)$ as

$$F^m := F - F^b. \tag{5.18}$$

In practice, $F$ is obtained from the measurements while $F^b$ has to be computed by numerically solving (5.16) for given $n_b$ and $u^i$. We remark that when $n_b = 1$, then $F^b = 0$ and we simply have $F^m = F$.

The transmission eigenvalues for the artificial background are defined as the values of $k > 0$ for which there exists a nontrivial $u^i \in L^2(D_b)$ satisfying $\Delta u^i + k^2 u^i = 0$ in $D_b$ and such that the corresponding scattered fields $u^s$ and $u^s_b$, respectively, defined by (1.27)–(1.29) and (5.16) are such that $u_\infty = u^b_\infty$. By Rellich's Lemma, this implies $u^s = u^s_b$ in $\mathbb{R}^3 \setminus \overline{D_b}$. Setting $w := u^s + u^i|_{D_b}$ and $w_b := u^s_b + u^i|_{D_b}$ we obtain

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D_b, \\ \Delta w_b + k^2 n_b w_b = 0 & \text{in } D_b, \\ w = w_b & \text{on } \partial D_b, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial w_b}{\partial \nu} & \text{on } \partial D_b, \end{cases} \tag{5.19}$$

which can be viewed as a generalized form of (3.2). *Modified transmission eigenvalues* are then defined as values of $k$ for which (5.19) has nontrivial solutions $(w, w_b) \in L^2(D_b) \times L^2(D_b)$. In this section we always assume that $n$ is real.

Note that the arguments and the results of Chapter 3 can be straightforwardly adapted to (5.19) by replacing $n - 1$ with $n - n_b$ and $D$ with $D_b$. For instance, $n - n_b \geq \alpha > 0$ or $n_b - n \geq \alpha > 0$ in a neighborhood of $\partial D_b$ for some constant $\alpha$; then the set of transmission eigenvalues of (5.19) is discrete.

The advantage of introducing the artificial background is to gain a degree of freedom in (5.19), which is the value assigned to $n_b$ inside $D_b$, which can be exploited in the solution to the inverse problem. This has been done in number of ways in the literature [8] (see also [39], [61], [62], [63], [69]).

For instance, if we choose $n_b = 0$ in $D_b$ (and $n_b = 1$ in $\mathbb{R}^3 \setminus \overline{D_b}$) as suggested in [9], the modified transmission eigenvalue problem is equivalent to solving for $u := w - w_b \in H^2_0(D_b)$ and $k^2$ satisfying the linear eigenvalue problem

$$\Delta(n^{-1}\Delta u) = -k^2 \Delta u \quad \text{in } D_b. \tag{5.20}$$

As opposed to the original transmission eigenvalue problem (3.2), one obtains a quite simple eigenvalue problem similar to the so-called plate buckling eigenvalue problem. Classical results concerning linear self-adjoint compact operators guarantee that the spectrum of (5.20) is made of real positive isolated eigenvalues of finite multiplicity $0 < k_1^2 \leq k_2^2 \leq \cdots \leq k_p^2 \leq \cdots$ (the numbering is chosen so that each eigenvalue is repeated according to its multiplicity). Moreover, $\lim_{p\to+\infty} k_p^2 = +\infty$ and we have the Courant–Fisher *min-max* formula

$$k_p^2 = \min_{E_p \in \mathcal{E}_p} \max_{u \in E_p \setminus \{0\}} \frac{(n^{-1}\Delta u, \Delta u)_{L^2(D_b)}}{\|\nabla u\|^2_{L^2(D_b)}}. \tag{5.21}$$

Here $\mathcal{E}_p$ denotes the sets of subspaces $E_p$ of $H^2_0(D_b)$ of dimension $p$. Observe that the characterization of the spectrum for problem (5.20) is much simpler than the one given in Chapter 4 for problem (3.2). Moreover, it holds under very general assumptions for $n$: we just need that $n|_{D_b} \in L^\infty(D_b)$ with $\inf_{D_b} n > 0$. In particular, $n$ can be equal to one inside $D_b$ and $n-1$ can change sign on the boundary. From (5.21), if we let $\tilde{k}_p^2$ be the eigenvalues of (5.20) for another real index $\tilde{n} \in L^\infty(\mathbb{R}^3)$ such that supp $(1-\tilde{n}) \subset D_p$, then

$$k_p^2 \le \tilde{k}_p^2 \quad \text{if} \quad n \ge \tilde{n} \quad \text{a.e. in } D_p.$$

Another advantage of (5.20) is that its spectrum is entirely real. As a consequence, no information on $n$ is lost in complex eigenvalues which may exist for the original transmission eigenvalue problem (3.2) and which cannot be determined from the knowledge of $F$ for real frequencies. We refer the reader to Section 5.3 for other choices of the background leading to a similar simplified spectral problem.

Our goal in the following is to show how to characterize the eigenvalues in (5.19) from a knowledge of $F^m$.

### 5.2.1 ▪ Factorization of the Modified Far Field Operator

In order to develop the previously introduced method for identifying transmission eigenvalues we first establish some properties of the operator $F^m$. The main ingredient is the factorization of the operator $\mathcal{F} = (S^b)^* F^m$ that has a structure similar to that of the operator $F$ (see [90], [91], [121], [138]). We follow the approach in [121]. We first introduce the Herglotz operator $H^b$ associated with the artificial background. For $g \in L^2(S^2)$ we define the function $v_g$ by

$$v_g(x) := \int_{S^2} g(d) w_b(x,d)\, ds(d), \tag{5.22}$$

where $w_b(\cdot,d) := u^i(\cdot,d) + u_b^s(\cdot,d)$. Then we define the operator $H^b : L^2(S^2) \to L^2(D_b)$ by

$$H^b g := v_g|_{D_b}. \tag{5.23}$$

**Lemma 5.10.** *The operator $H^b$ is compact, injective, and the closure of its range is given by*

$$H^b_{\mathrm{inc}}(D_b) := \{v \in L^2(D_b);\ \Delta v + k^2 n_b v = 0 \textit{ in } D_b\}.$$

*Moreover the adjoint operator $(H^b)^* : L^2(D_b) \to L^2(S^2)$ satisfies*

$$(H^b)^* v = 4\pi (S^b)^* \psi_\infty, \tag{5.24}$$

*where $\psi_\infty$ is the far field pattern of $\psi \in H^2_{loc}(\mathbb{R}^3)$ satisfying $\Delta\psi + k^2 n_b \psi = -v$ in $\mathbb{R}^3$, where $v$ is extended by $0$ outside $D_b$, together with the Sommerfeld radiation condition.*

***Proof.*** We have $H^b g = (u_g^i + u_{b,g}^s)|_{D_b}$, where $u_{b,g}^s$ is the scattered field of the solution of (5.16) with $u^i = u_g^i := \int_{S^2} g(d) e^{ikd\cdot x}\, ds(d)$. Since $g \mapsto u_g^i$ is compact from $L^2(S^2)$ into $L^2(D_b)$ and since $u^i \mapsto u_b^s$ is continuous from $L^2(D_b) \to H^2(D_b)$, we deduce that $H^b : L^2(S^2) \to L^2(D_b)$ is compact. Assume that $H^b(g) = 0$. Then $u_{b,g}^s$ satisfies (5.16) with $n_b = 1$ and $u^i = 0$. Therefore $u_{b,g}^s = 0$ and $u_g^i = 0$. The injectivity of the Herglotz operator (Lemma 2.1) implies $g = 0$.

We now prove (5.24). Using Green's formula in $B_R$ for $R > 0$ large enough such that $\mathrm{supp}(n_b - 1) \subset B_R$, using that $\Delta v_g + k^2 n_b v_g = 0$ in $\mathbb{R}^3$, and using (5.22), gives

$$
\begin{aligned}
(f, H^b g)_{L^2(D_b)} &= -\int\limits_{B_R} (\Delta\psi + k^2 n_b \psi)\, \overline{v_g}\, dx \\
&= \int\limits_{\partial B_R} \psi\, \partial_\nu \int\limits_{S^2} \overline{g(d)}(e^{-ikd\cdot x} + \overline{u_b^s(x,d)})\, ds(d)\, ds(x) \\
&\quad - \int\limits_{\partial B_R} \partial_\nu \psi \int\limits_{S^2} \overline{g(d)}(e^{-ikd\cdot x} + \overline{u_b^s(x,d)})\, ds(d)\, ds(x).
\end{aligned} \tag{5.25}
$$

On the one hand, formula (1.15) gives, for $\hat{x} \in S^2$,

$$
\psi_\infty(\hat{x}) = \frac{1}{4\pi} \int\limits_{\partial B_R} \left( \psi\, \partial_\nu (e^{-ik\hat{x}\cdot y}) - \partial_\nu \psi\ e^{-ik\hat{x}\cdot y} \right)\, ds(y).
$$

Using the Sommerfeld radiation condition and the asymptotic expansion (1.14), we have, on the other hand, for $d \in S^2$,

$$
\lim_{R\to\infty} \int\limits_{\partial B_R} \left( \psi \partial_\nu \overline{u_b^s(\cdot, d)} - \partial_\nu \psi \overline{u_b^s(\cdot, d)} \right)\, ds = 2ik \int\limits_{S^2} \psi_\infty \overline{u_\infty^b(\cdot, d)} ds.
$$

Therefore, interchanging the order of integration in (5.25) and taking the limit as $R \to +\infty$ yields

$$
\begin{aligned}
(v, H^b g)_{L^2(D_b)} &= 4\pi(\psi_\infty, g)_{L^2(S^2)} + 2ik(\psi_\infty, F^b g)_{L^2(S^2)} \\
&= 4\pi((S^b)^* \psi_\infty, g)_{L^2(S^2)}.
\end{aligned} \tag{5.26}
$$

This proves (5.24).

We now prove the denseness of the range in $H^b_{\mathrm{inc}}(D_b)$ which is a closed subspace of $L^2(D_b)$. It is sufficient to prove that $(H^b)^*$ is injective on $H^b_{\mathrm{inc}}(D_b)$. Assume that $(H^b)^* v = 0$ for some $v \in H^b_{\mathrm{inc}}(D_b)$. From (5.24), we deduce that $(S^b)^* \psi_\infty = 0$ and therefore $\psi^\infty = 0$ (since $(S^b)^*$ is invertible). From Rellich's Lemma, we infer that $\psi = 0$ in $\mathbb{R}^3 \setminus \overline{D_b}$. Then $\psi \in H_0^2(D_b)$ and, arguing as in Lemma 2.1,

$$
-\|v\|^2_{L^2(D_b)} = \int\limits_{D_b} (\Delta\psi + k^2 n_b \psi) \overline{v}\, dx = 0
$$

because $v \in H^b_{\mathrm{inc}}(D_b)$. Thus $v = 0$, and this proves the desired result. ◻

From the definitions of $F$ and $F^b$ we observe that $F^m g = u^m_\infty$, where $u^m_\infty$ is the far field pattern associated with $u^m \in H^2_{loc}(\mathbb{R}^3)$ being the unique solution to

$$
\begin{cases}
\Delta u^m + k^2 n\, u^m = k^2 (n_b - n) v \quad \text{in } \mathbb{R}^3, \\[2mm]
\displaystyle\lim_{R\to\infty} \int\limits_{|x|=R} |\partial u^m / \partial |x| - ik\, u^m|^2\ ds = 0
\end{cases} \tag{5.27}
$$

with $v = H^b g$. Define the operator $G^m : L^2(D_b) \to L^2(S^2)$ such that $G^m v = u^m_\infty$, where $u^m_\infty$ is the far field pattern $u^m \in H^2_{loc}(\mathbb{R}^3)$ satisfying (5.27). With such notation, we can factorize $F^m$ as

$$
F^m = G^m H^b. \tag{5.28}
$$

We note that, thanks to $S^b(S^b)^* = I$, the identity (5.24) can also be written as

$$\psi_\infty = \frac{1}{4\pi} S^b (H^b)^*,$$

which clearly shows that $G^m v = \frac{1}{4\pi} S^b (H^b)^* (k^2(n - n_b)(v + u^m))$. Therefore, if we define the operator $T^m : L^2(D_b) \to L^2(D_b)$ by

$$T^m v := \frac{1}{4\pi} k^2 (n - n_b)(v + u^m), \tag{5.29}$$

where $u^m \in H^2_{loc}(\mathbb{R}^3)$ is the unique solution of (5.27), then $G = S^b(H^b)^* T^m$, which implies the factorization

$$F^m = S^b (H^b)^* T^m H^b.$$

Finally, defining the operators $\mathcal{F} : L^2(S^2) \to L^2(S^2)$ and $\mathcal{S} : L^2(S^2) \to L^2(S^2)$ such that

$$\mathcal{F} := (S^b)^* F^m \qquad \text{and} \qquad \mathcal{S} := I + \frac{ik}{2\pi} \mathcal{F}, \tag{5.30}$$

we have the following statement.

**Proposition 5.11.** *The operator $\mathcal{F} : L^2(S^2) \to L^2(S^2)$ admits the factorization*

$$\mathcal{F} = (H^b)^* T^m H^b, \tag{5.31}$$

*where $H^b : L^2(S^2) \to L^2(D_b)$ and $T^m : L^2(D_b) \to L^2(D_b)$ are, respectively, defined as in* (5.23) *and* (5.29). *The operator $\mathcal{F}$ is normal and $\mathcal{S}$ is unitary. If $k$ is not a modified transmission eigenvalue, then $\mathcal{F}$ is injective and has dense range in $L^2(S^2)$.*

***Proof.*** We have

$$\mathcal{F} = (S^b)^*(F - F^b) = 2\pi(ik)^{-1}(S^b)^*(S - S^b) = 2\pi(ik)^{-1}((S^b)^* S - I)$$

and therefore

$$\mathcal{S} = (S^b)^* S.$$

Since $S$ and $S^b$ are unitary, we deduce that $\mathcal{S}$ is unitary. This is enough to guarantee that $\mathcal{F}$ is normal. The injectivity and denseness of the range of $\mathcal{F}$ when $k$ is not a modified transmission eigenvalue follow from the same properties satisfied by $F^m$ (the proof of which is the same as for $F$). □

We also observe that, following along the same lines as the proof of Lemma 2.25, one can show that if $n - n_b \geq \alpha > 0$ (respectively, $n_b - n \geq \alpha > 0$) in $D_b$ for some constant $\alpha > 0$ and if $k > 0$ is not a modified transmission eigenvalue, then the operator $T^m : L^2(D_b) \to L^2(D_b)$ defined by (5.29) (respectively, $-T^m$) satisfies Assumption 2.3 with $Y = Y^* = L^2(D_b)$.

**Remark 5.12.** *When we take $n_b = 1$ in $\mathbb{R}^3$, we have $F^b = 0$ and so $S^b = I$. In this case,* (5.31) *is nothing but the factorization of the operator $F$* (2.18).

For the identification of transmission eigenvalues, one already has sufficient ingredients to prove the equivalent of Theorem 5.1 with $F$ replaced by $\mathcal{F}$. For later use, we formulate the equivalent of Theorem 5.2 for GLSM that directly uses the operator $F^m$. The proof is based on Lemma 5.10 and (5.28) and is left to the reader.

**Theorem 5.13.** *Define*

$$J_\alpha(\phi_z; g) := \alpha \|H^b g\|^2_{L^2(D_b)} + \|F^m g - \phi_z\|^2_{L^2(S^2)},$$

*and let $g_z^\alpha$ be defined as in* (5.8). *Assume that $n - n_b \geq \alpha > 0$ or $n_b - n \geq \alpha > 0$ in a neighborhood of $\partial D_b$. Then for any ball $B \subset D$, $\|H^b g_z^\alpha\|_{L^2(D_b)}$ is bounded and $\|F g_z^\alpha - \phi_z\|_{L^2(S^2)} \to 0$ as $\alpha \to 0$ for almost every $z \in B$ if and only if $k$ is not a modified transmission eigenvalue of* (5.19).

## 5.2.2 ▪ The Inside-Outside Criterion for Modified Background

Now we study the behavior of the eigenvalues of $\mathcal{F}_k := \mathcal{F}$ as $k$ approaches a modified transmission eigenvalue $k_0$ of (5.19). As previously, we shall explicitly indicate in the notation the dependence on $k$ by adding a subscript. Given the discussion above, it is clear that the analysis of Section 5.1.3 straightforwardly generalizes to the operator $\mathcal{F}_k$, and we summarize the result in the following theorem. Moreover, in order to include the case of $n_b$ depending on $k$ as in (5.33), we make the additional assumption that the mapping $k \mapsto n_b(k)$ is continuously differentiable from $\mathbb{R}_+^*$ to $L^\infty(D_b)$, which does not affect the arguments of the proofs. We similarly introduce the orthonormal basis $(g_j(k))_{j=1,+\infty}$ of $L^2(S^2)$ such that

$$\mathcal{F}_k g_j(k) = \lambda_j(k) g_j(k), \tag{5.32}$$

where $\lambda_j(k) \neq 0$ form a sequence of complex numbers that goes to 0 as $j \to \infty$ and define $\hat{\lambda}_j(k) := \lambda_j(k)/|\lambda_j(k)|$ and $\delta_j(k)$ such that

$$\lambda_j(k) = \frac{2\pi}{ik}(e^{i\delta_j(k)} - 1), \quad 0 < \delta_j(k) < 2\pi,$$

so that $e^{i\delta_j(k)}$ are eigenvalues of the scattering operator $S_k$. These sequences are ordered such that

$$\begin{aligned} &2\pi > \delta_1(k) \geq \delta_2(k) \geq \cdots \geq \delta_j(k) \geq \cdots > 0 && \text{when } n - n_b(k) \geq \alpha > 0, \\ &0 < \delta_1(k) \leq \delta_2(k) \leq \cdots \leq \delta_j(k) \leq \cdots < 2\pi && \text{when } n_b(k) - n \geq \alpha > 0 \end{aligned}$$

for $k$ not being a modified transmission eigenvalue.

**Theorem 5.14.** *Let $k_0 > 0$ and $(k_\ell)$ be a sequence of positive numbers converging to $k_0$ as $\ell \to \infty$. Assume that $n - n_b(k_\ell) \geq \alpha > 0$ (respectively, $n_b(k_\ell) - n \geq \alpha > 0$) in $D_b$ for some constant $\alpha$. If the sequence $\delta_1(k_\ell) \to 2\pi$ or equivalently $\hat{\lambda}^\ell := \hat{\lambda}_1(k_\ell) \to -1$ (respectively, $\delta_1(k_\ell) \to 0$ or equivalently $\hat{\lambda}^\ell \to +1$) as $\ell \to \infty$, then $k_0$ is a modified transmission eigenvalue of* (5.19). *Moreover, the sequence $(w_b^j)$, with*

$$w_b^j := \frac{H^b_{k_j} g_1(k_j)}{\|H^b_{k_j} g_1(k_j)\|_{L^2(D_b)}},$$

*where $g_1$ is defined in* (5.32), *admits a subsequence which converges strongly to $w_b \in L^2(D_b)$, where $(w, w_b)$ is an eigenpair of* (5.19) *associated with $k_0$.*

### Artificial Backgrounds Leading to a Necessary Condition

As suggested in [10], in order to prove the converse of the previous theorem, we consider a specific choice of $n_b(k)$ given by

$$n_b = n_b(k) = \rho/k^2 \quad \text{in } D_b, \tag{5.33}$$

where $\rho \in \mathbb{R}$ is a constant independent of $k$ (this includes in particular the case $n_b = 0$ discussed earlier). The main point of this choice is that it leads to a space of incident fields

$$H_{\rm inc}(D_b) = \{v \in L^2(D_b);\ \Delta v + \rho v = 0 \text{ in } D_b\}$$

which is independent of $k$. As a consequence, a min-max criterion for transmission eigenvalues is derived on a Hilbert space independent of $k$ and a necessary and sufficient condition can be easily obtained. The first ingredient of the proof is the following expansion of $T_k^m$ at modified transmission eigenvalues.

**Proposition 5.15.** *Assume that $n_b$ satisfies* (5.33)*, and let $k_0 > 0$ be a modified transmission eigenvalue of* (5.19)*. Denote by $(w_0, v_0) \in L^2(D_b) \times L^2(D_b)$ an associated eigenpair with $v_0 \in H_{\rm inc}(D_b)$. Then there is $\varepsilon > 0$ such that we have the expansion*

$$4\pi(T_k^m v_0, v_0)_{L^2(D_b)} = 2k_0(k-k_0)(nw_0, w_0)_{L^2(D_b)} + (k-k_0)^2\eta(k), \tag{5.34}$$

*where the remainder $\eta(k)$ satisfies $|\eta(k)| \leq C\|v_0\|^2_{L^2(D_b)}$ with $C > 0$ independent of $k \in [k_0 - \varepsilon; k_0 + \varepsilon]$.*

***Proof.*** According to the definition of $T_k^m$ in (5.29), we have

$$4\pi(T_k^m v_0, v_0)_{L^2(D_b)} = k^2((n - n_b(k))(v_0 + u_k^m), v_0)_{L^2(D_b)}, \tag{5.35}$$

where $u_k^m$ is the solution of (5.27) with $v = v_0$. We remark that, according to the definition of modified transmission eigenvalues, the solution $u_0^m$ of (5.27) with $v = v_0$ and $k = k_0$ is such that $u_0^m = w_0 - v_0$ in $D_b$ and $u_0^m = 0$ outside $D_b$. We first need to compute the derivative $(u_0^m)'$ of $u_k^m$ at $k = k_0$. To proceed, we prove an expansion as $k \to k_0$ of the form

$$u_k^m - u_0^m = (k-k_0)(u_0^m)' + (k-k_0)^2\tilde{u}_k^m, \tag{5.36}$$

where $(u_0^m)'$ is independent of $k$ and where $\tilde{u}_k^m$ has a bounded norm as $k \to k_0$. Writing (5.27) in a variational form as in (1.55) we get, with $B_R$ a ball containing $D_b$ and $\Lambda_k$ the Dirichlet-to-Neumann map introduced in Definition 1.37,

$$(\nabla u_k^m, \nabla\varphi)_{L^2(B_R)} - k^2(nu_k^m, \varphi)_{L^2(B_R)} - \langle\Lambda_k u_k^m, \varphi\rangle = ((k^2 n - \rho)v_0, \varphi)_{L^2(D_b)},$$
$$(\nabla u_0^m, \nabla\varphi)_{L^2(B_R)} - k_0^2(nu_0^m, \varphi)_{L^2(B_R)} - \langle\Lambda_k u_0^m, \varphi\rangle = ((k_0^2 n - \rho)v_0, \varphi)_{L^2(D_b)}$$

for all $\varphi \in H^1(B_R)$. Note that we used that $u_0^m = 0$ outside $D_b$ to replace $\Lambda_{k_0}$ with $\Lambda_k$ in the second equation. Define $(u_0^m)' \in H^2_{loc}(\mathbb{R}^3)$ to be the solution of

$$(\nabla(u_0^m)', \nabla\varphi)_{L^2(B_R)} - k_0^2(n(u_0^m)', \varphi)_{L^2(B_R)} - \langle\Lambda_{k_0}(u_0^m)', \varphi\rangle = 2k_0(nw_0, \varphi)_{L^2(D_b)}$$

for all $\varphi \in H^1(B_R)$. Then one obtains that $\tilde{u}_k^m \in H^2_{loc}(\mathbb{R}^3)$ defined by (5.36) satisfies

$$\begin{aligned}(\nabla\tilde{u}_k^m, \nabla\varphi)_{L^2(B_R))} &- k^2(n\tilde{u}_k^m, \varphi)_{L^2(B_R))} - \langle\Lambda_k\tilde{u}_k^m, \varphi\rangle \\ &= (nw_0 + (k+k_0)(u_0^m)', \varphi)_{L^2(D_b)} - \left\langle \frac{\Lambda_k - \Lambda_{k_0}}{k-k_0}(u_0^m)', \varphi\right\rangle\end{aligned}$$

for all $\varphi \in H^1(B_R)$. Using that the mapping $k \mapsto \Lambda_k$ is real analytic from $\mathbb{R}_+^*$ into $\mathcal{L}(H^{1/2}(\partial B_R), H^{-1/2}(\partial B_R))$, we obtain (using the uniform bounds for scattering problems as in Lemma 5.8) that $\|\tilde{u}_k^m\|_{H^2(D_b)} \leq C\|v\|_{L^2(D_b)}$ for some constant $C$ independent of $k \in I$, $I$ being a given compact set of $\mathbb{R}_+^*$. Now inserting the expansion (5.36) into (5.35) and using that $(T_{k_0}^m v_0, v_0)_{L^2(D_b)} = 0$, we get

$$\begin{aligned}\frac{4\pi}{k-k_0}(T_k^m v_0, v_0)_{L^2(D_b)} &= 2k_0(nw_0, v_0)_{L^2(D_b)} \\ &+((k_0^2 n-\rho)(u_0^m)', v_0)_{L^2(D_b)} + (k-k_0)\eta(k)\end{aligned} \tag{5.37}$$

with

$$\eta(k) = \big((k^2 n-\rho)\tilde{u}_k^m + (k+k_0)n(u_0^m)' + nw_0, v_0\big)_{L^2(D_b)}.$$

Obviously, $\eta(k)$ satisfies the uniform estimate indicated in the proposition. We recall that $u_0^m \in H_0^2(D_b)$ and satisfies

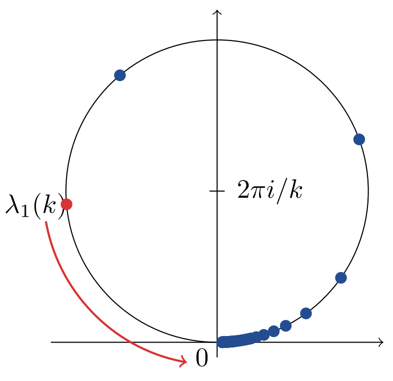
$$\Delta u_0^m + k_0^2 n u_0^m = -(k_0^2 n - \rho)v_0. \tag{5.38}$$

This allows us to write, since $n$ and $\rho$ are real,

$$\begin{aligned}((k_0^2 n-\rho)(u_0^m)', v_0)_{L^2(D_b)} &= ((u_0^m)', (k_0^2 n-\rho)v_0)_{L^2(D_b)} \\ &= -((u_0^m)', \Delta u_0^m + k_0^2 n u_0^m)_{L^2(D_b)} = -(\Delta(u_0^m)' + k_0^2 n(u_0^m)', u_0^m)_{L^2(D_b)} \\ &= 2k_0(nw_0, u_0^m)_{L^2(D_b)}.\end{aligned}$$

Substituting the latter identity into (5.37) gives the desired expansion (5.34). □

We now can state and prove the converse of Theorem 5.14 (illustrated by Figure 5.4).



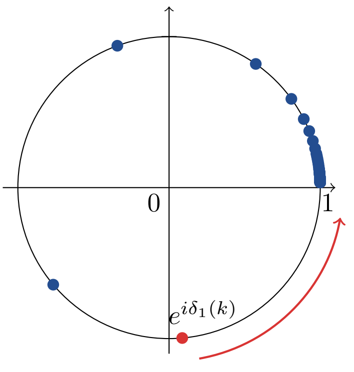




**Figure 5.4.** *Illustration of the behaviors of $k \mapsto \lambda_1(k)$ (left) and $k \mapsto e^{i\delta_1(k)}$ (right) when $k \to k_0$ in the situation where $n - n_b(k_0) \geq \alpha > 0$ in $D_b$ with $n_b(k) = \rho/k^2$.*

**Theorem 5.16.** *Assume that $n_b$ satisfies* (5.33)*, and let $k_0 > 0$ be a modified transmission eigenvalue of* (5.19)*. Assume that $n - n_b(k_0) \geq \alpha > 0$ (respectively, $n_b(k_0) - n \geq \alpha > 0$) in $D_b$ for some constant $\alpha$. Then $\delta_1(k) \to 2\pi$ or equivalently $\hat{\lambda}_1(k) \to -1$ as $k \to k_0$ and $k < k_0$ (respectively, $\delta_1(k) \to 0$ or equivalently $\hat{\lambda}_1(k) \to +1$) as $k \to k_0$ and $k > k_0$.*

***Proof.*** The proof uses the abstract result of Lemma 5.17 below. Let us consider the case $n - n_b(k_0) \geq \alpha > 0$ in $D_b$. Let $(w_0, v_0) \in L^2(D_b) \times L^2(D_b)$ be a corresponding eigenpair. According to Lemma 5.17 and the discussion of Section 5.2.1, we have

$$\cot\frac{\delta_1(k)}{2} = \inf_{\varphi \in H_{\mathrm{inc}}(D_b)} \frac{\Re\,(T_k^m \varphi, \varphi)_{L^2(D_b)}}{\Im\,(T_k^m \varphi, \varphi)_{L^2(D_b)}} \leq \frac{\Re\,(T_k^m v_0, v_0)_{L^2(D_b)}}{\Im\,(T_k^m v_0, v_0)_{L^2(D_b)}}.$$

From (2.35) we have that $\Im\,(T_k^m v_0, v_0)_{L^2(D_b)} > 0$ when $k$ is not a modified transmission eigenvalue. Using the expansion (5.34), we deduce that $\Im\,\eta(k) > 0$ for $k$ approaching $k_0$ and

$$\frac{\Re\,(T_k^m v_0, v_0)_{L^2(D_b)}}{\Im\,(T_k^m v_0, v_0)_{L^2(D_b)}} = \frac{2k_0(nw_0, w_0)_{L^2(D_b)} + (k-k_0)\,\Re(\eta(k))}{(k-k_0)\,\Im(\eta(k))} \to -\infty \tag{5.39}$$

as $k \to k_0$ and $k < k_0$. Note that we have $w_0 \not\equiv 0$ in $D_b$; otherwise $u_0^m = w_0 - v_0 \in H_0^2(D_b)$ would vanish and yield $(w_0, v_0) = (0,0)$. This proves that cot $\frac{\delta_1(k)}{2} \to -\infty$ and equivalently $\delta_1(k) = 2\pi$ as $k \to k_0$ and $k < k_0$. The case $n_b(k_0) - n \geq \alpha > 0$ in $D_b$ can be proved similarly by replacing $T_k^m$ by $-T_k^m$. ∎

For the proof of Theorem 5.16 we have used the following abstract result first shown in [83]. We follow here the presentation in [115].

**Lemma 5.17.** *Let $X$ be an infinite separable Hilbert space and $Y$ a reflexive Banach space. Let $F : X \to X$ be compact injective with dense range such that, $F = H^*TH$ with $H : X \to Y$ being injective and $T : Y \to Y^*$ satisfying Assumption* 2.3. *Assume in addition that $S = I + iF$ is unitary. Denote by $e^{i\delta_j}$, $j \in \mathbb{N}$, the eigenvalues of $S$ with $0 < \delta_j < 2\pi$ and $\delta_\star = \sup_j \delta_j$. Then*

$$\cot\frac{\delta_\star}{2} = \inf_{\varphi \in H_{\rm inc}} \frac{\Re\,\langle T\varphi,\,\varphi\rangle}{\Im\,\langle T\varphi,\,\varphi\rangle} \tag{5.40}$$

*with $\langle\cdot,\,\cdot\rangle$ denoting the $Y^* - Y$ duality product and where $H_{\rm inc}$ is the closure of the range of $H$.*

***Proof.*** We remark that since $F$ is injective, $1$ is not an eigenvalue of $S$, and by assumption the range of $F$ denoted $R(F)$ is dense in $X$. Denote by $\mathfrak{S}$ the Cayley transform of $S$ defined by

$$\mathfrak{S} := i(I+\mathcal{S})(I-\mathcal{S})^{-1} : R(F) \subset X \to X.$$

The operator $\mathfrak{S}$ is self-adjoint and its spectrum is discrete. Moreover, $e^{i\delta_j}$ is an eigenvalue of $S$ if and only if $-\cot(\delta_j/2) \in \mathbb{R}$ is an eigenvalue of $\mathfrak{S}$. Since $T$ satisfies Assumption 2.3, then $-\cot(\delta_\star/2)$ is the largest eigenvalue of $\mathfrak{S}$ (see Proposition 5.4). We then can apply the Courant–Fischer inf-sup principle to the self-adjoint operator $\mathfrak{S}$ and get

$$\begin{aligned}
-\cot\frac{\delta_\star}{2} &= \sup_{f\in R(F)} \frac{(\mathfrak{S}f, f)_X}{\|f\|_X^2} = \sup_{f\in R(F)} \frac{(i(I+\mathcal{S})(I-\mathcal{S})^{-1}f, f)_X}{\|f\|_X^2} \\
&= \sup_{g\in X} \frac{(i(I+\mathcal{S})g, (I-\mathcal{S})g)_X}{\|(I-\mathcal{S})g\|_X^2} \\
&= \sup_{g\in X} \frac{i(\|g\|_X^2 + 2i\Im\,(\mathcal{S}g, g)_X - \|\mathcal{S}g\|_X^2)}{\|g\|_X^2 - 2\Re\,(\mathcal{S}g, g)_X + \|\mathcal{S}g\|_X^2} \\
&= \sup_{g\in X} \frac{\Im\,(\mathcal{S}g, g)_X}{\Re\,(\mathcal{S}g, g)_X - \|g\|_X^2}.
\end{aligned}$$

Using the facts that $S = I + iF$ and that $F = H^*TH$ we obtain

$$-\cot\frac{\delta_\star}{2} = \sup_{g\in X} \frac{\Re\,(Fg, g)_X}{-\Im\,(Fg, g)_X} = \sup_{g\in X} \frac{\Re\,\langle THg,\,Hg\rangle}{-\Im\,\langle THg,\,Hg\rangle} = \sup_{\varphi\in H_{inc}} \frac{\Re\,\langle T\varphi,\,\varphi\rangle}{-\Im\,\langle T\varphi,\,\varphi\rangle}.$$

This proves (5.40). ∎

## 5.3 ▪ Other Modified Backgrounds and Some Applications

The use of modified backgrounds has attracted great interest for application purposes since it can simplify the analysis of the connection between transmission eigenvalues and the probed medium properties. This has already been discussed above for the case of zero index materials (see (5.20)–(5.21)). We hereafter give other possibilities that have been suggested in [9] with an application for the construction of an indicator function for "crack densities" [11]. We then present another point of view also relevant for applications, where the frequency is fixed and a spectral parameter is encoded in the definitions of the modified background [8], [39], [61], [62], [63]. Throughout this section we always assume that $n$ is real.

### 5.3.1 ▪ A Background with an Artificial Obstacle

Instead of zero index materials, one can use a background medium containing an artificial obstacle. More specifically we shall consider a background where $D_b$ is an obstacle with prescribed boundary conditions on $\partial D_b$. The system (5.16) is then replaced by $u_b^s \in H^1_{loc}(\mathbb{R}^3 \setminus D_b)$, the unique solution to

$$\begin{cases} \Delta u_b^s + k^2 u_b^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D_b}, \\ B(u_b^s) = -B(u^i) \quad \text{on } \partial D_b, \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} \left|\partial u_b^s/\partial |x| - ik\, u_b^s\right|^2 \, ds = 0. \end{cases} \tag{5.41}$$

The boundary operator $B$ is designed such that problem (5.41) is well-posed, for instance, $B(u) = u$ or $B(u) = \partial u/\partial\nu + \lambda u$, which respectively correspond to Dirichlet and Robin scattering problems, with $\lambda \in \mathbb{R}$ being a fixed impedance parameter. The well-posedness can be easily shown following the variational procedure presented in Section 1.4 for any incident field such that $B(u^i) \in H^{1/2}(\partial D_b)$ for the Dirichlet case and $B(u^i) \in H^{-1/2}(\partial D_b)$ for the Robin problem.

The modified transmission eigenvalues are defined as the values of $k > 0$ for which there exists a nontrivial $u^i \in H^1(D_b)$ satisfying $\Delta u^i + k^2 u^i = 0$ in $D_b$ and such that the corresponding scattered fields $u^s$ and $u_b^s$ respectively defined by (1.27)–(1.29) and (5.16), are such that $u^s = u_b^s$ in $\mathbb{R}^3 \setminus \overline{D_b}$. Setting $w := u^s + u^i|_{D_b}$ we obtain

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D_b, \\ B(w) = 0 & \text{on } \partial D_b, \end{cases} \tag{5.42}$$

and modified transmission eigenvalues can be equivalently defined as $k > 0$ for which (5.42) admits a nontrivial solution $w \in H^1(D_b)$. Similarly to (5.20), we obtain a simpler spectral problem than (3.2), associated to a linear self-adjoint operator made of isolated eigenvalues of finite multiplicity $0 < k_1^2 \le k_2^2 \le \cdots \le k_p^2 \le \cdots$ (the numbering is chosen so that each eigenvalue is repeated according to its multiplicity). The *min-max* formula also holds. For instance, in the case of Dirichlet boundary conditions ($B(u) = u$),

$$k_p^2 = \min_{E_p \in \mathcal{E}_p} \max_{u \in E_p \setminus \{0\}} \frac{(\nabla u, \nabla u)_{L^2(D_b)^3}}{(nu, u)_{L^2(D_b)}},$$

where $\mathcal{E}_p$ denotes the sets of subspaces $E_p$ of $H_0^1(D_b)$ of dimension $p$. The transmission eigenvalues can be identified from $F^m$ using GLSM in the same way as in Theorem 5.18 below with $P(g) = \|\mathcal{H}g\|^2_{H^1(D_b)}$.

### Application to Imaging of Cracks Network

The idea of introducing an artificial background with obstacles satisfying a Dirichlet boundary condition has been used to design an indicator function for the density of cracks in a highly fractured media [11]. Consider, for example, a network of cracks supported by surfaces $\Gamma \subset \partial D$ such that $\mathbb{R}^3 \setminus \Gamma$ is connected and $\partial D$ is the boundary of a smooth domain $D$. Assume for simplicity that the refractive index of the medium is $n = 1$. The scattering problem for cracks supported by $\Gamma$ can be modeled by $u^s \in H^1(\mathbb{R} \setminus \overline{\Gamma})$ such that

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Gamma}, \\ \partial u^s/\partial \nu = -\partial u^i/\partial \nu \quad \text{on } \Gamma, \\ \lim\limits_{R\to\infty} \displaystyle\int\limits_{|x|=R} \left| \partial u^s/\partial |x| - ik\, u^s \right|^2 \, ds = 0 \end{cases} \tag{5.43}$$

for some incident field $u^i$, where $\nu$ denotes an outward normal field defined on $\partial D$. We notice that we implicitly assume in (5.43) that the normal trace $\partial u^s/\partial \nu$ is the same on both sides of $\Gamma$. Using the variational procedure presented in Section 1.4, one can show by replacing the variational space $H^1(B_R)$ with $H^1(B_R \setminus \overline{\Gamma})$ that this problem is well-posed for $\partial u^i/\partial \nu \in H^{-1/2}(\partial D)$. Consider $u^s(\cdot, d)$ the scattered field associated with the incident plane wave $e^{ikd\cdot x}$ with $d \in S^2$ and denote by $u_\infty(\cdot, d)$ the associated far field. The far field operator $F$ associated with $u_\infty(\cdot, d)$ is defined by (5.1). The LSM and factorization methods have been studied for cracks in [25]. These methods provide satisfactory reconstructions in the case of well-separated cracks. However, in the case of multiple cracks that form a dense network, these methods only identify a domain that contains all the cracks (see Figure 5.6, right). This fact is expected to be the case for any inversion method that aims at reconstructing the exact shape of the cracks.

An alternative idea pursued in [11] is to exploit the monotonicity of some modified transmission eigenvalues with respect to the crack size in order to build an indicator function of the crack density (without reconstructing the exact shape). Consider the far field operator $F^b$ associated with the background media defined in equation (5.41) for some domain $D_b$ that does not necessarily contain (all of) $\Gamma$ in its interior. The modified transmission eigenvalues are defined similarly as above, being the values of $k > 0$ for which there exists a nontrivial $u^i \in H^1(D_b)$ satisfying $\Delta u^i + k^2 u^i = 0$ in $D_b$ and such that the corresponding far fields coincide. Rellich's Lemma and the unique continuation principle imply that $u^s$ and $u^s_b$, respectively defined by (5.43) and (5.16), are such that $u^s = u^s_b$ in $\mathbb{R}^3 \setminus \{\overline{D_b} \cup \overline{\Gamma}\}$. These modified transmission eigenvalues then correspond to the case when for $k > 0$ there exists a nontrivial solution $w \in H^1(D_b \setminus \Gamma)$ of

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } D_b, \\ \partial w/\partial \nu = 0 & \text{on } \Gamma \cap D_b, \\ B(w) = 0 & \text{on } \partial D_b. \end{cases} \tag{5.44}$$

Again, the *min-max* formula holds and for the Dirichlet boundary condition ($B(u) = u$), with $\Gamma_b := \Gamma \cap D_b$, we have

$$k_p^2(\Gamma_b) = \min_{E_p \in \mathcal{E}_p(\Gamma_b)} \max_{u \in E_p \setminus \{0\}} \frac{(\nabla u, \nabla u)_{L^2(D_b)^3}}{(u, u)_{L^2(D_b)}}, \tag{5.45}$$

where $\mathcal{E}_p(\Gamma_b)$ denotes the sets of subspaces $E_p$ of $H^1_0(D_b \setminus \Gamma_b)$ of dimension $p$. Since $H^1_0(D_b \setminus \Gamma_b) \subset H^1_0(D_b \setminus \tilde{\Gamma}_b)$ if $\Gamma_b \subset \tilde{\Gamma}_b$, we clearly obtain the monotonicity property

$$k_p^2(\Gamma_b) \geq k_p^2(\tilde{\Gamma}_b) \quad \text{if } \Gamma_b \subset \tilde{\Gamma}_b, \tag{5.46}$$

i.e., the modified transmission eigenvalues decrease with respect to the size of the crack. In order to determine these modified transmission eigenvalues, one can rely on the GLSM method by considering

$$J_\alpha(\phi_z; g) := \alpha P(g) + \|F^m g - \phi_z\|^2_{L^2(S^2)} \tag{5.47}$$

with

$$P(g) := \|\mathcal{H}g\|^2_{H^1(D_b)} + (F_\sharp g, g)_{L^2(S^2)}$$

and $F_\sharp$ defined by (2.49). From our previous analysis, we arrive at the following result.

**Theorem 5.18.** *Define $g_z^\alpha$ as in* (5.8) *with $J_\alpha$ given by* (5.47). *Assume that $g \mapsto v_g|_\Gamma$ is injective where $v_g$ is defined by* (1.31). *Then for any ball $B \subset D_b$, $P(g_z^\alpha)$ is bounded and $\|F^m g_z^\alpha - \phi_z\|_{L^2(S^2)} \to 0$ as $\alpha \to 0$ for almost every $z \in B$ if and only if $k$ is not a modified transmission eigenvalue associated with* (5.44).

### Indicator Function For Cracks Density

Theorem 5.18 and the monotonicity property (5.46) suggest the following procedure to build an indicator function sensitive to the local density of cracks in a highly fractured media. Let $R_0 > 0$ be chosen and denote by $k_p^0$ the Dirichlet eigenvalues defined by (5.45) for $\Gamma_b = \emptyset$ and $D_b = B_{R_0}$, the ball of radius $R_0$ centered at the origin. These eigenvalues coincide with the zeros of $x \mapsto j_n(xR_0)$, where $j_n$ is a spherical Bessel function. For $z \in \mathbb{R}^3$ we let $k_p(z)$ be the modified transmission eigenvalue defined by (5.45) with $D_b = D_b(z) := z + B_{R_0}$. The indicator function is then defined as, for some $N > 0$,

$$I(z) := \sum_{p=1}^{N} |k_p(z) - k_p^0|. \tag{5.48}$$

One therefore expects, according to (5.46), that $I(z)$ is monotonically increasing with respect to the size of cracks intersecting $D_b(z)$ as the sampling point $z$ varies inside the probed domain.

In practice, the values of $k_p(z)$ are evaluated from measurements based on Theorem 5.18. We refer the reader to [11] for a discussion of practical implementations in a 2-dimensional setting. The parameters $N$ and $R_0$ are related to the available frequency range in the measurements that should at least include $k_1^0$. Limiting the procedure to $N = 1$ turned out to be sufficient to obtain meaningful results as shown by Figures 5.5–5.6. In

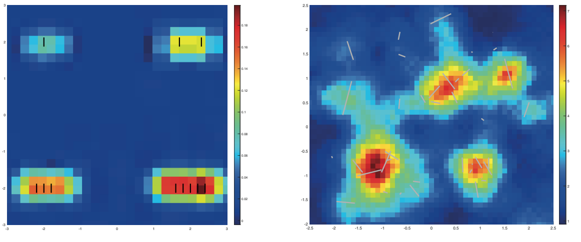


**Figure 5.5.** *Behavior of the indicator function* (5.48) *with $N = 1$. Left:* 11 *vertical cracks of length* 0.25 *arranged in* 4 *areas with different damage levels and reconstruction using $R_0 = 0.25$. Right:* 40 *cracks of different lengths arranged randomly and reconstruction using $R_0 = 0.1$. The data are corrupted with* 1% *noise. Reproduced from* [11] *with permission.*

this case, since $k_1^0 = \pi/R_0$ (in 3 dimensions), we see that $R_0 \geq \pi/k_{\max}$, where $k_{\max}$ denotes the maximum of the frequencies available in the measurements. Figure 5.5 illustrates a typical behavior of the indicator function for different configurations of the cracks. Figure 5.6 shows a comparison with the reconstruction provided by the $F_\sharp$ method for Neumann cracks as suggested in [25]. One clearly observes that the new indicator function is more representative of the local crack density of the medium.

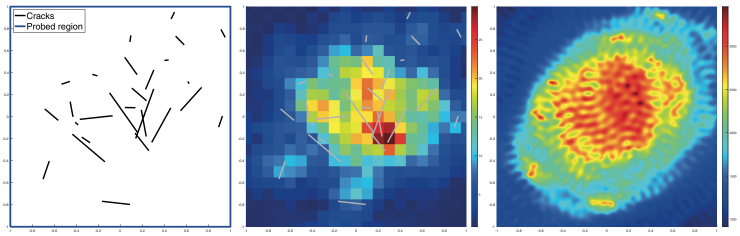



**Figure 5.6.** *Left: exact cracks configuration. Middle: indicator function* (5.48) *with* $N = 1$ *and* $R_0 = 0.1$. *Right: reconstruction provided by the* $F_\sharp$ *method. The data are corrupted with* $1\%$ *noise. Reproduced from* [11] *with permission.*

## 5.3.2 ▪ Spectral Parameters Encoded into the Background

The main drawback of using the indicator function (5.48) is the need for multifrequency measurements in an interval of frequencies that contains modified transmission eigenvalues. One can define a similar indicator function that can be used for a single frequency measurement by using a different spectral parameter. The latter is chosen in the definition of the background, for instance, the parameter $\lambda$ in the Robin boundary conditions of (5.42) or the value of $n_b|_{D_b}$ in the definition of problem (5.19). These two choices are discussed below.

### Modified Transmission Eigenvalues of Steklov Type

We define the modified transmission eigenvalues of Steklov type as the values of $\lambda$ for which problem (5.42), with $B(u) = \partial u/\partial\nu + \lambda u$, has a nontrivial solution $w \in H^1(D_b)$, the wave number $k$ being fixed. This eigenvalue problem can be written variationally as

$$\int_{D_b} \nabla w \cdot \nabla \overline{w}' \, dx - k^2 \int_{D_b} n \, w\overline{w}' \, dx = -\lambda \int_{\partial D_b} w\overline{w}' \, ds \qquad \text{for all} \quad w' \in H^1(D_b). \tag{5.49}$$

If $k^2$ is not a Robin eigenvalue, i.e., an eigenvalue of

$$\Delta w + k^2 n w = 0 \ \text{ in } D_b, \qquad \frac{\partial w}{\partial \nu} + \alpha w = 0 \ \text{ on } \partial D_b, \tag{5.50}$$

where $0 \leq \alpha$ is fixed ($\alpha = 0$ corresponds to a Neumann eigenvalue), we define the interior self-adjoint Robin-to-Dirichlet operator $R : L^2(\partial D_b) \to L^2(\partial D_b)$ by

$$R : \theta \mapsto w_\theta|_{\partial D_b},$$

where $w_\theta \in H^1(D)$ is the unique solution to

$$\int_{D_b} \nabla w_\theta \cdot \nabla \overline{w}' \, dx + \alpha \int_{\partial D_b} w_\theta \overline{w}' - k^2 \int_{D_b} n \, w_\theta \overline{w}' \, dx = \int_{\partial D_b} \theta \overline{w}' \, ds \qquad \text{for all } w' \in H^1(D_b).$$

The fact that $w_\theta|_{\partial D_b} \in H^{1/2}(\partial D_b)$ implies that $R : L^2(\partial D_b) \to L^2(\partial D_b)$ is compact. Then $\lambda$ is a Steklov eigenvalue if and only if

$$(-\lambda + \alpha)\, R\theta = \theta.$$

Note that from the analytic Fredholm theory (Theorem 1.12), a given $k^2$ cannot be a Robin eigenvalue for all $\alpha \geq 0$. Thus, choosing $\alpha$ appropriately, we have proven that for any fixed wave number $k > 0$ there exists an infinite set of Steklov eigenvalues and all the eigenvalues $\lambda_j$ are real without finite accumulation point. In the following lemma we actually show that they accumulate only at $-\infty$. To this end, let $(\cdot,\cdot)$ denote the $L^2(D_b)$-inner product and $\langle\cdot,\cdot\rangle$ the $L^2(\partial D_b)$-inner product.

**Assumption 5.1.** *The wave number $k > 0$ is such that $\eta := k^2$ is not a Dirichlet eigenvalue of the problem, $w \in H^1(D_b)$,*

$$\Delta w + \eta n w = 0 \quad \text{in } \; D_b, \qquad w = 0 \quad \text{on } \; \partial D_b. \tag{5.51}$$

**Theorem 5.19.** *For real $n$ and fixed $k > 0$ there exists at least one positive Steklov eigenvalue. If in addition $k > 0$ satisfies Assumption 5.1, then there are at most finitely many positive Steklov eigenvalues.*

***Proof.*** We assume to the contrary that all eigenvalues satisfy $\lambda_j \leq 0$. This means that

$$\int_{D_b} \nabla w \cdot \nabla \overline{w}\, dx - k^2 \int_{D_b} n\, |w|^2\, ds \geq 0$$

for all $w \in H^1(D_b)$ since the Steklov eigenfunctions form a Riesz basis for $H^1(D_b)$. Now taking $w = 1$ yields a contradiction, which proves the first statement.

Next we assume by contradiction that there exists a sequence of positive Steklov eigenvalues $\lambda_j > 0$, $j \in \mathbb{N}$, converging to $+\infty$ with eigenfunction $w_j$ normalized such that

$$\|w_j\|_{H^1(D_b)} + \|w_j\|_{L^2(\partial D_b)} = 1. \tag{5.52}$$

Then from

$$(\nabla w_j, \nabla w_j) - k^2\,(n w_j, w_j) = -\lambda_j\,\langle w_j, w_j\rangle\,, \tag{5.53}$$

since the left-hand side is bounded, we obtain that $w_j \to 0$ in $L^2(\partial D_b)$. Next, up to a subsequence, $w_j$ converges weakly in $H^1(D_b)$ to some $w \in H^1(D_b)$, and this weak limit satisfies $\Delta w + k^2 n w = 0$ in $D_b$. Hence from the above $w = 0$ on $\partial D_b$. Therefore, using Assumption 5.1, $w = 0$ in $D_b$. Hence, up to a subsequence, $w_j \to 0$ in $L^2(D_b)$ (strongly). From (5.53)

$$(\nabla w_j, \nabla w_j) - k^2\,(n w_j, w_j) < 0 \qquad \text{for all } j \in \mathbb{N},$$

and since the left-hand side is a bounded real sequence, we can conclude that, up to a subsequence,

$$(\nabla w_j, \nabla w_j) \to 0 \qquad \text{as } j \to \infty,$$

which implies that $\|\nabla w_j\|_{L^2(D_b)} \to 0$ in addition to $\|w_j\|_{L^2(\partial D_b)} \to 0$. This contradicts (5.52). $\square$

For the existence of Steklov eigenvalues for complex valued $C^\infty$ coefficient $n(x)$, see [39]. We now prove that we have a monotonicity property for the largest Steklov eigenvalue. We choose a positive constant $\tau > 0$ such that

$$\int_{D_b} \nabla w \cdot \nabla \overline{w}\, dx - k^2 \int_{D_b} n\, |w|^2\, dx + \tau \int_{\partial D_b} |w|^2\, ds \geq c\|w\|^2_{H^1(D_b)}, \quad c > 0. \tag{5.54}$$

The existence of such a $\tau$ is proved in the following theorem. Hence in this case our eigenvalue problem, which can be written as

$$\int_{D_b} \nabla w \cdot \nabla \overline{w}'\, dx - k^2 \int_{D_b} n\, w\overline{w}'\, dx + \tau \int_{\partial D_b} w\overline{w}'\, ds = -(\lambda - \tau) \int_{\partial D_b} w\overline{w}'\, ds, \tag{5.55}$$

becomes a generalized eigenvalue problem for a positive self-adjoint compact operator, and hence the eigenvalues $\tau - \lambda > 0$ satisfy the Courant–Fischer inf-sup principle. In particular, the largest positive Steklov eigenvalue $\lambda_1 = \lambda_1(n, k)$ satisfies

$$\lambda_1 = \sup_{w \in H^1(D_b), w \neq 0} \frac{k^2 \displaystyle\int_{D_b} n\, |w|^2\, dx - \int_{D_b} \nabla w \cdot \nabla w\, dx}{\displaystyle\int_{\partial D_b} |w|^2\, ds}, \tag{5.56}$$

whence it is monotonically increasing with respect to $n$ and monotonically decreasing with respect to $A$. The following theorem give the optimal conditions on $n$ and $k$ which ensure the coercivity property (5.54), whence the sup-condition (5.56).

**Theorem 5.20.** *Assume that $k^2 < \eta_0(n, D_b)$, where $\eta_0(n, D_b)$ is the first Dirichlet eigenvalue of* (5.51). *Then there is a $\tau > 0$ such that* (5.54) *holds. In particular, the largest positive Steklov eigenvalue satisfies* (5.56).

***Proof.*** Fix $k^2 < \eta_0(n, D_b)$ and assume to the contrary that there exists a sequence of positive constants $\tau_j = j$, $j \in \mathbb{N}$, and a sequence of functions $w_j \in H^1(D_b)$ normalized as $\|w_j\|_{H^1(D_b)} = 1$ such that

$$\int_{D_b} \nabla w_j \cdot \nabla \overline{w}_j\, dx - k^2 \int_{D_b} n\, |w_j|^2\, dx + j \int_{\partial D_b} |w_j|^2\, ds \leq 0. \tag{5.57}$$

From

$$\int_{D_b} \nabla w_j \cdot \nabla \overline{w}_j\, dx + j \int_{\partial D_b} |w_j|^2\, ds \leq k^2 \int_{D_b} n\, |w_j|^2\, dx$$

we see that $j \int_{\partial D_b} |w_j|^2\, ds$ is bounded, which implies that $w_j \to 0$ strongly in $L^2(\partial D_b)$ as $j \to +\infty$. On the other hand the boundedness implies that $w_j \rightharpoonup w$ in $H^1(D_b)$ and from the above $w = 0$ on $\partial D_b$, whence $w \in H^1_0(D_b)$. Next we have that, up to a subsequence, $w_j \to w$ strongly in $L^2(D_b)$. Since the norm of the weak limit is smaller than the lim-inf of the norm,

$$(\nabla w, \nabla w) \leq \liminf_{j \to \infty} \int_{D_b} \nabla w_j \cdot \nabla \overline{w}_j\, dx \leq \lim_{j \to \infty} k^2 \int_{D_b} n\, |w_j|^2\, dx = k^2(nw, w),$$

which contradicts the fact that

$$k^2 < \inf_{w \in H^1_0(D_b), w \neq 0} \frac{(\nabla w, w)}{(nw, w)} = \eta_0(n, D_b).$$

This ends the proof. ☐

In [8] it is shown that the largest Steklov eigenvalue blows up as $k$ approaches a Dirichlet eigenvalue defined in Assumption 5.1. More precisely we have the following theorem.

**Theorem 5.21.** *Assume that $k^2 < \eta_0(n, D_b)$, where $\eta_0(n, D_b)$ is the first Dirichlet eigenvalue of* (5.51). *Then the largest positive Steklov eigenvalue $\lambda_1 = \lambda_1(k)$ as a function of $k$ approaches $+\infty$ as $k^2 \to \eta_0(n, D_b)$.*

***Proof.*** Consider the first eigenvalue and eigenvector $(\eta_\delta, w_\delta)$, $\|w_\delta\|_{H^1(D_b)} = 1$, of the following Robin problem

$$\Delta w_\delta + \eta_\delta n w_\delta = 0 \ \text{ in } D_b, \qquad \frac{\partial w_\delta}{\partial \nu_A} + \frac{1}{\delta} w_\delta = 0 \ \text{ on } \partial D_b \tag{5.58}$$

for $\delta > 0$. If $\eta_0 := \eta_0(n, D_b)$ and $w_0$ denotes the first Dirichlet eigenvalue and eigenvector of (5.51), we notice that

$$\begin{aligned}\eta_\delta = \frac{(\nabla w_\delta, w_\delta) + \frac{1}{\delta}\langle w_\delta, w_\delta\rangle}{(n w_\delta, w_\delta)} &= \inf_{w \in H^1(D_b), w\neq 0} \frac{(\nabla w, w) + \frac{1}{\delta}\langle w, w\rangle}{(nw, w)} \\ &< \inf_{w \in H_0^1(D_b), w \neq 0} \frac{(\nabla w, w)}{(nw, w)} = \eta_0,\end{aligned}$$

i.e., $\eta_\delta < \eta_0$. Using the inf-criterion, one also easily observes that $\delta \mapsto \eta_\delta$ is decreasing, whence $\lim\limits_{\delta\to 0} \eta_\delta$ exits. On the other hand, (5.58) can be written as

$$\int_{D_b} \nabla w_\delta \cdot \nabla \overline{w}' \, dx + \frac{1}{\delta} \int_{\partial D_b} w_\delta \overline{w}' \, ds = \eta_\delta \int_{D_b} n w_\delta \overline{w}' \, dx, \tag{5.59}$$

and by taking $w' = w_\delta$ we see that $w_\delta \to 0$ strongly in $L^2(\partial D_b)$ as $\delta \to 0$. The $H^1(D_b)$-weak limit of $w_\delta$, denoted by $w$, satisfies $\Delta w + (\lim\limits_{\delta \to 0} \eta_\delta) n w = 0$ in $D_\delta$ and $w = 0$ on $\partial D_b$, which means $\lim\limits_{\delta\to 0} \eta_\delta = \eta_0$ (since $\eta_\delta < \eta_0$ and $\eta_0$ is the first Dirichlet eigenvalue) and $w = w_0$ the corresponding eigenfunction. From the compact embedding of $H^1(D_b)$ into $L^2(D_b)$ we have that (up to a subsequence) $w_\delta \to w_0$ strongly in $L^2(D_b)$. Now we consider the sequence $k_\delta^2 := \eta_\delta + \|w_\delta\|^2_{L^2(\partial D_b)} \to \eta_0$ as $\delta \to 0$. Then from (5.56)

$$\begin{aligned}\lambda_1(k_\delta) &\geq \frac{k_\delta^2 \displaystyle\int_{D_b} n\,|w_\delta|^2\,dx - \int_{D_b} \nabla w_\delta \cdot \nabla w_\delta \, dx}{\displaystyle\int_{\partial D_b} |w_\delta|^2 \, ds} \\ &= \frac{(k_\delta^2 - \eta_\delta) \displaystyle\int_{D_b} n\,|w_\delta|^2\,dx}{\displaystyle\int_{\partial D_b} |w_\delta|^2\,ds} + \frac{1}{\delta} = \int_{D_b} n\,|w_\delta|^2\,dx + \frac{1}{\delta}.\end{aligned}$$

Thus we have that

$$\lim_{\delta \to 0} \lambda_1(k_\delta) \geq \int_{D_b} n\,|w_0|^2\,dx + \lim_{\delta\to 0}\frac{1}{\delta} = +\infty,$$

which ends the proof. ∎

Indeed, the Steklov eigenvalues can be identified in the same way as modified transmission eigenvalues using the GLSM method.

**Theorem 5.22.** *Define*

$$J_\alpha(\phi_z; g) := \alpha\|\mathcal{H}g\|^2_{H^1(D_b)} + \|F^m g - \phi_z\|^2_{L^2(S^2)}\,,$$

*where $F^m = F - F^b$ and $F^b$ is the far field operator associated with the background scattering problem* (5.41)*, and let $g_z^\alpha$ be defined as in* (5.8)*. Then for any ball $B \subset D$, $\|\mathcal{H}g_z^\alpha\|_{H^1(D_b)}$ is bounded and $\|Fg_z^\alpha - \phi_z\|_{L^2(S^2)} \to 0$ as $\alpha \to 0$ for almost every $z \in B$ if and only if $\lambda$ is not a Steklov eigenvalue of* (5.49).

The application of Steklov eigenvalues to identify changes in the material properties of thin surfaces has been investigated in [53].

### Metamaterial Modified Transmission Eigenvalues

Another possibility to define a spectrum associated with the refractive index $n$ for a fixed frequency is to use a background medium as in (5.16) and set $n_b = \lambda$ in $D_b$. The resulting eigenvalue problem has a similar structure to (5.19) and therefore would inherit the same theoretical difficulties relating eigenvalues to the material properties. As suggested in [8], one can obtain an eigenvalue problem similar to the Steklov eigenvalue problem presented above by using a background containing a metamaterial in $D_b$. Given $a > 0$ as a fixed parameter such that $a \neq 1$ and $\lambda \in \mathbb{R}$, we replace (5.16) with the scattering problem for $u_b^s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D_b})$ and $u_b \in H^1(D_b)$ defined by

$$\begin{cases} \Delta u_b^s + k^2 u_b^s = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D_b}, \\ (-a)\Delta u_b + k^2\lambda u_b = 0 & \text{in } D_b, \\ u_b - u_b^s = u^i & \text{on } \partial D_b, \\ (-a)\dfrac{\partial u_b}{\partial \nu} - \dfrac{\partial u_b^s}{\partial \nu} = \dfrac{\partial u^i}{\partial \nu} & \text{on } \partial D_b, \\ \lim\limits_{R\to\infty} \displaystyle\int\limits_{|x|=R} |\partial u_b^s/\partial|x| - ik\,u_b^s|^2\, ds = 0, \end{cases} \tag{5.60}$$

where $u^i \in H^1(D_b)$ is some incident field satisfying $\Delta u^i + k^2 u^i = 0$ in $D_b$. The scattering problem (5.60) is well-posed as long as $\Im(\lambda) \geq 0$ [22], [60].

Arguing as in (5.19), the metamaterial transmission eigenvalues $\lambda$ correspond to the existence of a nontrivial $u^i \in H^1(D_b)$ satisfying $\Delta u^i + k^2 u^i = 0$ in $D_b$ such that the associated scattered fields $u^s$ and $u_b^s$, respectively defined by (1.27)–(1.29) and (5.60), are such that $u_\infty = u_\infty^b$. By Rellich's Lemma, this implies that $u^s = u_b^s$ in $\mathbb{R}^3 \setminus \overline{D_b}$. Setting $w := u^s + u^i|_{D_b}$ and $v := u_b^s + u^i|_{D_b}$ we obtain the eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n w = 0 &\quad \text{in } D_b, \\ (-a)\Delta v + k^2\lambda v = 0 &\quad \text{in } D_b, \\ w = v &\quad \text{on } \partial D_b, \\ \frac{\partial w}{\partial \nu_A} = -a\frac{\partial v}{\partial \nu} &\quad \text{on } \partial D_b \end{aligned} \tag{5.61}$$

for $v \in H^1(D_b)$ and $w \in H^1(D_b)$.

To study the eigenvalue problem (5.61), we first write it in the equivalent variational form

$$\int_{D_b} \nabla w \cdot \nabla \overline{w}' \, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v}' \, dx - k^2 \int_{D_b} n w \overline{w}' \, dx = -k^2 \lambda \int_{D_b} v \overline{v}' \, dx \qquad (5.62)$$

for $(w', v') \in \mathcal{H}(D_b)$, where

$$\mathcal{H}(D_b) = \left\{(w, v) \in H^1(D_b) \times H^1(D_b) \text{ such that } w = v \text{ on } \partial D_b\right\}.$$

Obviously, since $\Im(n) = 0$, this is an eigenvalue problem for a compact self-adjoint operator. To see this, one possibility is to fix a real $\beta$ such that $k$ is not a transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 n w &= 0 && \text{in } D_b, \\ (-a)\Delta v + k^2 \beta v &= 0 && \text{in } D_b, \\ w &= v && \text{on } \partial D_b, \\ \frac{\partial w}{\partial \nu_A} &= -a \frac{\partial v}{\partial \nu} && \text{on } \partial D_b. \end{aligned} \qquad (5.63)$$

This means that the self-adjoint operator $\mathbb{A} : \mathcal{H}(D_b) \to \mathcal{H}(D_b)$, defined by the Riesz representation as

$$(\mathbb{A}(w, v), (w', v'))_{\mathcal{H}(D_b)} = \int_{D_b} (\nabla w \cdot \nabla w' + a \nabla v \cdot \nabla v' \, dx - k^2 n w w' + k^2 \beta v v') \, dx$$

for all $(w', v') \in \mathcal{H}(D_b)$, is invertible. We remark that the operator $\mathbb{A}$ is of Fredholm type and depends analytically on $\beta$. Moreover, $\mathbb{A}$ is coercive for $k > 0$ and $\beta = i\tau$ with $\tau > 0$. This proves, by the Analytic Fredholm Theorem (Theorem 1.12), that for any fixed $k > 0$ there exists $\beta$ real such that $\mathbb{A}$ is invertible. Now consider the operator $\mathbb{T} : L^2(D) \to L^2(D)$ defined by

$$\mathbb{T} : f \in L^2(D) \mapsto v_f \in H^1(D_b), \quad \text{where} \quad (w_f, v_f) = \mathbb{A}^{-1}(0, f),$$

which is compact and self-adjoint. Therefore our eigenvalue problem for $\lambda$ becomes

$$\mathbb{T} v = -k^2(\lambda - \beta) v,$$

which is an eigenvalue problem for a self-adjoint compact operator. This implies in particular the existence of an infinite set of real eigenvalues $\lambda$, which, as we show in the next theorem, accumulate only at $-\infty$.

**Theorem 5.23.** *There exists at least one positive eigenvalue of* (5.61). *If in addition $k > 0$ satisfies Assumption* 5.1, *then there are at most finitely many positive eigenvalues.*

***Proof.*** Assume to the contrary that all eigenvalues are such that $\lambda_j \le 0$. This means that

$$\int_{D_b} \nabla w \cdot \nabla \overline{w} \, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v} \, dx - k^2 \int_{D_b} n \, |w|^2 \, ds \ge 0$$

for all $(w, v) \in \mathcal{H}(D_b)$ since due to self-adjointness all the eigenfunctions $(w, v)$ form a Riesz basis for $\mathcal{H}(D_b)$. Now taking $w = 1$ and $v = 1$ yields a contradiction, which proves the first statement.

Next we assume by contradiction that there exists a sequence of positive eigenvalues $\lambda_j > 0$, $j \in \mathbb{N}$, converging to $+\infty$ with eigenfunctions $(w_j, v_j) \in \mathcal{H}(D_b)$ normalized such that

$$\|w_j\|_{H^1(D_b)} + \|v_j\|_{H^1(D_b)} = 1. \tag{5.64}$$

Then from

$$(\nabla w_j, \nabla w_j) + a\,(\nabla v_j, \nabla v_j) - k^2\,(n w_j, w_j) = -k^2 \lambda_j\,(v_j, v_j)\,, \tag{5.65}$$

and since the left-hand side is bounded, we obtain that $v_j \to 0$ in $L^2(D_b)$. Next, up to a subsequence, $w_j \rightharpoonup w$ in $H^1(D_b)$; this weak limit satisfies $\Delta w + k^2 n w = 0$ in $D_b$ and $w = 0$ on $\partial D_b$. Our assumption on $k$ implies that $w = 0$, i.e., $w_j \rightharpoonup 0$ in $H^1(D_b)$ and up to a subsequence $w_j \to 0$ strongly in $L^2(D_b)$. From (5.65)

$$(\nabla w_j, \nabla w_j) + a\,(\nabla v_j, \nabla v_j) \le k^2\,(n w_j, w_j) \qquad \text{for all } j \in \mathbb{N}.$$

Since $(n w_j, w_j) \to 0$, we conclude that

$$(\nabla w_j, \nabla w_j) \to 0, \qquad \text{and} \qquad a\,(\nabla v_j, \nabla v_j) \to 0 \qquad \text{as } \; j \to \infty,$$

which implies that $\|\nabla w_j\|_{H^1(D_b)} \to 0$, $\|\nabla v_j\|_{H^1(D_b)} \to 0$. This contradicts (5.64), and the proof of the theorem is completed. $\Box$

For $(w, v) \in \mathcal{H}(D_b)$, since $w - v \in H^1_0(D_b)$, the Poincaré inequality

$$\|w - v\|^2 \le C_p \|\nabla w - \nabla v\|^2$$

holds with the optimal constant $C_p > 0$ being the first Dirichlet eigenvalue for $-\Delta$ in $D_b$. Thus

$$(w, w) \le C_p\,(\nabla w, \nabla w) + C_p\,(\nabla v, \nabla v) + (v, v)\,. \tag{5.66}$$

As with the Steklov eigenvalue problem discussed above, we would like to find a $\tau > 0$ such that

$$\begin{aligned}\int_{D_b} \nabla w \cdot \nabla \overline{w}\, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v}\, dx - k^2 \int_{D_b} n |w|^2\, dx + \tau \int_{D_b} |v|^2\, dx \\ \ge C \left( \|w\|^2_{H^1(D_b)} + \|v\|^2_{H^1(D_b)} \right).\end{aligned} \tag{5.67}$$

In this case, our eigenvalue problem

$$\begin{aligned}\int_{D_b} \nabla w \cdot \nabla \overline{w}'\, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v}'\, dx - k^2 \int_{D_b} n w \overline{w}'\, dx \\ + \tau \int_{D_b} v \overline{v}'\, dx = -k^2(\lambda + \tau) \int_{D_b} v \overline{v}'\, dx\end{aligned} \tag{5.68}$$

becomes a generalized eigenvalue problem for a positive compact self-adjoint operator and the eigenvalues $-(\lambda_j + \tau)$ satisfy the Courant–Fischer min-max principle. Consequently we obtain that our largest positive eigenvalue $\lambda_1 := \lambda_1(n, k)$ satisfies

$$\lambda_1 = \sup_{(w,v) \in \mathcal{H}(D_b), v \neq 0} \frac{k^2 \displaystyle\int_{D_b} n\, |w|^2\, dx - \int_{D_b} \nabla w \cdot \nabla w\, dx - a \int_{D_b} |\nabla v|^2\, dx}{\displaystyle\int_{D_b} |v|^2\, dx}. \tag{5.69}$$

Hence $\lambda_1$ is monotonically increasing with respect to $n$ and monotonically decreasing with respect to $A$. The following theorem indicates when (5.67) and (5.69) hold.

**Theorem 5.24.** *Assume that $k^2 < \eta_0(n, D_b)$, where $\eta_0(n, D_b)$ is the first Dirichlet eigenvalue of* (5.51). *Then there is a $\tau > 0$ such that* (5.67) *holds. In particular, in this case the largest positive eigenvalue satisfies* (5.69).

***Proof.*** Fix $k^2 < \eta_0(n, D_b)$ and assume to the contrary that there exists a sequence of positive constants $\tau_j = j$, $j \in \mathbb{N}$, and a sequence of functions $(w_j, v_j) \in \mathcal{H}(D_b)$ normalized as $\|w_j\|_{H^1(D_b)} + \|v_j\|_{H^1(D_b)} = 1$ such that

$$\int\limits_{D_b} \nabla w_j \cdot \nabla \overline{w}_j \, dx + a \int\limits_{D_b} |\nabla v_j|^2 \, dx - k^2 \int\limits_{D_b} n \, |w_j|^2 \, dx + j \int\limits_{D_b} |v_j|^2 \, ds \leq 0. \tag{5.70}$$

From

$$\int\limits_{D_b} \nabla w_j \cdot \nabla \overline{w}_j \, dx + a \int\limits_{D_b} |\nabla v_j|^2 \, d + j \int\limits_{D_b} |v_j|^2 \, ds \leq k^2 \int\limits_{D_b} n \, |w_j|^2 \, dx \tag{5.71}$$

we see that $j \int_{D_b} |v_j|^2 \, ds$ is bounded, which implies that $v_j \to 0$ strongly in $L^2(D_b)$. On the other hand, the boundedness implies that, up to a subsequence, $w_j \rightharpoonup w$ and $v_j \rightharpoonup 0$ in $H^1(D_b)$. Since $(w_j, v_j) \in \mathcal{H}(D_b)$ we get in particular that $w \in H_0^1(D_b)$. By going to a subsequence, one can also assume that $w_j \to w$ strongly in $L^2(D_b)$. Since the norm of the weak limit is smaller that the lim-inf of the norm

$$(\nabla w, \nabla w) \leq \liminf_{j \to \infty} \int\limits_{D_b} \nabla w_j \cdot \nabla \overline{w}_j \, dx \leq \lim_{j \to \infty} k^2 \int\limits_{D_b} n \, |w_j|^2 \, dx = k^2 (nw, w),$$

which contradicts the fact that

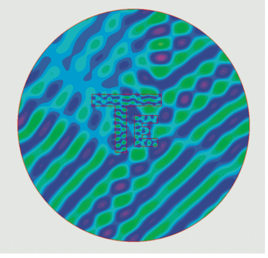
$$k^2 < \inf_{w \in H_0^1(D_b), w \neq 0} \frac{(\nabla w, w)}{(nw, w)} = \eta_1(n, D_b).$$

This ends the proof. $\square$

The metamaterial modified transmission eigenvalues can be identified from $F^m$ using GLSM in the same way as in Theorem 5.22 using

$$J_\alpha(\phi_z; g) := \alpha \|\mathcal{H}g\|^2_{L^2(D_b)} + \|F^m g - \phi_z\|^2_{L^2(S^2)},$$

where $F^m = F - F^b$ and $F^b$ is the far field operator associated with the background scattering problem (5.60).



# Chapter 6

# Inverse Spectral Problems for Transmission Eigenvalues

## 6.1 ▪ Spherically Stratified Media with Spherically Symmetric Eigenfunctions

The (normalized) transmission eigenvalue problem for an isotropic spherically stratified medium in $\mathbb{R}^3$ is to find a nontrivial solution $v, w \in L^2(B)$, $v - w \in H_0^2(B)$ to

$$\Delta w + k^2 n(r) w = 0 \quad \text{in } B, \tag{6.1}$$

$$\Delta v + k^2 v = 0 \quad \text{in } B, \tag{6.2}$$

$$v - w = 0 \quad \text{on } \partial B, \tag{6.3}$$

$$\frac{\partial v}{\partial r} - \frac{\partial w}{\partial r} = 0 \quad \text{on } \partial B \tag{6.4}$$

for $k \in \mathbb{C}$, where $B := \{x\colon |x| < 1\}$. We assume that $n \in C^3[0,1]$, although this condition can be weakened. If we look for spherically symmetric eigenfunctions

$$w(x) = a_0 \frac{y(r)}{r},$$
$$v(x) = b_0 \frac{\sin kr}{kr},$$

where $a_0$, $b_0$ are constants, then

$$y'' + k^2 n(r) y = 0,$$

$$y(0) = 0,\ y'(0) = 1,$$

where the second initial condition is a normalization condition. From this we see, after simplification, that $k$ is a transmission eigenvalue if and only if

$$d(k) := \det \begin{vmatrix} y(1) & \dfrac{\sin k}{k} \\ y'(1) & \cos k \end{vmatrix} = 0.$$

**Theorem 6.1.** *If $d(k)$ is identically zero, then $n(r)$ is identically equal to one.*

***Proof.*** ([3]) If $d(k) = 0$, then

$$\frac{\sin k}{k} y'(1) = y(1) \cos k. \tag{6.5}$$

Each of the four functions in (6.5) is an entire function of $k$ of order one. Furthermore, $y'(1)$ and $y(1)$ cannot vanish simultaneously and $\frac{1}{k} \sin k$ and $\cos k$ cannot vanish simultaneously. Thus (6.5) implies that $\frac{1}{k} \sin k$ and $y(1) = y(1; k)$ must have the same set of zeros including multiplicities and that $\cos k$ and $y'(1) = y'(1; k)$ must have the same zeros including multiplicities. Hence, by the Hadamard factorization theorem [158] and the fact that $\frac{1}{k} \sin k$ and $y(1; k)$ are even entire functions of order one, we can conclude that

$$y(1; k) = c_1 \frac{\sin k}{k}, \quad y'(1; k) = c_1 \cos k$$

for some nonzero constant $c_1$. But the zeros of $y(1; k)$ and $y'(1; k)$ correspond to two sets of spectra for $y'' + k^2 n(r) y = 0$, and it is well known that this information uniquely determines $n(r)$ for $r \in [0, 1]$ [14]. Thus $n(r)$ is uniquely determined by the combined knowledge of the zeros of $\frac{1}{k} \sin k$ and $\cos k$, and these zeros correspond to $n(r) = 1$ for $r \in [0, 1]$. $\square$

If $n(1) = 1$ and $n'(1) = 0$, then an elementary asymptotic analysis shows that [69]

$$d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin k \left( \int_0^1 \sqrt{n(\rho)}\, d\rho - 1 \right) + O\left(\frac{1}{k}\right) \right\} \tag{6.6}$$

as $k \to \infty$, and hence if

$$\delta := \int_0^1 \sqrt{n(\rho)}\, d\rho \neq 1,$$

there exist an infinite number of positive transmission eigenvalues. This can also be shown to be true of $n(1) \neq 1$ and $n'(1) \neq 0$ (see Section 4.2). However, as the following examples show, there can also exist complex eigenvalues.

**Example 6.2.** ([3]) When $n(r) = 1/4$ we have that

$$d(k) = -\frac{2}{k} \sin^3 \left( \frac{k}{2} \right),$$

and hence there exist an infinite number of real eigenvalues and no complex eigenvalues. On the other hand, if $n(r) = 4/9$, we have that

$$d(k) = -\frac{1}{k} \sin^3 \left( \frac{k}{3} \right) \left[ 3 + 2 \cos \left( \frac{2k}{3} \right) \right],$$

i.e., in this case there exist an infinite number of both real and complex eigenvalues.

The above examples are special cases of the following theorem [29], [112], [130].

**Theorem 6.3.** *Let $n(r) = n_0^2$, where $n_0$ is a positive constant not equal to one. Then if $n_0$ is an integer or the reciprocal of an integer, all transmission eigenvalues with spherically symmetric eigenfunction are real. If $n_0$ is not an integer or the reciprocal of an integer, then there are infinitely many real and complex transmission eigenvalues.*

We now note that

1. $d(k)$ is an even entire function of order one, i.e., $d(\sqrt{k})$ is an entire function of order $1/2$;

2. if $0 < n(r) < 1$, then $d(k)$ has a zero of order two at the origin.

Both of these facts can be seen by determining $y(r)$ by successive approximations as a perturbation from $y_0(r) = r$ and then substituting $y(1)$ and $y'(1)$ into the expression for $d(k)$ (cf. Section 7.6 in [112]). Hence, if $\delta \neq 1$ and the zeros $\{k_j\}$ of $d(k)$ are known (including multiplicity), then by Hadamard's factorization theorem [158]

$$d(k) = ck^2 \prod_{j=1}^{\infty} \left(1 - \frac{k^2}{k_j^2}\right),$$

where, from the asymptotic expansion (6.6), we can determine $cn(0)^{1/4}$. Thus, the transmission eigenvalues (real and complex and including multiplicity) determine $n(0)^{1/4}d(k)$.

Further results on transmission eigenvalues for a constant index of refraction can be found in [146], [147], and [160].

We now turn our attention to the inverse spectral problem of determining $n(r)$ from knowledge of the transmission eigenvalues. From the above discussion and assumptions this is equivalent to determining $n(r)$ from knowledge of the determinant $d(k)$. We first need an integral representation of the solution to

$$y'' + k^2 n(r) y = 0,$$

$$y(0) = 0, y'(0) = 1.$$

To this end, using the *Liouville transformation*

$$\xi := \int_0^r \sqrt{n(\rho)}\, d\rho,$$

$$z(\xi) := [n(r)]^{1/4}\, y(r),$$

we arrive at

$$z'' + \left[k^2 - p(\xi)\right] z = 0, \tag{6.7}$$

$$z(0) = 0, z'(0) = [n(0)]^{-1/4}, \tag{6.8}$$

where

$$p(\xi) := \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\frac{[n'(r)]^2}{[n(r)]^3}. \tag{6.9}$$

The solution of (6.7), (6.8) can be represented in the form [112]

$$z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin k\xi}{k} + \int_0^{\xi} K(\xi, t) \frac{\sin kt}{k}\, dt\right]$$

for $0 \le \xi \le \delta$ and $K(\xi, t)$ is the unique solution to the Goursat problem

$$K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \quad 0 < t < \xi < \delta,$$

$$K(\xi, 0) = 0, \quad 0 \le \xi \le \delta,$$

$$K(\xi, \xi) = \frac{1}{2}\int_0^\xi p(s)\,ds, \quad 0 \le \xi \le \delta.$$

The solution to the Goursat problem can be determined by successive approximations [112].

The following theorem, due to Rundell and Sacks [153], is fundamental to our investigation (see also Theorem 5.18 of [112]).

**Theorem 6.4.** *Let $K(\xi, t)$ satisfy the above Goursat problem. Then $p \in C^1[0, \delta]$ is uniquely determined by the Cauchy data $K(\delta, t)$ and $K_\xi(\delta, t)$.*

We can now establish our desired inverse spectral theorem [70]. We note that the condition on $n(1)$ and $n'(1)$ can be removed [69], [112].

**Theorem 6.5.** *Assume that $n \in C^3[0, 1]$, $n(1) = 1$, and $n'(1) = 0$. Then if $0 < n(r) < 1$, the transmission eigenvalues (including multiplicity) uniquely determine $n(r)$.*

***Proof.*** Recall the determinant

$$d(k) = \det \begin{vmatrix} y(1) & \dfrac{\sin k}{k} \\ y'(1) & \cos k \end{vmatrix} = 0.$$

From the above discussion we have that

$$y(1) = \frac{1}{[n(0)]^{1/4}}\left[\frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t)\frac{\sin kt}{k}\,dt\right],$$

$$y'(1) = \frac{1}{[n(0)]^{1/4}}\left[\cos k\delta + \frac{\sin k\delta}{2k}\int_0^\delta p(s)\,ds \right.$$
$$\left. + \int_0^\delta K_\xi(\delta, t)\frac{\sin kt}{k}\,dt\right],$$

where again

$$\delta := \int_0^1 \sqrt{n(\rho)\,d\rho}.$$

Note that $\delta$ can be determined from the asymptotic expansion (6.6). The above formulas now give for $\ell$ an integer

$$\ell\pi d\,(\ell\pi) = \frac{(-1)^\ell}{[n(0)]^{1/4}}\left[\sin \ell\pi\delta + \int_0^\delta K(\delta, t)\sin \ell\pi t\,dt\right] \tag{6.10}$$

and

$$\begin{aligned} \ell\pi d\left(\frac{\ell\pi}{\delta}\right) &= y(1)\frac{\ell\pi}{\delta}\cos\frac{\ell\pi}{\delta} \\ &\quad - \frac{\sin\frac{\ell\pi}{\delta}}{[n(0)]^{1/4}}\left[(-1)^{\ell} + \frac{\delta}{\ell\pi}\int_0^{\delta} K_{\xi}(\delta,t)\sin\frac{\ell\pi t}{\delta}\,dt\right]. \end{aligned} \tag{6.11}$$

We now note the following:

1. Since $\{\sin \ell\pi t\}$ is complete in $L^2[0,1]$, and hence in $L^2[0,\delta]$ if $\delta < 1$, we have from (6.10) that $K(\delta,t)$ is known.
2. Since $\left\{\sin\frac{\ell\pi t}{\delta}\right\}$ is complete in $L^2[0,\delta]$ we have from (6.11) that $K_{\xi}(\delta,t)$ is known.

Hence from Theorem 6.4 we can now conclude that $p(\xi)$ is uniquely determined for $0 \le \xi \le \delta$.

We now need to determine $n(r)$ from $p(\xi)$. Suppose $n_1(r)$ and $n_2(r)$ correspond to the same set of eigenvalues. Then $p(\xi_i)$ is uniquely determined where

$$\xi_i := \int_0^r \sqrt{n_i(\rho)}\,d\rho, \quad i = 1,2.$$

Since $n_i(1) = 1$ and $n_i'(1) = 0$ we have from (6.9) that $n_i(r(\xi_i))$ satisfies

$$\begin{gathered} \left(n_i^{1/4}\right)'' - p(\xi_i)n_i^{1/4} = 0, \quad 0 < \xi_i < \delta, \\ n_i^{1/4}(r(\delta)) = 1, \\ \left(n_i^{1/4}\right)'(r(\delta)) = 0 \end{gathered}$$

for $i = 1,2$. Hence by the uniqueness of the solution to the initial value problem for linear ordinary differential equations we have that $n_1(r(\cdot)) = n_2(r(\cdot))$. But $r_i = r(\xi_i)$ satisfies

$$\begin{gathered} \frac{dr_i}{d\xi_i} = \frac{1}{\sqrt{n_i(r(\xi_i))}}, \\ r_i(0) = 0 \end{gathered}$$

for $i = 1,2$ and hence $r_1(\cdot) = r_2(\cdot)$. This implies that $\xi_1 = \xi_2$ and hence $n_1(r) = n_2(r)$. $\square$

In view of Theorem 6.5, a natural question to ask is whether or not complex transmission eigenvalues exist. To this end, we define $\xi$, $z(\xi)$, and $\delta$ as before and set

$$\alpha := n(0)^{1/4}.$$

Then, under the assumption that $n \in C^2[0,1]$, we have that

$$z(\delta) = \frac{1}{\alpha k}\left[\sin(k\delta) + \int_0^{\delta} K(\delta,t)\sin(kt)dt\right], \tag{6.12}$$

$$z'(\delta) = \frac{1}{\alpha k}\left[k\cos(k\delta) + K(\delta,\delta)\sin(k\delta) + \int_0^{\delta} K_{\xi}(\delta,t)\sin(kt)\,dt\right] \tag{6.13}$$

and note the $z(\delta)$ and $z'(\delta)$ are both entire functions of type $\delta$ as a function of $k$.

Since $z(\xi) = n(r)^{1/4} y(r)$ we have that

$$y(1) = \frac{z(\delta)}{n(1)^{1/4}},$$
$$y'(1) = n(1)^{1/4} z'(\delta) - \frac{n'(1)}{4n(1)} y(1),$$

and hence

$$d(k) = \left[\frac{\cos(k)}{n(1)^{1/4}} + \frac{n'(1)}{4n(1)}\frac{\sin(k)}{k}\right] z(\delta) - n(1)^{1/4}\frac{\sin(k)}{k} z'(\delta).$$

Integrating by parts in (6.12) we have that

$$z(\delta) = \frac{1}{\alpha k}\left[\sin(k\delta) - K(\delta,\delta)\frac{\cos(k\delta)}{k} + \int_0^\delta K_t(\delta,t)\frac{\cos(kt)}{k}\,dt\right],$$

and thus in terms of the kernel function $K(\xi, t)$ we have from (6.13) that

$$\begin{aligned} d(k) = &\left(\frac{\cos(k)}{\alpha\, k\, n(1)^{1/4}} + \frac{n'(1)}{4\alpha n(1)}\frac{\sin(k)}{k^2}\right) \\ &\cdot\left(\sin(k\delta) - K(\delta,\delta)\frac{\cos(k\delta)}{k} + \int_0^\delta K_t(\delta,t)\frac{\cos(kt)}{k}\,dt\right) \\ &- \frac{n(1)^{1/4}\sin(k)}{\alpha k}\left[k\cos(k\delta) + K(\delta,\delta)\sin(k\delta) + \int_0^\delta K_\xi(\delta,t)\sin(kt)\,dt\right]. \end{aligned}$$

Setting

$$D(k) := \alpha\, n(1)^{1/4}\, k\, d(k)$$

we can now arrive at the formula

$$D(k) = \cos(k)\sin(k\delta) - \sqrt{n(1)}\sin(k)\cos(k\delta) + H(k), \tag{6.14}$$

where

$$\begin{aligned} H(k) := &\left(\frac{n'(1)}{4[n(1)]^{3/4}} - \sqrt{n(1)}K(\delta,\delta)\right)\frac{\sin(k)\sin(k\delta)}{k} - K(\delta,\delta)\frac{\cos(k)\cos(k\delta)}{k} \\ &- \frac{n'(1)}{4[n(1)]^{3/4}}K(\delta,\delta)\frac{\sin(k)\cos(k\delta)}{k^2} + \frac{\cos(k)}{k}\int_0^\delta K_t(\delta,t)\cos(kt)dt \\ &- [n(1)]^{1/2}\frac{\sin(k)}{k}\int_0^\delta K_\xi(\delta,t)\sin(kt)dt + \frac{n'(1)}{4[n(1)]^{3/4}}\frac{\sin(k)}{k^2}\int_0^\delta K_t(\delta,t)\cos(kt)dt. \end{aligned}$$

Using the representation (6.14) we intend to show that if $n(1) = 1, n'(1) = 0, n''(1) \neq 0$, and $\delta \neq 1$, then there exist an infinite number of complex transmission eigenvalues, i.e., an infinite number of complex zeros of $d(k)$. However, in order to do this we must first collect a number of results from the theory of entire functions of exponential type. Our first result is the celebrated Paley–Wiener Theorem [117], [173].

**Theorem 6.6 (Paley–Wiener Theorem).** *The entire function $f(z)$ is of exponential type less than or equal to $\tau$ and belongs to $L^2$ on the real axis if and only if*

$$f(z) = \int_{-\tau}^{\tau} \varphi(t) e^{izt}\, dt$$

*for some $\varphi \in L^2(-\tau, \tau)$. $f(z)$ is of type $\tau$ if $\varphi(t)$ does not vanish in a neighborhood of $\tau$ or $-\tau$.*

We say that an entire function belongs to the *Paley–Wiener class* if it has the representation given in the Paley–Wiener Theorem.

For future reference we note that

$$\int_0^{\tau} \psi(t) \sin(zt)\, dt$$

can be expressed as

$$\int_{-\tau}^{\tau} \phi(t) e^{izt}\, dt$$

for some function $\phi(t)$ defined for $t \in [-\tau, \tau]$ if $\psi(t)$ is extended onto the interval $[-\tau, 0]$ in an appropriate fashion.

Now let $n_+(r)$ denote the number of zeros of an entire function $f(z)$ in the right half-plane for $|z| \le r$ (one can also define a corresponding function $n_-(r)$ for zeros in the left half-plane). We then have the following theorem [117].

**Theorem 6.7 (Cartwright–Levinson Theorem).** *Let the entire function $f(z)$ of exponential type be such that*

$$\text{(a)} \qquad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2}\, dx < \infty$$

*and suppose that*

$$\text{(b)} \qquad \overline{\lim_{y \to \pm\infty}} \frac{\log |f(iy)|}{|y|} = \tau.$$

*Then*

$$\lim_{r \to \infty} \frac{n_+(r)}{r} = \frac{\tau}{\pi}.$$

The limit $\tau/\pi$ is called the *density* of the zeros of $f(z)$ in the right half-plane.

**Corollary 6.8.** *Let $f(z)$ be an entire function that is in the Paley–Wiener class of type at most $\tau$. Suppose $x f(x) = \sin(\tau x) + O\left(\frac{1}{x}\right)$ as $x$ tends to infinity on the real axis. Then $f(z)$ is of type $\tau$.*

***Proof.*** The density of the positive zeros of $f(z)$ is $\tau/\pi$. Therefore the type of $f(z)$ must be at least $\tau$ and so it equals $\tau$. ◻

Armed with the above tools from the theory of entire functions, we now return to (6.14) and use this representation to prove the following theorem [71].

**Theorem 6.9.** *Suppose the refractive index $n \in C^3[0,1]$ is such that $n'''$ is absolutely continuous with $n(1) = 1$, $n'(1) = 0$, $n''(1) \neq 0$, and $\delta \neq 1$. Then the entire function $D(k)$ has infinitely many nonreal zeros and infinitely many real zeros.*

***Proof.*** From (6.14) and the fact that $n(1) = 1$ and $n'(1) = 0$ we have that

$$\begin{aligned} D(k) &= \sin((\delta-1)k) - K(\delta,\delta)\frac{\cos((\delta-1)k)}{k} \\ &\quad + \frac{\cos k}{k}\int_0^\delta K_t(\delta,t)\cos(kt)\,dt - \frac{\sin k}{k}\int_0^\delta K_\xi(\delta,t)\sin(kt)\,dt. \end{aligned}$$

An integration by parts on the last two integrals and using the fact that $K_\xi(\delta,0) = 0$ shows that

$$\begin{aligned} D(k) &= \sin((\delta-1)k) - K(\delta,\delta)\frac{\cos((\delta-1)k)}{k} \\ &\quad + K_t(\delta,\delta)\frac{\cos k\sin(k\delta)}{k^2} + K_\xi(\delta,\delta)\frac{\sin k\cos(k\delta)}{k^2} \\ &\quad - \frac{\cos k}{k^2}\int_0^\delta K_{tt}(\delta,t)\sin(kt)\,dt - \frac{\sin k}{2k^2}\int_0^\delta K_{\xi t}(\delta,t)\cos(kt)\,dt. \end{aligned}$$

In the above expression the terms of order $1/k^2$ can be rewritten as

$$\begin{aligned} &\frac{K_t(\delta,\delta)}{2k^2}\left[\sin((\delta+1)k) + \sin((\delta-1)k)\right] \\ &\quad + \frac{K_\xi(\delta,\delta)}{2k^2}\left[\sin((\delta+1)k) - \sin((\delta-1)k)\right]. \end{aligned}$$

Hence, by Corollary 6.8, $kD(k)$, and hence $D(k)$, is an entire function of exponential type $\delta+1$ if the coefficient of $\sin((\delta+1)k)$ is nonzero. This coefficient is

$$\frac{K_t(\delta,\delta) + K_\xi(\delta,\delta)}{2k^2},$$

and since

$$K(\xi,\xi) = \frac{1}{2}\int_0^\xi p(s)\,ds$$

for $0 \le \xi \le \delta$ we have that

$$\frac{K_t(\delta,\delta) + K_\xi(\delta,\delta)}{2} = \frac{1}{4}p(\delta).$$

From (6.9) we see that $p(\delta) = \frac{1}{4}n''(1)$ since $n(1) = 1$ and $n'(1) = 0$. In summary, under the assumptions of the theorem, the asymptotic expansion of $D(k)$ has the form (for $k$ on the real axis)

$$D(k) = \sin((\delta-1)k) - \frac{1}{2k}\int_0^\delta p(s)\,ds\cos((\delta-1)k) + \frac{K_t(\delta,\delta)-K_\xi(\delta,\delta)}{2k^2}\sin((\delta-1)k) + \frac{n''(1)}{16k^2}\sin((\delta+1)k) + O\left(\frac{1}{k^3}\right).$$

If $n''(1) \neq 0$, then $D(k)$ is of exponential type $\delta+1$. Since the leading term $\sin((\delta-1)k)$ generates an infinite set of positive real zeros with density equal to $|1-\delta|/\pi$ while the density of all the zeros in the right half-plane equals $(\delta+1)/\pi$, we have by the Cartwright–Levinson theorem that in addition to the infinite set of positive real zeros there exist an infinite number of nonreal zeros in the right half-plane. □

## 6.2 ▪ Spherically Stratified Media with All Eigenvalues

We return to the inverse problem for (6.1)–(6.4) but no longer assume that the transmission eigenfunctions are spherically symmetric. In this case, we will show that the transmission eigenvalues uniquely determine $n(r)$, provided $n(0)$ is known but without assuming that $0 < n(r) \le 1$ as in Theorem 6.5. More specifically we consider the interior transmission eigenvalue problem (6.1)–(6.4), where $B := \{x : |x| < 1\}$, and assume that either $0 < n(r) \le 1$ or $n(r) \ge 1$ for $0 \le r \le 1$, and $n \in C^2[0, \infty)$.

Introducing spherical coordinates $(r, \theta, \varphi)$, we look for solutions of (6.1)–(6.4) in the form

$$v(r,\theta) = a_\ell j_\ell(kr)P_\ell(\cos\theta),$$
$$w(r,\theta) = b_\ell y_\ell(r)P_\ell(\cos\theta),$$

where $P_\ell$ is Legendre's polynomial, $j_\ell$ is a spherical Bessel function, $a_\ell$ and $b_\ell$ are constants, and $y_\ell$ is a solution of

$$y_\ell'' + \frac{2}{r}y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right)y_\ell = 0$$

for $r > 0$ such that $y_\ell(r)$ behaves like $j_\ell(kr)$ as $r \to 0$, i.e.,

$$\lim_{r\to 0} r^{-\ell}y_\ell(r) = \frac{\sqrt{\pi}k^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}.$$

From [69, Section 9.4, in particular Theorem 9.9], we can deduce that $k$ is a (possibly complex) transmission eigenvalue if and only if

$$d_\ell(k) = \det\begin{pmatrix} y_\ell(1) & -j_\ell(k) \\ y_\ell'(1) & -kj_\ell'(k)\end{pmatrix} = 0 \tag{6.15}$$

and that, for $k > 0$, $d_\ell(k)$ has the asymptotic behavior

$$d_\ell(k) = \frac{1}{k\,[n(0)]^{\ell/2+1/4}}\,\sin k\left(1 - \int_0^1 [n(r)]^{1/2}dr\right) + O\left(\frac{\ln k}{k^2}\right). \tag{6.16}$$

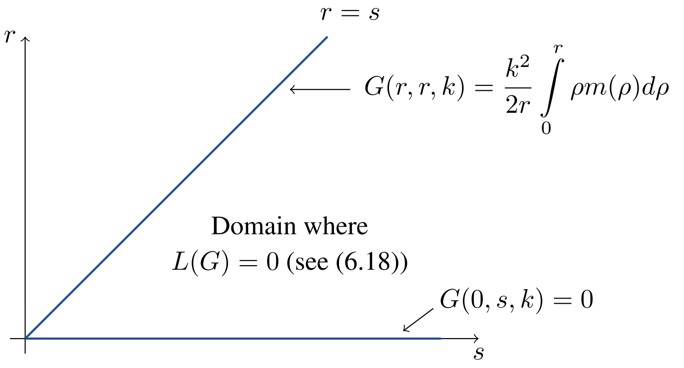



**Figure 6.1.** *Configuration of the Goursat problem.*

From [77, pp. 45–50], we can represent $y_\ell(r)$ in the form

$$y_\ell(r) = j_\ell(kr) + \int_0^r G(r,s,k) j_\ell(ks) ds, \tag{6.17}$$

where $G(r,s,k)$ satisfies the Goursat problem

$$r^2 \left[ \frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} + k^2 n(r) G \right] = s^2 \left[ \frac{\partial^2 G}{\partial s^2} + \frac{2}{s}\frac{\partial G}{\partial s} + k^2 G \right], \tag{6.18}$$

$$G(r,r,k) = \frac{k^2}{2r}\int_0^r \rho\, m(\rho) d\rho, \tag{6.19}$$

$$G(r,s,k) = O\left( (rs)^{1/2} \right), \tag{6.20}$$

and $m := 1 - n$ (see Figure 6.1). It is shown in [77] that $G$ can be solved by iteration, is an even function of $k$, and is an entire function of exponential type satisfying

$$G(r,s,k) = \frac{k^2}{2\sqrt{rs}} \int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho \left(1 + O(k^2)\right). \tag{6.21}$$

Note that, in contrast to the kernel $K(s,t)$ of Section 6.1, $G(r,s,k)$ depends on $k$.

We now return to the determinant (6.15) and compute the coefficient $c_{2\ell+2}$ of the term $k^{2\ell+2}$. A short computation using (6.15), (6.17), (6.21), and the order estimate

$$j_\ell(kr) = \frac{\sqrt{\pi}(kr)^\ell}{2^{\ell+1}\Gamma(\ell+3/2)} \left(1 + O(k^2 r^2)\right) \tag{6.22}$$

shows that

$$c_{2\ell+2} \left[ \frac{2^{\ell+1}\Gamma(\ell+3/2)}{\sqrt{\pi}} \right]^2 = \int_0^1 \frac{d}{dr} \left( \frac{1}{2\sqrt{rs}} \int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho \right)_{r=1} s^\ell\, ds \tag{6.23}$$

$$- \ell \int_0^1 \frac{1}{2\sqrt{as}} \int_0^{\sqrt{as}} \rho\, m(\rho)\, d\rho\, s^\ell\, ds + \frac{1}{2}\int_0^1 \rho\, m(\rho)\, d\rho.$$

After a rather tedious calculation involving a change of variables and interchange of orders of integration, the identity (6.23) remarkably simplifies to

$$c_{2\ell+2} = \frac{\pi a^2}{2^{\ell+1}\Gamma(\ell+3/2)} \int_0^1 \rho^{2\ell+2}\, m(\rho)\, d\rho. \tag{6.24}$$

We now note that $j_\ell(r)$ is odd if $\ell$ is odd and even if $\ell$ is even. Hence, since $G$ is an even function of $k$, we have that $d_\ell(k)$ is an even function of $k$. Furthermore, since both $G$ and $j_\ell$ are an entire function of $k$ of exponential type, so is $d_\ell(k)$. From the asymptotic behavior of $d_\ell(k)$ for $k \to \infty$, i.e., (6.16), we see that the rank of $d_\ell(k)$ is one, and hence by Hadamard's factorization theorem

$$d_\ell(k) = k^{2\ell+2} e^{a_\ell k + b_\ell} \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(1 - \frac{k}{k_{n\ell}}\right) e^{k/k_{n\ell}},$$

where $a_\ell, b_\ell$ are constants or, since $d_\ell$ is even,

$$d_\ell(k) = k^{2\ell+2} c_{2\ell+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{n\ell}^2}\right), \tag{6.25}$$

where $c_{2\ell+2}$ is a constant given by (6.24) and $k_{n\ell}$ are zeros in the right half-plane (possibly complex). In particular, $k_{n\ell}$ are the (possibly complex) *transmission eigenvalues* in the right half-plane. Thus if the transmission eigenvalues are known, so is

$$\frac{d_\ell(k)}{c_{2\ell+2}} = k^{2\ell+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{n\ell}^2}\right)$$

as well as (from (6.16)) a nonzero constant $\gamma_\ell$ independent of $k$ such that

$$\frac{d_\ell(k)}{c_{2\ell+2}} = \frac{\gamma_\ell}{k} \sin k \left(1 - \int_0^1 [n(r)]^{1/2}\, dr\right) + O\left(\frac{\ln k}{k^2}\right),$$

i.e.,

$$\frac{1}{c_{2\ell+2}\, [n(0)]^{\ell/2+1/4}} = \gamma_\ell.$$

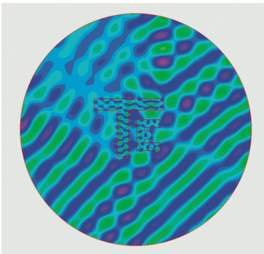
From (6.24) we now have

$$\int_0^1 \rho^{2\ell+2}\, m(\rho)\, d\rho = \frac{\left(2^{\ell+1}\Gamma(\ell+3/2)\right)^2}{[n(0)]^{\ell/2+1/4}\, \gamma_\ell \pi}.$$

If $n(0)$ is given, then $m(\rho)$ is uniquely determined by Müntz's theorem [173]. (Note that by Müntz's theorem the transmission eigenvalues are only needed for a subset $\{\ell_j\}$ such that $\sum_{j=1}^{\infty} \frac{1}{\ell_j} = \infty$.)

**Theorem 6.10.** *Assume that* $n(r) \in C^2[0, \infty)$*, that* $0 < n(r) \leq 1$ *or* $n(r) \geq 1$*, and that* $n(0)$ *is given. Then* $n(r)$ *is uniquely determined from a knowledge of the transmission eigenvalues corresponding to* (6.1)–(6.4) *for* $B := \{x : |x| < 1\}$.

**Remark 6.11.** In the case when $0 < n(r) \leq 1$ it can be shown that $n(r)$ can be uniquely determined from knowledge of the transmission eigenvalues for a single fixed $\ell$ without knowing $n(0)$ [172].



# Chapter 7
# Nonscattering Wave Numbers

The transmission eigenvalue problem has played a central role in all of the previous chapters. As the reader has seen by now, transmission eigenvalues are intrinsic to scattering phenomena and play a fundamental role in the solution of inverse scattering problems. The transmission eigenvalue problem first appeared in connection with the study of the injectivity of the far field operator $F_k \colon L^2(S^2) \to L^2(S^2)$ (see Section 1.2.1; note here we indicate the dependence on $k$ of the far field operator). The wave numbers $k > 0$ for which the far field operator is not injective are referred to as *nonscattering wave numbers* (see Theorem 1.16). In particular, at a nonscattering wave number $k > 0$ corresponding to an inhomogeneity $(n, D)$ there is a Herglotz wave function $v_g$ with some density $g \in L^2(S^2)$ given by (1.31) which does not scatter by this medium. The *nonscattering wave numbers form a subset of transmission eigenvalues*. A transmission eigenvalue $k$ is a nonscattering wave number if the $v$ part of the corresponding eigenfunction $(w, v)$ in (3.2) has the form of a Herglotz wave function. Thus a real transmission eigenvalue is not necessarily a nonscattering wave number, and it is desirable to understand which (if any) are. This chapter explores precisely this question for the case of isotropic media. For the case of anisotropic media we refer the reader to [56], [57], [171].

Conceptually, a nonscattering wave number is in relation to a probing experiment. For instance, in the definition of nonscattering frequencies one could consider other types of incident waves, such as plane waves, point sources, surface potentials, etc. For the given inhomogeneity $(n, D)$, one then specifies that $k$ is a nonscattering wave number associated with the particular incident wave. By this definition a nonscattering wave number for a given inhomogeneity $(n, D)$ would correspond to a transmission eigenvalue $k$ for which there exists an incident wave $u^i$ with wave number $k$ that renders the inhomogeneity invisible. We emphasize on the other hand that transmission eigenvalues are solely the property of the inhomogeneity $(n, D)$ (i.e., do not depend on any probing incident wave). In addition, there exist infinitely many transmission eigenvalues, and the real ones can be determined from appropriate scattering data. Thus, as has become clear throughout this monograph, transmission eigenvalues and the spectral properties of the transmission eigenvalue problem are of central importance in the solution of the inverse scattering problem. Nevertheless, to understand whether a given inhomogeneity $(n, D)$ admits an incident wave at a particular (nonscattering) wave number that is not scattered has its own mathematical interest, and it connects to corner singularity analysis as well as the regularity of free boundary problems. It will become clear in this chapter that a negative answer to the existence of nonscattering wave numbers can be given independently of the nature of the incident wave and is only related to a lack of sufficient regularity of the support $D$ and refractive index $n$. A negative answer also provides a mathematical tool to prove uniqueness theorems in inverse scattering.

## 7.1 ▪ The Case of Spherically Stratified Media

In this section we look at the particular case where the relation between transmission eigenvalues and nonscattering wave numbers becomes explicit. This is the case when the inhomogeneity $D := B_1(0)$ is the ball of radius 1 centered at the origin with radially symmetric real valued refractive index $n(r) > 0$, $r = |x|$, and assume throughout this section that $n \in C[0,1]$ and

$$\int_0^1 [n(r)]^{1/2} dr \neq 1.$$

Let us consider the incident fields that are entire solutions of the Helmholtz equation given by

$$u^i = j_\ell(k|x|)Y_\ell(\hat{x}), \tag{7.1}$$

where $j_\ell$ is a spherical Bessel function and $Y_\ell$ is a spherical harmonic of order $\ell \in \mathbb{N}$ (note that $Y_\ell$ denotes one of the $(2\ell+1)$ linearly independent spherical harmonics $Y_\ell^m$ for $m = -\ell, \dots, \ell$ corresponding to some $\ell \in \mathbb{N}$ (see, e.g., Section 2.3 in [69]). Note that these incident fields are examples of Herglotz wave functions given by (1.31). Straightforward calculation by separation of variables leads to the following expression for the scattered field outside $B$:

$$u^s(x) := \frac{C_\ell(k;n)}{W_\ell(k;n)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}), \qquad |x| > 1, \tag{7.2}$$

where $h_\ell^{(1)}(r)$ is a Hankel function of the first kind of order $\ell$ and

$$C_\ell(k;n) = \text{Det}\begin{pmatrix} y_\ell(1) & -j_\ell(k) \\ y_\ell'(1) & -kj_\ell'(k) \end{pmatrix}, \tag{7.3}$$

$$W_\ell(k;n) = \text{Det}\begin{pmatrix} y_\ell(1) & -h_\ell^{(1)}(k) \\ y_\ell'(1) & -kh_\ell^{(1)'}(k) \end{pmatrix} \tag{7.4}$$

with $y_\ell$ (depending on $k$ and $n$) being the solution to

$$y_\ell'' + \frac{2}{r}y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y_\ell = 0$$

that behaves like $j_\ell(kr)$ as $r \to 0$ (see Section 6.2).

Obviously, nonscattering wave numbers associated with the incident wave (7.1) correspond to those values of $k \in \mathbb{C}$ for which $C_\ell(k;n) = 0$. The asymptotic expression (6.16) shows that there are infinitely many real zeros of $C_\ell(k;n) = 0$ (note that $C_\ell(k;n)$ is denoted by $d_\ell(k)$ in (6.16)). Hence for given inhomogeneity $B_1(0), n(r)$ associated with each incident wave (7.1) there are infinitely many nonscattering wave numbers. In fact each of these nonscattering wave numbers is associated with at least $2\ell+1$ incident waves (corresponding to $2\ell+1$ different spherical harmonics $Y_\ell(\hat{x})$).

On the other hand, if $k \in \mathbb{C}$ is such that $C_\ell(k;n) = 0$, we observe that $v(x) := j_\ell(k|x|)Y_\ell(\hat{x})$ and $w_\ell(x) = y_\ell(|x|)Y_\ell(\hat{x})$ for $x \in B_1(0)$ solve the transmission eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n(r) w = 0 \qquad &\text{and} \qquad \Delta v + k^2 v = 0 \qquad &&\text{for } r < 1, \\ w = v \qquad &\text{and} \qquad \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad &&\text{for } r = 1, \end{aligned}$$

where $r = |x|$, i.e., such a $k$ is a transmission eigenvalue. Conversely, separating variables in the above transmission eigenvalue problem, we obtain that $k$ is a transmission eigenvalue if and only if $C_\ell(k;n) = 0$ with the corresponding transmission eigenfunctions being constant multiplications of $v(x) := j_\ell(k|x|)Y_\ell(\hat{x})$ and $w_\ell(x) = y_\ell(|x|)Y_\ell(\hat{x})$. Therefore we conclude that for *spherically stratified media the set of nonscattering wave numbers and transmission eigenvalues coincide*. Furthermore, since the $v$ part of the transmission eigenfunction corresponding to a transmission eigenfunction can only be finite linear combinations of $v(x) := j_\ell(k|x|)Y_\ell(\hat{x})$ (because due to the Fredholm property the transmission eigenspace is of finite dimension provided $n(1) \neq 1$; see Section 3.1), we conclude that in this case the nonscattering incident waves are only Herglotz wave functions (1.31). A nonscattering wave number is associated with only finitely many incident waves.

The above discussion is a feature of spherical symmetry of the scattering media. Indeed, in [168] it is shown that the existence of infinitely many nonscattering wave numbers associated with Herglotz wave functions is unstable with respect to perturbations of spherical symmetry. More specifically, it is proven that in $\mathbb{R}^2$ for any ellipse of sufficiently small (but nonzero) eccentricity with constant refractive index $0 < n \neq 1$, there exist at most (possibly none) finitely many positive wave numbers $k > 0$ associated with incident Herglotz wave functions $v_g$ with smooth densities $g$ that can be nonscattering. Note that from Theorem 4.12 we know that for such elliptic inhomogeneities there exist infinitely many transmission eigenvalues.

## 7.2 ▪ On the Nonexistence of Nonscattering Wave Numbers

Throughout this section we assume that the boundary $\partial D$ of the support of the inhomogeneity is Lipschitz and the refractive index $n \in L^\infty(D)$ is a positive real valued function. Let $\Omega$ be an open region in $\mathbb{R}^3$ such that $\Omega \supset \overline{D}$. To study the existence of nonscattering wave numbers in a framework independent of a particular probing wave it suffices to consider the following problem: Find a compactly supported solution $u$ to

$$\Delta u + k^2 n u = k^2(1-n)v \qquad \text{in } \Omega, \tag{7.5}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } \Omega, \tag{7.6}$$

$$u \equiv 0 \qquad \text{in } \Omega \setminus \overline{D}. \tag{7.7}$$

Note that in a scattering experiment $v := u^i|_\Omega$, where the $u^i$ is the incident wave and $v$ always satisfies the Helmholtz equation in a domain containing the inhomogeneity in its interior. Here $u \equiv 0$ in $\Omega \setminus \overline{D}$ means that the scattered field is zero outside of the inhomogeneity, and if the above problem has a solution, it means that $u^i$ is not scattered. If (7.5)–(7.7) does not have a solution for nonzero $v$ and any $k$, then this inhomogeneity does not admit nonscattering wave numbers.

Since $\partial D$ is Lipschitz the above problem can be reformulated as follows: Find $u \in H_0^2(D)$ that satisfies

$$\Delta u + k^2 n u = k^2(1-n)v \qquad \text{in } \; D, \tag{7.8}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } \; \Omega, \tag{7.9}$$

assuming $v$ is nontrivial. Note the fundamental point here is that $v$ satisfies the Helmholtz equation in an open region containing $\overline{D}$. For this reason (7.8)–(7.9) is overdetermined. We remind the reader that if $v \in L^2(D)$ satisfies (7.9) only in $D$, the above problem becomes the transmission eigenvalue problem.

In the following we explore sufficient conditions on $D$ and $n$ for which (7.8)–(7.9) does not have solutions. The approach we present next follows [55]. The analysis relies on viewing the boundary with vanishing Cauchy data as a free boundary, and applying the free boundary regularity results in [28], [107] for second order elliptic equations. Connecting the nonscattering configuration of a given inhomogeneity to the regularity of free boundary problems, we examine necessary conditions for an inhomogeneity to be nonscattering, or equivalently, by negation, sufficient conditions for it to be scattering. These conditions are formulated in terms of the regularity of the boundary and the refractive index of the inhomogeneity. There is a striking similarity in the mathematical structure of the problem of nonscattering inhomogeneities and the problem of domains that do not possess the Pompeiu property [15, 167]. Regularity properties of the latter are established in [170], and the analysis here follows part of this work.

### 7.2.1 ▪ A Free Boundary Regularity Result

We start by presenting two main classical results on the regularity of free boundary problems. With $a(x) = k^2 n(x)$ and $b(x) = k^2(1-n(x))v(x)$, the problem (7.5)–(7.7) becomes

$$\Delta u + a(x) u = b(x) \quad \text{in } \; D, \tag{7.10}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \; \partial D. \tag{7.11}$$

In the analysis below we make use of two classical free boundary regularity results. The first one is due to Kinderlehrer and Nirenberg in [107]. In [107] the theorem is proven for more general nonlinear second order elliptic partial differential operators, but in the following we state it as it applies to our linear equation (7.10).

**Theorem 7.1.** *Suppose that $0 \in \partial D$, and $\partial D \cap B_R(0)$ is of class $C^1$ for some ball $B_R(0)$ of radius $R$ centered at $0$. Suppose $a$ and $b$ are real valued functions in $C^1(\overline{D} \cap B_R(0))$ with $a(0) \neq 0$ and $b(0) \neq 0$. Furthermore suppose there exists a real valued solution $u$ to* (7.10)–(7.11)*, with $u \in C^2(\overline{D} \cap B_R(0))$. Then the following hold:*

1. *$\partial D \cap B_{R'}(0)$ is of class $C^{1,\alpha}$ for every positive $\alpha < 1$ and some $R' < R$.*

2. *If additionally $a \in C^{m,\mu}(\overline{D} \cap B_R(0))$ and $b \in C^{m,\mu}(\overline{D} \cap B_R(0))$ for $m \geq 1$, $0 < \mu < 1$, then $\partial D \cap B_{R'}(0)$ is of class $C^{m+1,\mu}$ for some $R' < R$.*

3. *If $a$ and $b$ are real analytic in $\overline{D} \cap B_R(0)$, then $\partial D \cap B_{R'}(0)$ is real analytic for some $R' < R$.*

**Remark 7.2.** The regularity of the free boundary is a local property. Correspondingly, the result of Theorem 7.1 holds for $u$ solving (7.10) in $D \cap B_R(0)$ with zero Cauchy data (7.11) only on $\partial D \cap B_R(0)$. However, in our particular applications the solution $u$ will be defined on all of $D$.

We initially assume that $\partial D$ is only Lipschitz regular. In order to apply Theorem 7.1 we must first show that the free boundary $\partial D \cap B_R(0)$ is indeed $C^1$ and then verify that the solution $u$ to (7.10)–(7.11) is in $C^2(\overline{D} \cap B_R(0))$. This intermediate regularity is achieved with the help of a classical result on regularity of the free boundary due to Caffarelli [28], which we state in the following theorem, modified to the framework of our problem. This result refers to a function $w$ that satisfies

$$\Delta w = g \quad \text{in } D \cap B_R(0), \qquad \text{such that } w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D \cap B_R(0), \tag{7.12}$$

where again $0 \in \partial D$ and $B_R(0)$ is some ball of radius $R$ centered at 0.

**Theorem 7.3.** *Suppose that $\partial D \cap B_R(0)$ is Lipschitz and the function $w$ satisfying* (7.12) *is in $C^{1,1}(\overline{D} \cap B_R(0))$. Furthermore, assume that $w \le 0$ in $D \cap B_R(0)$, and $g$ has a $C^1$-extension $g^*$ in a neighborhood of $\overline{D} \cap B_R(0)$ such that $g^* \le -\alpha < 0$. Then there exists $R' < R$ such that $\partial D \cap B_{R'}(0)$ is of class $C^1$ and all second derivatives of $w$ are continuous up to $\partial D \cap B_{R'}(0)$, i.e., $w \in C^2(\overline{D} \cap B_{R'}(0))$.*

The first obstacle to the application of Theorem 7.3 is to verify that the $H_0^2(D)$ solution $u$ to (7.5)–(7.7) has all second derivatives uniformly bounded in $D \cap B_R(0)$. For this purpose, we will make use of the following auxiliary result on the regularity of a volume potential. Let us consider

$$\Phi(x,y) := \frac{1}{4\pi|x-y|}$$

the free space fundamental solution to the Laplace operator. For later use we note the following estimates:

$$\left|\frac{\partial \Phi}{\partial x_j}(x,y)\right| \le \frac{C}{|x-y|^2}, \; 1 \le j \le 3, \quad \text{and} \quad \left|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\right| \le \frac{C}{|x-y|^3}, \; 1 \le i,j \le 3.$$

Regularity results for the volume potential

$$w_\psi(x) = \int\limits_D \psi(y)\Phi(x,y)\,dy$$

play an important role in our analysis. Lemma 3.7 in [114] proves that for $\psi \in L^\infty(D)$ we have that $w_\psi \in C^1(\mathbb{R}^3)$ and

$$\frac{\partial w_\psi}{\partial x_j}(x) = \int\limits_D \psi(y)\frac{\partial \Phi}{\partial x_j}(x,y)\,dy, \qquad x \in \mathbb{R}^3, \; j = 1,\dots,3.$$

Furthermore, all second derivatives of $w_\psi$ exist for $x \in \mathbb{R}^3 \setminus \overline{D}$ and one can differentiate twice inside the integral to obtain

$$\begin{aligned}\frac{\partial^2 w}{\partial x_i \partial x_j}(x) &= \int_D \psi(y)\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,dy \\ &= \int_D [\psi(y)-\psi(x)]\,\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,dy + \psi(x)\int_D \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,dy \\ &= \int_D [\psi(y)-\psi(x)]\,\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,dy - \psi(x)\int_{\partial D} \frac{\partial \Phi}{\partial x_j}(x,y)\nu_i(y)\,ds(y)\,,\end{aligned} \tag{7.13}$$

provided $\psi$ extends into $\mathbb{R}^3 \setminus \overline{D}$. Here the last integral over $\partial D$ is obtained by using the divergence theorem (the minus sign arises when one replaces an $x_i$ derivative with a $y_i$ derivative). Note that the unit outward normal vector $\nu = (\nu_i)_{i=1,3}$ is well defined for almost all $y \in \partial D$. We show next that if $\psi$, in addition to being bounded on $D$, is in $C^\alpha(B_R(0))$, then (7.13) holds true for $x \in D \cap B_R(0)$. To this end, we set

$$d_{ij}(x) := \int_D [\psi(y)-\psi(x)]\,\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,dy - \psi(x)\int_{\partial D} \frac{\partial \Phi}{\partial x_j}(x,y)\nu_i(y)\,ds(y)\,.$$

Note that $d_{ij}(x)$ is well defined for $x \in D \cap B_R(0)$, since for $\psi \in C^\alpha(B_R(0))$ the integrand inside the volume integral behaves as

$$\left|[\psi(y)-\psi(x)]\,\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\right| \le C|x-y|^{\alpha-3} \quad \text{for } y \text{ near } x \tag{7.14}$$

and is bounded for $y$ away from $x$; the surface integral exists since $x$ is not on $\partial D$. Now we choose $2\epsilon < \operatorname{dist}(x, \partial D)$ and again consider a smooth cutoff function $\xi$ such that $0 \le \xi(t) \le 1$, $\xi(t) = 1$ for $t \ge 2$, and $\xi(t) = 0$ for $t \le 1$. Set

$$d_{j,\epsilon}(x) := \int_D \psi(y)\xi(|x-y|/\epsilon)\frac{\partial \Phi}{\partial x_j}(x,y)\,dy\,.$$

We obtain

$$\begin{aligned}\frac{\partial d_{j,\epsilon}}{\partial x_i}(x) = &\int_D [\psi(y)-\psi(x)]\,\frac{\partial}{\partial x_i}\left(\xi(|x-y|/\epsilon)\frac{\partial \Phi}{\partial x_j}(x,y)\right)dy \\ &- \psi(x)\int_{\partial D} \frac{\partial \Phi}{\partial x_j}(x,y)\nu_i(y)\,ds(y)\,,\end{aligned}$$

and therefore

$$\begin{aligned}\left|d_{ij}(x) - \frac{\partial d_{j,\epsilon}}{\partial x_i}(x)\right| &\le C \int_{|y-x|\le 2\epsilon} \left(\frac{1}{|y-x|^3} + \frac{\|\xi'\|_\infty}{\epsilon|y-x|^2}\right)|y-x|^\alpha dy \\ &= C\int_0^{2\epsilon} \left(\frac{1}{r^{1-\alpha}} + \frac{\|\xi'\|_\infty}{\epsilon} r^\alpha\right) dr \le C\epsilon^\alpha\,.\end{aligned}$$

Hence, as $\epsilon \to 0$, $d_{j,\epsilon}(x)$ converges to $\frac{\partial w_\psi}{\partial x_j}(x)$ and $\frac{\partial d_{j,\epsilon}}{\partial x_i}(x)$ converges to $d_{ij}(x)$, both uniformly on compact subsets of $D \cap B_R(0)$. Thus $d_{ij}(x) = \frac{\partial^2 w_\psi}{\partial x_i \partial x_j}(x)$ for $x \in B_R(0) \cap D$.

Let us note that even for smooth $\psi$, but with $\psi \neq 0$ on $\partial D$, the second derivatives of $w_\psi$ may become unbounded as $x$ approaches a boundary point from either inside or outside $D$. Thus the volume potential is not necessarily in $C^2(\overline{D} \cap B_R(0))$. However, one can show that symmetric jumps of the second derivative (to become precise later) are uniformly bounded near $0 \in \partial D$ when $\psi \in C^\alpha(B_R(0))$ for $0 < \alpha < 1$. A similar result is proven in [170] for $\psi \equiv 1$. The proof of Lemma 7.4 below is in many ways very similar to that in [170].

First we introduce some notation. Denote $x := (\tilde{x}, x_3) \in \mathbb{R}^3$, where $\tilde{x} := (x_1, x_2) \in \mathbb{R}^2$, and consider a cylindrical neighborhood of 0 defined by $N := N(\rho, h) = \tilde{B}_\rho(0) \times [-h, h]$, where $\tilde{B}_\rho(0)$ is the $(m-1)$-dimensional ball of radius $\rho$ centered at the origin. We assume that $B_{2r}(0) \subset N \subset \overline{N} \subset B_R(0)$. Furthermore, we assume (by appropriate rotation and selection of $\rho$ and $h$) that $N \cap \partial D$ is the graph $x_3 = f(\tilde{x})$ of a Lipschitz continuous function $f : \tilde{B}_\rho(0) \to \mathbb{R}$ with Lipschitz constant $K$. We also assume that $h > K\rho$ and

$$N \cap D = \left\{ (\tilde{x}, x_3) \;:\; \tilde{x} \in \tilde{B}_\rho(0),\; f(\tilde{x}) < x_3 < h \right\}.$$

Finally we denote by $e_3$ the unit vector in the direction of the third variable. We can now prove the following lemma.

**Lemma 7.4.** *Assume that $\psi \in C^\alpha(B_R(0))$ for $0 < \alpha < 1$, in addition to being bounded on $D$. Then there exists $r > 0$ so that the symmetric jumps*

$$\frac{\partial^2 w_\psi}{\partial x_i \partial x_j}(x + \eta e_3) - \frac{\partial^2 w_\psi}{\partial x_i \partial x_j}(x - \eta e_3), \qquad 1 \leq i, j \leq 3,$$

*across the boundary at $x$ are uniformly bounded with respect to $0 < \eta \leq r$ and $x \in \partial D \cap B_r(0)$.*

***Proof.*** Using (7.13), outside and inside $D$, we write

$$\begin{aligned}
&\frac{\partial^2 w_\psi}{\partial x_i \partial x_j}(x + \eta e_3) - \frac{\partial^2 w_\psi}{\partial x_i \partial x_j}(x - \eta e_3) \\
&\quad = \int_D [\psi(y) - \psi(x + \eta e_3)] \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x + \eta e_3, y)\, dy \\
&\qquad - \int_D [\psi(y) - \psi(x - \eta e_3)] \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x - \eta e_3, y)\, dy \\
&\qquad - \psi(x + \eta e_3) \int_{\partial D} \frac{\partial \Phi}{\partial x_j}(x + \eta e_3, y) \nu_i(y)\, ds(y) \\
&\qquad + \psi(x - \eta e_3) \int_{\partial D} \frac{\partial \Phi}{\partial x_j}(x - \eta e_3, y) \nu_i(y)\, ds(y)
\end{aligned}$$

for $x \in \partial D \cap B_r(0)$. In the above integral expressions the part of the integrals taken over $D \setminus B_R(0)$ and $\partial D \setminus B_R(0)$ are uniformly bounded with respect to $\eta$ in $[0, r]$ and for all $x \in \partial D \cap B_r(0)$. So it suffices to consider only the integrals over $B_R(0) \cap D$ and $B_R(0) \cap \partial D$. Next we have the following estimates for the integrands of the volume integrals:

$$\left| [\psi(y) - \psi(x \pm \eta e_3)] \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x \pm \eta e_3, y) \right| \leq C |x \pm \eta e_3 - y|^{\alpha - 3}, \quad y \in B_R(0),$$

for $x \in \partial D \cap B_r(0)$ and $\eta < r$ (note that $x \pm \eta e_3 \in B_{2r}(0) \subset B_R(0)$). Therefore the integrals over $D \cap B_R(0)$ are bounded uniformly in $\eta \in [0, r]$ and $x \in \partial D \cap B_r(0)$. Let us consider the boundary integral term

$$\psi(x+\eta e_3) \int\limits_{\partial D \cap B_R(0)} \frac{\partial \Phi}{\partial x_j}(x+\eta e_3, y)\nu_i(y)\, ds(y)$$
$$-\psi(x-\eta e_3) \int\limits_{\partial D \cap B_R(0)} \frac{\partial \Phi}{\partial x_j}(x-\eta e_3, y)\nu_i(y)\, ds(y) \qquad \text{for } 1 \le i, j \le 3.$$

The above expression can be written as

$$\mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3,$$

where

$$\mathbb{I}_1 := \int\limits_{\partial D \cap B_R(0)} [\psi(x+\eta e_3) - \psi(y)]\, \frac{\partial \Phi}{\partial x_j}(x+\eta e_3, y)\nu_i(y)\, ds(y),$$
$$\mathbb{I}_2 := \int\limits_{\partial D \cap B_R(0)} [\psi(y) - \psi(x-\eta e_3)]\, \frac{\partial \Phi}{\partial x_j}(x-\eta e_3, y)\nu_i(y)\, ds(y),$$

and

$$\mathbb{I}_3 := \int\limits_{\partial D \cap B_R(0)} \left[\frac{\partial \Phi}{\partial x_j}(x+\eta e_3, y) - \frac{\partial \Phi}{\partial x_j}(x-\eta e_3, y)\right] \psi(y)\nu_i(y)\, ds(y).$$

Using the fact that $\psi \in C^\alpha(B_R(0))$, and that $z := x \pm \eta e_3 \in B_{2r}(0) \subset N$ for $x \in \partial D \cap B_r(0)$ and $\eta < r$, we obtain

$$\begin{aligned}
|\mathbb{I}_{1,2}| &\le C_1 \int\limits_{\partial D \cap B_R(0)} \frac{1}{|(x \pm \eta e_3) - y|^{2-\alpha}}\, ds(y) = C + C_1 \int\limits_{\partial D \cap N} \frac{1}{|z-y|^{2-\alpha}}\, ds(y) \\
&\le C + C_2 \int\limits_{\tilde{B}_\rho(0)} \frac{1}{|\tilde{z} - \tilde{y}|^{2-\alpha}} \sqrt{1 + |\nabla f(\tilde{y})|^2}\, d\tilde{y} \\
&\le C + C_3 \int\limits_{\tilde{B}_\rho(0)} \frac{1}{|\tilde{z} - \tilde{y}|^{2-\alpha}}\, d\tilde{y}.
\end{aligned}$$

Note that by Rademacher's theorem $\nabla f(\tilde{y})$ is well defined and is bounded at all points in $\tilde{y} \in \tilde{B}_\rho(0)$ except for a subset of Lebesgue measure zero. Hence $\mathbb{I}_{1,2}$ are also bounded uniformly in $\eta \in [0, r]$ and $x \in \partial D \cap B_r(0)$. To prove our lemma it thus suffices to estimate the term $\mathbb{I}_3$ with the symmetric jumps. We provide the details for $x = 0$. For $x$ near 0 (i.e., in $\partial D \cap B_r(0)$) the same approach works with obvious modifications. Since

$$\frac{\partial \Phi(x,y)}{\partial x_j} = \frac{-(x_j - y_j)}{2\pi |x-y|^3}, \qquad j = 1, \ldots, 3,$$

the integrals we need to study take the form

$$\int\limits_{\tilde{B}_\rho(0)} \left[\frac{y_j}{\left(|\tilde{y}|^2 + (f(\tilde{y}) - \eta)^2\right)^{3/2}} - \frac{y_j}{\left(|\tilde{y}|^2 + (f(\tilde{y}) + \eta)^2\right)^{3/2}}\right] F(\tilde{y}) d\tilde{y}$$

for $j = 1, \ldots, 2$ and

$$\int_{\tilde{B}_\rho(0)} \left[ \frac{(f(\tilde{y}) - \eta)}{(|\tilde{y}|^2 + (f(\tilde{y}) - \eta)^2)^{3/2}} - \frac{(f(\tilde{y}) + \eta)}{(|\tilde{y}|^2 + (f(\tilde{y}) + \eta)^2)^{3/2}} \right] F(\tilde{y}) d\tilde{y}$$

for $j = m$. Here

$$F(\tilde{y}) := \sqrt{1 + |\nabla f(y^{(m-1)})|^2}\, \psi(\tilde{y}, f(\tilde{y}))\, \nu_i(\tilde{y}, f(\tilde{y}))$$

is a function in $L^\infty(\tilde{B}_\rho(0))$, and hence there is a $C > 0$ such that $|F(\tilde{y})| \le C$ for almost all $\tilde{y} \in \tilde{B}_\rho(0)$. In order to estimate the above integrals it therefore suffices to estimate

$$\int_{\tilde{B}_\rho(0)} \left| \frac{y_j}{(|\tilde{y}|^2 + (f(\tilde{y}) - \eta)^2)^{3/2}} - \frac{y_j}{(|\tilde{y}|^2 + (f(\tilde{y}) + \eta)^2)^{3/2}} \right| d\tilde{y} \tag{7.15}$$

for $j = 1, 2$ and

$$\int_{\tilde{B}_\rho(0)} \left| \frac{(f(\tilde{y}) - \eta)}{(|\tilde{y}|^2 + (f(\tilde{y}) - \eta)^2)^{m/2}} - \frac{(f(\tilde{y}) + \eta)}{(|\tilde{y}|^2 + (f(\tilde{y}) + \eta)^2)^{m/2}} \right| d\tilde{y}. \tag{7.16}$$

In fact these are exactly the integrands estimated in [170] using simple algebraic manipulations (see also Appendix A.1 in [55]). Upon substitution of $\tilde{y} = \eta \tilde{u}$ these calculations imply that the integrals (7.15) and (7.16) are bounded by

$$\int_{\tilde{B}_{\rho/\eta}(0)} \frac{1}{(|\tilde{u}|^2 + 1)^{3/2}}\, d\tilde{u} < +\infty,$$

uniformly in $0 < \eta \le r$, and $x \in \partial D \cap B_r(0)$. This completes the proof of Lemma 7.4. $\square$

### 7.2.2 ▪ Regularity Results for Nonscattering Inhomogeneities

In the proof of the main results concerning regularity properties of inhomogeneities that may admit nonscattering wave numbers, we shall make use of a regularity result about $H_0^2(D)$ solutions to (7.5)–(7.7). A central ingredient in the proof of this regularity result is the regularity analysis for the volume potential found in the previous section.

**Proposition 7.5.** *Assume that $\partial D$ is Lipschitz, $0 \in \partial D$, and the refractive index is given by $n \in L^\infty(D)$. Furthermore, we assume that $n \in C^\alpha(\overline{D} \cap B_R(0))$ for some ball $B_R(0)$ of radius $R$ centered at $0$ and some $0 < \alpha \le 1$. Then $u \in H_0^2(D)$ that satisfies (7.5)–(7.7) lies in $C^1(\overline{D})$ and has all its second derivatives $\left\{ \frac{\partial^2 u}{\partial x_i \partial x_j} \right\}$, $i, j = 1, 2, 3$, uniformly bounded in $D \cap B_r(0)$ for some $r > 0$.*

***Proof.*** First we remark that the incident field $v$ is real analytic in $\overline{D}$, as it is an $L^2$-solution of the Helmholtz equation in a region containing $\overline{D}$. Introduce the function

$$U(x) = \begin{cases} u(x) & \text{for } x \in D\,, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus D\,. \end{cases}$$

This function is in $H^2(\mathbb{R}^3)$ (since $u \in H_0^2(D)$). Hence it follows from the Sobolev embedding theorem that $U \in C^\alpha(\mathbb{R}^3)$ for some $0 < \alpha < 1$. $U$ is a solution of

$$\Delta U = \Psi \text{ in } \mathbb{R}^3, \quad \text{where } \Psi(x) = \begin{cases} \psi(x) & \text{for } x \in D, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus D \end{cases}$$

with $\psi(x) = k^2(1-n(x))v(x) - k^2 n(x)u(x),\ x \in D$. The function $\psi$ is clearly in $L^\infty(D)$ and, due to the assumptions about $n$ and $v$ and the $C^\alpha$ extendability of $u$, it has an extension that lies in $C^\alpha(B_r(0))$. The solution $U$ is now given by the formula

$$U(x) = -\int_D \psi(y)\Phi(x,y)\,dy = -w_\psi(x)$$

with $\psi = k^2(1-n)v - k^2 nu \in L^\infty(D) \cap C^\alpha(B_r(0))$. Form the above we have that $U \in C^1(\mathbb{R}^3)$ and, since $U = 0$ outside $D$, Lemma 7.4 implies that all second derivatives of $u$ are uniformly bounded in $D \cap B_r(0)$ for some $r > 0$. $\square$

**Remark 7.6.** In the above proof of Proposition 7.5 it is shown that $U$ is in $C^1(\mathbb{R}^3)$; as a consequence $u$ has an extension (by zero) which is in $C^1(\mathbb{R}^3)$. We also note that the fact that all second derivatives of $u$ are shown to be uniformly bounded in $D \cap B_r(0)$ implies that $u$ is in $C^{1,1}(\overline{D} \cap B_r(0))$.

The application of Theorem 7.1 requires real valued functions. With this in mind, we note that the real valued function $w = \Re(u)$ is an $H^2(D)$ solution to

$$\Delta w + k^2 nw = -k^2(n-1)\Re(v) \quad \text{with } w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial D\,. \tag{7.17}$$

Since the incident wave is an $L^2$ solution to $\Delta v + k^2 v = 0$ in a neighborhood of $\overline{D}$, it follows that $\Re(v)$ is a real analytic function in $\overline{D}$. In particular, Proposition 7.5 also applies to $w$. Of course one could consider the imaginary part of the scattered field $u$ which satisfies the same equation as above with $\Re(v)$ replaced by $\Im(v)$. Accordingly, in what follows, everything holds true if we replace $\Re(v)$ by $\Im(v)$.

To apply Theorem 7.1 to (7.17) we must first appeal to Theorem 7.3 in order to establish that $w \in C^2(\overline{D} \cap B_r(0))$ and that $\partial D \cap B_r(0)$ is of class $C^1$. Proposition 7.5 (see also Remark 7.6) guaranties that $w \in C^{1,1}(\overline{D} \cap B_r(0))$ and that $g = -k^2(nw + (n-1)\Re(v))$ has a $C^1$ extension to all of $\mathbb{R}^3$. The essential missing step for the application of Theorem 7.3 is therefore to show that $w$ is of one sign. This is established by the following proposition.

**Proposition 7.7.** *Assume that $\partial D$ is Lipschitz, $0 \in \partial D$, and the refractive index is given by $n \in L^\infty(D)$. Furthermore, suppose $n$ lies in $C^{1,1}(\overline{D} \cap B_r(0))$ for some ball $B_r(0)$ of radius r centered at $0$ and suppose $(n(0)-1)\Re(v(0)) \neq 0$. Let $w \in H_0^2(D)$ be a solution to* (7.17)*. Then $w < 0$ in $D \cap B_r(0)$ for some $r > 0$ if $(n(0)-1)\Re(v(0)) > 0$, and $w > 0$ in $D \cap B_r(0)$ for some $r > 0$ if $(n(0)-1)\Re(v(0)) < 0$.*

Due to its technical nature we do not present the proof of Proposition 7.7. Instead we refer the reader to the proof of Proposition 5.3 in [55], which follows almost verbatim the analysis in [170].

We are now ready to state and prove the main results of this section. These results are stated in terms of sufficient conditions of nonsmoothness of $\partial D$ for scattering to occur for a given incident wave. By negation they could as well have been stated as necessary smoothness conditions that follow from nonscattering. In the formulation of the results we refer to the region $D_\delta \subset D$ defined by

$$D_\delta := \{x \in D,\ dist(x, \partial D) < \delta\} \qquad \text{for some fixed } \delta > 0.$$

**Theorem 7.8.** *Let $k > 0$ be a fixed wave number. Assume the positive refractive index $n$ is in $L^\infty(D)$ and that the boundary $\partial D$ is Lipschitz. Consider an incident field $v$ satisfying* (7.9). *Assume that $n$ is real analytic in $\overline{D_\delta}$ and there exists $x_0 \in \partial D$ such that $(n(x_0) - 1)v(x_0) \neq 0$. Assume furthermore that $\partial D \cap B_r(x_0)$ is not real analytic for any ball $B_r(x_0)$ of radius $r$ centered at $x_0$. Then the incident field $v$ is scattered by the inhomogeneity $(D, n)$. In other words there exists no function $u \in H_0^2(D)$ satisfying* (7.5)–(7.7).

***Proof.*** The proof proceeds by contradiction. Suppose the incident field $v$ is not scattered by $(D, n)$. Without loss of generality we assume that $(n(x_0) - 1)\Re(v(x_0)) \neq 0$ and we choose $x_0$ to be the origin of the coordinate system (the argument works similarly if $(n(x_0) - 1)\Im(v(x_0)) \neq 0$). The function $w = \Re(u)$ is a solution to (7.17). Since the incident wave is an $L^2$ solution to $\Delta v + k^2 v = 0$ in a neighborhood of $\overline{D}$, it follows that $\Re(v)$ is a real analytic function on $\overline{D}$. By assumption the refractive index $n$ is also real analytic on $\overline{D_\delta}$ and so the assumptions of Propositions 7.5 and 7.7 are satisfied. In particular, Proposition 7.5 (and the remark following) implies that $w \in C^{1,1}(\overline{D} \cap B_r(0))$ for some ball $B_r(0)$, and that it has a $C^1$ extension to all of $\mathbb{R}^3$. Proposition 7.7 implies that $w \geq 0$ or $w \leq 0$ in $\overline{D} \cap B_r(0)$, depending on whether $(n(0) - 1)\Re(v(0)) < 0$ or $(n(0) - 1)\Re(v(0)) > 0$, respectively. We now introduce $g := -k^2(nw + (n - 1)\Re(v))$. Thanks to the $C^1$ extendability of $w$, and the analyticity of $n$ and $\Re(v)$, the function $g$ has a $C^1$-extension $g^*$ in a neighborhood of $\overline{D} \cap B_R(0)$. Since $w$ vanishes at $\partial D$, $g(0) = -k^2(n(0) - 1)\Re(v(0))$, and so it follows that $g^* \geq \gamma > 0$ in $\overline{D} \cap B_r(0)$ or $g^* \leq -\gamma < 0$ in $\overline{D} \cap B_r(0)$ (with $r$ sufficiently small), depending on whether $(n(0) - 1)\Re(v(0)) < 0$ or $(n(0) - 1)\Re(v(0)) > 0$, respectively. Since $w$ satisfies $\Delta w = g$ in $D$, the assumptions of Theorem 7.3 are now satisfied for $w$ if $(n(0) - 1)\Re(v(0)) > 0$, and for $-w$ if $(n(0) - 1)\Re(v(0)) < 0$. In both cases we may therefore conclude that $w \in C^2(\overline{D} \cap B_r(0))$ and $\partial D \cap B_r(0)$ is of class $C^1$.

We now apply Theorem 7.1. We set $a(x) = k^2 n(x)$ and $b(x) = k^2(1 - n(x))\Re(v(x))$; then $a$ and $b$ are both real analytic. By assumption $a(0),\ b(0) \neq 0$ and $w \in C^2(\overline{D} \cap B_r(0))$ satisfies

$$\Delta w + aw = b \text{ in } D \qquad \text{with } w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial D\,,$$

where $\partial D \cap B_r(0)$ is known to be of class $C^1$. The third case in Theorem 7.1 yields that $\partial D \cap B_r(0)$ is real analytic for $r$ sufficiently small. However, this represents a contradiction and so we conclude that the incident field $v$ is scattered by $(D, n)$, thus completing the proof of Theorem 7.8. □

For less regular refractive index $n$ there is a similar result.

**Theorem 7.9.** *Let $k > 0$ be a fixed wave number. Assume the positive refractive index $n$ is in $L^\infty(D)$ and that the boundary $\partial D$ is Lipschitz. Consider an incident field $v$ satisfying* (7.9). *Assume that $n \in C^{m,\mu}(\overline{D_\delta}) \cap C^{1,1}(\overline{D_\delta})$ for $m \geq 1,\ 0 < \mu < 1$, and there exists $x_0 \in \partial D$ such that $(n(x_0) - 1)v(x_0) \neq 0$. Assume furthermore that $\partial D \cap B_r(x_0)$ is not of class $C^{m+1,\mu}$ for any ball $B_r(x_0)$ of radius $r$ centered at $x_0$. Then the incident field $v$ is scattered by the inhomogeneity $(D, n)$.*

***Proof.*** The proof of this result, which applies to less regular refractive index $n$, can be done in the exact same manner. The regularity result of Propositions 7.5 and 7.7 are still applicable since we have assumed that $n \in C^{1,1}(\overline{D}_\delta)$. For the free boundary regularity we rely on case 2 of Theorem 7.1. □

**Remark 7.10.** The smoothness assumptions on the refractive index $n$ in Theorems 7.8 and 7.9 are only needed locally in $\overline{D} \cap B_R(x_0)$ for some ball centered at $x_0$ of radius $R > 0$.

Of course Theorems 7.8 and 7.9 only add insight if the wave number $k$ is a real transmission eigenvalue (which is a necessary condition for the incident field to produce a vanishing scattered field). At any $k$ other than a transmission eigenvalue every incident field is scattered by the given inhomogeneity. However, it is important to emphasize that we do not need to know a priori that $k > 0$ is a transmission eigenvalue, and therefore the results hold under weaker conditions on the contrast than those (currently) needed to prove the existence of real transmission eigenvalues. If $k > 0$ is a transmission eigenvalue, the assumptions in Theorems 7.8 and 7.9 imply that the part $v$ of the transmission eigenfunction cannot be extended into the exterior of $D$ as a solution of the Helmholtz equation, provided $n \neq 1$ on $\partial D$ and that this eigenfunction does not vanish at the point $x_0 \in \partial D$. We also remark that a more relaxed starting regularity than Lipschitz in the study of nonscattering inhomogeneity is used in [155] based on the techniques in [4].

Theorems 7.8 and 7.9 can also be interpreted as up-to-the-boundary regularity of the $v$-part of the transmission eigenfunction. Without loss of generality, we may assume that the eigenfunction $(u, v)$ is real valued. In general, since $v$ assumes no boundary condition, it is not possible from the equations to conclude any regularity for $v$ up to the boundary. Our free boundary regularity results provide some insight into this issue. Recall that Theorems 7.8 and 7.9 state necessary regularity conditions on $\partial D$ in order that (7.9) can have an $H_0^2(D)$ solution ($v$ being defined and regular in an $\mathbb{R}^3$ neighborhood of $\partial D$). It is clear from the analysis that the statements of Theorems 7.8 and 7.9 are valid if $v$ is only defined on one side of $\partial D$ and the regularity of $v$ up to the boundary matches that of $n$. Also notice that the employed arguments rely only on the local regularity of the source term $(1-n)v$ in $\overline{D} \cap B_R(x_0)$. We thus have the following consequence of the proofs of Theorems 7.8 and 7.9.

**Corollary 7.11.** *Assume $k > 0$ is a real transmission eigenvalue with eigenfunction $(u, v)$, $\partial D$ is Lipschitz, $n \in L^\infty(D)$, and there exists $x_0 \in \partial D$ such that $n(x_0) - 1 \neq 0$. The following assertions hold:*

1. *If $n$ is real analytic in a neighborhood of $x_0$ and $\partial D \cap B_r(x_0)$ is not real analytic for any ball $B_r(x_0)$, then $v$ cannot be real analytic in any neighborhood of $x_0$, unless $v(x_0) = 0$.*

2. *If $n \in C^{m,\mu}(\overline{D} \cap B_R(x_0)) \cap C^{1,1}(\overline{D} \cap B_R(x_0))$ for $m \geq 1$, $0 < \mu < 1$, and some ball $B_R(x_0)$, and $\partial D \cap B_r(x_0)$ is not of class $C^{m+1,\mu}$ for any ball $B_r(x_0)$, then $v$ cannot lie in $C^{m,\mu}(\overline{D} \cap B_r(x_0)) \cap C^{1,1}(\overline{D} \cap B_r(x_0))$ for any ball $B_r(x_0)$, unless $v(x_0) = 0$.*

A direct consequence of Corollary 7.11 is that at a transmission eigenvalue the part $v$ of the transmission eigenfunction lacks sufficient regularity near a singular boundary point $x_0$ unless it vanishes at this point, thus recovering similar results for the case of corners in [16, 17].

**Remark 7.12.** The nondegeneracy condition, which for our nonscattering configuration amounts to $n(x_0) \neq 1$ and $v(x_0) \neq 0$ at the boundary point of interest $x_0 \in \partial D$, is inherent in the free boundary methods for obstacle problems. Unfortunately, this nonvanishing property of the incident field $v$ is not in general guarantied for solutions $\Delta v + k^2 v = 0$ in $\Omega \supseteq \overline{D}$. Obviously, incident plane waves $v = e^{ikx \cdot d}$ with $d$ a unit vector, or point sources $v = \frac{e^{ik|x-z|}}{4\pi|x-z|}$ with $z$ outside $\Omega$, are nonvanishing incident fields, and all the above nonscattering results apply to them. However, the superposition of plane waves, a.k.a. Herglotz wave functions (1.31) that are related to injectivity of the far field operator $F_k$, may vanish at boundary points. In the case when $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D$, in [155], using a denseness property of Herglotz wave functions in $C^k(\overline{D})$, it is shown that there are plenty of Herglotz wave functions that are positive on $\partial D$. Since nonscattering wave numbers are a subset of transmission eigenvalues, the latter assumption excludes $k^2$ that are simultaneously transmission and Dirichlet eigenvalues, and the size of such a set is not yet understood.

Note that in the above analysis we start with a Lipschitz domain and conclude higher regularity of the boundary $\partial D$ for the inhomogeneity to possibly be nonscattering. In [155], the authors relax the starting Lipschitz regularity of $D$ to merely so-called solid body, i.e., $\text{int}(\overline{D}) = D$. Then they conclude that a nonscattering inhomogeneity has to be sufficiently regular as stated in Theorems 7.8 and 7.9 or the complement of $D$ is thin near a boundary point $x_0$ (see Theorem 1.4 in [155] for a precise definition) such as a reentering cusp at $x_0$. In a follow-up paper [118], regions containing reentering cusp points on the boundary are examined in this context.

The above results do not make any statement on the existence of nonscattering wave numbers for inhomogeneities with analytic boundary $\partial D$ other than spheres. An attempt to construct nonspherical analytic inhomogeneities that do not scatter is made in [155]. However, from their construction it is not clear how to obtain a positive refractive index $n > 0$, which is a physical requirement. We conclude this section by remarking that in [168] it is shown that, given any smooth, strictly convex domain in $\mathbb{R}^2$, there exist at most finitely many positive wave numbers $k$ for which an (arbitrary but fixed) incident plane wave can be nonscattering. In [19] the authors prove that inhomogeneities containing a boundary point of high curvature (near which the inhomogeneity could be analytic) scatter any incident field whose modulus is bounded away from zero by a constant depending on the curvature and the value of the contrast $n - 1$ at this point.

### 7.2.3 ▪ Corners Always Scatter. A Uniqueness Theorem in Inverse Scattering

In the case when the support $D$ of the inhomogeneity contains a circular conical point, a vertex, or an edge, then the set of nonscattering wave numbers is empty, provided $n-1 \neq 0$ on the singular portion of the boundary and some local regularity on $n$. In other words such inhomogeneities scatter any incident wave without any nonvanishing property as opposed to the above results based on free boundary methods. This result was first proven in [20] for a right vertex, followed by [143] for an arbitrary convex corner. The analysis in these papers is specifically tailored to inhomogeneities exhibiting a corner singularity and employs the so-called complex geometrical optics (CGO) solutions for the Helmholtz equation. More precisely, in the aforementioned papers one starts with the transmission eigenvalue problem corresponding to a real transmission eigenvalue and shows that the part $v$ of the eigenfunctions cannot be real analytic in a neighborhood of the vertex. In particular, if $x_0 \in \partial D$ is a vertex and $B_\epsilon(x_0)$ the ball of radius $\epsilon$ centered at $x_0$, it is easy to show that for the transmission eigenfunctions $u$, $v$, and any solution of

$$\Delta\psi + k^2 n\psi = 0 \qquad \text{in } B_\epsilon(x_0) \cap D,$$

we have that

$$\int\limits_{\partial B_\epsilon(x_0)\cap D} \frac{\partial\psi}{\partial\nu}u - \frac{\partial u}{\partial\nu}\psi\, ds = \int\limits_{B_\epsilon(x_0)\cap D} k^2(1-n)v\psi\, dx,$$

where $\partial B_\epsilon(x_0) \cap D$ is the part of the sphere inside $D$ and we use that $u = \partial_\nu u = 0$ on $\partial D \cap B_\epsilon(x_0)$. Then using as test functions $\psi$ from a family of CGO solutions which decay quickly away from $x_0$ combined with asymptotic analysis, it is possible to arrive at a contradiction if one assumes that $v$ is analytic in $B_\epsilon(x_0)$. This approach is generalized in [57] to the scattering problem modeled by

$$\nabla \cdot a(x)\nabla u + k^2 n(x)u = 0$$

with $C^2(\overline{D})$-smooth positive scalar function $a$ and positive $n$ with $a - 1 \neq 0$ on the boundary $\partial D$. This case brings up some anomalous configurations where domains with corners can allow for nonscattering wave numbers (see [171]).

The most comprehensive analysis, implying that corner and edge singularities always scatter in $\mathbb{R}^3$, is given in [84] based on a refined corner singularity analysis of the solution to (7.5)–(7.7). In the following we state precisely this result and refer the reader to [84] for the proof.

**Theorem 7.13.** *Let $D, n$ be an inhomogeneity with refractive index $n \in L^\infty(D)$ and $\overline{D} = \mathrm{Supp}(n-1)$ is bounded in $\mathbb{R}^3$. Assume that there exists a boundary point $x_0 \in \partial D$ that is a vertex, an edge point, or a circular conic point (see Definitions 2 and 3 in [84] for the precise definition of these boundary singularities). Furthermore assume that there exist $m \in \mathbb{N}_0$, $\mu \in (0,1)$, and an $\epsilon > 0$ such that $n \in C^{m,\mu}(\overline{D} \cap B_\epsilon(x_0)) \cup W^{m,\infty}(B_\epsilon(x_0))$, $\nabla^m(n-1)(x_0) \neq 0$, and $n(x) > 0$ for $x \in \overline{D} \cap B_\epsilon(x_0)$. Then this inhomogeneity scatters every incident wave nontrivially.*

Note that the result of the above theorem is a local property in a neighborhood of the singular point $x_0$. The proof shows that it is not possible to have nontrivial $u$ and $v$ satisfying

$$\Delta u + k^2 n(x)u = 0 \qquad \text{in } D \cap B_\delta(x_0), \tag{7.18}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } B_\delta(x_0), \tag{7.19}$$

$$u = v \quad \text{and} \quad \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} \qquad \text{on } \partial D \cap B_\delta(x_0) \tag{7.20}$$

for some ball $B_\delta(x_0)$ centered at $x_0$ with radius $\delta$.

We conclude this section with an application of Theorem 7.13 to the inverse scattering problem. More specifically, we prove that the support of a convex polyhedral inhomogeneous medium is uniquely determined from scattering data corresponding to one single incident wave. This proof was first given in [100].

**Lemma 7.14.** *Let $D_j \in \mathbb{R}^3$, $j = 1,2$, be two inhomogeneities with respective refractive indices $n_j \in L^\infty(D_j)$. Assume that at each vertex, edge point, or circular conic point $x_0^j \in \partial D_j$, there exists $m \in \mathbb{N}_0$, $\mu \in (0,1)$, and an $\epsilon > 0$ such that*

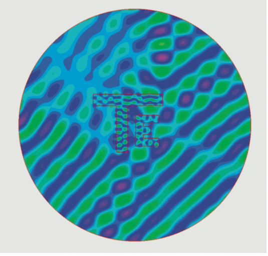
$n_j \in C^{m,\mu}(\overline{D_j} \cap B_\epsilon(x_0^j)) \cup W^{m,\infty}(B_\epsilon(x_0^j))$*,* $\nabla^m(n_j - 1)(x_0^j) \neq 0$*, and* $n_j(x) > 0$ *for* $x \in \overline{D_j} \cap B_\epsilon(x_0)$*. If* $\partial D_2$ *differs from* $\partial D_1$ *in the presence of a vertex, edge point, or circular conic point lying on the boundary of the unbounded component* $\mathbb{R}^3 \setminus \overline{(D_1 \cup D_2)}$*, then the far field patterns corresponding to* $(D_1, n_1)$ *and* $(D_2, n_2)$ *due to any incoming incident wave cannot coincide.*

***Proof.*** Let $w_j = u_j^s + u^i$, $j = 1, 2$, be the solutions of (1.20)–(1.22) corresponding to $(D_j, n_j)$, respectively, due to an incident field $u^i$ (which can, for example, be a plane wave or a point source). If the far field patterns satisfy $u_1^\infty(\hat{x}) = u_2^\infty(\hat{x})$, then by Rellich's Lemma (Lemma 1.6) the corresponding scattered fields satisfy $u_1^s(x) = u_2^s(x)$ for $x \in G$, which is the unbounded component of $\mathbb{R}^3 \setminus \overline{(D_1 \cup D_2)}$. Assume that the singular point $x_0 \in \partial D_1$ but $x_0 \notin \partial D_2$. Let $B_\delta(x_0)$ be a ball centered at $x_0$ with radius $\delta$ sufficiently small such that it lies entirely in the exterior of $\overline{D_2}$. Then the total fields satisfy $w_1 \equiv w_2$ in $B_\delta(x_0) \cap (\mathbb{R}^3 \setminus D_1)$, $w_2$ satisfies the Helmholtz equation in $B_\delta(x_0)$ as the total field in the exterior of $D_2$, and $w_1$ satisfies $\Delta w_1 + k^2 n_1 w_1 = 0$ in $D_1 \cap B_\delta(x_0)$ as the total field in the interior of $D_1$. This means that the nontrivial $u := w_1$ and $v := w_2$ satisfy (7.18)–(7.20) with $D := D_1$ and $n := n_1$, which is not possible. $\square$

Clearly, the geometrical assumptions in Lemma 7.14 are fulfilled if $D_1$ and $D_2$ are distinct convex polyhedra with piecewise flat boundaries. Hence, we obtain the following global uniqueness results for the inverse scattering problem.

**Theorem 7.15.** *Let* $(D, n)$ *be an inhomogeneity with refractive index* $n \in L^\infty(D)$*. Assume that* $D \subset \mathbb{R}^3$ *is a convex polyhedron such that for any of the vertices* $x_0$ *there exists* $m \in \mathbb{N}_0$*,* $\mu \in (0, 1)$*, and an* $\epsilon > 0$ *such that* $n \in C^{m,\mu}(\overline{D} \cap B_\epsilon(x_0)) \cup W^{m,\infty}(B_\epsilon(x_0))$ *and* $\nabla^m(n - 1)(x_0) \neq 0$ *and* $n(x) > 0$ *for* $x \in \overline{D} \cap B_\epsilon(x_0)$*. Then* $D$ *can be uniquely determined from knowledge of the far field pattern due to a single incident wave.*

Note that, based on Lemma 7.14, we can phrase the above theorem to state that the polyhedron convex hull of a collection of inhomogeneities is uniquely determined from the measurements due to one single incident wave, provided that at every vertex of this convex hull the refractive index $n$ satisfies the assumptions stated in Theorem 7.15. For a discussion on stability we refer the reader to [18].



# Chapter 8
# Transmission Eigenvalues versus Scattering Poles

While the set of transmission eigenvalues and the set of nonscattering wave numbers $k$ are related to the kernel of the far field operator $F_k$ defined in Section 1.2.1, the study of this operator (which in the literature [134] is also referred to as the relative scattering operator, or incoming-outgoing operator) brings up another spectral set of values of $k$, namely, the scattering poles or scattering resonances. The theory of scattering resonances is a rich and beautiful part of scattering theory and, although the notion of resonances is intrinsically dynamical, an elegant mathematical formulation comes from considering them as the poles of the meromorphic operator valued mapping $k \in \mathbb{C} \mapsto F_k$. We refer the reader to the comprehensive monograph [82] for an account of the vast literature on the subject. Scattering poles exist and are complex with negative imaginary part. They capture physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. For a given inhomogeneity, at a nonscattering wave number there is zero scattered field for a nonzero incident field, whereas at a scattering pole, there is a nonzero scattered field in the absence of an incident field.

Recalling that $\Phi_k(x,y)$ denotes the radiating fundamental solution of the Helmholtz equation defined by (1.8), it is shown in Theorem 1.9 that solving the scattering problem is equivalent to solving the Lippmann–Schwinger equation

$$(I - T(k))w = v, \qquad T(k)w := k^2 \int_{\mathbb{R}^3} \Phi_k(x,y)(n(y)-1)w(y)\,dy, \tag{8.1}$$

where the integral operator $T(k) : L^2(D) \to L^2(D)$ is compact. Here $w$ is the total field which is decomposed as $w = u + u^i$, where the scattered field $u \in H^2_{loc}(\mathbb{R}^3)$ satisfies (see Section 1.1)

$$\Delta u + k^2 n u = k^2(1-n)v \qquad \text{in } \mathbb{R}^3, \tag{8.2}$$

together with the Sommerfeld radiation condition (1.5), and $v := u^i|_D$ with $u^i$ being the incident field. The Fredholm alternative ensures that injectivity of $I - T(k)$ implies its bounded invertibility. Since by Rellich's Lemma (Lemma 1.6) if $\Im(k) \geq 0$, the kernel of $I - T(k)$ is trivial, and the fact that the mapping $k \in \mathbb{C} \mapsto T(k) \in \mathcal{L}(L^2(D))$ is analytic, we obtain by the Analytic Fredholm Theorem 1.12 that the kernel of $I - T(k)$ is at most discrete without finite accumulation points. Thus

$$w = (I - T(k))^{-1} v \in L^2(D)$$

is well defined for all $k \in \mathbb{C}$ except for possibly a discrete set of $k$ in the lower half-plane $\Im(k) < 0$. These are the poles of the meromorphic operator valued function $k \mapsto (I - T(k))^{-1}$, and are precisely what is referred to as the scattering poles. This leads to the following definition of the scattering poles.

**Definition 8.1.** *A value of $k \in \mathbb{C}$ with $\Im(k) < 0$ is a scattering pole if the kernel of $I - T(k) \in \mathcal{L}(L^2(D))$ is nontrivial.*

Note that at a scattering pole there is a nonzero scattered field $u$ corresponding to a trivial incident field. If $k$ is not a scattering pole, then the scattered field outside $D$ corresponding to $v := u^i|_D$ is defined by

$$u(x) = k^2 \int\limits_{\mathbb{R}^3} \Phi_k(x,y)(n(y)-1)\left[(I - T(k))^{-1} v\right](y)\, dy, \qquad x \in \mathbb{R}^3.$$

The goal of this chapter is to establish a characterization of scattering poles in a dual framework to the one for transmission eigenvalues.

## 8.1 ▪ Scattering Poles for Spherically Stratified Media

Similarly to Section 7.1 we consider the scattering problem by a ball $B$ of radius one centered at the origin with spherically stratified refractive index $n(r)$ which we assume to be in $C[0,1]$. Taking $v := j_\ell(k|x|)Y_\ell(\hat{x})$ for $|x| < 1$ (see below (7.1)) in (8.1) and using the addition formula for $\Phi_k(x,y)$ (see, e.g., [69]) it is easy to see that the corresponding scattered field is given by

$$u := \frac{C_\ell(k;n)}{W_\ell(k;n)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}), \qquad |x| > 1,$$

for any $k \in \mathbb{C}$ such that $W_\ell(k;n) \neq 0$, where

$$W_\ell(k;n) = \operatorname{Det}\begin{pmatrix} y_\ell(1) & -h_\ell^{(1)}(k) \\ y_\ell'(1) & -kh_\ell^{(1)'}(k) \end{pmatrix} \tag{8.3}$$

(see (7.2) for the definition of all expressions). It is easily verifiable that $C_\ell(k;n)$ and $W_\ell(k;n)$ do not vanish at the same $k$. Hence the values of $k$ for which $W_\ell(k;n) = 0$ correspond exactly to scattering poles. If $k$ is such that $W_\ell(k;n) = 0$, then

$$u = h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}) \quad \text{for} \quad |x| > 1 \qquad \text{and} \qquad u = y_\ell(|x|)Y_\ell(\hat{x}) \quad \text{for} \quad |x| < 1$$

satisfies (8.1) with $v = 0$; thus in this case there is a nonzero scattered field with zero incident field.

The existence of scattering poles in the lower half-complex-plane for this spherically stratified case can be obtained from more general results contained in Theorems 2.10 and 2.16 of [82] (see also [157] and references therein). We present here a simple argument that shows the existence of an infinite set of scattering poles that are the zeros of $W_0(k;n) = 0$, which can be rewritten as the zeros of

$$\widetilde{W}_0(k;n) := \operatorname{Det}\begin{pmatrix} y(1) & e^{ik}/k \\ y'(1) & ie^{ik} \end{pmatrix} = 0,$$

where we denote $y_0(r) := y(r)/r$, i.e., $y(r)$ satisfies $y'' + k^2 n(r) y = 0$. Hence for our purpose it suffices to analyze only $\widetilde{W}_0(k;n)$, which corresponds to the scattering poles with spherically symmetric eigenfunctions. For this case, assuming that $n(1) \neq 1$, we will show that there exist infinitely many scattering poles. From (6.17) and (6.21) one can see that for $k$ in a neighborhood of zero, $y_0(1)$ behaves like $j_0(k)$. Hence $k\widetilde{W}_0(k;n)$ is an entire function. Furthermore $k\widetilde{W}_0(k;n)|_{k=0} = -1$. Thus by Hadamard's factorizations theorem

$$k\widetilde{W}_0(k;n) = -e^{\alpha k} \prod_{j=1}^{\infty} \left(1 - \frac{k}{k_j}\right) e^{k/k_j}, \tag{8.4}$$

where $\alpha$ is a complex constant and $k_j$ are the zeros of $k\widetilde{W}_0(k;n)$, i.e., scattering poles, which we know are complex with negative imaginary part. Taking large $k > 0$, from Section 6.2 we have that $y$ behaves as

$$y(r) = \frac{1}{k\left[n(0)\, n(r)\right]^{1/4}} \sin\left(k \int_0^r \left[n(\rho)\right]^{1/2} d\rho\right) + O\left(\frac{1}{|k|^2}\right), \tag{8.5}$$

$$y'(r) = \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos\left(k \int_0^r [n(\rho)]^{1/2}\, d\rho\right) + O\left(\frac{1}{|k|}\right). \tag{8.6}$$

In particular, we have that $k\widetilde{W}_0(k;n)$ remains bounded oscillating as $k \to +\infty$. Obviously, if there were no zeros of $k\widetilde{W}_0(k;n)$, then from (8.4) $k\widetilde{W}_0(k;n) = -e^{\alpha k}$, which, since $n(1) \neq 1$, does not match this asymptotic behavior. On the other hand, if the product in (8.4) is finite, i.e., there are only finitely many zeros of $k\widetilde{W}_0(k;n)$, then $k\widetilde{W}_0(k;n)$ would either go to zero or become unbounded as $k \to +\infty$. This proves the existence of an infinite number of zeros of $\widetilde{W}_0(k;n)$, i.e., scattering poles.

## 8.2 ▪ Duality between Transmission Eigenvalues and Scattering Poles

In this section our goal is to establish a duality between the set of scattering poles and the set of transmission eigenvalues. In particular, these two sets are interchangeable if instead of the scattering operator for the scattering problem (8.2) we consider the scattering operator for an appropriate interior scattering problem. This concept of duality was first introduced in [36] (see also [37]).

We assume that the refractive index $n$ is a complex valued $L^\infty$ function with $\Re(n) > 0$ and $\Im(n) \geq 0$, such that $n - 1$ is supported in $\overline{D}$. Unless otherwise indicated, we assume that the boundary $\partial D$ is Lipschitz smooth. We start by providing an equivalent definition of the scattering poles. Given $v \in L^2(D)$ we define the scattering problem associated with an incident field $v$ as determining the scattered field $u \in H^2_{loc}(\mathbb{R}^3)$ such that

$$\left\{ \begin{array}{ll} \Delta u + k^2 n u = k^2(1-n)v & \text{in } \mathbb{R}^3, \\ u = \mathcal{S}_k^{\partial D}(\partial u/\partial \nu) - \mathcal{D}_k^{\partial D}(u) & \text{in } \mathbb{R}^3 \setminus \overline{D}, \end{array} \right. \tag{8.7}$$

where $\mathcal{S}_k^{\partial D}$ and $\mathcal{S}_k^{\partial D}$ are the single and double layer potentials defined by (3.71) and (3.72), respectively (to avoid any ambiguity going forward we indicate the surface where these potentials are defined). For the reader's convenience we recall the definition of the single layer potential $\mathcal{S}_k^{\partial D} : H^{s-1/2}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^3 \setminus \partial D)$ (see also [133] for the mapping properties),

$$\mathcal{S}_k^{\partial D}(\psi)(x) := \int_{\partial D} \psi(y)\Phi_k(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{8.8}$$

and double layer potential $\mathcal{D}_k^{\partial D} : H^{s+1/2}(\partial D) \to H^{s+1}_{loc}(\mathbb{R}^3 \setminus \partial D)$,

$$\mathcal{D}_k^{\partial D}(\psi)(x) := \int_{\partial D} \psi(y)\frac{\partial \Phi_k(x,y)}{\partial \nu_y}\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{8.9}$$

where $-1 \leq s \leq 1$ for $\partial D$ Lipschitz. In the above scattering problem $w = u + v$ denotes the total field, and $w = u$ if there is no incident field, which is the case in the context of the scattering poles.

**Proposition 8.2.** *$k \in \mathbb{C}$ is a scattering pole of the medium scattering problem* (8.7) *if and only if the homogeneous problem* (8.7)*, i.e., with $v = 0$, has a nontrivial solution $w \in H^2_{loc}(\mathbb{R}^3)$.*

***Proof.*** From Definition 8.1, there exists a $w \neq 0$ which satisfies

$$w = T(k)(w) \text{ in } L^2(D).$$

We extend $w$ to all of $\mathbb{R}^3$ using the representation

$$w(p) = \int_D k^2(n(y)-1)w(y)\Phi_k(y,p)dy, \qquad p \in \mathbb{R}^3.$$

Properties of volume potentials ensure that $w \in H^2_{loc}(\mathbb{R}^3)$ satisfies $\Delta w + k^2 n w = 0$ in $\mathbb{R}^3$ (see Theorem 1.8). We also have that for such $p$ the mapping

$$y \mapsto \Phi_k(y,p) = \Phi_k(p,y), \qquad y \in D,$$

satisfies the Helmholtz equation in $D$, where we use the symmetry of the fundamental solution. Hence for fixed $p \in \mathbb{R}^3 \setminus D$, Green's representation theorem implies that

$$\Phi_k(p,y) = \Phi_k(y,p) = \int_{\partial D} \left(\Phi_k(x,p)\frac{\partial \Phi_k(x,y)}{\partial \nu_x} - \frac{\partial \Phi_k(x,p)}{\partial \nu_x}\Phi_k(x,y)\right) ds(x) \tag{8.10}$$

for all $y \in D$. The integral representation of $w$ given in the second equation of (8.7) now is obtained by multiplying (8.10) by $k^2(n(y)-1)w(y)$ then integrating over $D$.

Conversely, consider $w \in H^2_{loc}(\mathbb{R}^3)$ satisfying (8.7) with $v = 0$. Let $B$ be a bounded domain with Lipschitz boundary containing $\overline{D}$ in its interior. Green's representation theorem in $B$ implies that

$$w(p) = \int_D k^2(n(y)-1)w(y)\Phi_k(y,p)dy - (\mathcal{S}_k^{\partial B}(\partial w/\partial \nu) - \mathcal{D}_k^{\partial B}(w))(p), \quad p \in B.$$

On the other hand, for $p \in \mathbb{R}^3 \setminus B$ and using the fact that $w$ and $\Phi(\cdot,p)$ satisfy the Helmholtz equation in the domain between $B$ and $D$, Green's formula yields

$$(\mathcal{S}_k^{\partial B}(\partial w/\partial \nu) - \mathcal{D}_k^{\partial B}(w))(p) = (\mathcal{S}_k^{\partial D}(\partial w/\partial \nu) - \mathcal{D}_k^{\partial D}(w))(p), \quad p \in \mathbb{R}^3 \setminus B.$$

Therefore,

$$w(p) = (\mathcal{S}_k^{\partial B}(\partial w/\partial \nu) - \mathcal{D}_k^{\partial B}(w))(p), \quad p \in \mathbb{R}^3 \setminus B.$$

We then infer from the continuity of $w$ across $\partial B$ and the jump relations for single and double layer potentials across $\partial B$ (3.79) and (3.80) that

$$w(p) = \int_D k^2(n(y)-1)w(y)\Phi_k(y,p)dy, \quad p \in \partial B. \tag{8.11}$$

Since $\partial B$ is an arbitrarily chosen boundary enclosing $D$, the latter identity holds for all $p \in \mathbb{R}^3 \setminus \overline{D}$. Both sides of the equality (8.11) satisfy $\Delta u + k^2 u = -k^2(n-1)w$ in $\mathbb{R}^3$. Hence unique continuation arguments imply that (8.11) holds for all $p \in \mathbb{R}^3$ and in particular $w = T(k)(w)$ in $D$, which concludes the proof. $\square$

We now consider a smooth closed surface $\partial\mathcal{C}$ circumscribing a simply connected region $\mathcal{C} \subset D$.

**Assumption 8.1.** *$k^2$ is not a Dirichlet eigenvalue of the negative Laplacian in $\mathcal{C}$, and $k$ is not a transmission eigenvalue of* (3.2).

Under the Assumption 8.1, for a point $z \in D$, let $u(\cdot,z), v(\cdot,z) \in L^2(D) \times L^2(D)$ be such that $u(\cdot,z) - v(\cdot,z) \in H^2(D)$ and satisfy

$$\begin{cases} \Delta u(\cdot,z) + k^2 n(x) u(\cdot,z) = 0 & \text{in } D, \\ \Delta v(\cdot,z) + k^2 v(\cdot,z) = 0 & \text{in } D, \\ u(\cdot,z) - v(\cdot,z) = \Phi_k(\cdot,z) & \text{on } \partial D, \\ \dfrac{\partial u(\cdot,z)}{\partial\nu} - \dfrac{\partial v(\cdot,z)}{\partial\nu} = \dfrac{\partial \Phi_k(\cdot,z)}{\partial\nu} & \text{on } \partial D. \end{cases} \tag{8.12}$$

This interior transmission problem, which is discussed extensively in Section 3.1, will play the role of a forward (interior) scattering problem that provides a new equivalent definition of scattering poles. Here we have assumed that the refractive index $n \in L^\infty(D)$, with $\Re(n) > 0$ and $\Im(n) \geq 0$, is such that the resolvent of (3.2) is Fredholm, i.e., (8.12) has a unique solution if $k$ is not a transmission eigenvalue. As is shown in Section 3.1, this is true, for example, if $\Re(n) - 1 \geq n_0 > 0$ or $1 - \Re(n) \geq n_0 > 0$ in a neighborhood of $\partial D$.

**Remark 8.3.** Complex transmission eigenvalues in the lower half-plane may exist in general, and in fact for spherically symmetric media it is proven that they do exist [69], [71]. It is not clear how to fully understand the intersection of the set of transmission eigenvalues and the scattering poles. However, in general there are infinitely many scattering poles that are not transmission eigenvalues. Indeed in [120, 149] it is proven that for inhomogeneous media there exist infinitely many scattering poles lying along the complex axis without a finite accumulation point. On the other hand, for media $(n, D)$ satisfying $\Re(n)-1 \geq n_0 > 0$ or $1-\Re(n) \geq n_0 > 0$ in a neighborhood of $\partial D$, we showed in Section 3.1.3 (see Remark 3.19) that $k := i\kappa$ for $|\kappa|$ large enough are not transmission eigenvalues

For any $k \in \mathbb{C}$, a function $w \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ that satisfies

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ w = \mathcal{S}_k^{\partial D}(\partial w/\partial\nu) - \mathcal{D}_k^{\partial D}(w) & \text{in } \mathbb{R}^3 \setminus \overline{D} \end{cases} \tag{8.13}$$

is referred to as a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$. We then denote the space of radiating solutions by

$$H^e_{inc}(D) = \{w \in H^2_{loc}(\mathbb{R}^3 \setminus D),\ w \text{ satisfies (8.13)}\}. \tag{8.14}$$

The interior scattering operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is now defined as

$$\mathcal{N}_k\varphi(x) = \int_{\partial\mathcal{C}} \varphi(z)v(x,z)\,ds(z), \qquad x \in \partial\mathcal{C}, \tag{8.15}$$

where $v$ is defined by (8.12). Obviously

$$\mathcal{N}_k : \varphi \mapsto \tilde{v}_\varphi|_{\partial\mathcal{C}}, \tag{8.16}$$

where $(\tilde{u}_\varphi, \tilde{v}_\varphi) \in L^2(D) \times L^2(D)$ is the solution to (8.12) with $\Phi_k(\cdot, z)$ replaced by $\mathcal{S}_k^{\partial\mathcal{C}}(\varphi)$. Hence

$$\mathcal{N}_k\varphi = \mathcal{G}_k\mathcal{S}_k^{\partial\mathcal{C}}(\varphi), \tag{8.17}$$

where $\mathcal{G}_k : H^e_{inc}(D) \to L^2(\partial\mathcal{C})$ is defined as the mapping

$$w \mapsto v_w|_{\partial\mathcal{C}} \tag{8.18}$$

with $(u_w, v_w) \in L^2(D) \times L^2(D)$ being the solution to (8.12), where $\Phi_k(\cdot, z)$ is replaced by $w$.

In what follows we shall keep using the notation $(\tilde{u}_\varphi, \tilde{v}_\varphi)$ and $(u_w, v_w)$ to refer to solutions of (8.12) where in the boundary data $\Phi_k(\cdot, z)$ is replaced by $\mathcal{S}_k^{\partial\mathcal{C}}(\varphi)$ and $w$, respectively.

In our discussion we use the following technical result.

**Lemma 8.4.** *Let $k \in \mathbb{C}$ and $\varphi \in L^2(\partial\mathcal{C})$. The single layer potential $w := \mathcal{S}^k_{\partial\mathcal{C}}(\varphi)$ is in $H^2_{loc}(\mathbb{R}^3 \setminus \overline{\mathcal{C}})$ and it satisfies $w = \mathcal{S}^k_{\partial D}(\partial w/\partial\nu) - \mathcal{D}^k_{\partial D}(w)$ in $\mathbb{R}^3 \setminus \overline{D}$.*

***Proof.*** The mapping property (8.8) implies that $w \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{\mathcal{C}})$. Next, for $p \in \mathbb{R}^3 \setminus \overline{D}$, we recall from the definition of $\mathcal{S}^k_{\partial\mathcal{C}}(\varphi)$ that

$$w(p) = \int_{\partial\mathcal{C}} \varphi(y)\Phi_k(p,y)\,ds(y).$$

Multiplying identity (8.10) by $\varphi \in L^2(\partial\mathcal{C})$, integrating over $\partial\mathcal{C}$, and exchanging the order of integration we obtain

$$w(p) = \mathcal{S}^k_{\partial D}\left(\frac{\partial w}{\partial\nu}\right)(p) - \mathcal{D}^k_{\partial D}(w)(p)$$

and the proof is complete. $\square$

**Theorem 8.5.** *Assume that $k \in \mathbb{C}$ is not a scattering pole of the medium scattering problem $(n, D)$ and satisfies Assumption* 8.1. *Then the operator $\mathcal{N}_k : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is symmetric and injective with dense range.*

***Proof.*** A simple exchange of integration yields that the transpose operator $\mathcal{N}_k{}^\top : L^2(\partial\mathcal{C}) \to L^2(\partial\mathcal{C})$ is given by

$$\left(\mathcal{N}_k{}^\top\varphi\right)(x) = \int_{\partial\mathcal{C}} \varphi(z)v(z,x)\,ds(z), \qquad x \in \partial\mathcal{C}. \tag{8.19}$$

Next we show that $v(x,z) = v(z,x)$ for all $x, z \in D$. Indeed, viewing $v(x,z)$ as a function $x \mapsto v(x,z)$ which solves the Helmholtz equation in $D$ we have

$$v(x,z) = -\int_{\partial D} \left( \Phi_k(y,x) \frac{\partial v(y,z)}{\partial \nu_y} - \frac{\partial \Phi_k(y,x)}{\partial \nu_y} v(y,z) \right) ds(y),$$

and viewing $v(z,x)$ as a function $z \mapsto v(z,x)$ which solves the Helmholtz equation in $D$ we have

$$v(z,x) = -\int_{\partial D} \left( \Phi_k(y,z) \frac{\partial v(y,x)}{\partial \nu_y} - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} v(y,x) \right) ds(y).$$

Therefore

$$\begin{aligned} v(x,z) - v(z,x) = &\int_{\partial D} \left( \Phi_k(y,z) \frac{\partial v(y,x)}{\partial \nu_y} - \Phi_k(y,x) \frac{\partial v(y,z)}{\partial \nu_y} \right) ds(y) \\ &+ \int_{\partial D} \left( \frac{\partial \Phi_k(y,x)}{\partial \nu_y} v(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} v(y,x) \right) ds(y). \qquad (8.20) \end{aligned}$$

Form the boundary conditions in (8.12) and Green's second identity applied to the two solutions of the Helmholtz equation in $D$, namely, $x \mapsto v(x,z)$ and $z \mapsto v(z,x)$, we have

$$\begin{aligned} &\int_{\partial D} \left( \Phi_k(y,z) \frac{\partial v(y,x)}{\partial \nu_y} - \Phi_k(y,x) \frac{\partial v(y,z)}{\partial \nu_y} \right) ds(y) \\ &\quad = \int_{\partial D} \left( u(y,z) \frac{\partial v(y,x)}{\partial \nu_y} - u(y,x) \frac{\partial v(y,z)}{\partial \nu_y} \right) ds(y) \\ &\quad\quad - \int_{\partial D} v(y,z) \left( \frac{\partial v(y,x)}{\partial \nu_y} - v(y,x) \frac{\partial v(y,z)}{\partial \nu_y} \right) ds(y) \\ &\quad = \int_{\partial D} u(y,z) \left( \frac{\partial v(y,x)}{\partial \nu_y} - u(y,x) \frac{\partial v(y,z)}{\partial \nu_y} \right) ds(y). \qquad (8.21) \end{aligned}$$

Using again the boundary conditions in (8.12) and Green's representation theorem for $x, z \in D$, $z \neq x$, we have that

$$\begin{aligned} &\int_{\partial D} \frac{\partial \Phi_k(y,x)}{\partial \nu_y} v(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} v(y,x)\, ds(y) \\ &\quad = \int_{\partial D} \left( \frac{\partial \Phi_k(y,x)}{\partial \nu_y} u(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} u(y,x) \right) ds(y) \\ &\quad\quad - \int_{\partial D} \left( \frac{\partial \Phi_k(y,x)}{\partial \nu_y} \Phi_k(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} \Phi_k(y,x) \right) ds(y) \\ &\quad = \int_{\partial D} \left( \frac{\partial \Phi_k(y,x)}{\partial \nu_y} u(y,z) - \frac{\partial \Phi_k(y,z)}{\partial \nu_y} u(y,x) \right) ds(y) \\ &\quad\quad + (\Phi_k(x,z) - \Phi_k(z,x)). \qquad (8.22) \end{aligned}$$

In the last equality we used the fact that one can find nonintersecting balls $B_\epsilon(z)$ and $B_\epsilon(x)$ inside $D$ since $z \neq x$, the fact that both $\Phi_k(\cdot, x)$ and $\Phi_k(\cdot, z)$ satisfy the Helmholtz equation in $D \setminus \left(\overline{B}_\epsilon(z) \cup \overline{B}_\epsilon(x)\right)$, and Green's representation theorem for the integrals over $\partial B_\epsilon(z)$ and $\partial B_\epsilon(x)$. Using the boundary conditions in (8.12) once again, the fact that the functions $x \mapsto u(x, z)$ and $z \mapsto u(z, x)$ solve $\Delta u + k^2 nu = 0$ in $D$, and the symmetry of $\Phi_k(x, z)$, we finally obtain

$$\begin{aligned} v(x,z) - v(z,x) &= (8.21) + (8.22) \\ &= \int\limits_{\partial D} \left( \frac{\partial u(y,x)}{\partial \nu_y} u(y,z) - \frac{\partial u(y,z)}{\partial \nu_y} u(y,x) \right) ds(y) = 0. \end{aligned}$$

Hence we have $v(x, z) = v(z, x)$ for all $x, z \in D$, which proves the symmetry of the operator $\mathcal{N}_k$.

Next, since injectivity of $\mathcal{N}_k$ implies the denseness of its range thanks to symmetry, we only need to prove injectivity. To this end, let $\mathcal{N}_k \varphi = 0$. This means that $\tilde{v}_\varphi = 0$ on $\partial\mathcal{C}$. Since $\Delta \tilde{v}_\varphi + k^2 \tilde{v}_\varphi = 0$ in $D$ and hence in $\mathcal{C}$, Assumption 8.1 guaranties that $\tilde{v}_\varphi = 0$ in $\mathcal{C}$. Therefore, by a unique continuation argument, $\tilde{v}_\varphi = 0$ in $D$. Consequently, the function $w$ defined as

$$w = \tilde{u}_\varphi \ \text{ in } D \quad \text{ and } \quad w = \mathcal{S}_k^{\partial\mathcal{C}}(\varphi) \ \text{ in } \mathbb{R}^3 \setminus D$$

is in $H^2_{loc}(\mathbb{R}^3)$ and satisfies (8.7). From the proof of Lemma 8.4 we see that $w$ satisfies the integral representation in (8.7).

Now, since $k$ is not a scattering pole, from Proposition 8.2 we conclude that $w = \mathcal{S}_k^{\partial\mathcal{C}}(\varphi) \equiv 0$ in $\mathbb{R}^3 \setminus \overline{D}$. Finally, the unique continuation principle, Assumption 8.1, and the jump relation for the normal derivative of the single layer potential across $\partial\mathcal{C}$ imply that $\varphi = 0$. This proves that $\mathcal{N}_k$ is injective and finishes the proof. ∎

**Lemma 8.6.** *Assume that $k \in \mathbb{C}$ is not a scattering pole and satisfies Assumption* 8.1. *Let $z \in \mathbb{R}^3 \setminus \overline{\mathcal{C}}$. Then $\Phi_k(\cdot, z)$ is in the range of $\mathcal{G}_k$ if and only if $z \in \mathbb{R}^3 \setminus \overline{D}$.*

***Proof.*** For $z \in \mathbb{R}^3 \setminus \overline{D}$ we define $w \in H^2_{loc}(\mathbb{R}^3)$ to be the solution of (8.7) with $v = \Phi_k(\cdot, z)|_D$. Since $\Phi_k(\cdot, z)$ satisfies the Helmholtz equation in $D$, we have $v_w = \Phi_k(\cdot, z)$ and therefore $\mathcal{G}_k w = \Phi_k(\cdot, z)|_{\partial\mathcal{C}}$.

Conversely, assume that for $z \in D \setminus \overline{\mathcal{C}}$ there exists $w \in H^e_{inc}(D)$ such that the solution $v_w$ satisfies $v_w = \Phi_k(\cdot, z)$ on $\partial\mathcal{C}$ and hence, thanks to Assumption 8.1 and unique continuation, $v_w = \Phi_k(\cdot, z)$ in $D$. This is a contradiction since $\Delta v_w \in L^2(D)$, while $\Delta\Phi_k(\cdot, z)$ is not. ∎

We now prove a denseness lemma. To this end, one needs to exclude exceptional values of $k$ that correspond to being both Dirichlet and Neumann scattering poles, simultaneously, i.e., the values of $k \in \mathbb{C}$ for which there exists a nonzero $w_d \in H^1_{loc}(\mathbb{R}^3 \setminus D)$ satisfying

$$\begin{cases} \Delta w_d + k^2 w_d = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ w_d = 0 & \text{on } \partial D, \\ w = \mathcal{S}_k^{\partial D}(\partial w_d/\partial\nu) - \mathcal{D}_k^{\partial D}(w_d) & \text{in } \mathbb{R}^3 \setminus \overline{D}, \end{cases} \tag{8.23}$$

and nonzero $w_n \in H^1_{loc}(\mathbb{R}^3 \setminus D)$ satisfying

$$\begin{cases} \Delta w_n + k^2 w_n = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \partial w_n/\partial\nu = 0 & \text{on } \partial D, \\ w = \mathcal{S}_k^{\partial D}(\partial w_n/\partial\nu) - \mathcal{D}_k^{\partial D}(w_n) & \text{in } \mathbb{R}^3 \setminus \overline{D}. \end{cases} \tag{8.24}$$

We refer the reader to [162, Chapter 7] for the above characterization of scattering poles of (8.23) and (8.24). Note that for a unit ball they correspond to common zeros of $h^{(1)}_{\ell_1}$ and $h^{(1)'}_{\ell_2}$ for some $\ell_1, \ell_2 \in \mathbb{N}$. In general the characterization of this set is not understood.

**Lemma 8.7.** *Let the boundary $\partial D$ be of class $C^{1,1}$. Assume that $k^2$ is not an eigenvalue of the negative Laplacian in $\mathcal{C}$ and assume that $k$ is not simultaneously both a scattering pole for the Dirichlet scattering problem and the Neumann scattering problem for $D$. Then the operator $\mathcal{S}_k^{\partial\mathcal{C}} : L^2(\partial\mathcal{C}) \to H^e_{inc}(D)$ is injective with dense range.*

***Proof.*** Assume first that $k$ is not a Dirichlet scattering pole and let $\mathcal{S}^k_{\partial\mathcal{C}}(\varphi) = 0$. This means that $\mathcal{S}^k_{\partial\mathcal{C}}(\varphi)$ satisfies (8.23), where we have used Lemma 8.4. Since $k$ is not a Dirichlet scattering pole we can conclude that $\mathcal{S}^k_{\partial\mathcal{C}}(\varphi) \equiv 0$ in $\mathbb{R}^3 \setminus \overline{D}$ (see also Proposition 2.2 in [36]). Finally, the unique continuation principle, Assumption 8.1, and the jump relation for the normal derivative of the single layer potential across $\partial\mathcal{C}$ imply that $\varphi = 0$. In the case that $k$ is a Dirichlet scattering pole, then from our assumption it is not a Neumann scattering pole and the same reasoning as above can be accordingly modified by considering $\frac{\partial}{\partial\nu}\mathcal{S}_k^{\partial\mathcal{C}}(\varphi) = 0$ on $\partial D$ to conclude in the same way based on (8.24).

As for the denseness of the range, according to Lemma 8.4 it is sufficient to prove that either the operator $S : L^2(\partial\mathcal{C}) \to H^{3/2}(\partial D)$ or the operator $K : L^2(\partial\mathcal{C}) \to H^{1/2}(\partial D)$ defined by

$$S(\varphi) := \mathcal{S}_k^{\partial\mathcal{C}}(\varphi)_{|\partial D} \qquad \text{and} \qquad K(\varphi) := \frac{\partial \mathcal{S}_k^{\partial\mathcal{C}}(\varphi)}{\partial\nu}|_{\partial D},$$

respectively, has dense range when $k$ is not a scattering pole for the Dirichlet (respectively, Neumann) scattering problem for $D$.

To this end, assume first that $k$ is not a scattering pole for the Dirichlet scattering problem for $D$. Let $\psi \in H^{-3/2}(\partial D)$ be such that $S^\top\psi = 0$ on $\partial\mathcal{C}$ where the transpose operator $S^\top : H^{-3/2}(\partial D) \to L^2(\partial\mathcal{C})$ is defined by

$$(S^\top\psi)(x) := \int_{\partial D} \psi(y)\Phi_k(x,y)\, ds(y), \qquad x \in \partial\mathcal{C}. \tag{8.25}$$

We observe that $S^\top\psi := \mathcal{S}_k^{\partial D}\psi|_{\partial\mathcal{C}}$ and $\mathcal{S}_{\partial D}\psi$ defines an $L^2(D)$-solution of the Helmholtz equation (see Section 3.1.4). By the uniqueness of the Dirichlet problem in $\mathcal{C}$, we have that $\mathcal{S}_k^{\partial D}\psi \equiv 0$ in $\mathcal{C}$ and, by unique continuation, in all of $D$. Thus the trace of $\mathcal{S}_k^{\partial D}\psi$ on $\partial D$ defined as an element in $H^{-1/2}(\partial D)$ vanishes (see, e.g., [133]). Let us now define $w =: \mathcal{S}_k^{\partial D}\psi$ in $\mathbb{R}^3 \setminus D$. Again from the discussion in Section 3.1.4, we obtain that $w$ is an $L^2$ solution of the Helmholtz equation in $\mathbb{R}^3 \setminus D$ with homogeneous Dirichlet boundary conditions on $\partial D$. Elliptic regularity implies that this solution is in $H^1_{loc}(\mathbb{R}^3 \setminus D)$. Let $B$ be a bounded domain with $C^{1,1}$ boundary such that $\overline{D} \subset B$. Lemma 8.4 (where $\partial D$ plays the role of $\partial\mathcal{C}$ and $B$ plays the role of $D$) implies that

$$w(p) = (\mathcal{S}_k^{\partial B}(\partial w/\partial\nu) - \mathcal{D}_k^{\partial B}(w))(p), \quad p \in \mathbb{R}^3 \setminus B.$$

(The application of Lemma 8.4 can be easily extended to densities that are only in $H^{-3/2}$ using a density argument.) Applying the second Green's formula in the domain between $B$ and $D$ yields

$$w(p) = (\mathcal{S}_k^{\partial D}(\partial w/\partial\nu) - \mathcal{D}_k^{\partial D}(w))(p), \quad p \in \mathbb{R}^3 \setminus B.$$

Since $B$ is arbitrary, we have that $w$ satisfies the integral representation in (8.23) and therefore $w = 0$ by our assumption on $k$. The jump relations for normal derivatives of single layer potentials with $H^{-3/2}(\partial D)$ densities in Section 3.1.4 imply that $\psi = 0$, and this finishes the proof for the first case.

We now consider the case where $k$ is not a scattering pole for the Neumann scattering problem for $D$ and shall prove that $K : L^2(\partial\mathcal{C}) \to H^{1/2}(\partial D)$ has dense range. The proof follows along the same lines as in the previous case and we will only give an outline. The transpose operator $K^\top : H^{-1/2}(\partial D) \to L^2(\partial\mathcal{C})$ is defined by

$$K^\top \psi := \mathcal{D}_k^{\partial D}(\psi)|_{\partial\mathcal{C}}.$$

Let us set $w = \mathcal{D}_k^{\partial D}\psi$. Properties of double layer potentials with densities in $H^{-1/2}$ again can be found in Section 3.1.4. Similar considerations to those above show that if $K^\top\psi = 0$, then $w = 0$ in $D$. Since the normal derivative of $\mathcal{D}_k^{\partial D}\psi$ is continuous across $\partial D$, we obtain that $w$ is an $L^2$ solution of the Helmholtz equation in $\mathbb{R}^3 \setminus D$ with homogeneous Neumann boundary conditions on $\partial D$. The result of Lemma 8.4 holds true (and can be proven exactly in the same way) if we replace the single layer potential with the double layer potential. Therefore, applying this lemma together with elliptic regularity for the Neumann problem and the same argument as above for justifying the integral representation of $w$ outside $D$, we get that $w$ is associated with a scattering pole for the Neumann problem. Hence $w = 0$ and the jump relation for the trace of the double layer potential on $\partial D$ implies that $\psi = 0$. $\square$

As a consequence of Lemma 8.7, combined with Lemma 8.6, we can prove the following theorem. In order to simplify the notation, for $w \in H^e_{inc}(D)$ we set

$$\|w\|_{\mathcal{H}^{3/2}(\partial D)} := \|w\|_{H^{3/2}(\partial D)} + \|\partial w/\partial\nu\|_{H^{1/2}(\partial D)},$$

which clearly defines an equivalent norm on $H^e_{inc}(D)$.

**Theorem 8.8.** *Let $z \in \mathbb{R}^3 \setminus \overline{D}$, and let $\partial D$ be of class $C^{1,1}$. Assume that $k \in \mathbb{C}$ is not a scattering pole of the medium scattering problem $(n, D)$, and $k$ satisfies Assumption* 8.1*, and in addition that $k$ is not simultaneously both a Dirichlet and a Neumann scattering pole for $D$. Then for every $\epsilon > 0$ there exists $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ such that*

$$\lim_{\epsilon\to 0} \|\mathcal{N}_k\varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0 \qquad \textit{and} \qquad \|\mathcal{S}_k^{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)} < C.$$

We now state the complementary result to the above theorem at a scattering pole.

**Theorem 8.9.** *Assume that $k \in \mathbb{C}$ is a scattering pole of the medium scattering problem $(n, D)$ and satisfies Assumption* 8.1*, and $\partial D$ is of class $C^{1,1}$. Let $\varphi^z_\epsilon \in L^2(\partial\mathcal{C})$ be a sequence such that*

$$\lim_{\epsilon\to 0} \|\mathcal{N}_k\varphi^z_\epsilon - \Phi_k(\cdot, z)\|_{L^2(\partial\mathcal{C})} = 0.$$

*Then $\|\mathcal{S}_k^{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)}$ cannot be bounded for all $z$ in a ball $B \subset \mathbb{R}^3 \setminus \overline{D}$.*

***Proof.*** Corresponding to the scattering pole $k$ there is a nonzero (the corresponding eigenfunction) $w_0 \in H^2_{loc}(\mathbb{R}^3)$ that satisfies (8.7) with $v = 0$. Assume to the contrary that there exists a sequence $\{\varphi^z_\epsilon\}$ in $L^2(\partial\mathcal{C})$ and a small ball $B \subset \mathbb{R}^3 \setminus \overline{D}$ such that $\mathcal{N}_k\varphi^z_\epsilon$ converges to $\Phi_k(\cdot, z)$ in $L^2(\partial\mathcal{C})$ and $\|\mathcal{S}^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)\|_{\mathcal{H}^{3/2}(\partial D)} < C$ for all $z \in B$. From the latter we can assume without loss of generality that $SL^k_{\partial\mathcal{C}}(\varphi^z_\epsilon)$ converges weakly to $w_z \in H^e_{inc}(D)$ as $\epsilon \to 0$ with $H^e_{inc}(D)$ given by (8.14). Let $v_z = \mathcal{G}_k w_z$, where $(v_z, u_z)$

solves the interior transmission problem (8.12) with $\Phi_k(\cdot, z)$ replaced by $w_z$, which for $\tilde{w}_z := u_z - v_z \in H^2(D)$ can be written as

$$\begin{cases} \Delta \tilde{w}_z + k^2 n \tilde{w}_z = k^2(1-n)v_z & \text{in } D, \\ \tilde{w}_z = w_z \quad \text{and} \quad \dfrac{\partial \tilde{w}_z}{\partial \nu} = \dfrac{\partial w_z}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{8.26}$$

It is clear from (8.17) and the convergence of $\mathcal{N}_k \varphi_\epsilon^z$ to $\Phi_k(\cdot, z)$ in $L^2(\partial\mathcal{C})$ that $v_z = \Phi_k(\cdot, z)$ on $\partial\mathcal{C}$, and hence $v_z = \Phi_k(\cdot, z)$ in $\mathcal{C}$ by the uniqueness of the Dirichlet problem in $\mathcal{C}$ and consequently in $D$ by analyticity. Considering $W_z := \tilde{w}_z$ in $D$ and $W_z := w_z$ in $\mathbb{R}^3 \setminus D$ from (8.26) and the facts that $w_z \in H^e_{inc}(D)$ in (8.14) and $v_z = \Phi(\cdot, z)$ we have that $W_z \in H^2_{loc}(\mathbb{R}^3)$ satisfies (8.7) with $v := \Phi(\cdot, z)$. The latter means that

$$(I - T(k))(W_z + \Phi(\cdot, z)) = \Phi(\cdot, z) \quad \text{in } D.$$

Multiplying this equation by $k^2(n-1)w_0$ and then integrating over $D$ and changing the order of integration implies that

$$\int_D k^2(n-1)(W_z + \Phi(\cdot, z))(I - T(k))w_0 dx = \int_D \Phi(y, z)k^2(n-1)w_0(y)\, dy.$$

Therefore

$$\int_D \Phi(y, z)k^2(n-1)w_0(y)\, dy = 0 \qquad \text{for } z \in B.$$

Unique continuation for solutions of the Helmholtz equation yields

$$P(z) := \int_D \Phi(y, z)k^2(n-1)w_0(y)\, dy = 0 \qquad \text{for } z \in \mathbb{R}^3 \setminus \overline{D},$$

and hence $P(z) = 0$ and $\partial P(z)/\partial \nu = 0$ on $\partial D$. Now inside $D$ we have that $P(z) \in H^2(D)$ satisfies

$$\Delta P + k^2 P = -k^2(n-1)w_0.$$

Since $w_0$ solves $\Delta w_0 + k^2 n w_0 = 0$ in $D$ we conclude that $(w_0, v)$ with $v := w_0 - P$ satisfies the homogeneous interior transmission problem, and from Assumption 8.1, i.e., $k$ is not a transmission eigenvalue, we conclude that $w_0 = 0$ in $D$ and therefore in $\mathbb{R}^3$ (by unique continuation), which is a contradiction. This proves the theorem. □

We can combine Theorems 8.8 and 8.9 to formulate the following criteria for the determination of the scattering poles using GLSM similarly as in Theorem 5.2.

**Corollary 8.10.** *Assume that the hypotheses of Theorems* 8.8 *and* 8.9 *hold. Define for* $g \in L^2(\partial\mathcal{C})$

$$J_\alpha(\phi_z; g) := \alpha \|\mathcal{S}^k_{\partial\mathcal{C}}(g)\|^2_{\mathcal{H}^{3/2}(D)} + \|\mathcal{N}_k g - \phi_z\|^2_{L^2(\partial\mathcal{C})},$$

*where* $\phi_z := \Phi_k(\cdot, z)$. *Let* $g_z^\alpha$ *be defined as in* (5.8). *Then we have that for any ball* $B \subset D$, $\|\mathcal{S}^k_{\partial\mathcal{C}}(g_z^\alpha)\|^2_{\mathcal{H}^{3/2}(D)}$ *is bounded and* $\|\mathcal{N}_k g_z^\alpha - \phi_z\|_{L^2(\partial\mathcal{C})} \to 0$ *as* $\alpha \to 0$ *for almost every* $z \in B$ *if and only if* $k$ *is not a scattering pole of the inhomogeneous medium* $(n, D)$.

We conclude this section by sketching the above approach for the special case when the operator $\mathcal{N}_k$ is defined on $\partial\mathcal{C} := \partial B_\delta$, where $B_\delta$ is a small ball inside $D$ centered at the origin [37]. This allows us to give an equivalent definition of the involved operators in terms of Fourier series representation which better reveals the duality between the exterior and interior operators (since there is no natural dual definition of the far field for the interior problem). To explain this, we first represent the far field operator introduced in Section 1.2.1 as a Fourier series in terms of spherical harmonics $\{Y_\ell^m(\hat{x})\}$. For $g \in L^2(S^2)$ we set

$$g(\hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} Y_\ell^m(\hat{x})$$

and define an isometry $\mathcal{I}$ between $L^2(S^2)$ and $l^2(\mathbb{Z})$ by the mapping $g \mapsto \tilde{g} := \{g_{\ell,m}\}$. Thus we now have a new representation of the far field operator, denoted by $\tilde{F}(k) : l^2(\mathbb{Z}) \mapsto l^2(\mathbb{Z})$,

$$\tilde{F}(k) = \mathcal{I}^{-1^*} F(k)\, \mathcal{I}^{-1},$$

where $\mathcal{I}^{-1^*}$ denotes the $L^2$-adjoint of $\mathcal{I}^{-1}$. The discussion on transmission eigenvalues in connection to the kernel of the operator $F(k)$ can now be carried over in exactly same way as before if we replace $F(k)$ by $\tilde{F}(k)$. To introduce the duality, we first observe that the operator $\tilde{F}(k)$ can be equivalently defined using scattered waves associated with incident spherical waves. More precisely, let $u^s_{\ell,m}(x)$ be the scattered field corresponding to the incident wave (which is a Herglotz wave function)

$$v(x) := j_\ell(k|x|) Y_\ell^m(\hat{x}).$$

Outside a ball $B_R$ of radius $R$ containing $D$, the scattered field can be expanded as

$$u^s_{\ell,m}(x) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} a^{p,q}_{\ell,m} h^{(1)}_p(k|x|) \overline{Y_p^q}(\hat{x})$$

(note that $\overline{Y_p^q} = Y_p^{-q}$). Then, up to a multiplicative constant, the Fourier coefficients of $\tilde{F}(k)\tilde{g}$ are

$$(\tilde{F}(k)\tilde{g})_{p,q} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} a^{p,q}_{\ell,m}. \tag{8.27}$$

Now reversing the role of the incident and scattered waves, we can define similarly to (8.27) an interior far field operator to characterize the scattering poles where in the following we assume that $k$ is complex with $\Im(k) < 0$. To this end, let $B_\delta \subset D$ be a ball centered at the origin, and for an outgoing solution to the Helmholtz equation,

$$w_{\ell,m}(x) = j_\ell(k\delta) h^{(1)}_\ell(k|x|) Y_\ell^m(\hat{x}),$$

we denote by $(u_{\ell,m}, v_{\ell,m}) \in L^2(D) \times L^2(D)$ the solution of the interior transmission problem

$$\begin{cases} \Delta u_{\ell,m} + k^2 n(x) u_{\ell,m} = 0 & \text{in } D, \\ \Delta v_{\ell,m} + k^2 v_{\ell,m} = 0 & \text{in } D, \\ u_{\ell,m} - v_{\ell,m} = w_{\ell,m} & \text{on } \partial D, \\ \dfrac{\partial u_{\ell,m}}{\partial \nu} - \dfrac{\partial v_{\ell,m}}{\partial \nu} = \dfrac{\partial w_{\ell,m}}{\partial \nu} & \text{on } \partial D. \end{cases}$$

Inside $B_\delta$, the field $v_{\ell,m}$ can be expanded as

$$v_{\ell,m}(x) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} b^{p,q}_{\ell,m} \frac{j_p(k|x|)}{j_p(k\delta)} \overline{Y_p^q}(\hat{x}).$$

Thus, we can define the interior far field operator $\tilde{F}_{int} : l^2(\mathbb{Z}) \mapsto l^2(\mathbb{Z})$ by its Fourier coefficients

$$(\tilde{F}_{int}(k)\tilde{g})_{p,q} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} b_{\ell,m}^{p,q}. \tag{8.28}$$

Observe that this operator can be defined for any $k \in \mathbb{C}$ with $\Im(k) < 0$ that does not coincide with a transmission eigenvalue. Now, let $k$ be such that there exists a $\tilde{g} \neq 0$ with $\tilde{F}_{int}(k)\tilde{g} = 0$. Then by unique continuation we have that

$$v_{\tilde{g}} := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_{n,m} v_{n,m} = 0 \qquad \text{in } D,$$

and consequently one can show that $w \in H^2_{loc}(\mathbb{R}^3)$, defined by

$$w = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} u_{\ell,m} \qquad \text{in } D$$

and

$$w = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell,m} w_{\ell,m} \qquad \text{in } \mathbb{R}^3 \setminus D,$$

is a nontrivial solution of

$$\Delta w + k^2 n w = 0 \quad \text{in } \mathbb{R}^3$$

and

$$w = \int_{\partial D} \left( \Phi(\cdot, y) \frac{\partial w(y)}{\partial \nu} - w(y) \frac{\partial \Phi(\cdot, y)}{\partial \nu} \right) dy \quad \text{in } \mathbb{R}^3 \setminus D.$$

This means that such a value of $k$ is a scattering pole. For the given inhomogeneity $(D, n)$, there is a duality between $\tilde{F}(k)$ whose kernel is related to the transmission eigenvalues and $\tilde{F}_{int}(k)$ whose kernel is related to the scattering poles. Both are defined by similar expressions, but $\tilde{F}(k)$ corresponds to the exterior scattering problem due to an incident Herglotz wave function, whereas $\tilde{F}_{int}(k)$ corresponds to the interior scattering problem due to an incident outgoing spherical wave. Note that the interior operator $\tilde{F}_{int}(k)$ coincides (up to an isometry) with the operator $\mathcal{N}_k$ (8.16) when the single layer potential is supported by $\partial\mathcal{C} := \partial B_\delta$.

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