REMARKS ON QUADRILATERAL REISSNER-MINDLIN ELEMENTS

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July 8, 2002

Outline of Talk

1. Formulation of the Reissner–Mindlin plate model

2. A Standard formulation of finite element approximation schemes

3. Low order rectangular elements

4. Extension to quadrilaterals:
A necessary condition for optimal order approximation

5. Applications to Reissner-Mindlin elements

6. Low order triangular elements

Reissner–Mindlin plate model

Determines functions $\boldsymbol{\theta}$ and ω, defined on middle surface Ω of plate (approximate rotation vector and transverse displacement), as minimizers of energy functional:

$$J(\boldsymbol{\theta},\omega) = \frac{1}{2}\int_\Omega \mathsf{C}\,\mathcal{E}\,\boldsymbol{\theta} : \mathcal{E}\,\boldsymbol{\theta} + \frac{\lambda t^{-2}}{2}\int_\Omega |\,\boldsymbol{\theta} - \mathbf{grad}\,\omega|^2 - \int_\Omega g\omega$$

over $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$ (clamped B.C.).

$\mathcal{E}\,\boldsymbol{\theta}$ denotes symmetric part of gradient of $\boldsymbol{\theta}$,
g, scaled transverse loading function,
t plate thickness,
$\lambda = Ek/2(1+\nu)$,
E Young's modulus,
ν Poisson ratio,
k shear correction factor.

For all 2×2 symmetric matrices $\mathcal{J}$,

$$\mathsf{C}\mathcal{J} = \{E/[12(1-\nu^2)]\}[(1-\nu)\mathcal{J} + \nu\,\mathrm{tr}(\mathcal{J})\mathcal{I}],$$

where $\mathcal{I}$ = 2×2 identity matrix.

Finite Element Approximation Schemes

To avoid "locking" problem – causing large errors for thin plates, consider schemes of following form:

Find $(\boldsymbol{\theta}_h, \omega_h) \in \boldsymbol{\Theta}_h \times W_h$ minimizing modified energy functional:

$$J_h(\boldsymbol{\theta}, \omega) = \frac{1}{2}\int_\Omega \mathsf{C}\,\mathcal{E}\,\boldsymbol{\theta} : \mathcal{E}\,\boldsymbol{\theta} + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{R}_h\,\boldsymbol{\theta} - \mathbf{grad}\,\omega|^2 - \int_\Omega g\omega.$$

Modification: incorporation of *reduction operator* $\boldsymbol{R}_h : \boldsymbol{\Theta}_h \to \boldsymbol{\Gamma}_h$.

$\boldsymbol{\Gamma}_h$ an auxiliary finite element space,
$\boldsymbol{R}_h$ typically either interpolation operator or L^2-projection operator.

To specify particular scheme, specify spaces $\boldsymbol{\Theta}_h, W_h$, and $\boldsymbol{\Gamma}_h$, and reduction operator $\boldsymbol{R}_h$.

Preliminaries

Assume Ω a polygonal domain.

Let $\{\tau_h\}_{0<h<1}$ a subdivision of Ω into rectangles or quadrilaterals, (h = diameter of largest element in subdivision).

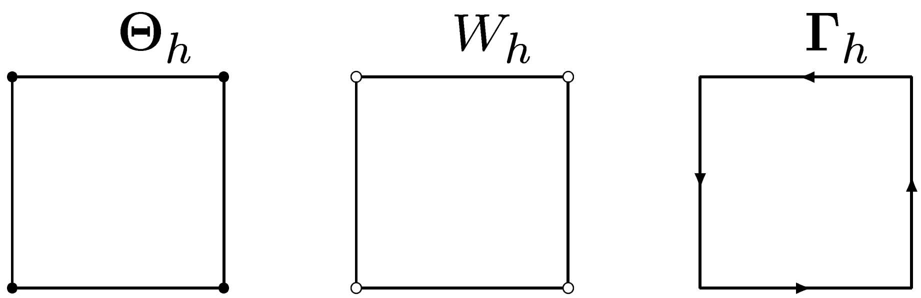
For any set K, define:

$P_k(K)$ polynomials of degree at most k on K,
$P_{k_1,k_2}(K)$ polynomials of degree at most k_1 in x_1 and k_2 in x_2 on K,
$Q_k(K) = P_{k,k}(K)$ polynomials of degree at most k in each variable on K.

λ_i = barycentric coordinates,
$\boldsymbol{\tau}_i$ = unit tangent vectors on element sides.

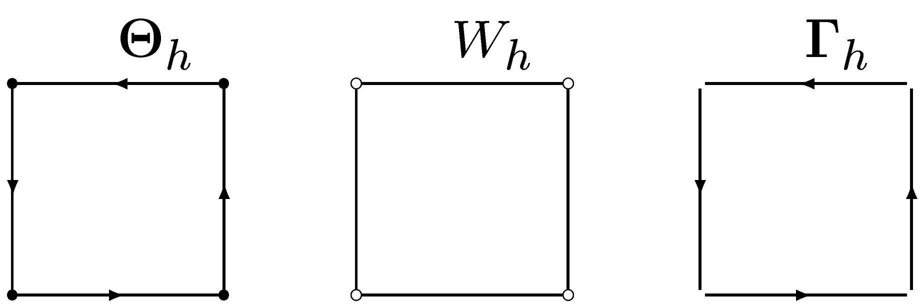
Rectangular Elements

MITC4 element:



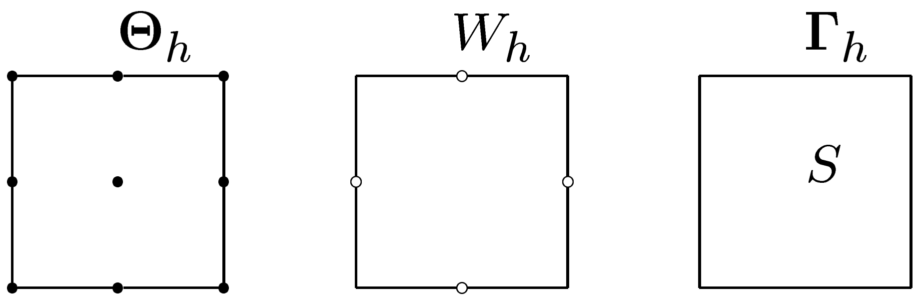
$\boldsymbol{\Theta}_h = C^0\, \boldsymbol{Q}_1$, $\qquad W_h = C^0\, Q_1$,
$\boldsymbol{\Gamma}_h =$ rotation of $\boldsymbol{RT}_0$, $(a + by, c + dx)$,
$\boldsymbol{R}_h =$ interpolation into $\boldsymbol{\Gamma}_h$ defined by 4 edge conditions $\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau} = \int_e \gamma \cdot \boldsymbol{\tau}$.

Durán–Liberman element:



Same as above, except add 4 edge bubbles:
$\hat{x}(1-\hat{x})(1-\hat{y})\,\boldsymbol{\tau}_1$, $\hat{x}\hat{y}(1-\hat{y})\,\boldsymbol{\tau}_2$, $\hat{x}(1-\hat{x})\hat{y}\,\boldsymbol{\tau}_3$, and $(1-\hat{x})\hat{y}(1-\hat{y})\,\boldsymbol{\tau}_4$ to $\boldsymbol{\Theta}_h$.

Ye element: (analogue of A-F triangular element)



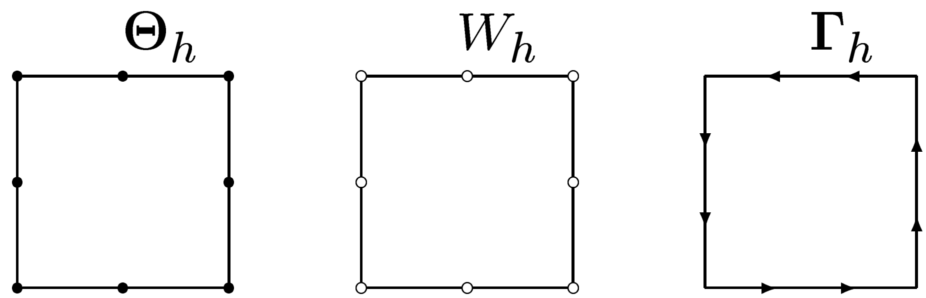
$\boldsymbol{\Theta}_h = C^0\, \boldsymbol{Q}_1$,
W_h = nonconforming (continuous average values) rotated $Q_1 = \text{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}$.

Usual bilinear elements not uniquely defined by average values on edges: $(\hat{x} - 1/2)(\hat{y} - 1/2)$ has zero average value on each side.

$\boldsymbol{\Gamma}_h =$ discontinuous vectors of form $(b + dx, c - dy)$
$\boldsymbol{R}_h = \boldsymbol{L}^2$ projection into $\boldsymbol{\Gamma}_h$.

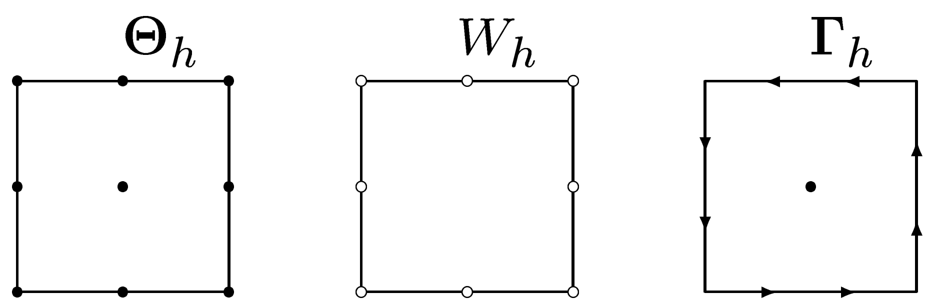
For next element: define serendipity Q_2 (Q_2/x^2y^2) (only edge degrees of freedom).

MITC8 element:



$\mathbf{\Theta}_h = C^0$ serendipity $\boldsymbol{Q}_2$, $W_h = C^0$ serendipity Q_2
$\mathbf{\Gamma}_h =$ rotation of $\boldsymbol{BDM}_1$, $(a_1 + b_1 x + c_1 y + 2dxy + ey^2, a_2 + b_2 x + c_2 y + dx^2 + 2exy)$.
$\boldsymbol{R}_h =$ interpolation into $\mathbf{\Gamma}_h$ defined by 8 edge conditions: $\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau}\, p_1 = \int_e \gamma \cdot \boldsymbol{\tau}\, p_1$.

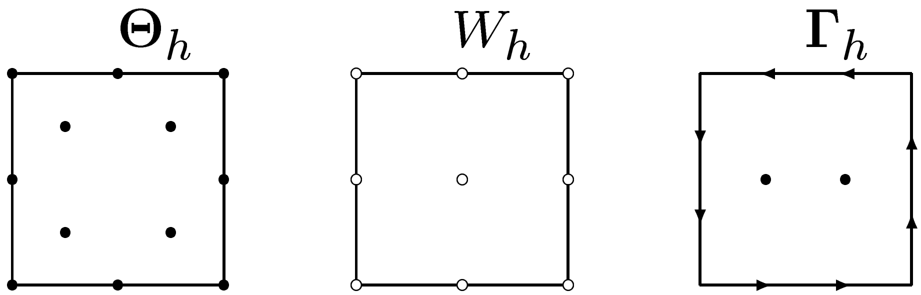
MITC9 element:



$\mathbf{\Theta}_h = C^0\, \boldsymbol{Q}_2$, $W_h = C^0$ serendipity Q_2,
$\mathbf{\Gamma}_h =$ rotation of $\boldsymbol{BDFM}_2$, $(a_1 + b_1 x + c_1 y + d_1 xy + e_1 y^2, a_2 + b_2 x + c_2 y + d_2 xy + e_2 x^2)$,
$\boldsymbol{R}_h$ interpolation into $\mathbf{\Gamma}_h$ defined by 9 conditions:
$\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau}\, p_1 = \int_e \gamma \cdot \boldsymbol{\tau}\, p_1, \quad \int_K \boldsymbol{R}_h \gamma = \int_K \gamma$.

Analogue of MITC9 based on rotated version of $\boldsymbol{RT}_1$.

MITC12 element:



$\boldsymbol{\Theta}_h = C^0\ \boldsymbol{Q}_2$ + bicubic bubble functions,
$(1-x)x(1-y)y(ax+by+cxy)$,
$W_h = C^0\ Q_2$, $\boldsymbol{\Gamma}_h =$ rotation of $\boldsymbol{RT}_1$ ($P_{1,2} \times P_{2,1}$),
$\boldsymbol{R}_h$ interpolation into $\boldsymbol{\Gamma}_h$ defined by conditions
$\int_e \boldsymbol{R}_h \boldsymbol{\gamma} \cdot \boldsymbol{\tau}\, p_1 = \int_e \boldsymbol{\gamma} \cdot \boldsymbol{\tau}\, p_1$,
$\int_K \boldsymbol{\gamma} \cdot \boldsymbol{q} = \int_K \boldsymbol{R}_h \boldsymbol{\gamma} \cdot \boldsymbol{q} \quad \forall \boldsymbol{q} \in P_{1,0} \times P_{0,1}$.

Consider extension to **quadrilateral** elements.

Let $\boldsymbol{F}$ = invertible bilinear mapping from reference element $\hat{K} = [0,1] \times [0,1]$ to convex quadrilateral K.

For scalar functions, if $\hat{v}(\hat{\boldsymbol{x}})$ defined on $\hat{K}$, define: $v(\boldsymbol{x})$ on K by $v = \hat{v} \circ \boldsymbol{F}^{-1}$.

Then, for $\hat{V}$ = shape functions given on $\hat{K}$, define

$$V_F(K) = \{v : v = \hat{v} \circ \boldsymbol{F}^{-1}, \hat{v} \in \hat{V}\}.$$

For all examples given previously, W_h defined in this way, beginning with shape functions denoted in figures.

Preserves appropriate interelement continuity for usual degrees of freedom.

Same mapping, applied to each component, used to define $\boldsymbol{\Theta}_h$.

Exception: Durán-Liberman edge bubbles

Can also use different definition for interior degrees of freedom – does not affect interelement continuity.

To define space $\boldsymbol{\Gamma}_h$, use rotated version of **Piola** transform.

Let $D\boldsymbol{F}$ denote Jacobian $\boldsymbol{F}$. For $\widehat{\boldsymbol{\eta}}$ a vector function defined on $\widehat{K}$, define $\boldsymbol{\eta}$ on K by

$$\boldsymbol{\eta}(\boldsymbol{x}) = \boldsymbol{\eta}(\boldsymbol{F}(\widehat{\boldsymbol{x}})) = [D\boldsymbol{F}(\widehat{\boldsymbol{x}})]^{-t}\,\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}),$$

where A^{-t} denotes transpose of inverse of A.

Then, if $\widehat{\boldsymbol{V}}$ is vector shape functions on $\widehat{K}$, define

$$\boldsymbol{V}_F(K) = \{\,\boldsymbol{\eta} :\ \boldsymbol{\eta} = [D\boldsymbol{F}]^{-t}\,\widehat{\boldsymbol{\eta}} \circ \boldsymbol{F}^{-1},\, \widehat{\boldsymbol{\eta}} \in \widehat{\boldsymbol{V}}\}.$$

For $\omega \in W_h$, $\mathbf{grad}\,\omega = D\boldsymbol{F}^{-t}\widehat{\mathbf{grad}}\,\widehat{\omega}$.

Hence, if on reference square $\widehat{\mathbf{grad}}\,\widehat{\omega} \subseteq \widehat{\boldsymbol{V}}$, $\mathbf{grad}\,\omega \subseteq \boldsymbol{\Gamma}_h$, (needed to guarantee uniqueness of solutions).

Extensions to quadrilaterals mostly straightforward to define, BUT:

Does element retain same order of approximation as in rectangular case?

Scalar approximation on rectangular meshes:

Given $\hat{S} \subseteq L^2(\hat{K})$, define associated subspace on arbitrary square K by

$$S(K) = \{\, u : K \to \mathbb{R} \,|\, \hat{u}_K \in \hat{S} \,\}.$$

Let τ_h be uniform mesh of unit square Ω into n^2 subsquares when $h = 1/n$, and define

$$S_h = \{\, u : \Omega \to \mathbb{R} \,|\, u|_K \in S(K) \text{ for all } K \in \tau_h \,\}.$$

Equivalent conditions for optimal order convergence:

Theorem: Let $\hat{S}$ be a finite dimensional subspace of $L^2(\hat{K})$, r a non-negative integer. The following conditions are equivalent:

There is a constant C such that for all $u \in H^{r+1}(\Omega)$

$$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} \le C h^{r+1} |u|_{H^{r+1}(\Omega)}.$$

$$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} = o(h^r) \quad \text{for all } u \in P_r(\Omega).$$

$$P_r(\hat{K}) \subset \hat{S}.$$

For optimal order approximation, need $P_r(\hat{K}) \subset \hat{S}$.

Need **stronger** condition on **quadrilaterals.**

τ_h: family of shape-regular quadrilateral meshes of Ω, i.e., all quadrilaterals convex and there exist constants $\sigma \ge 1$ and $0 < \rho < 1$ independent of h such that

$$h_K / h_K^{'} \le \sigma, \quad |\cos \theta_{iK}| \le \rho, \quad i = 1, 4, \quad K \in \tau_h,$$

where h_K, $h_K^{'}$, and θ_{iK} are diameter of K, smallest length of sides of K, and angles of K, respectively.

Theorem: Let V^{τ_h} be functions on Ω that $\in V_{F_K}(K)$ when restricted to $K \in \tau_h$. Then $Q_r \subseteq \hat{V}$ is necessary for condition:

$$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} = o(h^r) \quad \text{for all } u \in P_r(\Omega).$$

Approximation of vectorfields on quadrilaterals

Map spaces defined on reference square to general quadrilateral using rotated version of Piola transform.

Recall: rotated Raviart–Thomas space of order r on reference element: $\boldsymbol{RT}_r = P_{r,r+1} \times P_{r+1,r}$.

Define $\boldsymbol{S}_r$ by $\boldsymbol{RT}_r = \text{span}\,(\boldsymbol{S}_r, [\hat{x}^r \hat{y}^{r+1}, -\hat{x}^{r+1}\hat{y}^r])$. $\boldsymbol{S}_0 = (a + b\hat{y}, c + b\hat{x})$. Add $(\hat{y}, -\hat{x})$ to get $\boldsymbol{RT}_0$.

Define R_r as span of monomials $\hat{x}^i \hat{y}^j$, $i, j = 0, \ldots, r$, but omitting $\hat{x}^r \hat{y}^r$. $R_1 = P_1$.

Theorem: Let $\boldsymbol{V}^{\tau_h}$ denote vectorfields on Ω that $\in \boldsymbol{V}_{F_K}(K)$ when restricted to $K \in \tau_h$. A necessary condition for estimate:

$$\inf_{\boldsymbol{\eta}\in\boldsymbol{V}^{\tau_h}} \|\boldsymbol{\gamma} - \boldsymbol{\eta}\|_{L^2(\Omega)} = o(h^r) \quad \forall\, \boldsymbol{\gamma} \in P_r(\Omega, \mathbb{R}^2)$$

is $\boldsymbol{S}_r \subseteq \hat{\boldsymbol{V}}$.

A necessary condition for estimate

$$\inf_{\boldsymbol{\eta}\in\boldsymbol{V}^{\tau_h}} \|\mathrm{rot}[\boldsymbol{\gamma} - \boldsymbol{\eta}]\|_{L^2(\Omega)} = o(h^r)$$

for all $\boldsymbol{\gamma}$ with $\mathrm{rot}\,\boldsymbol{\gamma} \in P_r(\Omega)$ is $R_{r+1} \subseteq \hat{\mathrm{rot}}\,\hat{\boldsymbol{V}}$.

Key ingredient in establishing scalar theorem: first show for $P_r(K)$ to belong to $V_F(K)$, must have $Q_r(\hat{K}) \in \hat{V}$.

Key ingredient for vector theorem: first show for $P_r(K, \mathbb{R}^2)$ to belong to $\boldsymbol{V}_F(K)$, must have $\boldsymbol{S}_r \in \hat{\boldsymbol{V}}$.

Applications to Reissner-Mindlin elements

MITC4 and **Durán-Liberman:** use full Q_1 space to approximate both $\boldsymbol{\theta}$ and ω and full $\boldsymbol{RT}_0$ space to approximate $\boldsymbol{\gamma}$.

Since $\boldsymbol{RT}_0 \supseteq \boldsymbol{S}_0$, do not expect degradation of convergence rate from rectangular mesh to shape-regular quadrilateral mesh. Recent work of D-L shows D-L optimal order $O(h)$ on shape-regular meshes and MITC4 optimal order for asymptotically parallelogram meshes. No degradation of convergence rates in numerical experiments.

Nonconforming Ye element approximates ω by space for which $\widehat{V}$ does not contain all of Q_1. Expect degradation of convergence rate.

Possible remedy: add nonconforming bubble $(\hat{x} - 1/2)(\hat{y} - 1/2)$ to basis functions on reference element for space W_h and its gradient to basis for $\boldsymbol{\Gamma}_h$. Problem: Key properties valid on rectangles no longer hold, so extension of analysis not clear.

MITC8: approximates both $\boldsymbol{\theta}$ and ω by spaces obtained from mappings of quadratic serentipity space, which does not contain all of Q_2, (i.e, missing basis function x^2y^2). $\boldsymbol{\Gamma}_h$ obtained by mapping $\boldsymbol{BDM}_1$, which does not contain $\boldsymbol{S}_1$. Expect degradation of convergence rate.

MITC9: uses full $\boldsymbol{Q}_2$ approximation for $\boldsymbol{\theta}$, but Q_2 serendipity space to approximate ω and $\boldsymbol{BDFM}_2$ space (does not contain $\boldsymbol{S}_1$) to approximate $\boldsymbol{\gamma}$.
Expect degradation of convergence rate.

MITC12: Expect no degradation of convergence on shape-regular quadrilateral meshes as a result of sub-optimal approximation by subspaces. Still need analysis to check no other problems.

Asymptotically Parallelogram meshes:

1. Degradation of convergence rates noted for general quadrilateral meshes do NOT occur for parallelogram meshes or for elements sufficiently close to parallelograms (almost affine maps).

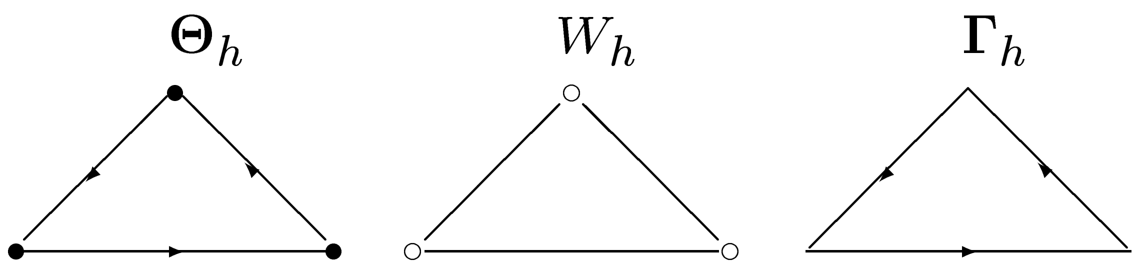
2. Asymptotically parallelogram and shape-regular meshes obtained beginning with mesh of convex quadrilaterals and refining by dividing each quadrilateral in four by connecting the midpoints of opposite edges.

3. For such meshes and scalar finite element approximation, if reference space contains only $P_r(\hat{K})$, rather than $Q_r(\hat{K})$, one still has optimal $O(h^{r+1})$ convergence in L^2.

Since such meshes occur commonly in practice, may explain why the degradation of convergence rate not observed more often.

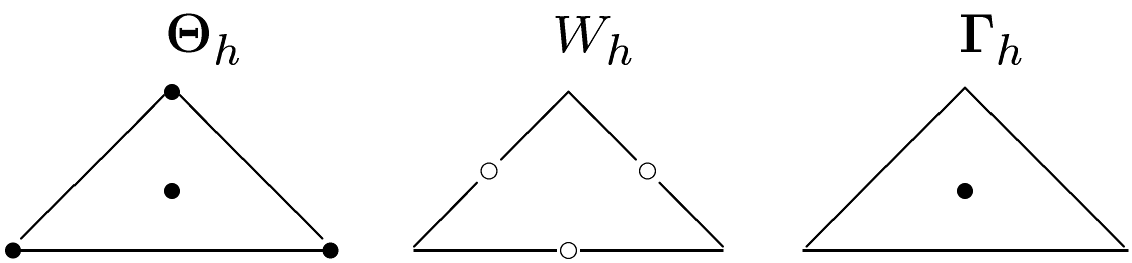
Low order Triangular Elements

Durán–Liberman element:



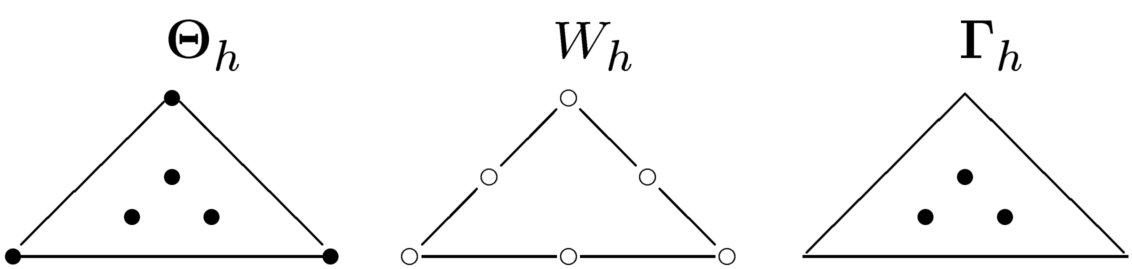
$\boldsymbol{\Theta}_h = C^0\, \boldsymbol{P}_1$ + span$\{\lambda_2\lambda_3\,\boldsymbol{\tau}_1$, $\lambda_3\lambda_1\,\boldsymbol{\tau}_2$, $\lambda_1\lambda_2\,\boldsymbol{\tau}_3\}$,
$W_h = C^0\, P_1$
$\boldsymbol{\Gamma}_h =$ rotation of $\boldsymbol{RT}_0$, $(a - by, c + bx)$.
$\boldsymbol{R}_h$ defined by 3 edge conditions:
$\int_e \boldsymbol{R}_h \boldsymbol{\gamma} \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\gamma} \cdot \boldsymbol{\tau}$.

Arnold–Falk element:



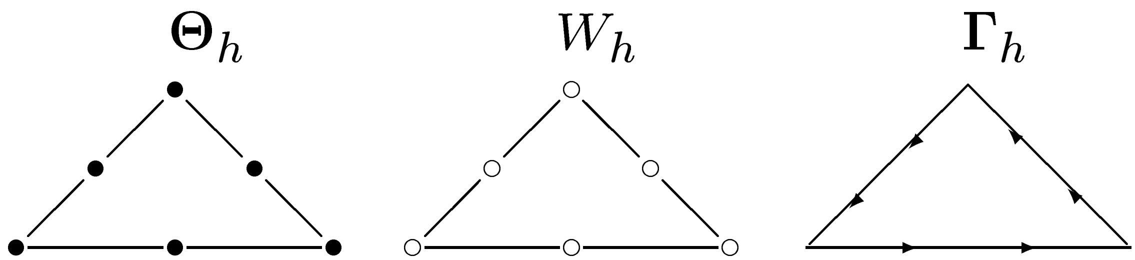
$\boldsymbol{\Theta}_h = C^0 \boldsymbol{P}_1$ + cubic bubble functions $(\lambda_1\lambda_2\lambda_3)$,
$W_h =$ nonconforming P_1,
$\boldsymbol{\Gamma}_h = \boldsymbol{P}_0$
$\boldsymbol{R}_h$ $\boldsymbol{L}^2$ projection defined by: $\int_T \boldsymbol{R}_h \boldsymbol{\phi} = \int_T \boldsymbol{\phi}$.

Falk–Tu element:



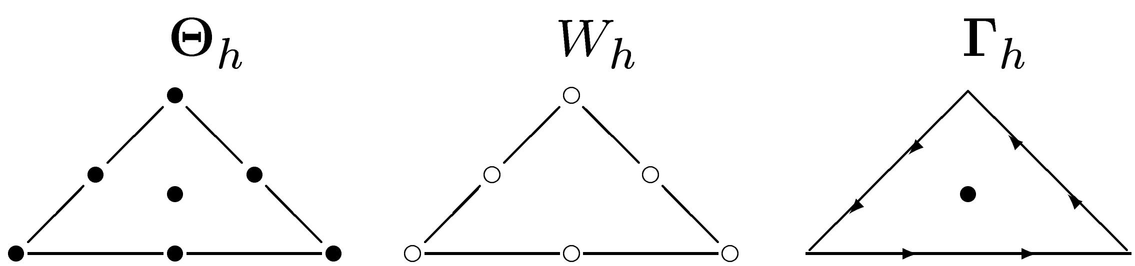
$\boldsymbol{\Theta}_h = \boldsymbol{P}_1$ + quartic bubbles, $\lambda_1\lambda_2\lambda_3 \sum_{i=1}^3 a_i\lambda_i$,
$W_h = C^0\ P_2$, $\boldsymbol{\Gamma}_h =$ discontinuous $\boldsymbol{P}_1$,
$\boldsymbol{R}_h = L^2$ projection into $\boldsymbol{\Gamma}_h$.

MITC6 element:



$\Theta_h = C^0\ \boldsymbol{P}_2$, $\qquad W_h = C^0\ P_2$,
$\Gamma_h =$ rotation of $\boldsymbol{BDM}_1$,
(vectors with components in $\boldsymbol{P}_1$),
$\boldsymbol{R}_h$ = interpolation operator into Γ_h defined by 6 edge conditions: $\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau}\, p_1 = \int_e \gamma \cdot \boldsymbol{\tau}\, p_1$.

MITC7 element:



$\Theta_h = \boldsymbol{P}_2$ + cubic bubble functions,
$W_h = P_2$, $\qquad \Gamma_h =$ rotation of $\boldsymbol{RT}_1$,
$(a + bx + cy - dxy - ey^2, f + gx + hy + exy + dx^2)$,
$\boldsymbol{R}_h =$ interpolation operator into Γ_h defined by:
$\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau}\, p_1 = \int_e \gamma \cdot \boldsymbol{\tau}\, p_1$, $\qquad \int_T \boldsymbol{R}_h \gamma = \int_T \gamma$.

Error Analysis: Generalization of D-L

Theorem: Suppose $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$ and let $\omega_I \in W_h$ and $\boldsymbol{\phi}_I \in \boldsymbol{\Theta}_h$. Define

$$\begin{aligned}\boldsymbol{\gamma} &= t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}),\\ \boldsymbol{\gamma}_h &= t^{-2}(\mathbf{grad}\,\omega_h - \boldsymbol{R}_h\,\boldsymbol{\phi}_h),\\ \boldsymbol{\gamma}_I &= t^{-2}(\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\,\boldsymbol{\phi}_I).\end{aligned}$$

Suppose for $s \geq 1$, $\|\,\boldsymbol{\gamma} - \boldsymbol{R}_h\,\boldsymbol{\gamma}\|_0 \leq Ch\|\,\boldsymbol{\gamma}\|_1$. Then

$$\begin{aligned}&\|\,\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\,\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0\\ &\qquad\leq C\Big(\|\,\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_1 + t\|\,\boldsymbol{\gamma} - \boldsymbol{\gamma}_I\|_0 + h\|\,\boldsymbol{\gamma}\|_0\Big).\end{aligned}$$

If $s \geq 2$ and $(\,\boldsymbol{\gamma} - \boldsymbol{R}_h\,\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in \boldsymbol{P}_{s-2}$, ($\boldsymbol{P}_k$ discontinuous piecewise polynomials vectors of degree $\leq k$), then

$$\begin{aligned}&\|\,\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\,\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0\\ &\leq C\Big(\|\,\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_1 + t\|\,\boldsymbol{\gamma} - \boldsymbol{\gamma}_I\|_0 + h\|\,\boldsymbol{\gamma} - \boldsymbol{\Pi}\,\boldsymbol{\gamma}\|_0\Big).\end{aligned}$$

$\boldsymbol{\Pi} = \boldsymbol{L}^2$ projection into $\boldsymbol{P}_{s-2}$.

To apply, find approximations ω_I, $\boldsymbol{\phi}_I$ satisfying:

$$\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\,\boldsymbol{\phi}_I = \boldsymbol{R}_h\,\mathbf{grad}\,\omega - \boldsymbol{R}_h\,\boldsymbol{\phi}$$

and

$$\|\phi - \phi_I\|_0 + h\|\phi - \phi_I\|_1 \le Ch^{s+1}\|\phi\|_{s+1},$$
$$\|\gamma - R_h\gamma\|_0 \le Ch^s\|\gamma\|_s.$$

If so, then

$$\gamma_I = t^{-2}(\mathbf{grad}\,\omega_I - R_h\phi_I)$$
$$= t^{-2}R_h(\mathbf{grad}\,\omega - \phi) = R_h\gamma.$$

Hence, for $s \ge 1$

$$\|\phi - \phi_h\|_1 + t\|\gamma - \gamma_h\|_0$$
$$\le Ch^s\Big(\|\phi\|_{s+1} + \|\gamma\|_{s-1} + t\|\gamma\|_s\Big),$$

C independent of h and t. Note:norms on right hand side NOT independent of t for $s \ge 3/2$.

Remarks: Estimate needs special interpolants.

Only upper bound, but indicates error in rotation ϕ and shear stress γ depend on simultaneous approximation of both variables.

Estimate for ω error indicates dependence on approximation of $\mathbf{\Theta}_h$ and $\mathbf{\Gamma}_h$, as well as W_h.