AN OVERVIEW OF REISSNER-MINDLIN PLATE ELEMENTS

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Outline of Talk:

- Unified approach to error analysis
- Historical perspective

Finite Element Approximation

Recall minimum energy formulation of R-M:

$$(\phi, \omega) \to \int_\Omega \frac{1}{2} \mathcal{C}\mathcal{E}(\phi) : \mathcal{E}(\phi) + \frac{1}{2}\lambda t^{-2} \int_\Omega |\, \mathbf{grad}\, \omega - \boldsymbol{\phi} \,|^2 - \int_\Omega g\omega$$

If low order finite element spaces used in the approximation: problem of locking: example P1:P1

Introducing the shear stress $\gamma = \lambda t^{-2}(\mathbf{grad}\, \omega - \boldsymbol{\phi})$, a weak formulation of this problem is given by:

Find $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega)$, $\omega \in H_0^1(\Omega)$, $\gamma \in \boldsymbol{L}^2(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\gamma, \mathbf{grad}\, v - \boldsymbol{\psi}) &= (g, v) \\ &\forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega), v \in H_0^1(\Omega), \\ (\mathbf{grad}\, \omega - \boldsymbol{\phi}, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\gamma, \boldsymbol{\eta}) &= 0, \\ &\forall \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) \end{aligned}$$

% error t=1.0 P1P1

Fcn	N=1	N=2	N=4	N=8	N=16
ϕ_1	71.05	21.80	6.00	1.56	0.39
ϕ_2	71.05	21.80	6.00	1.56	0.39
ω	30.89	9.60	2.67	0.70	0.18
$\partial\phi_1/\partial x$	74.64	38.33	19.11	9.52	4.75
$\partial\phi_2/\partial x$	108.57	58.04	30.28	15.42	7.75
$\partial\omega/\partial x$	32.82	17.60	9.28	4.76	2.40
$\partial\phi_1/\partial y$	108.57	58.04	30.25	15.40	7.74
$\partial\phi_2/\partial y$	74.64	38.33	19.12	9.52	4.75
$\partial\omega/\partial y$	32.82	17.60	9.28	4.76	2.40

% error t=0.01 P1P1

Fcn	N=1	N=2	N=4	N=8	N=16
ϕ_1	99.93	99.66	98.63	94.78	82.41
ϕ_2	99.93	99.66	98.63	94.78	82.42
ω	99.90	99.64	98.60	94.74	83.32
$\partial\phi_1/\partial x$	99.92	99.66	98.63	94.80	82.59
$\partial\phi_2/\partial x$	99.92	99.66	98.63	94.80	82.52
$\partial\omega/\partial x$	99.90	99.65	98.62	94.78	82.42
$\partial\phi_1/\partial y$	99.92	99.66	98.63	94.80	82.52
$\partial\phi_2/\partial y$	99.92	99.66	98.63	94.80	82.59
$\partial\omega/\partial y$	99.90	99.65	98.62	94.78	82.42

Many of the finite element methods which have been proposed to overcome the problem of “locking” have the following variational formulation.

Find $\boldsymbol{\phi}_h \in \boldsymbol{\Theta}_h$, $\omega_h \in W_h$, $\boldsymbol{\gamma}_h \in \boldsymbol{\Gamma}_h$ such that

$$a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + (\boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{R}_h \boldsymbol{\psi}) = (g, v) \qquad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, v \in W_h,$$

$$(\mathbf{grad}\, \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0 \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h,$$

where $\boldsymbol{\Theta}_h$, W_h, $\boldsymbol{\Gamma}_h$ finite dimensional subspaces of $\boldsymbol{H}_0^1(\Omega)$, $H_0^1(\Omega)$, and $\boldsymbol{L}^2(\Omega)$, respectively.

$\boldsymbol{R}_h =$ interpolation or projection operator mapping into $\boldsymbol{\Gamma}_h$.

In some cases, spaces are nonconforming and differential operators applied on each element.

This variational formulation equivalent to minimization problem over $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$, $\omega \in W_h$

$$(\boldsymbol{\phi}, \omega) \to \int_\Omega \frac{1}{2} \mathcal{C}\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\phi}) + \frac{1}{2} \lambda t^{-2} \int_\Omega \Big[|\, \mathbf{grad}\, \omega - \boldsymbol{R}_h \boldsymbol{\phi}\, |^2 - g\omega \Big] dx$$

Abstract Error Analysis

Assume Ω a convex polygon and $\{\mathcal{T}_h\}_{\{0<h<1\}}$ regular family of triangulations of Ω, where h refers to diameter of largest triangle.

Theorem: (Generalization of Durán–Liberman result) Suppose $\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h$.

Let $\omega_I \in W_h$, $\boldsymbol{\phi}_I \in \mathbf{\Theta}_h$, $\boldsymbol{\gamma}_I = t^{-2}(\mathbf{grad}\, \omega_I - \boldsymbol{R}_h \boldsymbol{\phi}_I)$. Suppose for $s \geq 1$

$$\boxed{\|\boldsymbol{\gamma} - \boldsymbol{R}_h \boldsymbol{\gamma}\|_0 \leq Ch\|\boldsymbol{\gamma}\|_1,}$$

$$\boxed{(\boldsymbol{\gamma} - \boldsymbol{R}_h \boldsymbol{\gamma}, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in \boldsymbol{P}_{s-2}}$$

($\boldsymbol{P}_k$ denotes discontinuous piecewise polynomials of degree $\leq k$.) Then, letting $\mathbf{\Pi}$ denote $\boldsymbol{L}^2$ projection into $\boldsymbol{P}_{s-2}$

$$\boxed{\begin{aligned} &\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \\ &\quad \leq C\Big(\|\boldsymbol{\phi}_I - \boldsymbol{\phi}\|_1 + t\|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}\|_0 + h\|\boldsymbol{\gamma} - \mathbf{\Pi}\boldsymbol{\gamma}\|_0\Big) \end{aligned}}$$

To apply theorem, find approximations which satisfy:

$$\boxed{\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I = \boldsymbol{R}_h\,\mathbf{grad}\,\omega - \boldsymbol{R}_h\boldsymbol{\phi}}$$

and

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_0 + h\|\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_1 \le Ch^{s+1}\|\boldsymbol{\phi}\|_{s+1},$$
$$\|\boldsymbol{\gamma} - \boldsymbol{R}_h\boldsymbol{\gamma}\|_0 \le Ch^s\|\boldsymbol{\gamma}\|_s$$

If so then

$$\boldsymbol{\gamma}_I = t^{-2}(\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I)$$
$$= t^{-2}\boldsymbol{R}_h(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \boldsymbol{R}_h\boldsymbol{\gamma}$$

Apply Theorem to get for $s \ge 1$

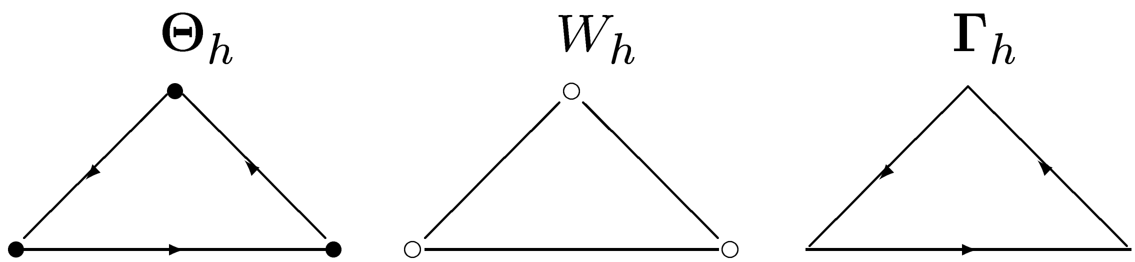
$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0$$
$$\le C\Big(\|\boldsymbol{\phi}_I - \boldsymbol{\phi}\|_1 + t\|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}\|_0 + h\|\boldsymbol{\gamma} - \boldsymbol{\Pi}\boldsymbol{\gamma}\|_0\Big)$$
$$\le Ch^s\Big(\|\boldsymbol{\phi}\|_{s+1} + \|\boldsymbol{\gamma}\|_{s-1} + t\|\boldsymbol{\gamma}\|_s\Big)$$

Note: Constant C independent of h and t.

But: norms on rhs NOT independent of t for $s \ge 3/2$

How do we construct such interpolants?

Example 1: Durán–Liberman element:



$\mathbf{\Theta}_h$: C^0 piecewise linear vectors plus span of $\lambda_2\lambda_3\boldsymbol{\tau}_1$, $\lambda_3\lambda_1\boldsymbol{\tau}_2$, $\lambda_1\lambda_2\boldsymbol{\tau}_3$

Degrees of freedom: $\boldsymbol{\phi}(v_i)$, $\int_{e_i}\boldsymbol{\phi}\cdot\boldsymbol{\tau}_i$, $\quad i = 1, \ldots, 3$

W_h: continuous piecewise linear functions

$\mathbf{\Gamma}_h$: Subset of linear vectors of the form $(a - by, c + bx)$ on each triangle

Degrees of freedom: $\boldsymbol{\gamma}\cdot\boldsymbol{\tau}_i$ on each edge e_i. Constant on each edge and continuous

Rotation of the lowest order Raviart-Thomas space

$\boldsymbol{R}_h$ defined by

$$\boxed{\int_e \boldsymbol{R}_h\boldsymbol{\gamma}\cdot\boldsymbol{\tau} = \int_e \boldsymbol{\gamma}\cdot\boldsymbol{\tau}}$$

Define $\boldsymbol{\phi}_I$ so that $\int_e \boldsymbol{\phi}_I \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\phi} \cdot \boldsymbol{\tau}$. Then

$$\int_e \boldsymbol{R}_h \boldsymbol{\phi}_I \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\phi}_I \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\phi} \cdot \boldsymbol{\tau} = \int_e \boldsymbol{R}_h \boldsymbol{\phi} \cdot \boldsymbol{\tau}$$

so $$\boxed{\boldsymbol{R}_h \boldsymbol{\phi}_I = \boldsymbol{R}_h \boldsymbol{\phi}}$$

Important property: $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$.

Also, if ω_I is the standard piecewise linear interpolant of ω, and e is the edge joining vertices v_a and v_b, then

$$\begin{aligned}\int_e \mathbf{grad}\, \omega_I \cdot \boldsymbol{\tau} &= \int_e \partial \omega_I / \partial s = w_I(v_b) - w_I(v_a) \\ &= w(v_b) - w(v_a) = \int_e \partial \omega / \partial s = \int_e \mathbf{grad}\, \omega \cdot \boldsymbol{\tau}\end{aligned}$$

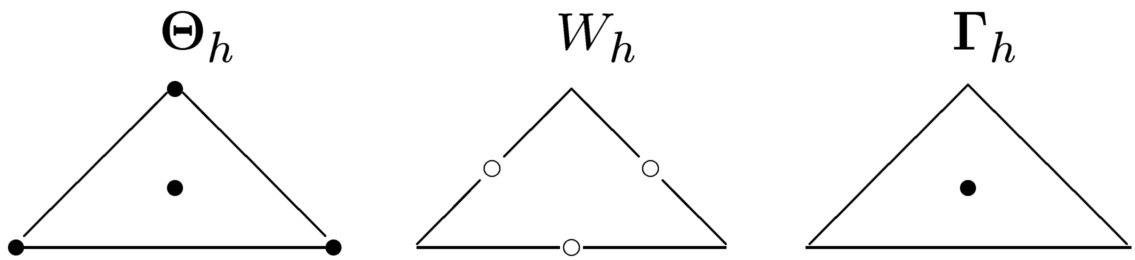
so

$$\boxed{\mathbf{grad}\, \omega_I = \boldsymbol{R}_h \, \mathbf{grad}\, \omega}$$

Error estimates:

$$\boxed{\begin{aligned}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|\omega - \omega_h\|_1 &\le Ch, \\ \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 &\le Ch^2\end{aligned}}$$

Example 2: Arnold–Falk element:



$\boldsymbol{\Theta}_h$: C^0 piecewise linear vectors + bubbles $\lambda_1\lambda_2\lambda_3$

Degrees of freedom: $\boldsymbol{\phi}(v_i)$, $\int_T \boldsymbol{\phi}$, $i = 1, \ldots, 3$

W_h: Nonconforming piecewise linear functions: continuous at midpoints of edges. Degrees of freedom: $\int_e w =$ value of w at the edge midpoint

$\boldsymbol{\Gamma}_h$: Piecewise constant vectors

$\boldsymbol{R}_h$ defined by: $\boxed{\int_T \boldsymbol{R}_h\boldsymbol{\phi} = \int_T \boldsymbol{\phi}}$

Important property: $\mathbf{grad}_h W_h \subset \boldsymbol{\Gamma}_h$. Hence, for $\omega \in W_h$, $\boldsymbol{R}_h \, \mathbf{grad}_h \, w = \mathbf{grad}_h \, w$. Note grad not defined on W_h, but defined piecewise on each triangle.

If ω_I nonconforming P_1 interpolant of ω defined by $\int_e w_I = \int_e w$, then for all constant vectors $\boldsymbol{q}$,

$$\int_T \mathbf{grad}\, \omega_I \cdot \boldsymbol{q} = \sum_{i=1}^{3} \int_{e_i} \omega_I \boldsymbol{q} \cdot \boldsymbol{n}$$
$$= \sum_{i=1}^{3} \int_{e_i} \omega \boldsymbol{q} \cdot \boldsymbol{n} = \int_T \mathbf{grad}\, \omega \cdot \boldsymbol{q}$$

so $$\boxed{\mathbf{grad}_h\, \omega_I = \boldsymbol{R}_h\, \mathbf{grad}\, \omega}$$

Since $\int_T \boldsymbol{\phi}$ degree of freedom of $\boldsymbol{\Theta}_h$: can choose $\boldsymbol{\phi}_I$ to satisfy

$$\boxed{\int_T \boldsymbol{\phi}_I = \int_T \boldsymbol{\phi}}$$

Hence: $$\int_T \boldsymbol{R}_h \boldsymbol{\phi}_I = \int_T \boldsymbol{\phi}_I = \int_T \boldsymbol{\phi} = \int_T \boldsymbol{R}_h \boldsymbol{\phi}$$

so $$\boxed{\boldsymbol{R}_h \boldsymbol{\phi}_I = \boldsymbol{R}_h \boldsymbol{\phi}}$$

Hence: $$\boxed{\mathbf{grad}\, \omega_I - \boldsymbol{R}_h \boldsymbol{\phi}_I = \boldsymbol{R}_h\, \mathbf{grad}\, \omega - \boldsymbol{R}_h \boldsymbol{\phi}}$$

Since W_h nonconforming space, additional terms arise in error estimates. Get:

Error estimates:

$$\|\phi - \phi_h\|_1 + \|\omega - \omega_h\|_{1,h} \leq Ch,$$
$$\|\phi - \phi_h\|_0 + \|\omega - \omega_h\|_0 \leq Ch^2$$

% error t=0.01 Arnold-Falk

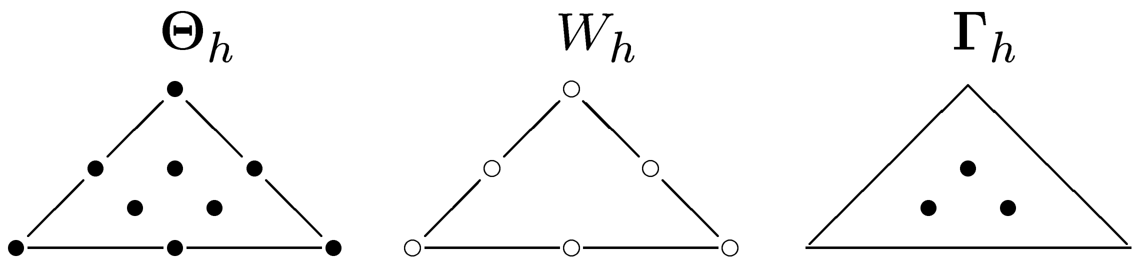
Fcn	N=1	N=2	N=4	N=8	N=16
ϕ_1	58.33	16.62	4.31	1.08	0.27
ϕ_2	58.33	16.62	4.31	1.08	0.27
ω	59.21	18.08	4.78	1.21	0.30
$\partial\phi_1/\partial x$	60.56	29.98	14.88	7.40	3.69
$\partial\phi_2/\partial x$	126.81	68.17	35.21	17.50	8.68
$\partial\omega/\partial x$	63.66	29.98	14.56	7.22	3.60
$\partial\phi_1/\partial y$	126.81	68.17	35.19	17.48	8.67
$\partial\phi_2/\partial y$	60.56	29.98	14.89	7.41	3.69
$\partial\omega/\partial y$	63.66	29.98	14.56	7.21	3.60

Do we really need to use nonconforming $P1$ for W_h?

% error t=0.01 P1BP1

Fcn	N=1	N=2	N=4	N=8	N=16
ϕ_1	63.00	61.17	63.56	61.50	52.05
ϕ_2	63.00	61.17	63.57	61.51	52.06
ω	65.48	61.05	62.65	60.58	51.10
$\partial\phi_1/\partial x$	68.51	73.86	77.07	74.60	63.13
$\partial\phi_2/\partial x$	142.65	109.37	100.66	92.48	75.67
$\partial\omega/\partial x$	67.83	64.42	64.37	61.74	52.15
$\partial\phi_1/\partial y$	142.65	109.37	100.59	92.42	75.62
$\partial\phi_2/\partial y$	68.51	73.86	77.10	74.62	63.15
$\partial\omega/\partial y$	67.83	64.42	64.38	61.75	52.16

Example 3: Zienkiewicz–Lefebvre



$\boldsymbol{\Theta}_h$: C^0 piecewise quadratic vectors plus quartic bubble functions $\lambda_1\lambda_2\lambda_3\boldsymbol{P}_1$

Degrees of freedom:

$\phi(v_i)$, $\phi(m_i)$, $\int_T \phi\boldsymbol{P}_1$, $i = 1, \ldots, 3$

W_h: Continuous piecewise quadratic functions

$\boldsymbol{\Gamma}_h$: Piecewise linear vectors

$\boldsymbol{R}_h = \boldsymbol{\Pi}$ ($\boldsymbol{L}^2$ projection) defined by:

$$\boxed{\int_T \boldsymbol{R}_h\phi\boldsymbol{P}_1 = \int_T \phi\boldsymbol{P}_1}$$

Important property: $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$.

Not easy to see how to construct ω_I so that for all $\boldsymbol{P}_1$

$$\int_T \mathbf{grad}\,\omega_I \cdot \boldsymbol{q} = \int_T \mathbf{grad}\,\omega \cdot \boldsymbol{q}$$

to get $\mathbf{grad}_h\,\omega_I = \boldsymbol{R}_h\,\mathbf{grad}\,\omega$.

Try another approach: Choose ω_I standard piecewise linear interpolant.

Choose $\boldsymbol{\phi}_I = \boldsymbol{\phi}_I^0 + \boldsymbol{\phi}_I^b$ where $\boldsymbol{\phi}_I^0$ is a standard piecewise quadratic interpolant of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_I^b \in [B^4]^2$ is defined by

$$\boldsymbol{\Pi}\boldsymbol{\phi}_I^b = \boldsymbol{\Pi}\boldsymbol{\phi} - \boldsymbol{\Pi}\boldsymbol{\phi}_I^0 - \boldsymbol{\Pi}\,\mathbf{grad}\,\omega + \mathbf{grad}\,\omega_I$$

Hence,

$$\begin{aligned}\boldsymbol{R}_h\gamma_I &= t^{-2}(\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I)\\ &= t^{-2}\boldsymbol{R}_h(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = t^{-2}\boldsymbol{R}_h\gamma\end{aligned}$$

New problem: Is $\boldsymbol{\phi}_I$ a good approximation to $\boldsymbol{\phi}$?

For $s = 0, 1$, can show

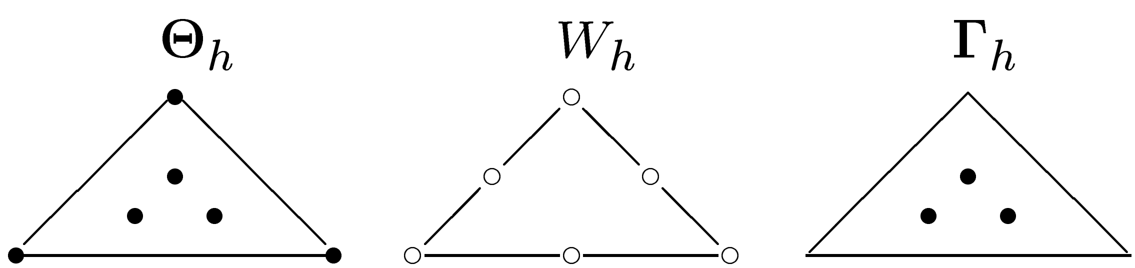
$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_s \le Ch^{2-s}(\|\boldsymbol{\phi}\|_2 + \|\omega\|_3)$$

Suboptimal estimate, but sufficient for our needs.

Error estimates:

$$
\begin{gathered}
\|\phi - \phi_h\|_1 \le Ch \min(1, ht^{-1}), \\
\|\phi - \phi_h\|_0 \le Ch^2 \min(1, ht^{-1}), \\
\|\omega - \omega_h\|_1 \le Ch^2
\end{gathered}
$$

Example 4: Falk–Tu

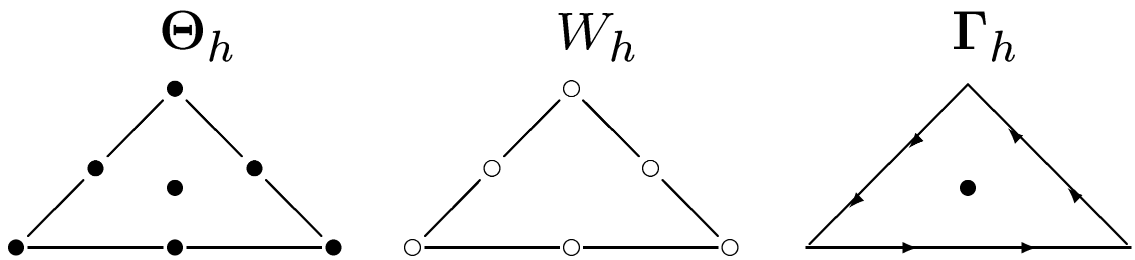


Similar to previous element, but $\mathbf{\Theta}_h$ is space of C^0 piecewise linears + quartic bubble functions.

Error estimates:

$$
\begin{gathered}
\|\phi - \phi_h\|_1 \le Ch \\
\|\phi - \phi_h\|_0 + \|\omega - \omega_h\|_1 \le Ch^2
\end{gathered}
$$

Example 5: Bathe–Brezzi–Fortin–Stenberg
(lowest order element of family)



$\mathbf{\Theta}_h$: C^0 piecewise P_2 vectors plus cubic bubbles

W_h: continuous piecewise quadratic functions

$\mathbf{\Gamma}_h$: Subset of quadratic vectors of the form $(a + bx + cy - dxy - ey^2, f + gx + hy + exy + dx^2)$ on each triangle (Rotation of the second lowest order R-T space)
Note: $\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h$.

$\boldsymbol{R}_h$ defined by

$$\boxed{\int_e \boldsymbol{R}_h \gamma \cdot \boldsymbol{\tau} p_1 = \int_e \gamma \cdot \boldsymbol{\tau} p_1} \qquad \boxed{\int_T \boldsymbol{R}_h \gamma = \int_T \gamma}$$

If ω_I^0 standard piecewise linear interpolant of ω, easy to show that

$$\boxed{\mathbf{grad}\, \omega_I^0 = \boldsymbol{R}_h \, \mathbf{grad}\, \omega}$$

In this case, not easy to find ϕ_I so that

$$\boldsymbol{R}_h\boldsymbol{\phi}_I = \boldsymbol{R}_h\boldsymbol{\phi}$$

Instead, find ϕ_I such that

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_I\|_1 \le Ch^2\|\boldsymbol{\phi}\|_3,$$
$$\operatorname{rot} \boldsymbol{R}_h[\boldsymbol{\phi}_I - \boldsymbol{\phi}] = 0$$

Then for some $z_h \in W_h$

$$\boldsymbol{R}_h[\boldsymbol{\phi}_I - \boldsymbol{\phi}] = \mathbf{grad}\, z_h$$

Hence, we may take $\omega_I = \omega_I^0 + z_h$ and the equation

$$\mathbf{grad}\,\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I = \boldsymbol{R}_h\,\mathbf{grad}\,\omega - \boldsymbol{R}_h\boldsymbol{\phi}$$

will be satisfied.

To find ϕ_I, use ideas from Stokes. $\phi =$ quadratics + cubic bubbles, p = discontinuous P_1 stable Stokes element. rot behaves like div so can find ϕ_I so that

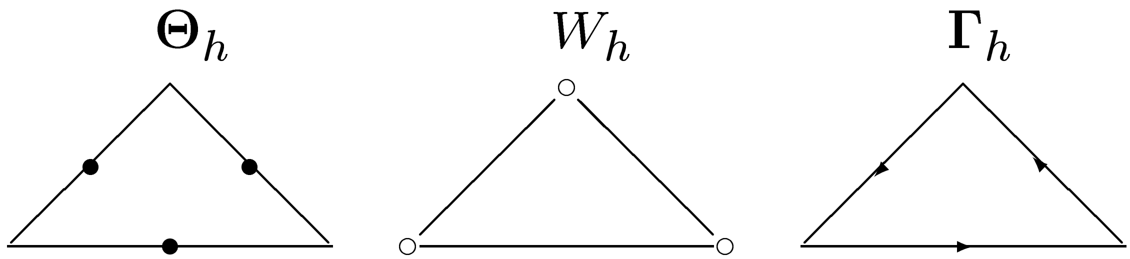
$$\int_T \operatorname{rot}[\boldsymbol{\phi}_I - \boldsymbol{\phi}]p_1 = 0$$

Hence

$$\int_T \operatorname{rot} \boldsymbol{R}_h[\boldsymbol{\phi}_I - \boldsymbol{\phi}]p_1 = \int_T \operatorname{rot}[\boldsymbol{\phi}_I - \boldsymbol{\phi}]p_1 = 0$$

so $\operatorname{rot} \boldsymbol{R}_h[\boldsymbol{\phi}_I - \boldsymbol{\phi}] = 0$.

Example 6: Onãte–Zarate–Flores



$\boldsymbol{\Theta}_h$: Nonconforming piecewise linear vectors

W_h: continuous piecewise linear functions

$\boldsymbol{\Gamma}_h$: Subset of linear vectors of the form $(a - by, c + dx)$ on each triangle

Rotation of lowest order Raviart-Thomas space, with $\boldsymbol{R}_h$ as in Durán–Liberman element

As shown previously,

$$\boxed{\mathbf{grad}\,\omega_I = \boldsymbol{R}_h\,\mathbf{grad}\,\omega}$$

Since the usual nonconforming piecewise linear interpolant satisfies

$$\boxed{\int_e \phi_I = \int_e \phi}$$

we immediately have

$$\int_e \boldsymbol{R}_h \boldsymbol{\phi}_I \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\phi}_I \cdot \boldsymbol{\tau} = \int_e \boldsymbol{\phi} \cdot \boldsymbol{\tau} = \int_e \boldsymbol{R}_h \boldsymbol{\phi} \cdot \boldsymbol{\tau}$$

so $\boxed{\boldsymbol{R}_h \boldsymbol{\phi}_I = \boldsymbol{R}_h \boldsymbol{\phi}.}$

However, another problem arises with this method:

Analysis depends on the use of Korn's inequality:

$\boxed{\|\boldsymbol{\phi}\|_1 \leq C \| \mathcal{E}(\boldsymbol{\phi})\|_0}$ for all $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega)$.

Nonconforming piecewise linear functions $\notin \boldsymbol{H}_0^1(\Omega)$ and obvious discrete version of this inequality fails. However, they satisfy

$$\|\boldsymbol{\phi}\|_{1,h} \leq C(\| \mathcal{E}(\boldsymbol{\phi})\|_{0,h} + \| \operatorname{rot}(\boldsymbol{\phi})\|_{0,h})$$

Analysis of Arnold-Falk shows:

Method is of optimal order if $t \leq h$, but fails to converge for t fixed as $h \to 0$.

Historical Perspective

How were these methods discovered?

A key idea (Brezzi-Fortin): Use Helmholtz decomposition and relation to Stokes problem.

Use Helmholtz decomposition on shear stress

$$\boxed{\lambda t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \mathbf{curl}\,p + \mathbf{grad}\,r}$$

where $r \in H_0^1(\Omega)$ and $p \in H^1(\Omega)$. Note

$$\boxed{\int_\Omega \mathbf{curl}\,p \cdot \mathbf{grad}\,r = 0}$$

Then

$$-\lambda t^{-2}\,\mathrm{div}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = g$$

becomes

$$\boxed{-\Delta\, r = g \text{ in } \Omega, \qquad r = 0 \text{ on } \partial\Omega}$$

so r is easily determined.

Applying rot to Helmholtz decomposition, get

$$\boxed{-\lambda t^{-2}\operatorname{rot}\phi = -\Delta\, p}$$

Taking the dot product with the unit tangent vector $\mathbf{s}$ on $\partial\Omega$, get boundary condition $\partial p/\partial n = 0$.

With r known, ϕ and p are determined as solutions of system:

$$\boxed{\begin{gathered} -\mathbf{div}\, C\,\mathcal{E}(\phi) - \mathbf{curl}\, p = \mathbf{grad}\, r \text{ in } \Omega \\ \operatorname{rot}\phi = \lambda^{-1}t^{2}\,\Delta\, p \text{ in } \Omega \\ \phi = \partial p/\partial n = 0 \text{ on } \partial\Omega \end{gathered}}$$

This is completely analogous to system obtained by replacing $\mathbf{curl}\, p$ by $\mathbf{grad}\, p$ and $\operatorname{rot}\phi$ by $\operatorname{div}\phi$ and is a singularly perturbed stationary Stokes equation.

Once ϕ and p are determined, ω can be recoved by solving

$$\boxed{\Delta\,\omega = \operatorname{div}\phi + \lambda^{-1}t^{2}\,\Delta\, r}$$

with zero boundary condition.

Brezzi-Fortin proposed FEM based on this reformulation: Choose ω_h, r_h C^0 piecewise linear.

Find $\boldsymbol{\phi}_h \in \boldsymbol{\Theta}_h \subset \boldsymbol{H}^1_0(\Omega)$, $p_h \in Q_h \subset \hat{H}^1(\Omega)$:

$$\begin{gathered}\int_\Omega C\,\mathcal{E}(\boldsymbol{\phi}_h) : \mathcal{E}(\boldsymbol{\psi}) - \int_\Omega \mathbf{curl}\, p_h \cdot \boldsymbol{\psi} \\ = \int_\Omega \mathbf{grad}\, r \cdot \boldsymbol{\psi}, \\ \int_\Omega \boldsymbol{\phi}_h \cdot \mathbf{curl}\, q + \lambda^{-1} t^2 \int_\Omega \mathbf{curl}\, p_h \cdot \mathbf{curl}\, q = 0\end{gathered}$$

for all $\boldsymbol{\psi} \in \boldsymbol{\Theta}_h$, $q \in Q_h$.

Well known that not every combination of spaces works for this problem. Spaces $\boldsymbol{\Theta}_h$ and Q_h must satisfy stability condition. One simple choice that does work is MINI element.

$$\boxed{\boldsymbol{\Theta}_h = [P_1 + B_3]^2, \qquad Q_h = P_1}$$

Method of Brezzi–Fortin: optimal order, locking-free

BUT: need additional variables and must reformulate system.

Arnold–Falk approach:

Question: Can we choose approximations in such a way that we can reverse the procedure and obtain a method only in the original variables?

$Q_h =$ continuous piecewise linear functions

$W_h =$ nonconforming piecewise linear functions

Use: Discrete Helmholtz decomposition

Lemma: Let $\boldsymbol{\Gamma}_h =$ piecewise constant vectors. Then $\forall \boldsymbol{\zeta}_h \in \boldsymbol{\Gamma}_h$, there exists $p_h \in S_h$, $r_h \in W_h$ such that

$$\boxed{\boldsymbol{\zeta}_h = \mathbf{curl}\, p_h + \mathbf{grad}_h\, r_h}$$

and

$$\boxed{\int_\Omega \mathbf{curl}\, p_h \cdot \mathbf{grad}_h\, r_h = 0}$$

Can eliminate r_h and p_h and write problem in original formulation by introducing $\boldsymbol{R}_h = \boldsymbol{L}^2$ projection.

Another idea: Bathe–Brezzi–Fortin–Stenberg:

Use mixed finite element methods to approximate the second order equation

$$\boxed{\operatorname{rot}\boldsymbol{\phi} = \lambda^{-1}t^2\,\Delta\, p}$$

Set $\boldsymbol{\alpha} = \mathbf{curl}\, p$. Get variational formulation:

Find $r \in H^1_0$, $\omega \in H^1_0$, $p \in L^2_0$, $\boldsymbol{\phi} \in \boldsymbol{H}^1_0$, $\boldsymbol{\alpha} \in \boldsymbol{H}_0(\operatorname{rot})$ such that

$$\begin{gathered}
(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu) \\
a(\boldsymbol{\phi}, \boldsymbol{\psi}) - (p, \operatorname{rot}\boldsymbol{\psi}) = (\mathbf{grad}\, r, \boldsymbol{\psi}) \\
\lambda^{-1}t^2(\operatorname{rot}\alpha, q) + (\operatorname{rot}\boldsymbol{\phi}, q) = 0 \\
(\boldsymbol{\alpha}, \boldsymbol{\beta}) - (p, \operatorname{rot}\boldsymbol{\beta}) = 0 \\
(\mathbf{grad}\,\omega, \mathbf{grad}\, s) - (\boldsymbol{\phi}, \mathbf{grad}\, s) \\
\quad -\lambda^{-1}t^2(\mathbf{grad}\, r, \mathbf{grad}\, s) = 0
\end{gathered}$$

for all $s \in H^1_0$, $\mu \in H^1_0$, $q \in L^2_0$, $\boldsymbol{\psi} \in \boldsymbol{H}^1_0$, $\boldsymbol{\beta} \in \boldsymbol{H}_0(\operatorname{rot})$

Discretization: Let $r_h, \omega_h \in W_h$, $\boldsymbol{\phi}_h \in \boldsymbol{\Theta}_h$, $p_h \in Q_h$, $\boldsymbol{\alpha}_h \in \boldsymbol{\Gamma}_h$.

Choose subspaces to satisfy:

1. $\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h$
2. $\operatorname{rot} \mathbf{\Gamma}_h \subset Q_h$
3. $\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\phi} = \mathbf{\Pi}_h \operatorname{rot} \boldsymbol{\phi}$, $\quad \mathbf{\Pi}_h$: L^2 projection $\to Q_h$
4. If $\boldsymbol{\gamma} \in \mathbf{\Gamma}_h$ and $\operatorname{rot} \boldsymbol{\gamma} = 0$, then $\boldsymbol{\gamma} = \mathbf{grad}\, \omega_h$
for some $\omega_h \in W_h$.
5. $(\mathbf{\Theta}_h^{\perp}, Q_h)$ is a stable pair for the Stokes problem

$$\sup_{\boldsymbol{\psi} \in \mathbf{\Theta}_h} \frac{(\operatorname{rot} \boldsymbol{\psi}, q)}{\|\boldsymbol{\psi}\|_1} \geq C \|q\|_0, \quad q \in Q_h$$

Example 1:

$\mathbf{\Theta}_h = C^0$ piecewise quadratics + cubic bubbles
$Q_h =$ discontinuous piecewise linears
$\mathbf{\Gamma}_h =$ second lowest order rotated R-T element

$(\mathbf{\Theta}_h^{\perp}, Q_h)$ stable Stokes pair

These choices satisfy (2) and (3)

$W_h =$ continuous piecewise quadratics

This satisfies (1) and (4).

Reason for (4): Space $\mathbf{\Gamma}_h$ has the discrete Helmholtz decomposition:

$$\boxed{\mathbf{\Gamma}_h = \mathbf{grad}\, W_h \oplus \mathbf{curl}_h\, Q_h}$$

where for $q \in Q_h$, $\boldsymbol{\alpha}_h \equiv \mathbf{curl}_h\, q \in \mathbf{\Gamma}_h$ defined by

$$(\boldsymbol{\alpha}_h, \boldsymbol{\beta}_h) = (q, \operatorname{rot} \boldsymbol{\beta}_h) \quad \forall \boldsymbol{\beta}_h \in \mathbf{\Gamma}_h$$

Hence,

$$(\boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h) = (q, \operatorname{rot} \boldsymbol{\alpha}_h)$$

so $\operatorname{rot} \boldsymbol{\alpha}_h = 0$ implies $\boldsymbol{\alpha}_h = 0$, i.e., no rotation free elements in $\mathbf{curl}_h\, Q_h$.

Note: follows trivially from definition that decomposition is orthogonal.

$$\boxed{(\mathbf{grad}\, w_h, \mathbf{curl}_h\, q) = 0}$$

Example 2: Durán–Liberman element

$\mathbf{\Theta}_h$: Continuous piecewise linear vectors plus span of $\lambda_2\lambda_3\boldsymbol{\tau}_1$, $\lambda_3\lambda_1\boldsymbol{\tau}_2$, $\lambda_1\lambda_2\boldsymbol{\tau}_3$

$Q_h =$ piecewise constant vectors

$(\mathbf{\Theta}^\perp, Q_h)$ stable Stokes pair

$\mathbf{\Gamma}_h$: Rotation of lowest order Raviart-Thomas space

Subset of linear vectors of the form $(a - by, c + bx)$ on each triangle

These choices satisfy:

2. $\operatorname{rot} \mathbf{\Gamma}_h \subset Q_h$
3. $\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\phi} = \mathbf{\Pi}_h \operatorname{rot} \boldsymbol{\phi}$

W_h: continuous piecewise linear functions. Satisfies:

1. $\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h$
4. If $\boldsymbol{\gamma} \in \mathbf{\Gamma}_h$ and $\operatorname{rot} \boldsymbol{\gamma} = 0$, then $\boldsymbol{\gamma} = \mathbf{grad}\, \omega_h$ for some $\omega_h \in W_h$.