REMARKS ON IRREGULAR HEXAHEDRAL FINITE ELEMENTS

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Motivation

Two disadvantages of DG methods for simple problems for which standard conforming Galerkin methods work well (e.g., approximating second order elliptic problems using triangular or tetrahedral finite elements) are:

(1) more degrees of freedom needed

(2) bilinear forms for DG method more complicated because of jump and penalty terms.

However, in some applications, construction of conforming elements not so easy or requires many more degrees of freedom than needed for optimal order approximation.

$\mathcal{P}_r$ on an element enough to get approx of $O(h^{r+1})$. When spaces discontinuous, can define functions directly on physical element.

When some continuity needed, usually define spaces by mapping from reference element.

Explore difficulties of this procedure.

In particular, what functions do we need on reference element to produce $\mathcal{P}_r$ on the physical element?

For triangles (and tetrahedrons), shape functions usually $\mathcal{P}_r$. For affine maps, $\mathcal{P}_r$ maps to $\mathcal{P}_r$. For bilinear and trilinear maps, this is not the case.

Additional complications when functions defined by Piola transform.

$\boxed{\textbf{Outline of Talk}}$

1. Scalar quadrilateral finite elements in 2D

2. $\boldsymbol{H}(\mathrm{div}, \Omega)$ quadrilateral finite elements in 2D

3. $\boldsymbol{H}(\mathrm{div}, \Omega)$ hexahedral finite elements in 3D

4. $\boldsymbol{H}(\mathrm{curl}, \Omega)$ hexahedral finite elements in 3D

Construction: Scalar Finite Elements - 2D

Let $\widehat{K}$ be reference element (unit triangle or square).

Let $\widehat{V}$ be finite dimensional space of functions on $\widehat{K}$, typically polynomial:
$\mathcal{P}_r(\widehat{K})$, polynomials of total degree $\leq r$ on $\widehat{K}$
or $\mathcal{Q}_r(\widehat{K})$, polynomials of degree $\leq r$ on $\widehat{K}$ in each variable separately.

Let $\boldsymbol{F}$ be an isomorphism of $\widehat{K}$ onto element K, i.e., $\boldsymbol{x} \in K = \boldsymbol{F}(\widehat{\boldsymbol{x}})$, $\widehat{\boldsymbol{x}} \in \widehat{K}$.

For triangular or rectangular finite elements:
$\boldsymbol{F}$ is affine, so $\boldsymbol{x} = \boldsymbol{B}\widehat{\boldsymbol{x}} + \boldsymbol{b}$.

When K is a quadrilateral, $\boldsymbol{F}$ is bilinear map, i.e.,

$$(x, y) = \boldsymbol{F}(\widehat{x}, \widehat{y})$$

$$= (a_1 + b_1\widehat{x} + c_1\widehat{y} + d_1\widehat{x}\widehat{y}, a_2 + b_2\widehat{x} + c_2\widehat{y} + d_2\widehat{x}\widehat{y})$$

For scalar functions, if $\widehat{v}(\widehat{\boldsymbol{x}})$ defined on $\widehat{K}$, define: $v(\boldsymbol{x})$ on K by $v = \widehat{v} \circ \boldsymbol{F}^{-1}$.

Then for $\widehat{V}$ = shape functions on $\widehat{K}$, define

$$\boxed{V_F(K) = \{v : v = \widehat{v} \circ \boldsymbol{F}^{-1}, \widehat{v} \in \widehat{V}\}.}$$

For triangular meshes, if $\boldsymbol{F}$ is affine and $\widehat{V}$ is $\mathcal{P}_r(\widehat{K})$, then $V_F(K) = \mathcal{P}_r(K)$.

For rectangular meshes, if $\boldsymbol{F}$ has form $(a_1\widehat{x} + b_1, a_2\widehat{y} + b_2)$ and $\widehat{V}$ is $\mathcal{Q}_r(\widehat{K})$, then $V_F(K) = \mathcal{Q}_r(K)$.

For quadrilateral meshes, space $V_F(K)$ more complicated.

Example:

$$\boldsymbol{F}(\widehat{x}, \widehat{y}) = (\widehat{x}, \widehat{y}(\widehat{x} + 1))$$

maps unit square to quadrilateral K with vertices: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 2)$.

So

$$\hat{x} = x, \qquad \hat{y} = y/(1+x).$$

Suppose $\hat{V}$ is linear functions on $\hat{K}$, i.e., span of $1, \hat{x}, \hat{y}$. Since $u(\boldsymbol{x}) = \hat{u}(\hat{\boldsymbol{x}})$, $V_F(K)$ is span of

$$1, x, y/(1+x).$$

If $\hat{V}$ is bilinear functions on $\hat{K}$, i.e., span of $1, \hat{x}, \hat{y}, \hat{x}\hat{y}$, then $V_F(K)$ is span of

$$1, x, y/(1+x), xy/(1+x).$$

Key fact: Span of $1, x, y/(1+x)$ does not contain y. But span of $1, x, y/(1+x), xy/(1+x)$ does contain y since

$$y = y/(1+x) + xy/(1+x).$$

So in first case, $V_F(K)$ does not contain $\mathcal{P}_1$.
In second case it does.

Define finite element space as

$$S_h = \{\, v \in L^2 \,|\, v|_K \in V_F(K), \forall K \in \mathcal{T}_h \,\}.$$

Interested in best approximation of u by functions in S_h as a quantity involving powers of h, the maximum element diameter.

Scalar approximation on rectangular meshes:

Let $\mathcal{T}_h$ be uniform mesh of unit square Ω into n^2 subsquares when $h = 1/n$.

Equivalent conditions for optimal order convergence:

Theorem: Let $\hat{V}$ be a finite dimensional subspace of $L^2(\hat{K})$, r a non-negative integer. The following conditions are equivalent:

There is a constant C such that for all $u \in H^{r+1}(\Omega)$
$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} \le Ch^{r+1}|u|_{H^{r+1}(\Omega)}$.
$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} = o(h^r)$ for all $u \in P_r(\Omega)$.
$\hat{V} \supseteq \mathcal{P}_r(\hat{K})$.

Thus, for optimal order approximation, need $\hat{V} \supseteq \mathcal{P}_r(\hat{K})$.

Need **stronger** condition on **quadrilaterals.**

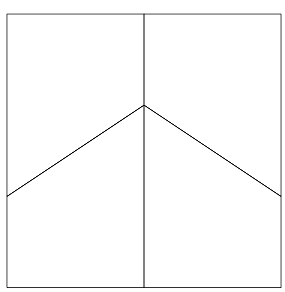
Now let $\mathcal{T}_h$ be family of shape-regular quadrilateral meshes of Ω.

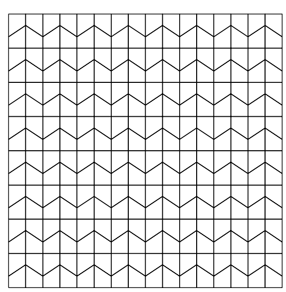
Theorem: Let $V^{\mathcal{T}_h}$ be functions on Ω that $\in V_{F_K}(K)$ when restricted to $K \in \mathcal{T}_h$. Then $\hat{V} \supseteq \mathcal{Q}_r$ is necessary for condition:

$$\inf_{v \in S_h} \|u - v\|_{L^2(\Omega)} = o(h^r) \quad \text{for all } u \in \mathcal{P}_r(\Omega).$$

Proof: Show that for Ω = unit square and simple sequence of meshes (all elements similar to a single right trapezoid) and u a polynomial, whenever $\mathcal{Q}_r$ is not contained in $\hat{V}$, then above estimate violated.

Divide unit square into four congruent trapezoids and define mesh $\mathcal{T}_h$ of scaled and translated versions of these trapezoids.





For error estimate to hold, need $V_F(K)$ to contain $\mathcal{P}_r(K)$ for each element K. Relate this to condition on $\hat{K}$.

Key lemma:

Consider following choices of mapping $\boldsymbol{F}$.

$$\boldsymbol{F}^1: \qquad x = \hat{x}, \qquad y = (1+\hat{x})\hat{y},$$

$$\boldsymbol{G}^1: \qquad x = \hat{y}, \qquad y = (1+\hat{y})\hat{x},$$

from $\hat{K}$ to quadrilateral K^1 with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,2)$.

Lemma: Let $\hat{V}$ be a vectorspace of functions on $\hat{K}$. Suppose that $V_{F^1}(K^1) \supseteq \mathcal{P}_r(K^1)$ and $V_{G^1}(K^1) \supseteq \mathcal{P}_r(K^1)$. Then $\hat{V} \supseteq \mathcal{Q}_r(\hat{K})$.

Proof: case $r = 1$. Since $V_{F^1}(K^1) \supseteq 1, x, y$, $\hat{V} \supseteq 1, \hat{x}, (1+\hat{x})\hat{y}$. Since $V_{G^1}(K^1) \supseteq 1, x, y$, $\hat{V} \supseteq 1, \hat{y}, (1+\hat{y})\hat{x}$. Hence:
$\hat{V} \supseteq 1, \hat{x}, \hat{y}, \hat{x}\hat{y} = \mathcal{Q}_1(\hat{K})$.

Implication for optimal order approximation.

Roughly speaking, for optimal $O(h^{r+1})$ approximation, require space of functions defined on K to contain polynomials of degree $\leq r$. By above, this requires condition on space $\hat{V}$.

Using discontinuous quadrilateral elements, only need $\mathcal{P}_r$ defined directly on K for optimal order approximation.

Mapping from reference element (to construct continuous elements), need to start with $\mathcal{Q}_r$ on reference element to get $\mathcal{P}_r$ on physical element K.

Note: for continuity, do not need all of $\mathcal{Q}_r$.

$\boxed{\textbf{Serendipity spaces } S_r(\hat{K})\textbf{:}}$

$S_r(\hat{K}) =$ span of $\mathcal{P}_r(\hat{K})$ together with monomials $\hat{x}^r\hat{y}$ and $\hat{x}\hat{y}^r$.

$$S_2 = \mathcal{P}_2 + \hat{x}^2\hat{y} + \hat{x}\hat{y}^2 = \mathcal{Q}_2 - \hat{x}^2\hat{y}^2$$

Eliminate interior degree of freedom, but maintain C^0 elements. On rectangles, have $O(h^3)$ approximation. Result implies only $O(h^2)$ approximation on quadrilaterals.

Stationary Stokes equations:

Choice of bilinearly mapped piecewise continuous $\mathcal{Q}_2$ for the two components of velocity and discontinuous piecewise linear elements (unmapped) for pressure known to be stable and $O(h^2)$ in H^1 for velocities and in L^2 for pressure. If pressure elements defined by composing linear functions on $\hat{K}$ with bilinear mappings, would only get $O(h)$ for the pressure.

Nonconforming elements:

For triangular element, simplest is $\mathcal{P}_1$ with dof at midpoints of triangles edges. For rectangles, bilinears with dof at midpoints of edges not unisolvent.

$(\hat{x} - 1/2)(\hat{y} - 1/2) = 0$ at midpoints of edges.

Alternative: replace basis function $\hat{x}\hat{y}$ by $\hat{x}^2 - \hat{y}^2$. OK on rectangles, but now $\hat{V}$ does not contain $\mathcal{Q}_1$, so degradation of convergence.

$\boldsymbol{H}(\mathrm{div},\Omega)$ **Finite Elements**

To construct finite element subspaces of $\boldsymbol{H}(\mathrm{div},\Omega)$, need continuity of $\boldsymbol{u}\cdot\boldsymbol{n}$ across element interfaces. Starting from reference element (unit square) $\widehat{K}$, this is done using Piola transform defined by:

$$\boldsymbol{u} = \boldsymbol{P}_K \hat{\boldsymbol{u}} = J\boldsymbol{F}(\hat{\boldsymbol{x}})^{-1} D\boldsymbol{F}(\hat{\boldsymbol{x}})\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}})$$

where $\boldsymbol{x} = \boldsymbol{F}(\hat{\boldsymbol{x}})$ and $D\boldsymbol{F}(\hat{\boldsymbol{x}})$ is Jacobian matrix of $\boldsymbol{F}$ and J its determinant.

Transform has property: if $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ related as above and $p(x) = \widehat{p}(\widehat{x})$, then

$$\int_{\partial K} \boldsymbol{u}\cdot\boldsymbol{n}\, p\, ds = \int_{\partial \widehat{K}} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\, \widehat{p}\, d\widehat{s}.$$

Best known example of shape functions on reference square are Raviart-Thomas elements of index r,
$\widehat{\boldsymbol{V}} = \boldsymbol{\mathcal{RT}}_r := \mathcal{P}_{r+1,r} \times \mathcal{P}_{r,r+1}$.

Other examples are $\boldsymbol{\mathcal{BDM}}_r$ and $\boldsymbol{\mathcal{BDFM}}_{r+1}$.

$\boldsymbol{\mathcal{BDM}}_1 =$ span of $\boldsymbol{\mathcal{P}}_1$ and vectorfields $\mathrm{curl}(\widehat{x}_1^2\widehat{x}_2)$ and $\mathrm{curl}(\widehat{x}_1\widehat{x}_2^2)$.

$\boldsymbol{\mathcal{BDFM}}_2 =$ span of $\boldsymbol{\mathcal{P}}_1$ and $(x_1x_2, 0)$, $(0, x_1x_2)$, $(x_1^2, 0)$, and $(0, x_2^2)$.

By previous results, if $\mathcal{T}_h =$ sequence of meshes of unit square into congruent subsquares of side length $h = 1/n$, then

$$\inf_{\boldsymbol{v}\in\boldsymbol{V}^{\mathcal{T}_h}} ||\boldsymbol{u}-\boldsymbol{v}||_{L^2(\Omega)} = o(h^r) \text{ for all } \boldsymbol{u} \in \boldsymbol{\mathcal{P}}_r(\Omega) \tag{1}$$

is valid only if $\hat{\boldsymbol{V}} \supseteq \boldsymbol{\mathcal{P}}_r(\hat{K})$ and the estimate

$$\inf_{\boldsymbol{v}\in\boldsymbol{V}^{\mathcal{T}_h}} \|\operatorname{div}\boldsymbol{u} - \operatorname{div}\boldsymbol{v}\|_{L^2(\Omega)} = o(h^r) \tag{2}$$

for all $\boldsymbol{u}$ with $\operatorname{div}\boldsymbol{u} \in \mathcal{P}_r(\Omega)$
is valid only if $\widehat{\operatorname{div}}(\hat{\boldsymbol{V}}) \supseteq \mathcal{P}_r(\hat{K})$.

For these estimates to hold for more general quadrilateral mesh sequences $\mathcal{T}_h$, stronger conditions on $\hat{\boldsymbol{V}}$ required.

Define:

$\boldsymbol{\mathcal{S}}_r$ = subspace of codimension one of $\boldsymbol{\mathcal{RT}}_r$ spanned by vector fields in $\boldsymbol{\mathcal{RT}}_r$ except two fields $(\hat{x}_1^{r+1}\hat{x}_2^r, 0)$ and $(0, \hat{x}_1^r\hat{x}_2^{r+1})$ replaced by single vector field $(\hat{x}_1^{r+1}\hat{x}_2^r, -\hat{x}_1^r\hat{x}_2^{r+1})$.

$\boldsymbol{\mathcal{S}}_0 = (a+bx_1, c-bx_2), \boldsymbol{\mathcal{RT}}_0 = (a+bx_1, c+dx_2)$.

R_r = subspace of codimension one of $\mathcal{Q}_{r+1}$, space of polynomials of degree $\leq r+1$ in each variable separately, spanned by monomials in $\mathcal{Q}_{r+1}$ except $\hat{x}_1^{r+1}\hat{x}_2^{r+1}$. ($R_0 = \mathcal{P}_1$).

Thm: Suppose that estimate (1) holds whenever $\mathcal{T}_h$ is a shape-regular sequence of quadrilateral meshes of a two-dimensional domain Ω. Then $\hat{\boldsymbol{V}} \supseteq \boldsymbol{\mathcal{S}}_r$.

Thm: Suppose that estimate (2) holds whenever $\mathcal{T}_h$ is a shape-regular sequence of quadrilateral meshes of a two-dimensional domain Ω. Then $\widehat{\mathrm{div}}\hat{\boldsymbol{V}} \supseteq R_r$.

Proof: Take Ω and sequence $\mathcal{T}_h$ of meshes as before and prove that whenever $\widehat{V}$ does not contain $\boldsymbol{\mathcal{S}}_r$, there exists $\boldsymbol{u}$ (again a polynomial) such that

$$\inf_{\boldsymbol{v}\in S_h} ||\boldsymbol{u}-\boldsymbol{v}||_{L^2(\Omega)} \neq o(h^r).$$

Key lemmas:

Consider following choices of mapping $\boldsymbol{F}$. For $\alpha > 0$, define

$$\boldsymbol{F}^\alpha : \qquad x = \hat{x}, \qquad y = (\alpha + \hat{x})\hat{y},$$

$$\boldsymbol{G}^\alpha : \qquad x = \hat{y}, \qquad y = (\alpha + \hat{y})\hat{x},$$

from $\widehat{K}$ to quadrilaterals K^α with vertices $(0,0)$, $(1,0)$, $(1,1+\alpha)$, and $(0,\alpha)$.

Then, applying Piola transform associated to $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{F}^\alpha : \quad \hat{u}_1 = (\alpha + \hat{x})u_1, \quad \hat{u}_2 = -\hat{y}u_1 + u_2,$$

$$\boldsymbol{F} = \boldsymbol{G}^\alpha : \quad \hat{u}_1 = \hat{x}u_1 - u_2, \quad \hat{u}_2 = -(\alpha + \hat{y})u_1.$$

Lemma: Let $\hat{\boldsymbol{V}}$ be a space of vectorfields on $\hat{K}$. Suppose $\boldsymbol{V}_{F^1}(K^1)$ and $\boldsymbol{V}_{G^1}(K^1)$ each $\supseteq \boldsymbol{\mathcal{P}}_r(K^1)$ and $\boldsymbol{V}_{F^2}(K^2)$ and $\boldsymbol{V}_{G^2}(K^2)$ each $\supseteq \boldsymbol{\mathcal{P}}_r(K^2)$. Then $\hat{\boldsymbol{V}} \supseteq \hat{\boldsymbol{\mathcal{S}}}_r(\hat{K})$.

Proof by induction on r. The case $r = 0$ checked by taking $\boldsymbol{u}(x, y) = (1, 0)$ and $(0, 1)$. Then $\hat{\boldsymbol{V}}$ must contain

$$(\alpha + \hat{x}, -\hat{y}), \quad (0, 1), \quad (\hat{x}, -(\alpha + \hat{y})), \quad (-1, 0),$$

$\alpha = 1, 2$. Spanned by $(\hat{x}, -\hat{y})$, $(0, 1)$, $(1, 0)$. Precisely $\boldsymbol{\mathcal{RT}}_0(\hat{K})/(\hat{x}, \hat{y}) = \mathcal{P}_{1,0} \times \mathcal{P}_{0,1}/(\hat{x}, \hat{y})$.

Next consider divergence. Note $\widehat{\operatorname{div}}\, \hat{\boldsymbol{u}} = J \operatorname{div} \boldsymbol{u}$.

Recall $\hat{R}_r = \hat{\mathcal{Q}}_{r+1}/\hat{x}^{r+1}\hat{y}^{r+1}$.

Lemma: Let $\hat{\boldsymbol{V}}$ be a space of vectorfields on $\hat{K}$. Suppose $\operatorname{div} \boldsymbol{V}_{F^1}(K^1)$ and $\operatorname{div} \boldsymbol{V}_{G^1}(K^1)$ each $\supseteq \mathcal{P}_r(K^1)$ and $\operatorname{div} \boldsymbol{V}_{F^2}(K^2)$, $\operatorname{div} \boldsymbol{V}_{G^2}(K^2)$ each $\supseteq \mathcal{P}_r(K^2)$. Then $\widehat{\operatorname{div}} \hat{\boldsymbol{V}} \supseteq \hat{R}_r(\hat{K})$.

Proof: A simple calculation shows that

$$J_{F^\alpha} = \alpha + \hat{x}, \qquad J_{G^\alpha} = \alpha + \hat{y}.$$

Only look at case $r = 0$. Taking $\operatorname{div} \boldsymbol{u} = 1$. Easily follows $\widehat{\operatorname{div}} \hat{\boldsymbol{V}}$ must contain $\alpha + \hat{x}$ and $\alpha + \hat{y}$ for $\alpha = 1, 2$. This space is precisely $\hat{R}_0(\hat{K})$.

Can show necessary conditions are sufficient for optimal order approximation with usual seminorm.

For quadrilateral mesh in which quadrilaterals not necessarily parallelograms, previous best known estimate, proved by Thomas was

$$\inf_{v \in V^{\mathcal{T}_h}} ||\boldsymbol{u} - \boldsymbol{v}||_{L^2(\Omega)} \\ \leq C h^{r+1} [|\boldsymbol{u}|_{r+1} + h| \operatorname{div} \boldsymbol{u}|_{r+1}]$$

Corollary: for $\mathcal{RT}$ elements, no longer need second term on right side of Thomas estimate for optimal order L^2 approximation.

Note, however, Raviart-Thomas elements do not produce optimal order approximation in $\boldsymbol{H}(\operatorname{div}, \Omega)$ on quadrilaterals.

Situation for $\mathcal{BDM}$ and $\mathcal{BDFM}$ even worse. Recall $\mathcal{BDFM}_1 = \mathcal{RT}_0$. For $r \geq 1$

$$\mathcal{BDM}_r = \mathcal{P}_r^2 + a \text{ curl}(\hat{x}^{r+1}\hat{y}) + b \text{ curl}(\hat{x}\hat{y}^{r+1})$$

$$\mathcal{BDFM}_{r+1} = \mathcal{P}_{r+1}^2 - (0, \hat{x}^{r+1}) - (\hat{y}^{r+1}, 0)$$

$$\mathcal{RT}_r = \mathcal{P}_{r+1,r} \times \mathcal{P}_{r,r+1}$$

$$\mathcal{S}_r = \mathcal{RT}_r - (\hat{x}^{r+1}\hat{y}^r, \hat{x}^r\hat{y}^{r+1})$$

$$\dim \mathcal{BDM}_r = (r+1)(r+2) + 2,$$

$$\dim \mathcal{BDFM}_{r+1} = (r+2)(r+3) - 2.$$

Both $< \dim S_r = 2(r+1)(r+2) - 1$, so neither gives optimal order L^2 approximation.

Example: $r = 1$. $\dim \mathcal{S}_1 = 11$, $\dim \mathcal{RT}_1 = 12$, $\dim \mathcal{BDM}_1 = 8$, $\dim BDFM_2 = 10$.

Spaces $\mathcal{BDM}_1$, $\mathcal{BDFM}_2$, $\mathcal{RT}_1$, and $\mathcal{S}_1$ all contain $\mathcal{P}_1$, so all give optimal L^2 approximation on rectangles.

Construction of spaces with optimal order $\boldsymbol{H}(\mathrm{div}, \Omega)$ approximation:

For $O(h^{r+1})$ L^2 approximation, require $\hat{\boldsymbol{V}} \supseteq \hat{\boldsymbol{\mathcal{S}}}_r$.

Now $\widehat{\mathrm{div}}\, \hat{\boldsymbol{\mathcal{S}}}_r = \hat{\mathcal{Q}}_r / \hat{x}^r \hat{y}^r$.

Need to have $\hat{\mathcal{Q}}_{r+1} / \hat{x}^{r+1} \hat{y}^{r+1}$.

Achieved for space $\hat{\boldsymbol{V}} = \boldsymbol{\mathcal{P}}_{r+2,r} \times \boldsymbol{\mathcal{P}}_{r,r+2}$.

For $r = 0$, this is $\boldsymbol{\mathcal{RT}}_0$ plus two degrees of freedom:

$$(\hat{x}(1-\hat{x}), 0), \qquad (0, \hat{y}(1-\hat{y}))$$

Hexahedral finite elements in 3D

Element K constructed by mapping reference unit cube $\hat{K}$ using trilinear mapping $\boldsymbol{F}$ given by:

$$a_1 + b_1\hat{x} + c_1\hat{y} + d_1\hat{z} + e_1\hat{x}\hat{y} + f_1\hat{y}\hat{z} + g_1\hat{z}\hat{x} + h_1\hat{x}\hat{y}\hat{z},$$

$$a_2 + b_2\hat{x} + c_2\hat{y} + d_2\hat{z} + e_2\hat{x}\hat{y} + f_2\hat{y}\hat{z} + g_2\hat{z}\hat{x} + h_2\hat{x}\hat{y}\hat{z},$$

$$a_3 + b_3\hat{x} + c_3\hat{y} + d_3\hat{z} + e_3\hat{x}\hat{y} + f_3\hat{y}\hat{z} + g_3\hat{z}\hat{x} + h_3\hat{x}\hat{y}\hat{z}.$$

In general, faces of resulting solid are hyperboloids – not flat.

Results for scalar elements analogous to those in 2D.

Results for $\boldsymbol{H}(\mathrm{div}, \Omega)$ elements considerably more complicated.

To define subspaces of $\boldsymbol{H}(\mathrm{div}, \Omega)$, again use Piola transform $\boldsymbol{P}_F$ associated to map $\boldsymbol{F}$ defined by:

$$\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{P}_F\hat{\boldsymbol{q}}(\hat{\boldsymbol{x}}) = J_F^{-1}(\hat{\boldsymbol{x}})D\boldsymbol{F}(\hat{\boldsymbol{x}})\hat{\boldsymbol{q}}(\hat{\boldsymbol{x}}).$$

Inverting this relationship, get

$$\hat{\boldsymbol{q}}(\hat{\boldsymbol{x}}) = J_F(\hat{\boldsymbol{x}})D\boldsymbol{F}^{-1}(\hat{\boldsymbol{x}})\boldsymbol{q}(\boldsymbol{F}(\hat{\boldsymbol{x}})).$$

Determine set of functions $\hat{\boldsymbol{V}}$ on $\widehat{K}$ such that $\boldsymbol{P}_F(\hat{\boldsymbol{V}}) \supseteq \boldsymbol{\mathcal{P}}_r$. Insures approximation of order h^{r+1}.

Consider only case $r = 0$.

Choose $\boldsymbol{q} = (1,0,0), (0,1,0), (0,0,1)$.

Find that $\widehat{\boldsymbol{q}}(\widehat{\boldsymbol{x}})$ has general form

$$A_1 + (D_3 - C_2)\widehat{x} + C_1\widehat{y} + D_1\widehat{z} + G_1\widehat{x}^2$$

$$+(E_2 - G_2)\widehat{x}\widehat{y} + (E_3 - G_3)\widehat{x}\widehat{z} + H_3\widehat{x}^2\widehat{y} + H_2\widehat{x}^2\widehat{z},$$

$$A_2 + B_2\widehat{x} + (B_1 - D_3)\widehat{y} + D_2\widehat{z} + G_2\widehat{y}^2$$

$$+(E_1 - G_1)\widehat{y}\widehat{x} - (E_3 + G_3)\widehat{y}\widehat{z} - H_3\widehat{x}\widehat{y}^2 + H_1\widehat{y}^2\widehat{z},$$

$$A_3 + B_3\widehat{x} + C_3\widehat{y} + (C_2 - B_1)\widehat{z} + G_3\widehat{z}^2$$

$$-(E_1 + G_1)\widehat{z}\widehat{x} - (E_2 + G_2)\widehat{z}\widehat{y} - H_2\widehat{x}\widehat{z}^2 - H_1\widehat{y}\widehat{z}^2.$$

21 dimensional subspace of $\boldsymbol{\mathcal{RT}}_1$ (dimension = 36) on $\widehat{K}$ insures mapped space contains all constant vectors. Recall: dim $\boldsymbol{\mathcal{RT}}_0 = 6$.

Coefficients not all independent, so lower dimensional space may suffice (these 21 coefficients are functions of only 14 parameters).

Note $\boldsymbol{q} \cdot \boldsymbol{n}$ is polynomial of degree ≤ 1 on each face and $\operatorname{div} \boldsymbol{q} = 0$. To insure continuity of $\boldsymbol{q} \cdot \boldsymbol{n}$, specify 3 degrees of freedom on each face, i.e.,

$$\int_F (\boldsymbol{q} \cdot \boldsymbol{n})\, p\, ds, \quad p \in \mathcal{P}_1(F).$$

If $\boldsymbol{q} = 0$ at these degrees of freedom, then $\boldsymbol{q}$ will have form:

$$q_1 = \widehat{x}(1 - \widehat{x})[-H_3(\widehat{y} - 1/2) - H_2(\widehat{z} - 1/2)],$$

$$q_2 = \widehat{y}(1 - \widehat{y})[H_3(\widehat{x} - 1/2) - H_1(\widehat{z} - 1/2)],$$

$$q_3 = \widehat{z}(1 - \widehat{z})[H_2(\widehat{x} - 1/2) + H_1(\widehat{y} - 1/2)].$$

Defining vectors

$$\boldsymbol{r}_1 := (1/2 - \widehat{y}, x - 1/2, 0),$$

$$\boldsymbol{r}_2 := (1/2 - \widehat{z}, 0, \widehat{x} - 1/2),$$

$$\boldsymbol{r}_3 := (0, 1/2 - \widehat{z}, \widehat{y} - 1/2),$$

final 3 degrees of freedom can be defined by:

$$\int_K \boldsymbol{q} \cdot \boldsymbol{r}_i\, dx, \quad i = 1, \cdots, 3.$$

Next consider simpler situation:

Require both boundary faces and secondary faces (corresponding to mappings of planes $\widehat{x} = 1/2$, $\widehat{y} = 1/2$, and $\widehat{z} = 1/2$) to be flat.

To simplify, write $\boldsymbol{F} = \boldsymbol{A} \circ \boldsymbol{B}$, where $\boldsymbol{A}$ is an affine map and $\boldsymbol{B}(\widehat{\boldsymbol{x}})$ is given by

$$\bar{x} = B_1 = \widehat{x} + E_1\widehat{y}\widehat{z} + F_1\widehat{x}\widehat{z} + G_1\widehat{x}\widehat{y} + H_1\widehat{x}\widehat{y}\widehat{z},$$

$$\bar{y} = B_2 = \widehat{y} + E_2\widehat{y}\widehat{z} + F_2\widehat{x}\widehat{z} + G_2\widehat{x}\widehat{y} + H_2\widehat{x}\widehat{y}\widehat{z},$$

$$\bar{z} = B_3 = \widehat{z} + E_3\widehat{y}\widehat{z} + F_3\widehat{x}\widehat{z} + G_3\widehat{x}\widehat{y} + H_3\widehat{x}\widehat{y}\widehat{z}.$$

$\boldsymbol{B}$ has property that it maps points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to themselves. Composing with general affine map $\boldsymbol{A}$, maps these points arbitrarily.

Since $\boldsymbol{P}_F = \boldsymbol{P}_A \circ \boldsymbol{P}_B$, and $\boldsymbol{P}_A$ is constant, the space $\boldsymbol{P}_F^{-1}$ applied to constant vectors is same as space $\boldsymbol{P}_B^{-1}$ applied to constant vectors. Hence, sufficient to consider $\boldsymbol{B}$ instead of more general $\boldsymbol{F}$.

Plane $\widehat{x} = 0$ maps to:

$$\bar{x} = E_1\widehat{y}\widehat{z}, \quad \bar{y} = \widehat{y} + E_2\widehat{y}\widehat{z}, \quad \bar{z} = \widehat{z} + E_3\widehat{y}\widehat{z}$$

These points lie in a plane if there exists constants α, β, γ, and δ, not all zero, such that

$$\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \delta.$$

This requires:

$$\beta = 0, \quad \gamma = 0, \quad \alpha E_1 + \beta E_2 + \gamma E_3 = 0.$$

Hence $E_1 = 0$.

Similarly, plane $\widehat{y} = 0$ maps to a plane if $F_2 = 0$ and plane $\widehat{z} = 0$ maps to a plane if $G_3 = 0$.

Plane $\widehat{x} = 1$ maps to:

$$\bar{x} = 1 + F_1\widehat{z} + G_1\widehat{y} + H_1\widehat{y}\widehat{z},$$

$$\bar{y} = \widehat{y} + E_2\widehat{y}\widehat{z} + G_2\widehat{y} + H_2\widehat{y}\widehat{z},$$

$$\bar{z} = \widehat{z} + E_3\widehat{y}\widehat{z} + F_3\widehat{z} + H_3\widehat{y}\widehat{z}.$$

Result is a plane if:

$$H_1(G_2+1)(F_3+1) - G_1(H_2+E_2)(F_3+1)$$

$$-F_1(G_2+1)(H_3+E_3) = 0.$$

Get similar nonlinear conditions on the coefficients to ensure $\widehat{y} = 1$, $\widehat{z} = 1$, and $\widehat{x} = 1/2$, $\widehat{y} = 1/2$, $\widehat{z} = 1/2$ all map to planes.

Can show if:

$$G_1E_2F_3(1+G_1)(1+G_1/2)$$

$$\cdot(1+E_2)(1+E_2/2)(1+F_3)(1+F_3/2)$$

$$+F_1G_2E_3(1+F_1)(1+F_1/2)$$

$$\cdot(1+G_2)(1+G_2/2)(1+E_3)(1+E_3/2) \neq 0$$

then

$$H_1 = F_1G_1, \quad H_2 = E_2G_2, \quad H_3 = E_3F_3.$$

Mapping $\boldsymbol{B}$ then simplifies to:

$$\bar{x} = \widehat{x}(1 + G_1\widehat{y})(1 + F_1\widehat{z}),$$

$$\bar{y} = \widehat{y}(1 + G_2\widehat{x})(1 + E_2\widehat{z}),$$

$$\bar{z} = \widehat{z}(1 + F_3\widehat{x})(1 + E_3\widehat{y}).$$

where remaining coefficients satisfy:

$$E_3F_1 + E_2G_1 - F_1G_1 = 0,$$

$$E_2F_3 + F_1G_2 - E_2G_2 = 0,$$

$$G_1F_3 + G_2E_3 - E_3F_3 = 0.$$

In this case, find that to produce any constant vector (α, β, γ) for any constrained choices of coefficients $(E_2, E_3, F_1, F_3, G_1, G_2)$, shape functions on $\widehat{K}$ must contain:

$$
\begin{aligned}
&(1 + E_3\hat{y} + E_2\hat{z}) \cdot [\alpha(1 + G_2\hat{x})(1 + F_3\hat{x}) \\
&\qquad - \beta_1\hat{x}(1 + F_3\hat{x}) - \gamma F_1\hat{x}(1 + G_2\hat{x})], \\
&(1 + F_1\hat{z} + F_3\hat{x}) \cdot [-\alpha G_2\hat{y}(1 + E_3\hat{y}) \\
&\qquad + \beta(1 + G_1\hat{y})(1 + E_3\hat{y}) - \gamma E_2\hat{y}(1 + G_1\hat{y})], \\
&(1 + G_2\hat{x} + G_1\hat{y}) \cdot [-\alpha F_3\hat{z}(1 + E_2\hat{z}) \\
&\qquad - \beta E_3\hat{z}(1 + F_1\hat{z}) + \gamma(1 + F_1\hat{z})(1 + E_2\hat{z})]
\end{aligned}
$$

9 parameters subject to 3 constraints.

Not so easy to see how to choose basis functions independent of coefficients.

Work of Russell et al: chooses a local space of dimension 6, but *basis functions depend on particular mapping* $\boldsymbol{F}$ to physical element K.

$\boxed{\boldsymbol{H}(\mathrm{curl}, \Omega)\ \textbf{Finite Elements}}$

These elements are defined using the transformation:

$$\boldsymbol{u}(\boldsymbol{F}(\widehat{\boldsymbol{x}})) = (D\boldsymbol{F})^{-T}(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}).$$

Inverting this relation, have

$$\widehat{\boldsymbol{u}} = D\boldsymbol{F}^{T}\boldsymbol{u}$$

Using this formula, easily determine space $\widehat{\boldsymbol{V}}$ on $\widehat{K}$ so that corresponding $\boldsymbol{V}_F(K)$ contains all constant vectors.

Span of:

$$(1,0,0),(0,1,0),(0,0,1),(\widehat{y},\widehat{x},0),$$
$$(\widehat{z},0,\widehat{x}),(0,\widehat{z},\widehat{y}),(\widehat{y}\widehat{z},\widehat{x}\widehat{z},\widehat{x}\widehat{y})$$

Note that lowest order Nédélec space

$$\boldsymbol{N}_0 = \boldsymbol{\mathcal{P}}_{0,1,1} \times \boldsymbol{\mathcal{P}}_{1,0,1} \times \boldsymbol{\mathcal{P}}_{1,1,0}$$

contains necessary functions so is first order accurate on elements obtained by trilinear mapping.

Can also show that Nédélec space

$$\mathcal{P}_{1,2,2} \times \mathcal{P}_{2,1,2} \times \mathcal{P}_{2,2,1}$$

is second order convergent in L^2.

For approximation of Maxwell's equations, also want first order approximation of $\operatorname{curl} \boldsymbol{u}$. Now

$$\widehat{\operatorname{curl}} \hat{\boldsymbol{u}} = J(D\boldsymbol{F})^{-1} \operatorname{curl} \boldsymbol{u}.$$

Hence $\widehat{\operatorname{curl}} \hat{\boldsymbol{V}}$ must contain $J(D\boldsymbol{F})^{-1} \mathcal{P}_0$.

Precisely same calculation made for $\boldsymbol{H}(\operatorname{div}, \Omega)$
(21 dimensional space in general case).

But $\operatorname{curl} \boldsymbol{N}_0$ is span of:

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (x, -y, 0), (-x, 0, z).$$

So many more degrees of freedom must be added for hexahedrons.