Finite Element Methods for the Reissner-Mindlin Plate Problem

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Consider approximation of equations of linear elasticity when body is an isotropic, homogeneous, linearly elastic plate.

To describe geometry, change notation and consider plate occupying region $P_t = \Omega \times (-t/2, t/2)$, where Ω a smoothly bounded domain in $\mathbb{R}^2$ and $t \in (0, 1]$.

Consider thin plate, so thickness t small.

Denote union of top and bottom surfaces of plate by $\partial P_t^{\pm} = \Omega \times \{-t/2, t/2\}$ and lateral boundary by $\partial P_t^{\mathsf{L}} = \partial\Omega \times (-t/2, t/2)$.

Suppose plate loaded by surface force density $\underset{\sim}{g} \colon \partial P_t^{\pm} \to \mathbb{R}^3$ and volume force density $\underset{\sim}{f} \colon P_t \to \mathbb{R}^3$, and is clamped along lateral boundary.

Resulting stress $\underline{\underline{\sigma}}^*\colon P_t \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and displacement $\underline{u}^*\colon P_t \to \mathbb{R}^3$ then satisfy BVP

$$\mathcal{A}\underline{\underline{\sigma}}^* = \underline{\underline{\varepsilon}}(\underline{u}^*), \quad -\underline{\mathrm{div}}\,\underline{\underline{\sigma}}^* = \underline{f} \text{ in } P_t,$$
$$\underline{\underline{\sigma}}^*\underline{n} = \underline{g} \text{ on } \partial P_t^{\pm}, \quad \underline{u}^* = 0 \text{ on } \partial P_t^{\mathsf{L}}.$$

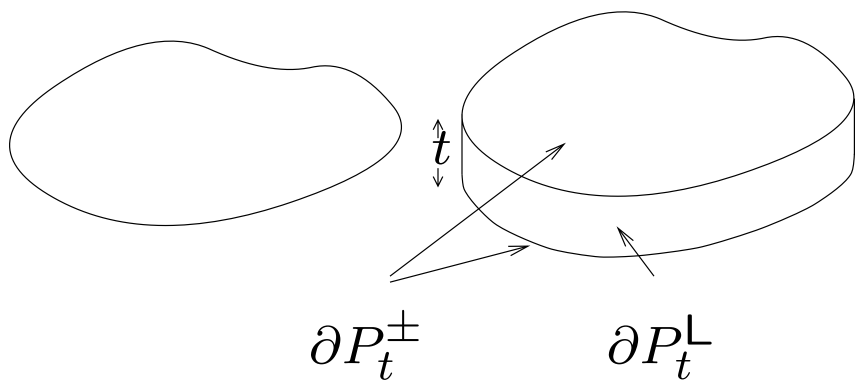


Figure 1: The two-dimensional domain Ω and plate domain P_t.

$\underline{\underline{\varepsilon}}(\underline{u}^*)$ again denotes strain tensor (symmetric part of gradient)

$\underline{\mathrm{div}}\,\underline{\underline{\sigma}}$ again denotes vector divergence of symmetric matrix $\underline{\underline{\sigma}}$ taken by rows.

Compliance tensor $\mathcal{A}$ given by

$$\mathcal{A}\underline{\underline{\tau}} = \frac{1}{E}\left[(1+\nu)\underline{\underline{\tau}} - \nu\,\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}\right],$$

with $E > 0$ Young's modulus, $\nu \in [0, 1/2)$ Poisson's ratio, and $\underline{\underline{\delta}}$ the 3×3 identity matrix.

Plate model seeks to approximate solution of elasticity problem in terms of solution of system of PDEs on 2-D domain Ω without requiring solution of 3-D problem. Passage from 3-D problem to plate model known as *dimensional reduction*.

Taking odd and even parts with respect to x_3, 3-D problem splits into two decoupled problems corresponding to *stretching* and *bending* of plate. system with $\underline{g}$ replaced by $\underline{g}^{\mathsf{s}}$ and $\underline{f}$ replaced by $\underline{f}^{\mathsf{s}}$, and bending portion of solution analogously.

Plate stretching models variants of equations of generalized plane stress. Plate bending models variants of Kirchhoff-Love biharmonic plate model or Reissner-Mindlin plate model.

Variations due to different specification of forcing functions for 2-D model in terms of 3-D loads $\underline{g}$ and $\underline{f}$ and different specification of approximate 3-D stresses and displacements in terms of solutions of 2-D boundary-value problems. No universally accepted basic two-dimensional model of plate stretching or bending.

A Variational Approach to Dimensional Reduction

Hellinger-Reissner principle gives variational characterization of solution to 3-D elasticity problem. Consider two forms of this principle.

To state first form (HR), define

$$\underline{\underline{\Sigma}}^\bullet = \underline{\underline{L}}^2(P_t), \quad \underline{V}^\bullet = \left\{ \underline{v} \in \underline{H}^1(P_t) \colon \underline{v} = 0 \text{ on } \partial P_t^{\mathsf{L}} \right\}.$$

Then HR characterizes $(\underline{\underline{\sigma}}^*, \underline{u}^*)$ as unique critical point (namely a saddle point) of HR functional

$$J(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2} \int_{P_t} \mathcal{A} \underline{\underline{\tau}} \colon \underline{\underline{\tau}} \, d\underline{x} - \int_{P_t} \underline{\underline{\tau}} \colon \underline{\underline{\varepsilon}}(\underline{v}) \, d\underline{x} + \int_{P_t} \underline{f} \cdot \underline{v} \, d\underline{x} + \int_{\partial P_t^\pm} \underline{g} \cdot \underline{v} \, d\utilde{x}$$

on $\underline{\underline{\Sigma}}^\bullet \times \underline{V}^\bullet$.

Equivalently, $(\underset{=}{\sigma}^*, \underset{-}{u}^*)$ is unique element of $\underset{=}{\Sigma}^\bullet \times \underset{-}{V}^\bullet$ satisfying:

$$\int_{P_t} \mathcal{A}\underset{=}{\sigma}^* : \underset{=}{\tau}\, d\underset{-}{x} - \int_{P_t} \underset{=}{\varepsilon}(\underset{-}{u}) : \underset{=}{\tau}\, d\underset{-}{x} = 0 \quad \text{for all } \underset{=}{\tau} \in \underset{=}{\Sigma}^\bullet,$$
$$\int_{P_t} \underset{=}{\sigma} : \underset{=}{\varepsilon}(\underset{-}{v})\, d\underset{-}{x} = \int_{P_t} \underset{-}{f} \cdot \underset{-}{v}\, d\underset{-}{x} + \int_{\partial P_t^\pm} \underset{-}{g} \cdot \underset{-}{v}\, d\underset{\sim}{x} \quad \text{for all } \underset{-}{v} \in \underset{-}{V}^\bullet.$$

Plate models derived by replacing $\underset{=}{\Sigma}^\bullet$ and $\underset{-}{V}^\bullet$ in HR with subspaces $\underset{=}{\Sigma}$ and $\underset{-}{V}$ with polynomial dependence on x_3 and then defining $(\underset{=}{\sigma}, \underset{-}{u})$ as unique critical point of J over $\underset{=}{\Sigma} \times \underset{-}{V}$.

Equivalent to restricting trial and test spaces in weak formulation to $\underset{=}{\Sigma} \times \underset{-}{V}$.

Insure unique solution by requiring $\underset{=}{\varepsilon}(\underset{-}{V}) \subset \underset{=}{\Sigma}$. Consider only simplest model, denoted by HR(1).

Define 2-D analogue of compliance tensor by:

$$A\underset{\approx}{\tau} = (1+\nu)\underset{\approx}{\tau}/E - \nu\,\mathrm{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}/E.$$

Can show that HR(1) solution given by:

$$\underline{u}(\underline{x}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) \\ 0 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x})x_3 \\ \omega(\underset{\sim}{x}) \end{pmatrix},$$

$$\underline{\sigma}(\underline{x}) = \begin{pmatrix} A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\phi})x_3 & \frac{E}{2(1+\nu)}(\underset{\sim}{\nabla}\omega - \underset{\sim}{\phi}) \\ \frac{E}{2(1+\nu)}(\underset{\sim}{\nabla}\omega - \underset{\sim}{\phi})^T & 0 \end{pmatrix},$$

where $\underset{\sim}{\eta}$ determined by classical generalized plane stress problem

$$-t\,\underset{\sim}{\mathrm{div}}\,A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) = 2\underset{\sim}{g}^0 + \underset{\sim}{f}^0 \text{ in } \Omega, \qquad \underset{\sim}{\eta} = 0 \text{ on } \partial\Omega,$$

and $\underset{\sim}{\phi}$ and ω by Reissner-Mindlin problem:

$$-\frac{t^3}{12}\,\underset{\sim}{\mathrm{div}}\,A^{-1}\underset{\approx}{\varepsilon}(\underset{\sim}{\phi}) + t\frac{E}{2(1+\nu)}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = -t(\underset{\sim}{g}^1 + \underset{\sim}{f}^1) \text{ in } \Omega,$$

$$t\frac{E}{2(1+\nu)}\,\mathrm{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla}\omega) = 2g_3^0 + f_3^0 \text{ in } \Omega,$$

$$\underset{\sim}{\phi} = 0, \quad \omega = 0 \text{ on } \partial\Omega.$$

In above,

$$f_3^0(\underset{\sim}{x}) = \int_{-t/2}^{t/2} f_3(\underset{\sim}{x}, x_3)\,dx_3, \quad f_3^1(\underset{\sim}{x}) = \int_{-t/2}^{t/2} f_3(\underset{\sim}{x}, x_3)\frac{x_3}{t}\,dx_3.$$

Denote even and odd parts of g_3 by g_3^0, g_3^1, resp., and define $\underset{\sim}{g}^0$, $\underset{\sim}{g}^1$, $\underset{\sim}{f}^0$, and $\underset{\sim}{f}^1$ analogously.

Verification of these equations straightforward, but tedious.

For purely transverse bending load, above system is classical R-M system with shear correction factor 1. So HR(1) method simple approach to deriving generalized plane stress and R-M type models.

Next consider alternative approach: produces models that are more accurate and more amenable to rigorous justification.

Second form of Hellinger-Reissner principle (HR$'$)

Define

$$\underset{=}{\Sigma}_g^* = \Big\{ \underset{=}{\sigma} \in \underset{=}{H}(\underset{-}{\mathrm{div}}, P_t) \mid \underset{=}{\sigma}\underset{-}{n} = \underset{-}{g} \text{ on } \partial P_t^{\pm} \Big\}, \qquad \underset{-}{V}^* = \underset{-}{L}^2(P).$$

HR$'$ characterizes $(\underset{=}{\sigma}^*, \underset{-}{u}^*)$ as unique critical point (again a saddle point) on $\underset{=}{\Sigma}_g^* \times \underset{-}{V}^*$ of HR$'$ functional

$$J'(\underset{=}{\tau}, \underset{-}{v}) = \frac{1}{2}\int_{P_t} \mathcal{A}\underset{=}{\tau} : \underset{=}{\tau}\, d\underset{-}{x} + \int_{P_t} \underset{-}{\mathrm{div}}\, \underset{=}{\tau} \cdot \underset{-}{v}\, d\underset{-}{x} + \int_{P_t} \underset{-}{f} \cdot \underset{-}{v}\, d\underset{-}{x}$$

Equivalently, $(\underset{=}{\sigma}^*, \underset{-}{u}^*)$ is unique element of $\underset{=}{\Sigma}_g^* \times \underset{-}{V}^*$ satisfying

$$\int_{P_t} \mathcal{A}\underset{=}{\sigma}^* : \underset{=}{\tau}\, d\underset{-}{x} + \int_{P_t} \underset{-}{u} \cdot \underset{-}{\mathrm{div}}\, \underset{=}{\tau}\, d\underset{-}{x} = 0 \quad \text{for all } \underset{=}{\tau} \in \underset{=}{\Sigma}_0^*,$$

$$\int_{P_t} \underset{-}{\mathrm{div}}\, \underset{=}{\sigma} \cdot \underset{-}{v}\, d\underset{-}{x} = -\int_{P_t} \underset{-}{f} \cdot \underset{-}{v}\, d\underset{-}{x} \quad \text{for all } \underset{-}{v} \in \underset{-}{V}^*,$$

where $\underset{=}{\Sigma}_0^* = \Big\{ \underset{=}{\sigma} \in \underset{=}{H}(\underset{-}{\mathrm{div}}, P_t) \mid \underset{=}{\sigma}\underset{-}{n} = 0 \text{ on } \partial P_t^{\pm} \Big\}$.

Note: displacement BC, essential BC in formulation HR, natural in HR′, while reverse true for traction BC.

By restricting J' to subspaces of $\underset{=}{\Sigma}_g^*$ and $\underline{V}^*$ with specified polynomial dependence on x_3, obtain variety of plate models.

HR′(1) model gives

$$\underline{u}(\underline{x}) = \begin{pmatrix} \underset{\sim}{\eta}(\underset{\sim}{x}) \\ \rho(\underset{\sim}{x})x_3 \end{pmatrix} + \begin{pmatrix} -\underset{\sim}{\phi}(\underset{\sim}{x})x_3 \\ \omega(\underset{\sim}{x}) + \omega_2(\underset{\sim}{x})r(x_3) \end{pmatrix}$$

and more complicated expressions for $\underset{=}{\sigma}$, where coefficient fcns $\underset{\sim}{\eta}$, ρ, $\underset{\sim}{\phi}$, ω, ω_2, functions of $\underset{\sim}{x}$, and $r(z) = 6z^2/t^2 - 3/10$.

Stretching portion of solution determined by BVP

$$-t \,\underset{\sim}{\operatorname{div}}\, A^{-1} \underset{\approx}{\varepsilon}(\underset{\sim}{\eta}) = \underset{\sim}{l}_1 + t \frac{\nu}{1-\nu} \underset{\sim}{\nabla} l_2 \text{ in } \Omega, \qquad \underset{\sim}{\eta} = 0 \text{ on } \partial\Omega,$$

$$\text{where} \quad \underset{\sim}{l}_1 = 2 \underset{\sim}{g}^0 + \underset{\sim}{f}^0, \quad l_2 = g_3^1 + \frac{t}{6} \operatorname{div} \underset{\sim}{g}^0 + f_3^1.$$

Note only loading term $\underset{\sim}{l}_1$ appeared in HR(1) model.

Bending portion of solution determined by solution of BVP

$$-\frac{t^3}{12} \underset{\sim}{\operatorname{div}}\, A^{-1} \underset{\approx}{\varepsilon}(\underset{\sim}{\phi}) + t \frac{5}{6} \frac{E}{2(1+\nu)} (\underset{\sim}{\phi} - \underset{\sim}{\nabla} \omega) = t \underset{\sim}{k}_1 - \frac{t^2}{12} \underset{\sim}{\nabla} k_2 \text{ in } \Omega,$$

$$t \frac{5}{6} \frac{E}{2(1+\nu)} \operatorname{div}(\underset{\sim}{\phi} - \underset{\sim}{\nabla} \omega) = k_3 \text{ in } \Omega, \qquad \underset{\sim}{\phi} = 0, \quad \omega = 0 \text{ on } \partial\Omega,$$

where $\underset{\sim}{k}_1$, k_2, k_3 functions of $\underset{\sim}{g}^1$, $\underset{\sim}{f}^1$, g_3^0, f_3^0, and f_3^2.

BVP determining bending solution different version of R-M equations than HR(1) model.

Formulas for applied load and couple more involved, but shear correction factor of 5/6 has been introduced. With $\underset{\sim}{\phi}$ and ω determined above, get different expressions for stresses.

For this model, possible to use "two-energies principle" to derive rigorous error estimates between solution of 3-D model and 2-D reduced model as function of t.

The Reissner–Mindlin Model

Introduce tensor $\mathcal{C} = A^{-1}$ and scale right hand side. Then R-M equations become:

$$-\,\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\theta}) - \lambda t^{-2}(\mathbf{grad}\,w - \boldsymbol{\theta}) = -\boldsymbol{f},$$
$$-\,\mathrm{div}(\mathbf{grad}\,w - \boldsymbol{\theta}) = \lambda^{-1}t^{2}g,$$

with λ constant depending on particular version of model. Define Reissner-Mindlin energy $J(\boldsymbol{\theta}, w)$

$$= \frac{1}{2}\int_\Omega \mathcal{C}\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\theta}) + \frac{1}{2}\lambda t^{-2}\int_\Omega |\,\mathbf{grad}\,w - \boldsymbol{\theta}|^2 - \int_\Omega gw + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\theta},$$

for which above equations are Euler equations.

Useful theoretical and computational tool: introduce shear stress $\boxed{\boldsymbol{\gamma} = \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\theta})}$. Get equivalent R-M system:

$$-\,\mathbf{div}\, \mathcal{C}\, \mathcal{E}(\boldsymbol{\theta}) - \boldsymbol{\gamma} = -\boldsymbol{f},$$
$$-\operatorname{div} \boldsymbol{\gamma} = g,$$
$$\mathbf{grad}\, w - \boldsymbol{\theta} - \lambda^{-1} t^2 \boldsymbol{\gamma} = 0,$$

Restrict attention to clamped plate, i.e., BC $\boldsymbol{\theta} = 0$ and $w = 0$ on $\partial\Omega$. Weak formulation is:

Find $\boldsymbol{\theta} \in \mathring{\boldsymbol{H}}^1(\Omega)$, $w \in \mathring{H}^1(\Omega)$, $\boldsymbol{\gamma} \in \boldsymbol{L}^2(\Omega)$ such that

$$a(\boldsymbol{\theta}, \boldsymbol{\phi}) + (\boldsymbol{\gamma}, \mathbf{grad}\, v - \boldsymbol{\phi}) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \;\; \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega)$$
$$(\mathbf{grad}\, w - \boldsymbol{\theta}, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0, \;\; \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega),$$

where $a(\boldsymbol{\theta}, \boldsymbol{\phi}) = (\mathcal{C}\, \mathcal{E}(\boldsymbol{\theta}), \mathcal{E}(\boldsymbol{\phi}))$.

Properties of the Solution

As $t \to 0$, $\boldsymbol{\theta} \to \boldsymbol{\theta}^0$ and $w \to w^0$, where $\boldsymbol{\theta}^0 = \mathbf{grad}\, w^0$. Can show w^0 satisfies limit problem:
Find $w^0 \in \mathring{H}^2(\Omega) = \{v \in H^2(\Omega) : v = \partial v/\partial n = 0 \text{ on } \partial\Omega\}$:

$$a(\mathbf{grad}\, w^0, \mathbf{grad}\, v) = (g, v) - (\boldsymbol{f}, \mathbf{grad}\, v), \qquad v \in \mathring{H}^2(\Omega).$$

Weak form of: $\operatorname{div} \mathbf{div}\, \mathcal{C}\, \mathcal{E}(\mathbf{grad}\, w^0) = g + \operatorname{div} \boldsymbol{f}$, which after application of calculus identities becomes:

$$D\, \Delta^2 w^0 = g + \operatorname{div} \boldsymbol{f}, \qquad D = \frac{E}{12(1-\nu^2)}.$$

Hence, limiting problem is biharmonic problem.

To understand limiting behavior and derive regularity results, introduce Helmholtz decomposition

$$\boldsymbol{\gamma} = \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\theta}) = \mathbf{grad}\, r + \mathbf{curl}\, p, \quad r \in \mathring{H}^1(\Omega), p \in \hat{H}^1(\Omega).$$

Rewrite R-M system as equivalent system:

Find $(r, \boldsymbol{\theta}, p, w) \in \mathring{H}^1(\Omega) \times \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu), \quad \mu \in \mathring{H}^1(\Omega), \tag{1}$$

$$a(\boldsymbol{\theta}, \boldsymbol{\phi}) - (\mathbf{curl}\, p, \boldsymbol{\phi}) = (\mathbf{grad}\, r, \boldsymbol{\phi}) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), \tag{2}$$

$$-(\boldsymbol{\theta}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0, \quad q \in \hat{H}^1(\Omega), \tag{3}$$

$$(\mathbf{grad}\, w, \mathbf{grad}\, s) = (\boldsymbol{\theta} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s), \quad s \in \mathring{H}^1(\Omega). \tag{4}$$

Define $(\boldsymbol{\theta}^0, p^0, w^0) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ as solution of (1)-(4) with $t = 0$. Note for r known, (2)-(3) is ordinary Stokes system for $(\theta_2^0, -\theta_1^0, p^0)$ and perturbed Stokes for $t > 0$.

Regularity Results

Key issue in approximation of R-M plate problem: dependence of solution on plate thickness t. There is a boundary layer, whose strength depends on particular boundary condition.

$$
\begin{aligned}
\boldsymbol{\theta} \cdot \boldsymbol{n} = \boldsymbol{\theta} \cdot \boldsymbol{s} = w = 0 &\qquad \text{hard clamped,} \\
\boldsymbol{\theta} \cdot \boldsymbol{n} = M_s(\boldsymbol{\theta}) \cdot \boldsymbol{s} = w = 0 &\qquad \text{soft clamped,} \\
M_n(\boldsymbol{\theta}) = \boldsymbol{\theta} \cdot \boldsymbol{s} = w = 0 &\qquad \text{hard simply supported,} \\
M_n(\boldsymbol{\theta}) = M_s(\boldsymbol{\theta}) = w = 0 &\qquad \text{soft simply supported,} \\
M_n(\boldsymbol{\theta}) = M_s(\boldsymbol{\theta}) = \partial w/\partial n - \boldsymbol{\theta} \cdot \boldsymbol{n} = 0 &\qquad \text{free,}
\end{aligned}
$$

where $\boldsymbol{n}$ and $\boldsymbol{s}$ denote unit normal and counterclockwise unit tangent vectors, respectively, and $M_n(\boldsymbol{\theta}) = \boldsymbol{n} \cdot \mathcal{C}\varepsilon(\boldsymbol{\theta})\boldsymbol{n}$, $M_s(\boldsymbol{\theta}) = \boldsymbol{s} \cdot \mathcal{C}\varepsilon(\boldsymbol{\theta})\boldsymbol{n}$.

If $\partial\Omega$ smooth, no boundary layer in w, i.e., $\|w\|_s \le C$, $s \in \mathbb{R}$.

Weakest boundary layer: soft clamped plate.

$$\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,7/2-s)}, \qquad \|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,3/2-s)}, \quad s \in \mathbb{R}.$$

Next weakest: hard clamped and hard simply supported plates.

$$\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,5/2-s)}, \qquad \|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,1/2-s)}, \quad s \in \mathbb{R}.$$

Strongest boundary layer: soft simply supported and free plates.

$$\|\boldsymbol{\theta}\|_s \le Ct^{\min(0,3/2-s)}, \qquad \|\boldsymbol{\gamma}\|_s \le Ct^{\min(0,-1/2-s)}, \quad s \in \mathbb{R}.$$

Also need estimates showing precise dependence on data that are valid when Ω is convex polygon.

Theorem: Let Ω be a convex polygon or a smoothly bounded domain in the plane. For any $t \in (0,1]$, $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, and $g \in H^{-1}(\Omega)$, there exists a unique solution $(r, \boldsymbol{\theta}, p, w) \in \mathring{H}^1(\Omega) \times \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ satisfying (1)-(4). Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, then $\boldsymbol{\theta} \in \boldsymbol{H}^2(\Omega)$ and there exists a constant C independent of t, $\boldsymbol{f}$, and g, such that

$$\|\boldsymbol{\theta}\|_2 + \|r\|_1 + \|p\|_1 + t\|p\|_2 + \|w\|_1 + \|\boldsymbol{\gamma}\|_0 \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}),$$

If, in addition, $g \in L^2(\Omega)$, then r and $w \in H^2(\Omega)$ and

$$\|r\|_2 + \|w\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\operatorname{div}\boldsymbol{\gamma}\|_0 \le C(\|g\|_0 + \|\boldsymbol{f}\|_0).$$

Finally, if $(\boldsymbol{\theta}^0, w^0)$ denotes solution of (1)-(4) with $t = 0$, then

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^0\|_1 \le Ct(\|\boldsymbol{f}\|_0 + \|g\|_{-1}), \qquad \|w^0\|_3 \le C(\|\boldsymbol{f}\|_0 + \|g\|_{-1}),$$
$$\|w - w^0\|_2 \le Ct(\|\boldsymbol{f}\|_0 + \|g\|_{-1} + t\|g\|_0).$$

Finite element Discretizations

Challenge: find schemes whose approximation accuracy does not deteriorate as plate thickness becomes small ("locking").

Recall: as $t \to 0$, minimizer $(\boldsymbol{\theta}, w)$ of R-M energy approaches $(\boldsymbol{\theta}^0, w^0)$, where $\boldsymbol{\theta}^0 = \mathbf{grad}\, w^0$.

If we discretize directly by seeking $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$ and $w_h \in W_h$ minimizing $J(\boldsymbol{\theta}, w)$ over $\boldsymbol{\Theta}_h \times W_h$, then as $t \to 0$ will have $(\boldsymbol{\theta}_h, w_h) \to (\boldsymbol{\theta}_h^0, w_h^0)$ where, again, $\boldsymbol{\theta}_h^0 = \mathbf{grad}\, w_h^0$.

Locking problem occurs because, for low order finite element spaces, last condition too restrictive to allow for good approximations of smooth functions.

If $\boldsymbol{\Theta}_h$ and W_h taken to be C^0 P_1 functions, then $\boldsymbol{\theta}_h^0 \equiv \mathbf{grad}\, w_h^0$ would be C^0 *and* piecewise constant, with zero BCs: Implies $\boldsymbol{\theta}_h^0 = 0$. Need careful choice of FE spaces to avoid "locking."

Many locking-free finite element schemes use following approach. Let $\boldsymbol{\Theta}_h \subset \mathring{\boldsymbol{H}}^1(\Omega)$, $W_h \subset \mathring{H}^1(\Omega)$, $\boldsymbol{\Gamma}_h \subset \boldsymbol{L}^2(\Omega)$, where $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$. Let $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ be an interpolation operator mapping $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{\Gamma}_h$. Then consider finite element approximation schemes of form:

Find $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$, $w_h \in W_h$, $\boldsymbol{\gamma}_h \in \boldsymbol{\Gamma}_h$ such that

$$a(\boldsymbol{\theta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h,$$
$$(\mathbf{grad}\, w_h - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_h, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h. \tag{5}$$

By introducing $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$, as $t \to 0$, get $\mathbf{grad}\, w_{h,0} \to \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta}_{h,0}$. If $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ chosen properly, this condition easier to satisfy, while still maintaining good approximation properties of each subspace.

Abstract Error Analysis

To analyze schemes using common framework, first establish several abstract approximation results. Results use following assumptions about approximation properties of finite dimensional subspaces and operator $\mathbf{\Pi}^{\mathbf{\Gamma}}$ that define various methods.

$$\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h,$$
$$\|\boldsymbol{\eta} - \mathbf{\Pi}^{\mathbf{\Gamma}}\eta\| \le ch\|\boldsymbol{\eta}\|, \qquad \boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega),$$

for some constant c independent of h. We also define $r_0 \ge -1$ as the greatest integer r for which

$$(\boldsymbol{\eta} - \mathbf{\Pi}^{\mathbf{\Gamma}}\boldsymbol{\eta}, \boldsymbol{\zeta}) = 0, \qquad \boldsymbol{\zeta} \in \boldsymbol{M}^r.$$

Of course this relation trivially holds for $r = -1$. We then let $\mathbf{\Pi}^0$ denote the $\boldsymbol{L}^2$ projection into $\boldsymbol{M}^{r_0}$.

Following basic result close to Lemma 3.1 of Durán-Liberman.

Theorem: Let $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$, $w^I \in W_h$ be arbitrary, and define $\boldsymbol{\gamma}^I = \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I) \in \boldsymbol{\Gamma}_h$. Then

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \le C(\|\boldsymbol{\theta}-\boldsymbol{\theta}^I\|_1 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_0 + h\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0).$$

Proof: Subtracting equations, we get the error equation

$$a(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\phi}) + (\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\phi}) = (\boldsymbol{\gamma}, [\boldsymbol{I}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\phi}),$$

for all $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$ and $v \in W_h$. Hence

$$\begin{aligned} a(\boldsymbol{\theta}^I-\boldsymbol{\theta}_h,\boldsymbol{\phi}) + (\boldsymbol{\gamma}^I-\boldsymbol{\gamma}_h, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\phi}) &= a(\boldsymbol{\theta}^I-\boldsymbol{\theta},\boldsymbol{\phi}) \\ &+ (\boldsymbol{\gamma}^I-\boldsymbol{\gamma}, \mathbf{grad}\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\phi}) + (\boldsymbol{\gamma}, [\boldsymbol{I}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\phi}). \end{aligned}$$

Taking $\phi = \phi^I - \phi_h$ and $v = w^I - w_h$, noting that $\mathbf{grad}\, w^I - \mathbf{\Pi}^{\mathbf{\Gamma}} \boldsymbol{\theta}^I = \lambda^{-1} t^2 \boldsymbol{\gamma}^I$ and $\mathbf{grad}\, w_h - \mathbf{\Pi}^{\mathbf{\Gamma}} \boldsymbol{\theta}_h = \lambda^{-1} t^2 \boldsymbol{\gamma}_h$, get

$$a(\boldsymbol{\theta}^I - \boldsymbol{\theta}_h, \boldsymbol{\theta}^I - \boldsymbol{\theta}_h) + \lambda^{-1} t^2 (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h, \boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h) = a(\boldsymbol{\theta}^I - \boldsymbol{\theta}, \boldsymbol{\theta}^I - \boldsymbol{\theta}_h) + \lambda^{-1} t^2 (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}, \boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h) + (\gamma, [\boldsymbol{I} - \mathbf{\Pi}^{\mathbf{\Gamma}}][\boldsymbol{\theta}^I - \boldsymbol{\theta}_h]).$$

Bound last term by:

$$|(\boldsymbol{\gamma}, [\boldsymbol{I} - \mathbf{\Pi}^{\mathbf{\Gamma}}][\boldsymbol{\theta}^I - \boldsymbol{\theta}_h])| \le Ch \|\boldsymbol{\gamma} - \mathbf{\Pi}^0 \boldsymbol{\gamma}\|_0 \|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1.$$

The theorem then follows easily.

Note: applying theorem in naive way, error estimates blow up as $t \to 0$, i.e., if we use simple estimate

$$t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\| = \lambda t^{-1} \|\mathbf{grad}(w - w^I) - (\boldsymbol{\theta} - \Pi^{\mathbf{\Gamma}} \boldsymbol{\theta}^I)\| \le \lambda t^{-1} (\|\mathbf{grad}(w - w^I)\| + \|\boldsymbol{\theta} - \Pi^{\mathbf{\Gamma}} \boldsymbol{\theta}^I\|),$$

and use approximation theory to bound each term on right separately, then bound will contain term t^{-1}.

Key idea to using theorem to obtain error estimates independent of t: find functions $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ satisfying

$$\boldsymbol{\gamma}^I = \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I) = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}. \tag{6}$$

We then have the following corollary.

Corollary: If $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ satisfy (6), then

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1+t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \le C(\|\boldsymbol{\theta}-\boldsymbol{\theta}^I\|_1+t\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\|_0+h\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0).$$

From assumptions about approximation properties of $\boldsymbol{\theta}^I$, w^I, and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}$, obtain order of convergence estimates. One such result is the following.

Theorem: Let $n \geq 1$ and assume for each $\boldsymbol{\theta} \in \boldsymbol{H}^{n+1}(\Omega) \cap \mathring{\boldsymbol{H}}^1(\Omega)$ and $w \in H^{n+2}(\Omega) \cap \mathring{H}^1(\Omega)$, there exists $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ and $w^I \in W_h$ satisfying (6). If for $1 \leq r \leq n$,

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \leq C h^r \|\boldsymbol{\theta}\|_{r+1},$$
$$\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma}\|_0 \leq C h^r \|\boldsymbol{\gamma}\|_r,$$

then

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 \leq C\left(h^r \|\boldsymbol{\theta}\|_{r+1} + h^r t \|\boldsymbol{\gamma}\|_r + h^{r_0+2} \|\boldsymbol{\gamma}\|_{r_0+1}\right).$$

To obtain L^2 errors for rotation and transverse displacement, need to define appropriate dual problem.

Given $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$ and $G \in L^2(\Omega)$, define $\boldsymbol{\psi}$, u, and $\boldsymbol{\zeta}$ to be solution to auxiliary problem

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\mathbf{grad}\, v - \boldsymbol{\phi}, \boldsymbol{\zeta}) = (\boldsymbol{\phi}, \boldsymbol{F}) + (v, G), \;\; \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1, v \in \mathring{H}^1(\Omega),$$
$$(\boldsymbol{\eta}, \mathbf{grad}\, u - \boldsymbol{\psi}) - \lambda^{-1} t^2 (\boldsymbol{\eta}, \zeta) = 0, \;\; \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega).$$

Then by previous regularity results,

$$\|\boldsymbol{\psi}\|_2 + \|u\|_2 + \|\boldsymbol{\zeta}\| + t\|\boldsymbol{\zeta}\|_1 + \|\operatorname{div} \boldsymbol{\zeta}\|_0 \leq c(\|\boldsymbol{F}\|_0 + \|G\|_0).$$

With these definitions, get following estimate.

Theorem: If hypotheses of previous theorems satisfied, then

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|^2/2 + \|w - w_h\|_0^2/2$$
$$\leq Ch^2(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1^2 + t^2\|\gamma - \gamma_h\|_0^2) + ([\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}_h, \boldsymbol{\zeta}) + (\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\psi^I).$$

Remark: Bounds on last two terms depend on particular method.

Next, give abstract estimate for approximation of derivatives of transverse displacement.

Theorem: For all $w_I \in W_h$, we have

$$\begin{aligned}&\|\,\mathbf{grad}[w - w_h]\|_0 \\ &\le C(\|\,\mathbf{grad}[w - w_I]\|_0 + \|[\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\theta}\|_0 + h\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0).\end{aligned}$$

In some cases, also possible to establish improved estimates for shear stress γ in negative norms.

Next: Applications of the Abstract Error Estimates

Some Triangular Reissner–Mindlin elements

Assume Ω a convex polygon. Let $\mathcal{T}_h$ denote triangulation of Ω and $\mathbf{V}$ and $\mathbf{E}$ set of vertices and edges, respectively in $\mathcal{T}_h$.

We will use the following finite element spaces (expressed in usual notation).

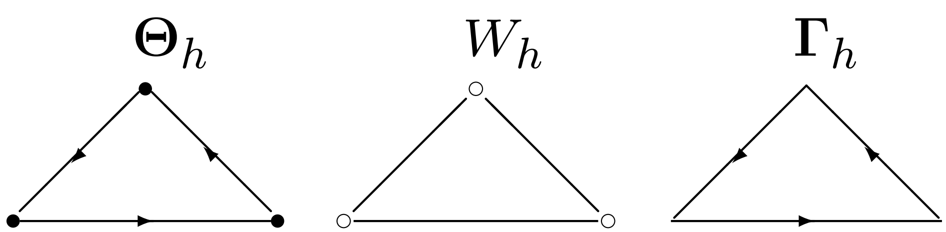
$M_k(\mathcal{T}_h)$:	piecewise polynomials degree $\leq k$,
$M_k^l(\mathcal{T}_h)$:	$M_k \cap C^l(\Omega)$,
$M_k^*(\mathcal{T}_h)$:	elements of M_k continuous at k Gauss-points of each edge,
$B_k(\mathcal{T}_h)$:	elements of M_k^0 which vanish on interelement edges,
$\boldsymbol{RT}_k^\perp(\mathcal{T}_h)$:	Raviart–Thomas approx of order k to $\boldsymbol{H}(\text{rot}, \Omega)$,
$\boldsymbol{BDM}_k^\perp(\mathcal{T}_h)$:	Brezzi-Douglas-Marini approx of order k to $\boldsymbol{H}(\text{rot}, \Omega)$,
$\boldsymbol{BDFM}_k^\perp(\mathcal{T}_h)$:	Brezzi-Douglas-Fortin-Marini approx of order k to $\boldsymbol{H}(\text{rot}, \Omega)$,

The Durán–Liberman element

$$\boldsymbol{\Theta}_h = \{\, \boldsymbol{\phi} \in \mathring{\boldsymbol{M}}_2^0 \,|\, \boldsymbol{\phi}\cdot\boldsymbol{n} \in P_1(e), e \in \mathbf{E}\,\}, \quad W_h = \mathring{M}_1^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_0^{\perp}.$$

$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is usual interpolant into $\boldsymbol{RT}_0^{\perp}$ defined for $\boldsymbol{\gamma} \in \boldsymbol{H}^1(\Omega)$ by

$$\int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\cdot\boldsymbol{s} = \int_e \boldsymbol{\gamma}\cdot\boldsymbol{s}, \qquad e \in \mathbf{E}.$$



We then get the following error estimate.

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Key steps in Proof: Can find $\boldsymbol{\theta}^I \in \boldsymbol{\Theta}_h$ satisfying $\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch\|\boldsymbol{\theta}\|_2$ and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_h$ (R-T interpolant) satisfying $\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\|_0 \le Ch\|\boldsymbol{\gamma}\|_1$.

Result follows from approximation theorems and regularity results if we can find $w^I \in W_h$ such that

$$\boxed{\boldsymbol{\gamma}^I = \lambda t^{-2}(\mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I) = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} = \lambda t^{-2}(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}).}$$

First show: $\boxed{\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}.}$ Standard interpolant $\boldsymbol{\theta}^I$ satisfies $\int_e \boldsymbol{\theta}^I \cdot \boldsymbol{s} = \int_e \boldsymbol{\theta} \cdot \boldsymbol{s}$. on each edge e. Then

$$\int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I \cdot \boldsymbol{s} = \int_e \boldsymbol{\theta}^I \cdot \boldsymbol{s} = \int_e \boldsymbol{\theta} \cdot \boldsymbol{s} = \int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta} \cdot \boldsymbol{s},$$

so

$$\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}.$$

Next show: $\boxed{\mathbf{grad}\, w^I = \mathbf{\Pi}^{\mathbf{\Gamma}}\, \mathbf{grad}\, w.}$ Choose $w^I = \Pi^W w$, standard piecewise linear interpolant of w. If e edge joining vertices v_a and v_b, then

$$\int_e \mathbf{grad}\, \Pi^W w \cdot \boldsymbol{s} = \int_e \partial \Pi^W w / \partial s = \Pi^W w(v_b) - \Pi^W w(v_a)$$
$$= w(v_b) - w(v_a) = \int_e \partial w / \partial s = \int_e \mathbf{grad}\, w \cdot \boldsymbol{s}.$$

Since $\mathbf{grad}\, \Pi^W w \in \mathbf{\Gamma}_h$,

$$\mathbf{grad}\, w^I = \mathbf{grad}\, \Pi^W w = \mathbf{\Pi}^{\mathbf{\Gamma}}\, \mathbf{grad}\, w.$$

Also possible to obtain following estimates:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \le Ch^2(\|\boldsymbol{f}\|_0 + \|g\|_0).$$
$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

MITC Triangular Families

Three triangular families considered in Brezzi-Fortin-Stenberg, defined for integer $k \geq 2$. For each of these families, $\mathbf{\Theta}_h$ is chosen to be

$$\mathbf{\Theta}_h = \begin{cases} \mathring{\boldsymbol{M}}_k^0 + \boldsymbol{B}_{k+1} & k = 2, 3 \\ \mathring{\boldsymbol{M}}_k^0 & k \geq 4 \end{cases}.$$

Then define

$$\begin{aligned} \text{Family I:} \qquad & W_h = \mathring{M}_k^0, \quad \mathbf{\Gamma}_h = \boldsymbol{RT}_{k-1}^{\perp}, \\ \text{Family II:} \qquad & W_h = \mathring{M}_k^0 + B_{k+1}, \quad \mathbf{\Gamma}_h = \boldsymbol{BDFM}_k^{\perp}, \\ \text{Family III:} \qquad & W_h = \mathring{M}_{k+1}^0 \quad \mathbf{\Gamma}_h = \boldsymbol{BDM}_k^{\perp}. \end{aligned}$$

Choose $\mathbf{\Pi}^{\mathbf{\Gamma}}$ to be usual interpolant into each $\mathbf{\Gamma}_h$ space.

Basic idea: combine known results on approximation of Stokes problems and mixed methods for linear elliptic problems.

Construction based on five properties relating spaces $\mathbf{\Theta}_h$, W_h, $\mathbf{\Gamma}_h$, and auxiliary space Q_h (not part of method).

P1: $\mathbf{grad}\, W_h \subset \mathbf{\Gamma}_h$.
P2: $\operatorname{rot} \mathbf{\Gamma}_h \subset Q_h$.
P3: $\operatorname{rot} \mathbf{\Pi}^{\mathbf{\Gamma}} \boldsymbol{\phi} = \Pi^0 \operatorname{rot} \boldsymbol{\phi}$, for $\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega)$, with $\Pi^0 : L^2_0(\Omega) \mapsto Q_h$ denoting L^2-projection ($L^2_0(\Omega)$ denotes functions in $L^2(\Omega)$ with mean value zero.)
P4: If $\eta \in \mathbf{\Gamma}_h$ satisfies $\operatorname{rot} \boldsymbol{\eta} = 0$, then $\boldsymbol{\eta} = \mathbf{grad}\, v$ for some $v \in W_h$.
P5: $(\mathbf{\Theta}_h^{\perp}, Q_h)$ a stable pair for Stokes problem, i.e.,

$$\sup_{\mathbf{0} \neq \boldsymbol{\phi} \in \mathbf{\Theta}_h} \frac{(\operatorname{rot} \boldsymbol{\phi}, q)}{\|\boldsymbol{\phi}\|_1} \geq C \|q\|_0, \qquad q \in Q_h.$$

For each of the three families described above,

$$Q_h = \{q \in L^2_0(\Omega) : q_T \in P_{k-1}(T),\ T \in \mathcal{T}_h\}.$$

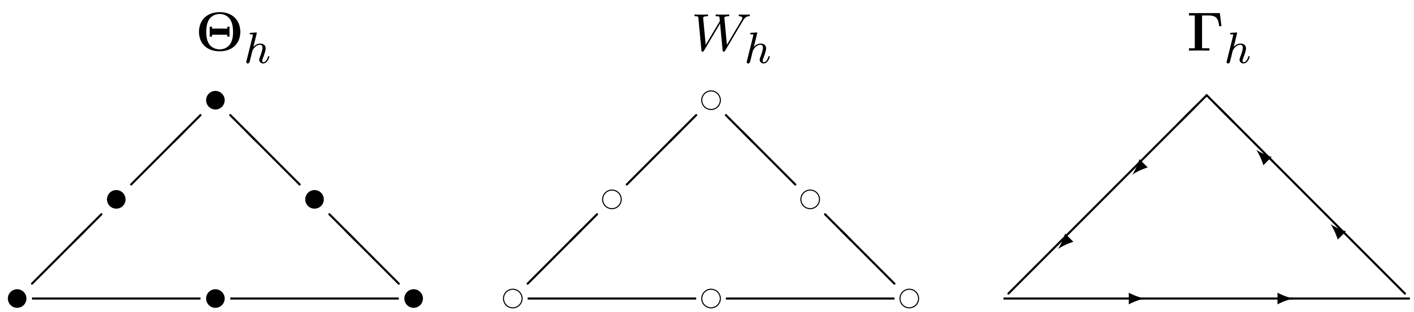
For this choice, fact that pair of spaces $(\boldsymbol{\Theta}_h, Q_h)$ satisfies **P5** follows from corresponding results known for Stokes problem.

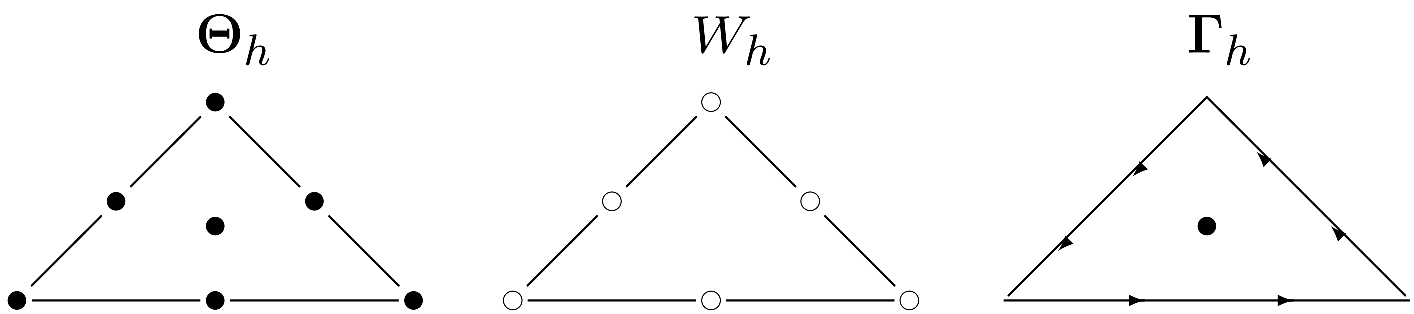
Families only defined for $k \geq 2$. Difficulties in extending to $k = 1$?

$\boldsymbol{B}_{k+1}$ only defined for $k \geq 2$, so this space must be replaced. Suitable replacement space for $\boldsymbol{\Theta}_h$ in Family I is space chosen in Durán–Liberman element. With this choice, Durán–Liberman element also fits this general framework, with $k = 1$.

For Family II, similar problem occurs for choice of W_h and in addition $\boldsymbol{BDFM}_1^{\perp} = \boldsymbol{RT}_0^{\perp}$, so method needs substantial change and does not give anything new.

For Family III, choices $W_h = \mathring{M}_2^0$ and $\boldsymbol{\Gamma}_h = \boldsymbol{BDM}_1^{\perp}$ make sense. Can choose $\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_2^0$. Corresponds to choice of piecewise constants for Q_h and $\boldsymbol{P}_2 - P_0$ Stokes element (element mentioned in Bathe-Brezzi-Fortin). This element, MITC6, depicted below followed by MITC7, $k = 2$ element of Family II.





Give analysis only for Family I:

$$\boldsymbol{\Theta}_h = \begin{cases} \mathring{\boldsymbol{M}}_k^0 + \boldsymbol{B}_{k+1} & k = 2, 3 \\ \mathring{\boldsymbol{M}}_k^0 & k \geq 4 \end{cases}, \quad W_h = \mathring{M}_k^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_{k-1}^{\perp}.$$

Analysis of other two families done in similar manner.

Theorem: For the MITC family of index $k \geq 2$, we have for $1 \leq r \leq k$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \leq Ch^r \left(\|\boldsymbol{\theta}\|_{r+1} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1} \right)$$

Key steps in Proof: Using standard results about stable Stokes elements, can find interpolant $\boldsymbol{\theta}^I$ of $\boldsymbol{\theta} \in \boldsymbol{\Theta}_h$ satisfying $\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \leq Ch^r \|\boldsymbol{\theta}\|_{r+1}$, $1 \leq r \leq k$ and

$$\int_\Omega \text{rot}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) q = 0 \quad \forall q \in M_{k-1}^{-1}.$$

By definition of $\mathbf{\Pi}^{\mathbf{\Gamma}}$, have $\forall q \in M_{k-1}^{-1}$,

$$0 = \int_\Omega \text{rot}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) q = \int_\Omega \text{rot}\, \mathbf{\Pi}^{\mathbf{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) q.$$

Choosing $q = \text{rot}\, \mathbf{\Pi}^{\mathbf{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I)$ implies $\text{rot}\, \mathbf{\Pi}^{\mathbf{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) = 0$. Hence,

$$\mathbf{\Pi}^{\mathbf{\Gamma}}(\boldsymbol{\theta} - \boldsymbol{\theta}^I) = \mathbf{grad}\, v^I, \qquad \text{for some } v^I \in W_h.$$

Let $\Pi^W w \in M_0^k$ be the interpolant of w defined for each vertex x, edge e and triangle T by

$$\Pi^W w(x) = w(x), \qquad \int_e \Pi^W w\, p = \int_e w\, p, \quad \text{for all } p \in P_{k-2}(e),$$
$$\int_T \Pi^W wp = \int_T wp, \quad \text{for all } p \in P_{k-3}(T).$$

Easy to check that $\mathbf{\Pi}^{\mathbf{\Gamma}}(\mathbf{grad}\, w) = \mathbf{grad}\, \Pi^W w$.

Choosing $w^I = \Pi^W w - v^I$, get

$$\begin{aligned}\boldsymbol{\gamma}^I = \lambda t^{-2}(\mathbf{grad}\, w^I - \mathbf{\Pi}^{\Gamma}\boldsymbol{\theta}^I) &= \lambda t^{-2}(\mathbf{grad}\, \Pi^W w - \mathbf{grad}\, v^I - \mathbf{\Pi}^{\Gamma}\boldsymbol{\theta}^I) \\ &= \lambda t^{-2}(\mathbf{\Pi}^{\Gamma}(\mathbf{grad}\, w) - \mathbf{\Pi}^{\Gamma}[\boldsymbol{\theta} - \boldsymbol{\theta}^I] - \mathbf{\Pi}^{\Gamma}\boldsymbol{\theta}^I) \\ &= \lambda t^{-2}\mathbf{\Pi}^{\Gamma}(\mathbf{grad}\, w - \boldsymbol{\theta}) = \mathbf{\Pi}^{\Gamma}\boldsymbol{\gamma}.\end{aligned}$$

L^2 estimates: For the MITC family of index $k \geq 2$, we have for $1 \leq r \leq k$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \leq Ch^{r+1}\left(\|\boldsymbol{\theta}\|_{r+1} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

The Falk-Tu elements with discontinuous shear stresses

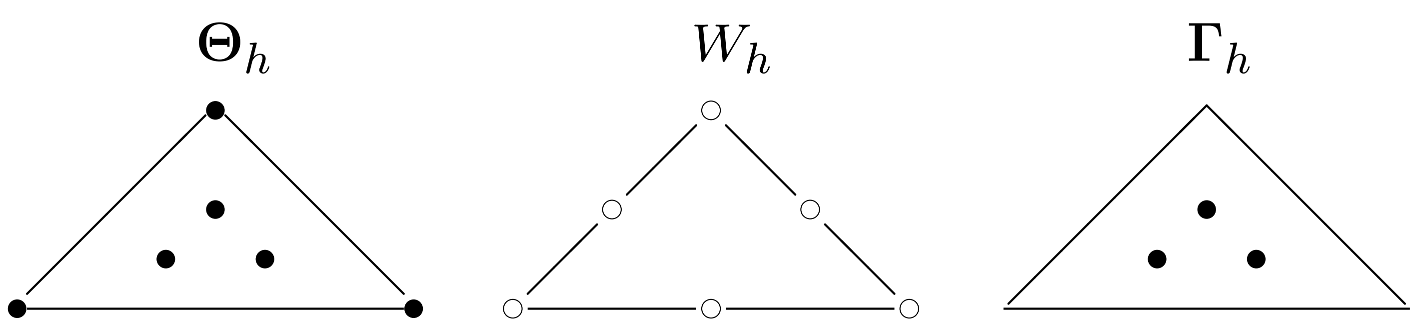
For $k = 2, 3, \ldots$, choose

$$\mathbf{\Theta}_h = \mathring{\boldsymbol{M}}^0_{k-1} + \boldsymbol{B}_{k+2}, \quad W_h = \mathring{M}^0_k, \quad \mathbf{\Gamma}_h = \boldsymbol{M}_{k-1},$$

and $\mathbf{\Pi}^{\mathbf{\Gamma}}$ to be $\boldsymbol{L}^2$ projection into $\mathbf{\Gamma}_h$.

(Also a related element of Zienkiewicz–Lefebvre.)

Lowest order ($k = 2$) element depicted below.



Theorem: For discontinuous shear stress family of index $k \geq 2$, and $1 \leq r \leq k-1$,

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1+t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \leq Ch^r \left(\|\boldsymbol{\theta}\|_{r+1}+\|w\|_{r+2}+t\|\boldsymbol{\gamma}\|_r+\|\boldsymbol{\gamma}\|_{r-1}\right).$$

For $k=2$ and $r=1$, also have estimate

$$\begin{aligned}\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1 &+ t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \\ &\leq Ch\left(\|\boldsymbol{\theta}\|_2+\|w^0\|_3+\|\boldsymbol{\gamma}\|_0+t\|\boldsymbol{\gamma}\|_1+t^{-1}\|w-w^0\|_2\right) \\ &\leq Ch\left(\|\boldsymbol{f}\|_0+\|g\|_0\right).\end{aligned}$$

Key points of Proof: For $1 \leq r \leq k-1$, let $\Pi^W w$ be standard interpolant of w satisfying

$$\|w-\Pi^W w\|_0+h\|w-\Pi^W w\|_1 \leq Ch^{r+2}\|w\|_{r+2}.$$

Let $\boldsymbol{\Pi}^M\boldsymbol{\theta} \in \mathring{\boldsymbol{M}}_0^{k-1}$ be a standard interpolant of $\boldsymbol{\theta}$ satisfying

$$\|\boldsymbol{\theta} - \boldsymbol{\Pi}^M\boldsymbol{\theta}\|_0 + h\|\boldsymbol{\theta} - \boldsymbol{\Pi}^M\boldsymbol{\theta}\|_1 \le Ch^{r+1}\|\boldsymbol{\theta}\|_{r+1}.$$

Define $\boldsymbol{\Pi}^B(\boldsymbol{\theta}, w^*) \in \boldsymbol{B}^{k+3}$ by

$$\boldsymbol{\Pi}^\Gamma\boldsymbol{\Pi}^B(\boldsymbol{\theta}, w^*) = \boldsymbol{\Pi}^\Gamma\boldsymbol{\theta} - \boldsymbol{\Pi}^\Gamma\boldsymbol{\Pi}^M\boldsymbol{\theta} - \boldsymbol{\Pi}^\Gamma\,\mathbf{grad}\,w^* + \mathbf{grad}\,\Pi^W w^*,$$

where w^* chosen as either w or w^0, (limiting transverse displacement from R-M system when $t = 0$).

Set $w^I = \Pi^W w$ and $\boldsymbol{\theta}^I = \boldsymbol{\Pi}^M\boldsymbol{\theta} + \boldsymbol{\Pi}^B(\boldsymbol{\theta}, w^*)$. Note $\boldsymbol{\theta}^I$ not an interpolant of $\boldsymbol{\theta}$, since it depends on w^* also. However, can show for $1 \le r \le k-1$,

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1 \le Ch^r\Big(\|\boldsymbol{\theta}\|_{r+1} + \|w^*\|_{r+2}\Big).$$

With this choice,

$$
\begin{aligned}
\lambda^{-1}t^2\boldsymbol{\gamma}^I &= \mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I \\
&= \mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{M}}\boldsymbol{\theta} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\Pi}^{\boldsymbol{B}}(\boldsymbol{\theta}, w^*) \\
&= \mathbf{grad}\, w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta} + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w^* - \mathbf{grad}\,\Pi^W w^* \\
&= \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}(\mathbf{grad}\, w - \boldsymbol{\theta}) + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}(w^* - w) \\
&\qquad - \mathbf{grad}\,\Pi^W(w^* - w) \\
&= \lambda^{-1}t^2\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} + \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}([I - \Pi^W][w^* - w].
\end{aligned}
$$

If $w^* = w$, then $\boldsymbol{\gamma}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}$, while choice $w^* = w^0$ does not satisfy this equation.

Need for second choice technical: on convex polygon, do not have a priori bound for $\|w\|_3$, but do have bound for $\|w^0\|_3$. Requires modified analysis. On domain with smooth boundary, simpler choice $w^* = w$ sufficient.

$\boldsymbol{L}^2$ error estimates: For discontinuous shear stress family of index $k \geq 2$, and $1 \leq r \leq k-1$

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_1 \\ \leq Ch^{r+1}\left(\|\boldsymbol{\theta}\|_{r+1} + \|w\|_{r+2} + t\|\boldsymbol{\gamma}\|_r + \|\boldsymbol{\gamma}\|_{r-1}\right).$$

For $k = 2$ and $r = 1$, we also have the estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_1 \leq Ch^2\left(\|\boldsymbol{f}\|_0 + \|g\|_0\right).$$

We do not obtain a higher order of convergence for $\|w - w_h\|_0$.

Linked Interpolation Methods

One approach: use mixed formulation (5), but replace space $\boldsymbol{\Theta}_h \times W_h$ by space $\boldsymbol{V}_h$ linking $\boldsymbol{\Theta}_h$ and W_h by a constraint.

Simplest example: method introduced by Xu and Auricchio and Taylor, and analyzed in Lyly, Lovadina, and Auricchio-Lovadina.

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_1^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0,$$
$$\boldsymbol{V}_h = \{(\boldsymbol{\phi}, v + L\boldsymbol{\phi}) : \boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h\},$$

where $L_T = L|_T$ is mapping from $\boldsymbol{H}^1(T)$ onto $P_{2,-}(T)$ given by

$$\int_e [(\mathbf{grad}\, L_T\boldsymbol{\phi} - \boldsymbol{\phi}) \cdot \boldsymbol{s}] \frac{\partial v}{\partial s} = 0, \quad v \in P_{2,-}(T),$$

for every edge e of T, where $P_{2,-}(T)$ is space of C^0 piecewise quadratics which vanish at vertices of T.

Then seek $(\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h) \in \boldsymbol{V}_h \times \boldsymbol{\Gamma}_h$ such that (5) holds for all $(\boldsymbol{\phi}, v^*;) \in \boldsymbol{V}_h \times \boldsymbol{\Gamma}_h$.

Equivalently, write method in terms of usual spaces, but with modified bilinear form, i.e., seek $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\Theta}_h \times W_h \times \boldsymbol{\Gamma}_h$:

$$a(\boldsymbol{\theta}_h, \boldsymbol{\phi}) + \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \mathbf{grad}(v + L\boldsymbol{\phi}) - \boldsymbol{\phi}) = (g, v + L\boldsymbol{\phi}) - (\boldsymbol{f}, \boldsymbol{\phi}),$$
$$\boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h,$$
$$(\mathbf{grad}(w_h + L\boldsymbol{\theta}_h) - \boldsymbol{\theta}_h, \boldsymbol{\eta}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h.$$

Note: can write above as slight perturbation of formulation (5), by defining $\boldsymbol{\Pi}^{\Gamma} = \boldsymbol{\Pi}^0 (I - \mathbf{grad}\, L)$ (where $\boldsymbol{\Pi}^0$ denotes $\boldsymbol{L}^2$ projection onto $\boldsymbol{\Gamma}_h$), and replacing term (g, v) by $(g, v + L\boldsymbol{\phi})$.

Term $(g, L\phi)$ higher order perturbation (and can be dropped), i.e., Lyly shows:

$$|(g, L\boldsymbol{\phi})|_T \le \|g\|_{0,T}\|L_T\boldsymbol{\phi}\|_{0,T} \le Ch_T\|g\|_{0,T}\|\nabla L_T\boldsymbol{\phi}\|_{0,T} \\ \le Ch_T^2\|g\|_{0,T}\|\boldsymbol{\phi}\|_{1,T}.$$

To apply approximation theorems, define $w^I = \Pi^W w$, C^0 piecewise linear interpolant of w, and $\boldsymbol{\theta}^I = \mathbf{\Pi}^M\boldsymbol{\theta} + \mathbf{\Pi}^B\boldsymbol{\theta}$, where $\mathbf{\Pi}^M\boldsymbol{\theta}$ denotes an interpolant of $\boldsymbol{\theta}$ satisfying

$$\|\boldsymbol{\theta} - \mathbf{\Pi}^M\boldsymbol{\theta}\|_0 + \|\boldsymbol{\theta} - \mathbf{\Pi}^M\boldsymbol{\theta}\|_1 \le Ch^s\|\boldsymbol{\theta}\|_s, \quad s = 1, 2,$$

and $\mathbf{\Pi}^B\boldsymbol{\theta} \in \boldsymbol{B}_3$ is defined by:

$$\mathbf{\Pi}^0\mathbf{\Pi}^B\boldsymbol{\theta} = \mathbf{\Pi}^0[(\boldsymbol{I} - \mathbf{grad}\, L)(\boldsymbol{\theta} - \mathbf{\Pi}^M\boldsymbol{\theta})]. \tag{7}$$

Can show that for $s = 1, 2$:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_0 \le C(\|\boldsymbol{\theta} - \mathbf{\Pi}^M\boldsymbol{\theta}\|_0 + h\|\boldsymbol{\theta} - \mathbf{\Pi}^M\boldsymbol{\theta}\|_1 \le Ch^s\|\boldsymbol{\theta}\|_s.$$

Now check that $\boldsymbol{\gamma}^I = \mathbf{\Pi}^\Gamma \boldsymbol{\gamma}$, where $\mathbf{\Pi}^\Gamma = \mathbf{\Pi}^0(I - \mathbf{grad}\, L)$.

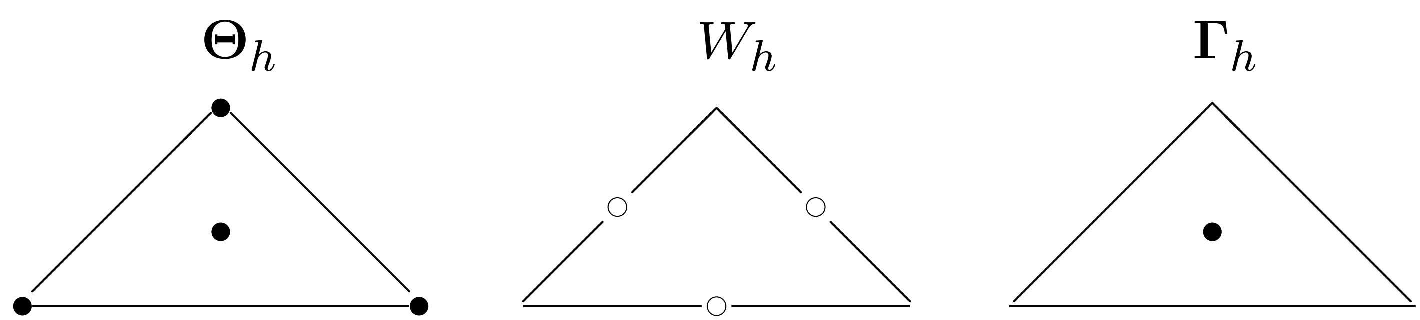
If we drop term $(g, L\boldsymbol{\phi})$ from right hand side of method, get from approximation theorems:

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 + \|w-w_h\|_1 \le Ch(\|\boldsymbol{\theta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}_0 + \|w\|_2)$$
$$\le Ch(\|g\|_0 + \|\boldsymbol{f}\|_0).$$

Simple extension of this argument gives same final result with term $(g, L\boldsymbol{\phi})$ included.

Nonconforming Element of Arnold and Falk

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_1^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0, \quad \boldsymbol{\Pi}^\Gamma = \boldsymbol{L}^2 - \text{proj.}$$



Note W_h not contained in $\mathring{H}^1$ and so $\mathbf{grad}$ must be replaced by $\mathbf{grad}_h$. See also Franca-Stenberg and Duràn-Ghioldi-Wolanski for a modification of this element, and Arnold for a relationship between these two approaches.

Use of nonconforming space W_h requires modifications in basic error estimates.

Theorem:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|\,\mathbf{grad}_h[w - w_h]\|_0 \leq Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Key ideas of Proof: Error equation will now contain additional term for consistency error.

$$\begin{aligned} a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\phi}) &+ (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \mathbf{grad}_h\, v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi}) \\ &= (\gamma, [\boldsymbol{I} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}]\boldsymbol{\phi}) + \sum_{T \in \tau} \int_{\partial T} v \boldsymbol{\gamma} \cdot \boldsymbol{n}_T, \quad \boldsymbol{\phi} \in \boldsymbol{\Theta}_h, v \in W_h. \end{aligned}$$

Following previous proof of Theorem, get

$$\|\boldsymbol{\theta}^I - \boldsymbol{\theta}_h\|_1^2 + t^2\|\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_h\|_0^2 \le C\Big(\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_1^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_0^2 + h^2\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0^2 + \Big|\sum_{T\in\tau}\int_{\partial T}(w^I - w_h)\boldsymbol{\gamma}\cdot\boldsymbol{n}_T\Big|\Big).$$

Use trivial estimate $\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_0 \le \|\boldsymbol{\gamma}\|_0$ and ideas from non-conforming methods to estimate $\sum_{T\in\tau}\int_{\partial T}(w^I - w_h)\boldsymbol{\gamma}\cdot\boldsymbol{n}_T$ (technical).

Choose $\boldsymbol{\theta}^I$ to be interpolant used for MINI element for Stokes problem. Satisfies $\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_{1,h} \le Ch\|\boldsymbol{\theta}\|_2$ and condition:

$$\boldsymbol{\Pi}^{\Gamma}\boldsymbol{\theta}^I = \boldsymbol{\Pi}^{\Gamma}\boldsymbol{\theta}.$$

Hence, to satisfy $\boldsymbol{\gamma}^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}$, only need to find w^I such that

$$\mathbf{grad}_h\, w^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w.$$

Choose w^I to satisfy $\int_e w^I = \int_e w$ on each edge e. Then for all $\eta \in P_0(T)$,

$$\int_T \mathbf{grad}\, w \cdot \boldsymbol{\eta} = \int_{\partial T} w\eta \cdot \boldsymbol{n}_T = \int_{\partial T} w^I \eta \cdot \boldsymbol{n}_T = \int_T \mathbf{grad}\, w^I \cdot \boldsymbol{\eta},$$

which implies $\mathbf{grad}_h\, w^I = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\,\mathbf{grad}\, w$.

To obtain error estimate for transverse displacement, need nonconforming version of previous result.

Also using non-conforming version of $\boldsymbol{L}^2$ result, get:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \le Ch^2(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Some Rectangular Reissner–Mindlin elements

Let $\mathcal{T}_h$ denote rectangular mesh of Ω and R an element of $\mathcal{T}_h$.

Denote by Q_{k_1,k_2} set of polynomials of separate degree $\leq k_1$ in x and $\leq k_2$ in y and set $Q_k = Q_{k,k}$.

Define serendipity polynomials $Q_k^s = P_k \oplus x^k y \oplus x y^k$.

Need rotated versions of rectangular Raviart-Thomas, Brezzi-Douglas-Marini, and Brezzi-Douglas-Fortin-Marini spaces, defined locally for $k \geq 1$ as follows.

$$\boldsymbol{RT}_{k-1}^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = (Q_{k-1,k}(R), Q_{k,k-1}(R))\},$$
$$\boldsymbol{BDM}_{k}^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} \in \boldsymbol{P}_k(R) \oplus \nabla(xy^{k+1}) \oplus \nabla(x^{k+1}y)\},$$
$$\boldsymbol{BDFM}_{k}^{\perp}(R) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = (P_k(R) \setminus \{x^k\}, P_k(R) \setminus \{y^k\})\}.$$

Rectangular MITC elements and generalizations

In original MITC family, choose for $k \geq 1$,

$$\begin{aligned}
\boldsymbol{\Theta}_h &= \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}_k(R)\}, \\
W_h &= \{v \in \mathring{H}^1(\Omega) : v|_R \in Q_k^s(R)\}, \\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{BDFM}_k^{\perp}(R)\}.
\end{aligned}$$

Auxiliary pressure space

$$Q_h = \{q \in L_0^2(\Omega) : q|_R \in P_{k-1}\}$$

and reduction operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ defined by

$$\begin{aligned}
&\int_e (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{s}\, p_{k-1}(s)\, ds = 0, \quad \forall e, \quad \forall p_{k-1} \in P_{k-1}(e), \\
&\int_R (\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma} - \boldsymbol{\gamma}) \cdot \boldsymbol{p}_{k-2}\, dx\, dy = 0, \quad \forall R, \quad \forall \boldsymbol{p}_{k-2} \in \boldsymbol{P}_{k-2}(R).
\end{aligned}$$

Lowest order element ($k = 1$) called MITC4. $\boldsymbol{BDFM}_1^{\perp}(R)$ has form $(a + by, c + dx)$ and coincides with lowest order rotated rectangular Raviart-Thomas element $\boldsymbol{RT}_0^{\perp}(R)$. Space $Q_1^s(R) = Q_1(R)$.

MITC4 element was proposed in Bathe-Dvorkin and analyzed by Bathe-Brezzi, and Duran and collaborators, (where proof extended to more general quadrilateral meshes using macro-element technique and results obtained under less regularity than previously required). For rectangular meshes, method coincides with T1 method of Hughes and Tezuyar.

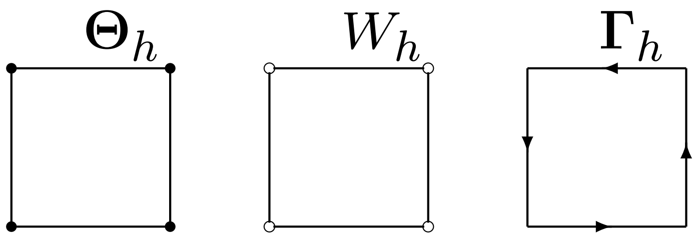
$k = 2$ method known as MITC9; analyzed in Bathe-Brezzi-Fortin and Duran-Liberman.

Shown in Stenberg-Suri and Perugia-Scapolla possible to reduce number of degrees of freedom in rotation space $\mathbf{\Theta}_h$ without affecting locking-free convergence. Several possibilities.
For example, for $k \geq 3$, replace

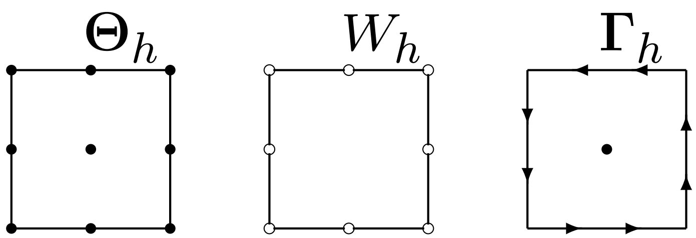
$$\mathbf{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \phi|_R \in \boldsymbol{Q}_k(R)\}$$

by

$$\mathbf{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \phi|_R \in [\boldsymbol{Q}_k(R) \cap \boldsymbol{P}_{k+2}(R)]\}.$$



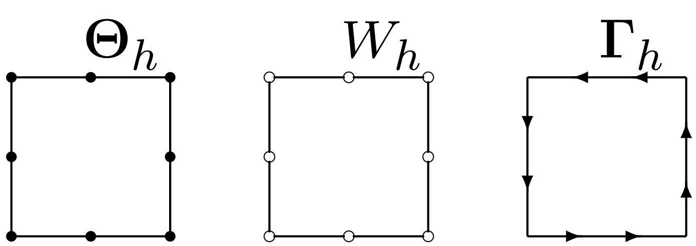
MITC4



MITC9

Can also consider MITC8 (Bathe-Dvorkin)

$$W_h = \{v \in \mathring{H}^1(\Omega) : v|_R \in Q_2^s(R)\},$$
$$\boldsymbol{\Gamma}_h = \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R \in \boldsymbol{BDM}_1^{\perp}(R)\},$$
$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}_2^s(R)\}.$$



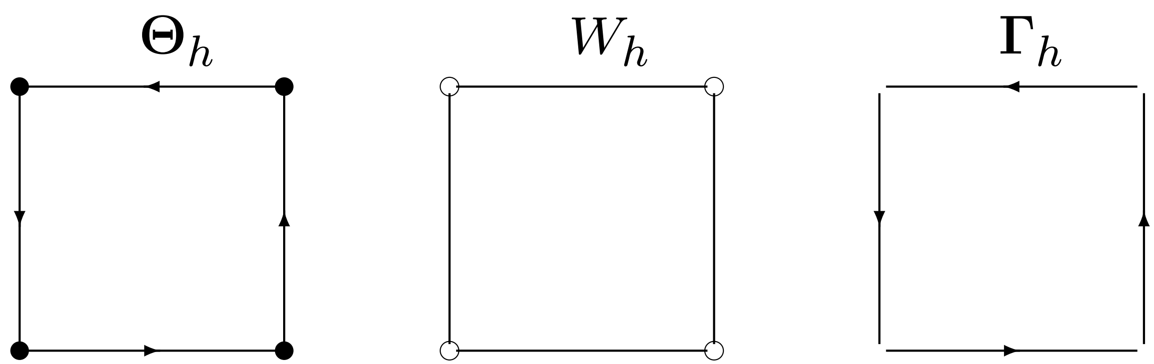
MITC8

DL4 method (Durán et al)

Extension to rectangles of Durán–Liberman triangular element defined previously. W_h and $\boldsymbol{\Gamma}_h$ are same as MITC4 method.

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_K \in \boldsymbol{Q}_1(K) \oplus \langle \boldsymbol{b}_1, , \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4 \rangle, \ \forall K \in \mathcal{T}_h\},$$

where $\boldsymbol{b}_i = b_i \boldsymbol{s}_i$, with $\boldsymbol{s}_i$ counterclockwise unit tangent vector to edge e_i of K and $b_i \in Q_2(K)$ vanishes on edges e_j, $j \neq i$.



DL4

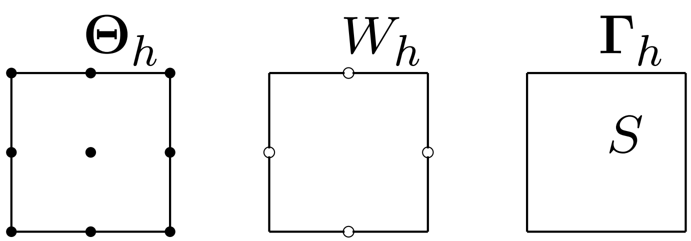
Ye's Method

Extension to rectangles of Arnold-Falk element. Not straightforward, since values at midpoints of edges of rectangle not a unisolvent set of degrees of freedom for a bilinear function (consider $(x-1/2)(y-1/2)$ on the unit square).

Nonconforming space W_h must be chosen differently.

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega) : \boldsymbol{\phi}|_R \in \boldsymbol{Q}_2(R)\},$$
$$\boldsymbol{\Gamma}_h = \{\boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\eta}|_R = (b + dx, c - dy) \equiv S\}.$$
$$W_h = \{v \in \mathring{H}^1(\mathcal{T}_h) : v|_R = a + bx + cy + d(x^2 - y^2)/2\},$$

and $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ is the $\boldsymbol{L}^2$ projection.



Ye

Extension to Quadrilaterals

Meshes of rectangular elements very restrictive, so would like to extend the elements defined above to quadrilaterals.

To do so, let $\boldsymbol{F}$ be an invertible bilinear mapping from reference element $\widehat{K} = [0,1] \times [0,1]$ to convex quadrilateral K.

For scalar functions, if $\widehat{v}(\widehat{\boldsymbol{x}})$ defined on $\widehat{K}$, define $v(\boldsymbol{x})$ on K by $v = \widehat{v} \circ \boldsymbol{F}^{-1}$. Then, for $\widehat{V}$ set of shape functions on $\widehat{K}$, define

$$V_F(K) = \{v : v = \widehat{v} \circ \boldsymbol{F}^{-1}, \widehat{v} \in \widehat{V}\}.$$

W_h defined in this way, beginning with shape functions denoted in figures. Preserves interelement continuity.

Same mapping, componentwise, used (with minor exceptions) to define $\boldsymbol{\Theta}_h$. Exception: D-L element. Define edge bubbles $\boldsymbol{b}_i = (\widehat{b}_i \circ \boldsymbol{F}^{-1})\boldsymbol{s}_i$ ($\boldsymbol{s}_i$ denotes unit tangent on ith edge of K). Also possible to use different mapping to define interior degrees of freedom for $\boldsymbol{\Theta}_h$ (doesn't affect interelement continuity).

$\boldsymbol{\Gamma}_h$ defined by rotated version of Piola transform. Letting $D\boldsymbol{F} =$ Jacobian of $\boldsymbol{F}$, if $\widehat{\boldsymbol{\eta}}$ vector function on $\widehat{K}$, define $\boldsymbol{\eta}$ on K by

$$\boldsymbol{\eta}(\boldsymbol{x}) = \boldsymbol{\eta}(\boldsymbol{F}(\widehat{\boldsymbol{x}})) = [D\boldsymbol{F}(\widehat{\boldsymbol{x}})]^{-t}\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}),$$

where A^{-t} denotes transpose of inverse of A. Then if $\widehat{\boldsymbol{V}}$ is a set of vector shape functions given on $\widehat{K}$, we define

$$\boldsymbol{V}_F(K) = \{\boldsymbol{\eta} : \boldsymbol{\eta} = [D\boldsymbol{F}]^{-t}\widehat{\boldsymbol{\eta}} \circ \boldsymbol{F}^{-1}, \widehat{\boldsymbol{\eta}} \in \widehat{\boldsymbol{V}}\}.$$

For $w \in W_h$, $\mathbf{grad}\, w = D\boldsymbol{F}^{-t}\widehat{\mathbf{grad}}\, \widehat{w}$. Hence, if on reference square $\widehat{\mathbf{grad}}\, \widehat{w} \subseteq \hat{\boldsymbol{V}}$, get $\mathbf{grad}\, w \subseteq \boldsymbol{\Gamma}_h$, key condition in analysis.

Extensions to quadrilaterals straightforward to define. Question: Does method retain same order of accuracy as for rectangles?

Problem: approximation properties of some elements can deteriorate, depending on way mesh is refined. Much of existing analysis for quadrilateral elements restricted to parallelograms (e.g., Stenberg-Suri), where $\boldsymbol{F}$ is affine, or to elements that are $O(h^2)$ perturbations of parallelograms.

If refinement strategy restricted to produce asymptotically affine meshes, deterioration in approximation avoided.

Error estimates for DL4 method for shape-regular quadrilateral meshes and for MITC4 method for asymptotically parallelogram meshes in Durán-etal. Numerical experiments do not indicate any deterioration of convergence rates for MITC4, even for more general shape regular meshes.

MITC8 approximates both $\boldsymbol{\theta}$ and w by spaces obtained from mappings of quadratic serendipity space. Since this space does not contain all of Q_2 , (missing basis function x^2y^2), expect to see only $O(h)$ convergence. $\boldsymbol{\Gamma}_h$ obtained by mapping $\boldsymbol{BDM}_1^{\perp}$ space, which also degrades in convergence after bilinear mapping.

MITC9 uses full Q_2 approximation for $\boldsymbol{\theta}$, but use of Q_2 serendipity space to approximate w and the $\boldsymbol{BDFM}_2^{\perp}$ space to approximate $\boldsymbol{\gamma}$ will cause degradation in convergence rate on general quadrilateral meshes.

Other Approaches for Locking-Free Schemes

So far, all finite element methods discussed modified original formulation only by introduction of reduction operator $\mathbf{\Pi}^{\mathbf{\Gamma}}$.

Now consider other modifications of variational formulation.

Expanded mixed formulations

One of first approaches: method proposed by Brezzi and Fortin based on expanded mixed formulation arising from introduction of Helmholtz decomposition:

$$\gamma = \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\theta}) = \mathbf{grad}\, r + \mathbf{curl}\, p, \quad r \in \mathring{H}^1(\Omega), p \in \hat{H}^1(\Omega).$$

Find $(r, \boldsymbol{\theta}, p, w) \in \mathring{H}^1(\Omega) \times \mathring{\boldsymbol{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$
\begin{aligned}
(\mathbf{grad}\, r, \mathbf{grad}\, \mu) = (g, \mu), &\quad \mu \in \mathring{H}^1(\Omega), \\
a(\boldsymbol{\theta}, \boldsymbol{\phi}) - (\mathbf{curl}\, p, \boldsymbol{\phi}) = (\mathbf{grad}\, r, \boldsymbol{\phi}) - (\boldsymbol{f}, \boldsymbol{\phi}), &\quad \boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), \\
-(\boldsymbol{\theta}, \mathbf{curl}\, q) - \lambda^{-1} t^2 (\mathbf{curl}\, p, \mathbf{curl}\, q) = 0, &\quad q \in \hat{H}^1(\Omega), \\
(\mathbf{grad}\, w, \mathbf{grad}\, s) = (\boldsymbol{\theta} + \lambda^{-1} t^2\, \mathbf{grad}\, r, \mathbf{grad}\, s), &\quad s \in \mathring{H}^1(\Omega).
\end{aligned}
$$

Key idea: middle two eqns perturbations of Stokes equations, so stable conforming approximation obtained by Stokes elements with continuous pressures (3rd eqn requires $p \in H^1(\Omega)$). Used Mini element.

Hence: $C^0 P_1$ used to approximate r, p, and w, $C^0 P_1 + B_3$ used to approximate $\boldsymbol{\theta}$.

Arnold-Falk method developed as modification with added feature that finite element method also equivalent to method using only primitive variables $\boldsymbol{\theta}$ and w.

New idea in A-F: use discrete Helmholtz decomposition:

$$\boldsymbol{M}_0(\mathcal{T}_h) = \mathbf{grad}_h\, M_1^*(\mathcal{T}_h) \oplus \mathbf{curl}\, M_1^0(\mathcal{T}_h).$$

to reduce discrete expanded mixed formulation back to discrete formulation using only primitive variables.

Simple Modification of Reissner-Mindlin Energy

Arnold and Brezzi modify definition of variable $\boldsymbol{\gamma}$ to be

$$\boldsymbol{\gamma} = \lambda(t^{-2} - 1)(\boldsymbol{\theta} - \mathbf{grad}\, w)$$

and new bilinear form defined:

$$a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = (\mathcal{C}\varepsilon\boldsymbol{\theta}, \varepsilon\boldsymbol{\phi}) + \lambda(\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\psi} - \mathbf{grad}\, v).$$

Modified weak formulation of R-M is:
Find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ such that

$$a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) + \lambda^{-1}t^2(\boldsymbol{\gamma}, \boldsymbol{\phi} - \mathbf{grad}\, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}),$$
$$\boldsymbol{\phi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega),$$
$$(\mathbf{grad}\, w - \boldsymbol{\theta}, \boldsymbol{\eta}) - \frac{t^2}{\lambda(1 - t^2)}(\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{L}^2(\Omega).$$

When discretized by finite elements, no longer need condition $\mathbf{grad}\, W_h \subset \boldsymbol{\Gamma}_h$, since form $a(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v)$ coercive over $\mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$. Hence, greater flexibility allowed.

Using this formulation, choice

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0 + \boldsymbol{B}_3, \quad W_h = \mathring{M}_2^0, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0$$

gives stable discretization and error estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \le Ch(\|\boldsymbol{f}\|_0 + \|g\|_0).$$

Least-squares Stabilization Schemes

Approach by Hughes-Franca and Stenberg. Bilinear forms modified by adding least-squares type stabilization terms.

Recall formulation of R-M equations without shear stress: Find $(\boldsymbol{\theta}, w) \in \mathring{\boldsymbol{H}}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$B(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1(\Omega), v \in \mathring{H}^1(\Omega),$$

where

$$B(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) + \lambda t^{-2}(\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v).$$

In stabilized scheme, define

$$B_h(\boldsymbol{\theta}, w; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) - \alpha \sum_{T \in \mathcal{T}_h} h_T^2 (\boldsymbol{L\theta}, \boldsymbol{L\psi})_T$$

$$+ \sum_{T \in \mathcal{T}_h} (\lambda^{-1} t^2 + \alpha h_T^2)^{-1} (\boldsymbol{\theta} - \mathbf{grad}\, w + \alpha h_T^2 \boldsymbol{L\theta}, \boldsymbol{\phi} - \mathbf{grad}\, v + \alpha h_T^2 \boldsymbol{L\phi})_T,$$

where $\boldsymbol{L\theta} = \operatorname{div} \mathcal{C}\varepsilon(\boldsymbol{\theta})$, and then seek an approximate solution $(\boldsymbol{\theta}_h, w_h) \in \boldsymbol{\Theta}_h \times W_h$ such that

$$B_h(\boldsymbol{\theta}_h, w_h; \boldsymbol{\phi}, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \quad \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, v \in W_h,$$

B_h constructed so that new formulation is both consistent and stable independent of choice of finite element spaces.

Choices $\boldsymbol{\Theta}_h = \boldsymbol{M}_k^0$, $W_h = M_{k+1}^0$ considered for $k \geq 1$. When $k = 1$, $\boldsymbol{L\phi}|_T = 0$ for all $T \in \mathcal{T}_h$ and all $\boldsymbol{\phi} \in \boldsymbol{\Theta}_h$.

Bilinear form reduces to:

$$B_h(\boldsymbol{\theta}_h, w_h; \boldsymbol{\phi}, v) = a(\boldsymbol{\theta}, \boldsymbol{\phi}) \\ + \sum_{T \in \mathcal{T}_h} (\lambda^{-1} t^2 + \alpha h_T^2)^{-1} (\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v)_T,$$

(method proposed by Pitkäranta). Under hypothesis $0 < \alpha < C_I$ (for an appropriate constant C_I), shown that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_1 + \|w - w_h\|_1 \leq C h^k (\|w\|_{k+2} + \|\boldsymbol{\theta}\|_{k+1}),$$

Modification considered in Brezzi-Fortin-Stenberg. $\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^0$, $W_h = \mathring{M}_1^0$, and term $(\boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\phi} - \mathbf{grad}\, v)$ modified to $(\boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\theta} - \mathbf{grad}\, w, \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\phi} - \mathbf{grad}\, v)$ by adding interpolation operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ into space $\boldsymbol{RT}_0^{\perp}$. Method uses only linear elements.

Lyly shows linked interpolation method close (and sometimes equivalent) to stabilized method of B-F-S and to stabilized linked method of Tessler and Hughes.

Discontinuous Galerkin Methods (Arnold-Brezzi-Marini, Arnold-Brezzi-Falk-Marini).

Bilinear forms modified to include terms allowing use of totally discontinuous elements.

Let $H^s(\mathcal{T}_h)$ denote functions whose restrictions to T belong to $H^s(T)$ for all $T \in \mathcal{T}_h$.

If φ belongs to $H^1(\mathcal{T}_h)$, define average $\{\varphi\}$ and jump $[\![\varphi]\!]$ on edge e shared by T^+ and T^- by:

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2}, \qquad [\![\varphi]\!] = \varphi^+ \boldsymbol{n}^+ + \varphi^- \boldsymbol{n}^-.$$

Need analogous definition for jump of vector $\boldsymbol{\phi} \in \boldsymbol{H}^1(\mathcal{T}_h)$ and definitions on boundary edges.

To obtain DG discretization, choose finite dimensional subspaces $\boldsymbol{\Theta}_h \subset \boldsymbol{H}^2(\mathcal{T}_h)$, $W_h \subset H^1(\mathcal{T}_h)$, and $\boldsymbol{\Gamma}_h \subset \boldsymbol{H}^1(\mathcal{T}_h)$. Method is:

Find $(\boldsymbol{\theta}_h, w_h) \in \boldsymbol{\Theta}_h \times W_h$ and $\boldsymbol{\gamma}_h \in \boldsymbol{\Gamma}_h$ such that

$$
\begin{aligned}
(\mathcal{C}\varepsilon_h(\boldsymbol{\theta}_h), \varepsilon_h(\boldsymbol{\phi})) - \langle \{\mathcal{C}\varepsilon_h(\boldsymbol{\theta}_h)\}, [\![\boldsymbol{\phi}]\!]\rangle - \langle [\![\boldsymbol{\theta}_h]\!], \{\mathcal{C}\varepsilon_h(\boldsymbol{\phi})\}\rangle \\
+(\boldsymbol{\gamma}_h, \mathbf{grad}_h\, v - \boldsymbol{\phi}) - \langle \{\boldsymbol{\gamma}_h\}, [\![v]\!]\rangle \\
+p_{\Theta}(\boldsymbol{\theta}_h, \boldsymbol{\phi}) + p_W(w_h, v) = (g, v) - (\boldsymbol{f}, \boldsymbol{\phi}), \\
(\boldsymbol{\phi}, v) \in \boldsymbol{\Theta}_h \times W_h, \\
(\mathbf{grad}_h\, w_h - \boldsymbol{\theta}_h, \boldsymbol{\eta}) - \langle [\![w_h]\!], \{\boldsymbol{\eta}\}\rangle - t^2(\boldsymbol{\gamma}_h, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \boldsymbol{\Gamma}_h.
\end{aligned}
$$

Make standard choice for interior penalty terms p_{Θ} and p_W:

$$
p_{\Theta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^{\Theta}}{|e|} \int_e [\![\boldsymbol{\theta}]\!] : [\![\boldsymbol{\phi}]\!]\, ds,\ p_W(w, v) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^{W}}{|e|} \int_e [\![w]\!] \cdot [\![v]\!]\, ds,
$$

$p_\Theta(\boldsymbol{\phi},\boldsymbol{\phi})$ and $p_W(v,v)$ viewed as measure of deviation of $\boldsymbol{\phi}$ and v from being continuous. Parameters κ^Θ and κ^W positive constants chosen sufficiently large to ensure stability. When $W_h \in C^0$, penalty term p_W not needed.

Simplest method: $k \geq 1$, $W_h = \mathring{M}^0_{k+1}$. Choose $w^I = \Pi^W w$.

Since $\mathbf{\Theta}_h$ need not be continuous, choose $\mathbf{\Theta}_h$ so that condition $\boldsymbol{\gamma}^I = \mathbf{\Pi}^{\mathbf{\Gamma}}\boldsymbol{\gamma}$ satisfied without using reduction operator $\mathbf{\Pi}^{\mathbf{\Gamma}}$.

Simplest choice: $\mathbf{\Theta}_h = \boldsymbol{BDM}^{\perp}_{k-1}$. Note $\mathbf{grad}\, W_h \subset \mathbf{\Theta}_h$.

Next define $\boldsymbol{\theta}^I = \mathbf{\Pi}^{\mathbf{\Theta}}\boldsymbol{\theta}$, where $\mathbf{\Pi}^{\mathbf{\Theta}} : \boldsymbol{H}^1(\Omega) \mapsto \mathbf{\Theta}_h$ defined by:

$$\int_e (\boldsymbol{\phi} - \Pi^{\boldsymbol{\Theta}} \boldsymbol{\phi}) \cdot \boldsymbol{s} q \, ds = 0, \quad q \in P_{k-1}(e),$$
$$\int_T (\boldsymbol{\phi} - \Pi^{\boldsymbol{\Theta}} \boldsymbol{\phi}) \cdot \boldsymbol{q} \, dx = 0, \quad \boldsymbol{q} \in \boldsymbol{RT}_{k-3}(T),$$

where $\boldsymbol{RT}_{k-3}$ is unrotated Raviart-Thomas space of index $k-3$. Note: interior degrees of freedom not original ones. However, natural interpolant defined by these modified degrees of freedom satisfies additional key property:

$$\boldsymbol{\Pi}^{\boldsymbol{\Theta}} \, \mathbf{grad} \, w = \mathbf{grad} \, \Pi^W w.$$

From this condition, get

$$\begin{aligned} \boldsymbol{\gamma}^I = \lambda t^{-2} (\mathbf{grad} \, w^I - \boldsymbol{\theta}^I) &= \lambda t^{-2} (\mathbf{grad} \, \Pi^W w - \boldsymbol{\Pi}^{\boldsymbol{\Theta}} \boldsymbol{\theta}) \\ &= \lambda t^{-2} \boldsymbol{\Pi}^{\boldsymbol{\Theta}} (\mathbf{grad} \, w - \boldsymbol{\theta}) = \boldsymbol{\Pi}^{\boldsymbol{\Theta}} \boldsymbol{\gamma}. \end{aligned}$$

Methods using Nonconforming Finite Elements

Method of Oñate, Zarate, and Flores. Choose

$$\mathbf{\Theta}_h = \mathring{\boldsymbol{M}}_1^*, \quad W_h = \mathring{M}_1^0, \quad \mathbf{\Gamma}_h = \boldsymbol{RT}_0^\perp.$$

$\mathbf{\Theta}_h$ not contained in $\mathring{\boldsymbol{H}}^1(\Omega)$, so replace $\mathcal{E}$ by $\mathcal{E}_h$.

Problem: $\| \mathcal{E}_h(\boldsymbol{\theta}_h)\|_0^2$ not a norm on $\mathbf{\Theta}_h$ because Korn's inequality fails for nonconforming piecewise linear functions.

To partially compensate for this, use following result established by A-F. Define

$$\boldsymbol{Z}_h = \left\{(\boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathring{\boldsymbol{M}}_1^* \times \mathbf{\Gamma}_h : \lambda^{-1} t^2 \operatorname{rot} \boldsymbol{\eta} = \operatorname{rot}_h \boldsymbol{\psi}\right\}.$$

Lemma: There exists constant c independent of h and t:

$$a_h(\boldsymbol{\psi},\boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\eta},\boldsymbol{\eta}) \ge c[\min(1, h^2/t^2)\|\boldsymbol{\psi}\|_{1,h}^2 + \|\,\mathcal{E}_h\,\boldsymbol{\psi}\|_0^2 + t^2\|\boldsymbol{\eta}\|_0^2 + h^2t^2\|\operatorname{rot}\boldsymbol{\eta}\|_0^2] \quad \text{for all } (\boldsymbol{\psi},\boldsymbol{\eta}) \in \boldsymbol{Z}_h.$$

So form not uniformly coercive. Then obtain:

Theorem: There exists constant C independent of h and t:

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{1,h} + t^2\|\operatorname{rot}(\boldsymbol{\gamma} - \gamma_h)\|_0^2 \le Ch\max(1, t^2/h^2)\|g\|_0,$$
$$\|\,\mathcal{E}(\boldsymbol{\theta} - \boldsymbol{\theta}_h)\|_0^2 + t\|\boldsymbol{\gamma} - \gamma_h\|_0 \le Ch\max(1, t/h)\|g\|_0.$$
$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0 + \|w - w_h\|_0 \le C\max(h^2, t^2)\|g\|_0.$$

Does not give convergence, but small error if $h \sim t$.

Method of Lovadina

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^*, \quad W_h = \mathring{M}_1^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0,$$

so two spaces non-conforming. Replace $\mathcal{E}$ and $\mathbf{grad}$ by element-wise counterparts. Bilinear form $a(\boldsymbol{\theta}, \boldsymbol{\phi})$ replaced by:

$$a_h(\boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_{T \in \mathcal{T}_h} a_T(\boldsymbol{\theta}, \boldsymbol{\phi}) + p_{\Theta}(\boldsymbol{\theta}, \boldsymbol{\phi}), \; a_T(\boldsymbol{\theta}, \boldsymbol{\phi}) = \int_T \mathcal{C}\varepsilon\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\phi})\, dx,$$

where p_{Θ} has same definition as in D-G method. By adding term p_{Θ}, can prove discrete Korn's inequality.

Method is simplified version of earlier method of Brezzi-Marini:

$$\boldsymbol{\Theta}_h = \mathring{\boldsymbol{M}}_1^* + \boldsymbol{B}_2^*, \quad W_h = \mathring{M}_1^* + B_2^*, \quad \boldsymbol{\Gamma}_h = \boldsymbol{M}_0 + \mathbf{grad}_h\, B_2^*,$$

where B_2^* is nonconforming quadratic bubble vanishing at two Gauss points of each triangle edge.

A negative-norm least squares method (Bramble-Sun)

Uses expanded mixed formulation of Brezzi-Fortin.

Reformulated as least squares method using special minus one norm developed previously by Bramble, Lazarov, and Pasciak.

Only C^0 finite elements needed to approximate all variables, and piecewise linears can be used.

Optimal order error estimates established uniformly in t.

Stability result also gives natural block diagonal preconditioner, for solution of least squares system, using only standard preconditioners for second order elliptic problem.