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STABILITY OF HIGHER-ORDER HOOD–TAYLOR METHODS*

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Abstract. The stability of a higher-order Hood–Taylor method for the approximation of the stationary Stokes equations using continuous piecewise polynomials of degree 3 to approximate velocities and continuous piecewise polynomials of degree 2 to approximate the pressure is proved. This result implies that the standard finite element method using these spaces satisfies a quasi-optimal error estimate. The technique used may also be applied to prove the stability of Hood–Taylor rectangular elements of arbitrary degree k for velocities and $k-1$ for pressure in each variable.

Key words. Stokes, finite element

AMS(MOS) subject classification. 65N30

1. Introduction. The purpose of this note is to prove the stability of a higher-order Hood–Taylor method for the approximation of the stationary Stokes equations

$$\begin{aligned} &-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in } \Omega,\\ &\operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega,\\ &\mathbf{u}=0 \quad \text{on } \partial\Omega. \end{aligned}$$

The original Hood–Taylor method, proposed in [6], seeks to approximate the velocity $\mathbf{u}$ by continuous piecewise quadratic functions and the pressure p by continuous piecewise linear functions. The approximate problem has the standard form:

Find $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{V} = (H_0^1(\Omega))^2$ and $p_h \in Q_h \subset Q = \left\{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\right\}$ such that

$$\begin{aligned} &a(\mathbf{u}_h,\mathbf{v}_h)-(p,\operatorname{div}\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h) \quad \text{for all } \mathbf{v}_h\in\mathbf{V}_h,\\ &(\operatorname{div}\mathbf{u}_h,q)=0 \quad \text{for all } q_h\in Q_h. \end{aligned}$$

The first error analysis of this method was given by Bercovier and Pironneau [1]. Their approach was to show that the Hood–Taylor spaces satisfied the condition

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\int_\Omega \operatorname{div}\mathbf{v}_h\, q_h\,dx}{\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}} \ge \gamma\|\nabla q_h\|_{L^2(\Omega)} \quad \text{for all } q_h\in Q_h, \tag{1.1}$$

where γ is a constant independent of the meshsize h. This is a modified form of the standard stability condition

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\int_\Omega \operatorname{div}\mathbf{v}_h\, q_h\,dx}{\|\mathbf{v}_h\|_{\mathbf{H}^1(\Omega)}} \ge \gamma\|q_h\|_{L^2(\Omega)} \quad \text{for all } q_h\in Q_h, \tag{1.2}$$

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which from the general theory of saddlepoint problems developed by Babuška and Brezzi (cf. [2]), implies the quasi-optimal error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \le C\inf(\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}^1(\Omega)} + \|p-q_h\|_{L^2(\Omega)}), \tag{1.3}$$

where the inf is taken over all $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$. Based on their stability result, Bercovier and Pironneau were able to prove optimal order error estimates of the form

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\mathbf{L}^2(\Omega)} + h\|\nabla(p-p_h)\|_{L^2(\Omega)} \le Ch^2\left(\|\mathbf{u}\|_{\mathbf{H}^3(\Omega)} + \|p\|_{H^2(\Omega)}\right).$$

Later, Verfürth [9] proved that if the modified stability condition (1.1) holds, then (1.2) also holds, thus fitting the analysis of this method into the standard theory.

The obvious first generalization of the Hood–Taylor method is to consider the case where $\mathbf{V}_h$ consists of continuous piecewise cubics and Q_h consists of continuous piecewise quadratics. It is the purpose of this paper to show that this pair of spaces also satisfies the stability condition (1.2) and hence the quasi-optimal error estimate (1.3). The proof uses the ideas of [9] and [1] and follows the presentation in Brezzi and Fortin [3] for the original Hood–Taylor method. Another proof of the stability of the original method based on the macro element technique is given in [5], and a proof of the main result of this paper based on the macro element technique has been contemporaneously given by Stenberg [8]. We note that for $n \ge 4$ Scott and Vogelius [7] have shown that, except for some exceptional meshes, the combination of continuous piecewise polynomials of degree $\le n$ for velocities and discontinous piecewise polynomials of degree $\le n-1$ for pressure satisfy the stability condition (1.2). In the final section of the paper, we discuss how the mesh restrictions they require may be weakened when continuous pressure elements are used.

In the next section, we state and prove a numerical integration formula which will be needed for the proof of our main result in §3. We shall use the standard notation that the subscript $0,\Omega$ applied to a norm denotes the norm in $L^2(\Omega)$ and the subscript $1,\Omega$ denotes the norm in $H^1(\Omega)$. When the norm is applied to a vector, the corresponding vector norm is used.

2. A numerical integration formula. The proof of our main result depends on a quadrature formula which is exact for quartic polynomials on a triangle. To state this formula, we let a_1, a_2, a_3 denote the vertices of a triangle T, a_{123} the centroid, and define on each side a_ia_j the points

$$a_{iij} = (\tfrac{1}{2}+\theta)a_i + (\tfrac{1}{2}-\theta)a_j, \tag{2.1}$$

where $\theta = 1/\sqrt{12}$.

LEMMA 2.1. *The quadrature formula*

$$\int_T \phi\,dx = |T|\left(\omega_1\phi(a_{123}) + \omega_2\sum_{i=1}^{3}\phi(a_i) + \omega_3\sum_{i\ne j}\phi(a_{iij})\right), \tag{2.2}$$

with $|T|$ *the area of* T *and* $\omega_1 = 9/20$, $\omega_2 = -1/60$, *and* $\omega_3 = 1/10$, *is exact for all polynomials of degree* ≤ 4 *on* T.

Proof. Since the quadrature points are easily seen to be a unisolvent set for polynomials of degree ≤ 3 on T, we obtain a formula exact for polynomials of degree ≤ 3 by choosing the weights as the integrals of the Lagrange basis functions associated with those points. By symmetry, the weights at the three vertices will be equal and the weights at the six points a_{iij} will be equal. Choosing $\phi = \lambda_1\lambda_2\lambda_3$ implies that $\omega_1 = 9/20$. Next, choosing $\phi = \lambda_1\lambda_2$, we find that $\omega_3 = 1/10$. The choice $\phi = 1$ then implies that $\omega_2 = -1/60$. It thus remains to show that the formula is also exact for polynomials of degree 4. To accomplish this, we need only show that the formula holds for the functions λ_i^4 and $\lambda_1\lambda_2\lambda_3(\lambda_i - \lambda_j)$. Note that the second of these functions is zero at all integration points and, by symmetry, is easily seen to have zero integral. Hence the integration formula is valid for these functions. For the function λ_i^4, a simple computation (most easily done on the standard reference triangle) shows that both sides of the formula give the value $1/30$.

3. Main results. Let τ_h be a regular sequence of triangulations of Ω (cf. Ciarlet [4]). To prove our stability result, we shall place a mild restriction on the triangulations, which we describe as follows. Consider h fixed and let S_2 denote the set of triangles with two sides lying on $\partial\Omega$ and let S_3 denote the set of triangles sharing a common edge with triangles of S_2. We shall assume that τ_h consists of more than one triangle, that $S_2 \cap S_3$ is empty, and that triangles of S_3 share a common edge with only one triangle of the set S_2. For future reference, we shall denote by S_1 the set of remaining triangles.

THEOREM 3.1. *Under the assumptions on τ_h described above, the spaces*

$$\mathbf{V}_h = \{\mathbf{v}_h \in (H_0^1(\Omega))^2 : \mathbf{v}_h|_T \in (P_3)^2 \text{ for all } T \in \tau_h\}, \tag{3.1}$$

$$Q_h = \left\{q_h \in H^1(\Omega) : q_h \in P_2 \text{ for all } T \in \tau_h, \int_\Omega q_h\,dx = 0\right\} \tag{3.2}$$

satisfy the stability condition (1.2).

Proof. We will prove that there exist positive constants γ_1 and γ_2 such that for every $q_h \neq 0$ in Q_h, we can find $\mathbf{v}_h \in \mathbf{V}_h$ satisfying

$$\int_\Omega \operatorname{div} \mathbf{v}_h q_h\,dx \geq \gamma_1 \|q_h\|_{0,\Omega}^2 \tag{3.3}$$

$$\|\mathbf{v}_h\|_{1,\Omega} \leq \gamma_2 \|q_h\|_{0,\Omega}. \tag{3.4}$$

This will imply (1.2) with $\gamma = \gamma_1/\gamma_2$. In order to construct $\mathbf{v}_h$, we consider first the L^2-projection $\bar{q}_h$ of q_h on the space of piecewise constants. It is well known that the stability condition (1.2) holds if we take a piecewise quadratic continuous velocity field and a piecewise constant pressure field (e.g., see [5]). This implies that there exists $\mathbf{w}_h \in \mathbf{V}_h$ such that

$$\int_\Omega \operatorname{div} \mathbf{w}_h \bar{q}_h\,dx \geq c_1 \|\bar{q}_h\|_{0,\Omega}^2, \tag{3.5}$$

$$\|\mathbf{w}_h\|_{1,\Omega} \leq c_2 \|\bar{q}_h\|_{0,\Omega}. \tag{3.6}$$

Hence we have

$$\begin{aligned}\int_\Omega \operatorname{div} \mathbf{w}_h q_h\,dx &= \int_\Omega \operatorname{div} \mathbf{w}_h \bar{q}_h\,dx + \int_\Omega \operatorname{div} \mathbf{w}_h (q_h - \bar{q}_h)\,dx \\ &\geq c_1 \|\bar{q}_h\|_{0,\Omega}^2 - c_2 \|\bar{q}_h\|_{0,\Omega} \|q_h - \bar{q}_h\|_{0,\Omega}.\end{aligned} \tag{3.7}$$

We now construct a function $\mathbf{z}_h \in \mathbf{V}_h$ which takes care, in a suitable sense, of the "nonconstant" part $q_h - \bar{q}_h$. Then $\mathbf{v}_h$ will be chosen as $\mathbf{v}_h = \mathbf{w}_h + \beta\mathbf{z}_h$, for a convenient choice of β. We first define $\mathbf{z}_h$ to be zero at all vertices of τ_h and $\mathbf{z}_h \cdot \mathbf{n}$ to be zero at all the nodes on the edges of τ_h (there are two such nodes on every edge) with the exception of the nodes on the common edge of triangles in $S_2 \cup S_3$. Note that this implies that $\mathbf{z}_h \cdot \mathbf{n} = 0$ on every triangle edge, except the edges that are common to triangles in S_2 and S_3. Since $\mathbf{z}_h$ must be in $(H_0^1(\Omega))^2$, we need $\mathbf{z}_h \cdot \mathbf{t}$ (the tangential component of $\mathbf{z}_h$) to vanish on $\partial\Omega$. We are still free to prescribe $\mathbf{z}_h$ at the centroids of the triangles of τ_h, $\mathbf{z}_h \cdot \mathbf{n}$ at the nodes on the common edge of triangles in $S_2 \cup S_3$, and $\mathbf{z}_h \cdot \mathbf{t}$ at the nodes on the internal edges. We first choose, for every internal edge e, a unit tangent vector $\boldsymbol{\tau}^e$. Note that, by convention, $\mathbf{t}$ is oriented in the counterclockwise direction, and therefore it reverses its direction when seen from an adjacent triangle. This will not be the case with $\boldsymbol{\tau}^e$. For triangles $T \in S_2$, let e_3 denote the interior edge of T and $\boldsymbol{\eta}$ the unit outward normal along e_3. We are now able to complete the definition of $\mathbf{z}_h$. On every triangle T, we set

$$\mathbf{z}_h(a_{123}) = -|T|\nabla q_h(a_{123}) \tag{3.8}$$

and for every edge e of T internal to Ω, we define

$$\mathbf{z}_h \cdot \boldsymbol{\tau}^e(a_{iij}) = -|e|^2(\nabla q_h \cdot \boldsymbol{\tau}^e)(a_{iij}), \tag{3.9}$$

where the a_{iij} are the nodes, on e, of the integration formula (2.2) defined by (2.1). Finally, we choose at the nodes on the common edge e_3 of triangles in $S_2 \cup S_3$,

$$\mathbf{z}_h \cdot \boldsymbol{\eta}(a_{iij}) = -\epsilon|e|^2(\nabla q_h^2 \cdot \boldsymbol{\eta})(a_{iij}), \tag{3.10}$$

where by q_h^2, we mean q_h as defined on $T \in S_2$ (the normal derivative of q_h will in general not be continuous across e_3) and ϵ is a constant to be chosen later.

We now show by a standard argument that

$$\|\mathbf{z}_h\|_{1,\Omega} \le c_3\|q_h - \bar{q}_h\|_{0,\Omega}. \tag{3.11}$$

Since the points a_{123}, a_i, and a_{iij} are a unisolvent set of points for polynomials of degree ≤ 3 on T, we write $\mathbf{z}_h$ in terms of the Lagrange basis functions corresponding to these points and use standard inverse estimates to obtain for triangles $T \in S_1$,

$$\begin{aligned}
\|\mathbf{z}_h\|_{1,T} &\le C\left(|\mathbf{z}_h(a_{123})| + \sum_{i=1}^{3}|\mathbf{z}_h(a_i)| + \sum_{i\neq j}|\mathbf{z}_h(a_{iij})|\right)\\
&\le C\left(|\nabla q_h(a_{123})||T| + \sum_{i\neq j}|\nabla q_h \cdot \boldsymbol{\tau}^e(a_{iij})||e|^2\right)\\
&\le C|T|\|\nabla(q_h - \bar{q}_h)\|_{\infty,T}\\
&\le C\|q_h - \bar{q}_h\|_{0,T}.
\end{aligned}$$

For a pair of triangles $T_2 \in S_2$ and $T_3 \in S_3$ with a common edge, a similar argument, also incorporating the additional term involving the normal component of $\mathbf{z}_h$, gives the same result for $\|\mathbf{z}_h\|_{1,T_2}$, and for T_3 we get

$$\|\mathbf{z}_h\|_{1,T_3} \le C\|q_h - \bar{q}_h\|_{0,T_2\cup T_3}.$$

The desired result follows by squaring and summing the results over all the triangles.

Now by Lemma 2.1, we have for every T in S_1

$$\begin{aligned}\int_T \operatorname{div} \mathbf{z}_h q_h \, dx &= -\int_T \mathbf{z}_h \cdot \nabla q_h \, dx \\ &= -|T| \Big(\omega_1 \mathbf{z}_h \cdot \nabla q_h(a_{123}) + \omega_2 \sum_{i=1}^{3} \mathbf{z}_h \cdot \nabla q_h(a_i) \\ &\qquad + \omega_3 \sum_{i \neq j} \mathbf{z}_h \cdot \nabla q_h(a_{iij}) \Big) \\ &= |T| \left(\omega_1 |\nabla q_h|^2(a_{123})|T| + \omega_3 \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h \cdot \boldsymbol{\tau}^e|^2(a_{iij})|e|^2 \right), \end{aligned} \tag{3.12}$$

where the notation has the obvious meaning. We next show by an easy scaling argument that

$$\int_T \operatorname{div} \mathbf{z}_h q_h \, dx \geq C \|q_h - \bar{q}_h\|_{0,T}^2. \tag{3.13}$$

Let $F_T(\hat{x})$ denote the affine mapping from the standard reference triangle $\hat{T}$ to T and define $\hat{v} = v \circ F_T$ for functions v defined on T. Since at least two edges in each triangle in S_1 are internal, it is easy to check that the expression

$$\left(|\nabla q_h|^2(a_{123}) + \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h \cdot \boldsymbol{\tau}^e|^2(a_{iij}) \right)^{1/2}$$

vanishes only if q_h is constant and hence is a norm on quadratic polynomials modulo constants on T. Thus,

$$\begin{aligned}\|q_h - \bar{q}_h\|_{0,T}^2 &\leq C|T| \|\hat{q}_h - \hat{\bar{q}}_h\|_{0,\hat{T}}^2 \\ &\leq C|T| \left(|\hat{\nabla} \hat{q}_h(\hat{a}_{123})|^2 + \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\hat{\nabla} \hat{q}_h \cdot \boldsymbol{\tau}^e(\hat{a}_{iij})|^2 \right) \\ &\leq C|T| \left(|\nabla q_h(a_{123})|^2 |T| + \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h \cdot \boldsymbol{\tau}^e(a_{iij})|^2 |e|^2 \right) \\ &\leq C \int_T \operatorname{div} \mathbf{z}_h q_h \, dx. \end{aligned}$$

Now let $T_2 \in S_2$ and $T_3 \in S_3$ share the common edge e_3 and denote by $\boldsymbol{\tau}^3$ and $\boldsymbol{\eta}^3$ the counterclockwise unit tangent and outward unit normal vectors to T_2 along e_3. We use the superscripts 2 and 3 to denote the values of various quantities on the triangles T_2 and T_3, respectively, and the notation $\sum_{e_3}$ to denote that the quantity which follows is evaluated only on the edge e_3. Now since $\mathbf{z}_h \cdot \mathbf{n}$ vanishes along all the edges of $T_2 \cup T_3$ except e_3,

$$\int_{T_2} \operatorname{div} \mathbf{z}_h q_h \, dx + \int_{T_3} \operatorname{div} \mathbf{z}_h q_h \, dx = -\int_{T_2} \mathbf{z}_h \nabla q_h \, dx - \int_{T_3} \mathbf{z}_h \nabla q_h \, dx.$$

By Lemma 2.1, we get that

$$
\begin{aligned}
-\int_{T_2} \mathbf{z}_h \nabla q_h \, dx = |T_2| \Bigg(& \omega_1 |\nabla q_h^2|^2 (a_{123}^2)|T_2| \\
& + \omega_3 \sum_{e_3} \sum_{i \neq j} \big[|\nabla q_h^2 \cdot \boldsymbol{\tau}^3|^2 (a_{iij}) |e_3|^2 \\
& + \epsilon |\nabla q_h^2 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \big] \Bigg)
\end{aligned} \tag{3.14}
$$

and

$$
\begin{aligned}
-\int_{T_3} \mathbf{z}_h \nabla q_h \, dx = {} & |T_3| \Bigg(\omega_1 |\nabla q_h^3|^2 (a_{123}^3)|T_3| + \omega_3 \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h^3 \cdot \boldsymbol{\tau}^e|^2 (a_{iij}^3) |e|^2 \\
& + \omega_3 \sum_{e_3} \sum_{i \neq j} \epsilon (\nabla q_h^3 \cdot \boldsymbol{\eta}^3)(\nabla q_h^2 \cdot \boldsymbol{\eta}^3)(a_{iij}) |e_3|^2 \Bigg) \\
\geq {} & |T_3| \Bigg(\omega_1 |\nabla q_h^3|^2 (a_{123}^3)|T_3| + \omega_3 \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h^3 \cdot \boldsymbol{\tau}^e|^2 (a_{iij}^3) |e|^2 \\
& - \omega_3 \sum_{e_3} \sum_{i \neq j} \Bigg[\frac{\epsilon |T_3|}{2|T_2|} |\nabla q_h^3 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \\
& + \frac{\epsilon |T_2|}{2|T_3|} |\nabla q_h^2 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \Bigg] \Bigg).
\end{aligned} \tag{3.15}
$$

Hence,

$$
\begin{aligned}
\int_{T_2} \operatorname{div} \mathbf{z}_h q_h \, dx + \int_{T_3} \operatorname{div} \mathbf{z}_h q_h \, dx \geq {} & |T_2| \Bigg(\omega_1 |\nabla q_h^2|^2 (a_{123}^2)|T_2| \\
& + \omega_3 \sum_{e_3} \sum_{i \neq j} \Big[|\nabla q_h^2 \cdot \boldsymbol{\tau}^3|^2 (a_{iij}) |e_3|^2 + \frac{\epsilon}{2} |\nabla q_h^2 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \Big] \Bigg) \\
& + |T_3| \Bigg(\omega_1 |\nabla q_h^3|^2 (a_{123}^3)|T_3| + \omega_3 \sum_{e \notin \partial\Omega} \sum_{i \neq j} |\nabla q_h^3 \cdot \boldsymbol{\tau}^e|^2 (a_{iij}^3) |e|^2 \\
& - \omega_3 \sum_{e_3} \sum_{i \neq j} \frac{\epsilon |T_3|}{2|T_2|} |\nabla q_h^3 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \Bigg).
\end{aligned}
$$

Employing an argument similar to that used for triangles in S_1, we get that

$$
\begin{aligned}
& |T_2| \Bigg(\omega_1 |\nabla q_h^2|^2 (a_{123}^2)|T_2| + \omega_3 \sum_{e_3} \sum_{i \neq j} \Big[|\nabla q_h^2 \cdot \boldsymbol{\tau}^3|^2 (a_{iij}) |e_3|^2 + \frac{\epsilon}{2} |\nabla q_h^2 \cdot \boldsymbol{\eta}^3|^2 (a_{iij}) |e_3|^2 \Big] \Bigg) \\
& \qquad \geq C \|q_h - \bar{q}_h\|_{0,T_2}^2
\end{aligned}
$$

and

$$|T_3|\left(\omega_1|\nabla q_h^3|^2(a_{123}^3)|T_3| + \omega_3 \sum_{e\notin\partial\Omega}\sum_{i\neq j}|\nabla q_h^3\cdot\boldsymbol{\tau}^e|^2(a_{iij}^3)|e|^2 - \omega_3\sum_{e_3}\sum_{i\neq j}\frac{\epsilon|T_3|}{2|T_2|}|\nabla q_h^3\cdot\boldsymbol{\eta}^3|^2(a_{iij})|e_3|^2\right)$$
$$\geq c_5\|q_h-\bar{q}_h\|_{0,T_3}^2 - |T_3|\omega_3\sum_{e_3}\sum_{i\neq j}\frac{\epsilon|T_3|}{2|T_2|}|\nabla q_h^3\cdot\boldsymbol{\eta}^3|^2(a_{iij})|e_3|^2$$
$$\geq (c_5-c_6\epsilon)\|q_h-\bar{q}_h\|_{0,T_3}^2.$$

Hence, for ϵ sufficiently small,

$$\int_{T_2}\operatorname{div}\mathbf{z}_h q_h\,dx + \int_{T_3}\operatorname{div}\mathbf{z}_h q_h\,dx \geq C\|q_h-\bar{q}_h\|_{T_2\cup T_3}^2. \tag{3.16}$$

Consider now a $\mathbf{v}_h$ of the form $\mathbf{v}_h = \mathbf{w}_h + \beta\mathbf{z}_h$. From (3.7), (3.13), and (3.16), we have

$$\int_\Omega \operatorname{div}\mathbf{v}_h q_h\,dx \geq c_1\|\bar{q}_h\|_{0,\Omega}^2 - c_2\|\bar{q}_h\|_{0,\Omega}\|q_h-\bar{q}_h\|_{0,\Omega} + \beta c_4\|q_h-\bar{q}_h\|_{0,\Omega}^2$$
$$\geq \frac{c_1}{2}\|\bar{q}_h\|_{0,\Omega}^2 + \left(\beta c_4 - \frac{c_2^2}{2c_1}\right)\|q_h-\bar{q}_h\|_{0,\Omega}^2 \geq \frac{c_1}{2}\|q_h\|_{0,\Omega}^2$$

for $\beta = (c_1^2+c_2^2)/(2c_1c_4)$ (since $\bar{q}_h$ and $q_h-\bar{q}_h$ are orthogonal in $L^2(\Omega)$). Finally, (3.6) and (3.11) give

$$\|\mathbf{v}_h\|_{1,\Omega} \leq (c_2+\beta c_3)\|q_h\|_{0,\Omega},$$

so that (3.3) and (3.4) hold.

Remark. It is easy to see that if instead of (3.8), (3.9), and (3.10), we choose the scaling

$$\mathbf{z}_h(a_{123}) = -\nabla q_h(a_{123}), \quad \mathbf{z}_h\cdot\boldsymbol{\tau}^e(a_{iij}) = -(\nabla q_h\cdot\boldsymbol{\tau}^e)(a_{iij}),$$
$$\mathbf{z}_h\cdot\boldsymbol{\eta}(a_{iij}) = -\epsilon(\nabla q_h^2\cdot\boldsymbol{\eta})(a_{iij}),$$

we obtain the Bercovier–Pironneau type inequality

$$\int_\Omega \operatorname{div}\mathbf{z}_h q_h\,dx \geq c_7\|\nabla q_h\|_{0,\Omega}\|\mathbf{z}_h\|_{0,\Omega}.$$

We note that the same proof applies to the case of rectangular elements on a mesh τ_h in which we take continuous velocities which are locally Q_k = {polynomials of degree $\leq k$ in each variable} and a continuous pressure which is locally in Q_{k-1}. The corresponding quadrature formula will be the tensor product of the one-dimensional Gauss–Lobatto formula exact for polynomials of degree $\leq 2k-1$. Thus, we have the following theorem.

THEOREM 3.2. *The spaces*

$$\mathbf{V}_h = \{\mathbf{v}_h \in (H_0^1(\Omega))^2 : \mathbf{v}_h|_K \in Q_k \text{ for all rectangles } K\in\tau_h\},$$
$$Q_h = \left\{q_h \in H^1(\Omega) : q_h \in Q_{k-1} \text{ for all rectangles } K\in\tau_h, \int_\Omega q_h\,dx = 0\right\}$$

satisfy the stability condition (1.2).

It is also easy to check that the following negative result holds.

THEOREM 3.3. *The combination of continuous Q_k velocities and discontinuous Q_{k-1} pressure does not satisfy the stability condition* (1.2).

To see this, consider the checkerboard function which is locally of the form

$$q_h = (x - l_1)(x - l_2)\cdots(x - l_{k-1})(y - l_1)(y - l_2)\cdots(y - l_{k-1}),$$

where $l_1, l_2, \cdots, l_{k-1}$ are the internal Gauss–Lobatto quadrature points. An explicit analysis of the difficulties in the $Q_1 - Q_0$ case may be found in [5].

4. Remarks on higher-order triangular Hood–Taylor methods. We close the paper with some remarks about triangular Hood–Taylor methods using the combination of continuous piecewise polynomials of degree $\leq n$ for velocities and continuous piecewise polynomials of degree $\leq n-1$ for pressure, with $n \geq 4$. As mentioned previously, Scott and Vogelius have established stability in this range for the combination of continuous velocities and discontinuous pressures, under certain restrictions on the mesh. To describe their results, we recall their definition that a vertex is said to be singular if the edges meeting at that vertex fall on two straight lines. There are four types of such singular boundary vertices, shown in Fig. 3 of [7]. Following [7], we define $\mathring{\mathcal{P}}^{[r],0}$ to be the space of functions in $C^0(\Omega)$, which are polynomials of degree $\leq r$ on each triangle T and vanish on $\partial\Omega$ and $\mathcal{P}^{[r],-1}$ to be the space of functions ϕ, which are polynomials of degree $\leq r$ on each triangle T and satisfy at any internal singular vertex the condition

$$\sum_{i=1}^{4}(-1)^i\phi(x_0) = 0, \tag{4.1}$$

where $\phi_i(x_0) = \phi|_{T_i}(x_0)$ and $T_1, \cdots, T_4$ are the triangles meeting at x_0, numbered consecutively. We then let $\tilde{\mathcal{P}}^{[r],-1}$ denote the subspace of $\mathcal{P}^{[r],-1}$ consisting of functions ϕ which satisfy the following two additional requirements.

$$\text{At any singular boundary vertex } x_0, \sum_{i=1}^{k}(-1)^i\phi_i(x_0) = 0, \tag{4.2}$$

where $T_1, \cdots, T_k$ are the triangles meeting at x_0, numbered consecutively and

$$\int_\Omega \phi\, dx = 0. \tag{4.3}$$

The results of Scott and Vogelius then require a certain nondegeneracy condition on the triangulations which they decribe as follows. Let x_0 denote any nonsingular vertex and let $\theta_i, 1 \leq i \leq k$, be the angles of the triangles $T_i, 1 \leq i \leq k$, meeting at x_0. Define

$$R(x_0) = \max\{|\theta_i + \theta_j - \pi| : 1 \leq i, j \leq k \text{ and } i - j = 1 \mod k\},$$

where the term $\mod k$ is dropped if x_0 is a boundary vertex. $R(x_0)$ measures how close x_0 is to being singular. Set

$$R_I(\tau_h) = \min\{R(x_0) : x_0 \text{ is a nonsingular internal vertex of } \tau_h\},$$
$$R_B(\tau_h) = \min\{R(x_0) : x_0 \text{ is a nonsingular boundary vertex of } \tau_h\},$$

and

$$\mathring{R}(\tau_h) = \min[R_I(\tau_h), R_B(\tau_h)].$$

The main result of [7] which is relevant to this paper is the following theorem (cf. Theorem 5.2 of [7]).

THEOREM 4.1. *Assume that all internal angles at corners of the polygonal domain Ω are less than 2π. Let τ_h be a quasi-uniform family of triangulations of Ω, and let n be an integer ≥ 4. Assume that $\mathring{R}(\tau_h) \geq \delta > 0$, with δ independent of h. Then*

$$\mathrm{div}(\mathring{\mathcal{P}}^{[n],0}(\tau_h) \times \mathring{\mathcal{P}}^{[n],0}(\tau_h)) = \tilde{\mathcal{P}}^{[n-1],-1}(\tau_h),$$

and there exists a linear operator

$$\mathcal{L}_n^h : \tilde{\mathcal{P}}^{[n-1],-1}(\tau_h) \to \mathring{\mathcal{P}}^{[n],0}(\tau_h) \times \mathring{\mathcal{P}}^{[n],0}(\tau_h),$$

such that

$$\mathrm{div}\, \mathcal{L}_n^h \phi = \phi \quad \forall \phi \in \tilde{\mathcal{P}}^{[n-1],-1}(\tau_h)$$

and

$$\|\mathcal{L}_n^h \phi\|_{1,\Omega} \leq C(n-1)^K \|\phi\|_{0,\Omega}$$

with constants C and K independent of h, n, and ϕ.

Since this result immediately implies the stability condition (1.2) for the spaces of the theorem, we wish to examine it more closely in the case of continuous pressures to see if the mesh restriction $\mathring{R}(\tau_h) \geq \delta > 0$ can be weakened. Examining the proof of this theorem, we note that the only place the mesh restriction is needed is in the proof of Lemma 4.1 of [7] which constructs for every $\phi \in \tilde{\mathcal{P}}^{[n-1],-1}(\tau_h)$ a function $\mathbf{v} \in \mathring{\mathcal{P}}^{[n],0}(\tau_h) \times \mathring{\mathcal{P}}^{[n],0}(\tau_h)$ satisfying

$$\phi - \mathrm{div}\, \mathbf{v} = 0 \quad \text{for all vertices of } \tau_h$$

and

$$\|\mathbf{v}\|_{1,\Omega} \leq C n^K \|\phi\|_{0,\Omega},$$

with constants C and K that are independent of h, n, and ϕ. The mesh restriction is needed since otherwise the constant C given above will become unbounded as $\delta \to 0$. A special case of the proof (cf. (4.2) of [7]) shows in fact that for continuous pressures, the construction given produces a constant C which remains bounded with only the condition $R_B \geq \delta > 0$. In fact, the minimal angle condition implies that $R(x_0) \geq \delta$ at any boundary vertex at which more than four triangles meet and, in the case in which exactly two or four triangles meet at a nonsingular boundary vertex, the construction of the proof (considering triangles pairwise) also produces a bounded constant without requiring $R(x_0) \geq \delta$. Since a boundary vertex involving only one triangle is singular, we only need require the condition $R(x_0) \geq \delta$ at nonsingular boundary vertices at which exactly three triangles meet. We note that from the remarks in [7], this condition can be eliminated if we require $\phi(x_0) = 0$ at such a vertex.

The implication of the above is that under this mild restriction on the mesh and for $n \geq 4$, the stability result (1.2) is valid for all pressures $\phi \in C^0(\Omega) \cap \tilde{\mathcal{P}}^{[n-1],-1}(\tau_h)$. Since (4.3) is already satisfied by functions in Q_h and (4.1) is automatically satisfied for continuous pressures, we need only consider the condition (4.2). For singular vertices with $k = 2$ or 4, (4.2) is again automatically satisfied for continuous pressures. When $k = 1$ or 3, (4.2) implies, in the case of a continuous pressure ϕ, that $\phi(x_0) = 0$. If we seek to satisfy (1.2) by finding a function $\mathbf{v}_h \in \mathbf{V}_h$ for which Theorem 4.1 holds, then this restriction cannot be eliminated, since $\mathbf{v}_h = 0$ on $\partial\Omega$ implies in these cases that $\mathrm{div}\, \mathbf{v}_h = 0$ at x_0.

If we let $X(\tau_h)$ denote the set of boundary vertices x_0, such that x_0 is either a vertex of only one triangle or is a boundary vertex at which exactly three triangles meet and which is either singular or nearly singular in the sense that it satisfies $R(x_0) \leq \delta$ (for some fixed δ independent of h), then the preceding arguments establish the following stability result.

THEOREM 4.2. *Let τ_h be a regular sequence of triangulations of Ω and assume that $X(\tau_h) = \emptyset$. Then for $n \geq 4$, the spaces*

$$\mathbf{V}_h = \{\mathbf{v}_h \in (H_0^1(\Omega))^2 : \mathbf{v}_h|_T \in (P_n)^2 \text{ for all } T \in \tau_h\},$$

$$Q_h = \left\{q_h \in H^1(\Omega) : q_h \in P_{n-1} \text{ for all } T \in \tau_h, \int_\Omega q_h\, dx = 0\right\}$$

satisfy the stability condition (1.2).

Finally, we remark that since for continuous pressures, the construction of a $\mathbf{v}_h$ for which Theorem 4.1 holds is not necessary to satisfy (1.2), it is possible that the hypothesis $X(\tau_h) = \emptyset$ can be further weakened.

REFERENCES

[1] M. BERCOVIER AND O. PIRONNEAU, *Error estimates for finite element solution of the Stokes problem in the primitive variables*, Numer. Math., 33 (1979), pp. 211–224.

[2] F. BREZZI, *On the existence, uniqueness, and the approximation of saddle point problems arising from Lagrangian multipliers*, RAIRO Anal. Numér., 8 (1974), pp. 129–151.

[3] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, in preparation.

[4] P. G. CIARLET, *The Finite Element Method for Elliptic Equations*, North-Holland, Amsterdam, 1978.

[5] V. GIRAULT AND P.-A. RAVIART, *Finite Element Methods for Navier–Stokes Equations*, Springer-Verlag, Berlin, 1986.

[6] P. HOOD AND C. TAYLOR, *A numerical solution of the Navier–Stokes equations using the finite element technique*, Comp. and Fluids, 1 (1973), pp. 73–100.

[7] L. R. SCOTT AND M. VOGELIUS, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials*, RAIRO Anal. Numér., 19 (1985), pp. 111–143.

[8] R. STENBERG, *Error analysis of some finite element methods for the Stokes problem*, INRIA Rapports de Recherche No. 948, 1988; Math. Comp., 54 (1990), pp. 495–548.

[9] R. VERFÜRTH, *Error estimates for a mixed finite element approximation of the Stokes equations*, RAIRO Anal. Numér., 18 (1984), pp. 175–182.