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Approximation of a Class of Optimal Control Problems with Order of Convergence Estimates*

RICHARD S. FALK

Department of Mathematics, Brown University, Providence, Rhode Island 02912†

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An approximation scheme for a class of optimal control problems is presented. An order of convergence estimate is then developed for the error in the approximation of both the optimal control and the solution of the control equation.

1. INTRODUCTION

In this work, we consider the approximation of a class of optimal control problems. More specifically, the control problems considered will be those of systems governed by partial differential equations of elliptic type. The general approach taken will be to approximate the optimal control and the solution of the control equation in such a way that the approximating control problem can be solved by mathematical programming. An error estimate for the approximation of the optimal control and the solution of the control equation is then given in an appropriate norm.

A general outline of the paper is as follows. In Section 2 we give a general formulation of the problem and define the notation to be used. In Section 3 we present an approximation technique and prove a general approximation theorem. The remainder of the paper contains the application of this estimate to a specific problem. In Section 4 we define some function spaces, and in Section 5 prove an a priori estimate for an optimal control problem set in these spaces. Section 6 contains the description of the construction of an approximate problem and the application of the general error estimate developed in Section 3. Finally, in Section 7, we make some comments about the order of convergence estimate obtained in Section 6, and discuss some conditions under which it can be improved.

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† Now at Dept. of Math., Rutgers University, New Brunswick, N.J. 08903.

2. General Formulation of the Problem

The general problem will be formulated in notation similar to that used by J. L. Lions [5]. Let $V \subset H$ be two Hilbert spaces, V dense in H and the injection of V into H continuous. Identify H with its dual space. If V' denotes the dual space of V, then H may be identified with a subspace of V' and we may write $V \subset H \subset V'$, where H is dense in V' and the injection of H into V' continuous. Let U be a Hilbert space of controls, and let $B \in \mathscr{L}(U, V')$, the set of linear operators mapping U into V'. For each $u \in U$, let $y \in V$ be the solution of

$$Ay = f + Bu$$

where $A \in \mathscr{L}(V, V')$ and $f \in V'$. Since y depends on u, write y as $y(u)$. Let $Z(u) = Cy(u)$ be an observation of $y(u)$, where $C \in \mathscr{L}(V, \mathscr{H})$, $\mathscr{H}$ a Hilbert space of observations. Let $N \in \mathscr{L}(U, U)$, N self-adjoint, positive definite, and satisfying $(Nu, u)_U \geqslant \nu \| u \|_U^2$ for all $u \in U$ and some constant $\nu > 0$.

For each $u \in U$, we associate the functional

$$J(u) = \| Cy(u) - Z_d \|_{\mathscr{H}}^2 + (Nu, u)_U ,$$

where Z_d is given in $\mathscr{H}$. The control problem is then: Find $u \in K$ such that $J(u) = \inf_{v \in K} J(v)$, where K is a closed convex subset of U. Lions has shown in [5] that this problem is equivalent to the variational inequality: Find $u \in K$ such that

$$(Cy(u) - Z_d , C[y(v) - y(u)])_{\mathscr{H}} + (Nu, v - u)_U \geqslant 0 \qquad \forall v \in K.$$

We denote these equivalent formulations of the control problem as Problem (P).

Once again following Lions, we let $\mathscr{H}'$ be the dual of $\mathscr{H}$ and set $\Lambda = \Lambda_{\mathscr{H}}$, the canonical isomorphism of $\mathscr{H} \to \mathscr{H}'$. Let $C^* \in \mathscr{L}(\mathscr{H}', V')$ denote the adjoint of C, and $A^* \in \mathscr{L}(V, V')$ denote the adjoint of A. Finally, for each control $v \in U$, we define the function $p(v) \in V$ as the solution of

$$A^* p(v) = C^* \Lambda (Cy(v) - Z_d).$$

Using this notation, we are now ready to proceed with the development of the approximation result.

3. An Approximation Technique, An Approximation Theorem, and A Priori Estimates

The method we will employ to approximate the solution to this problem proceeds as follows. Let U_k be a finite-dimensional subspace of U. Write an arbitrary element $v_k \in U_k$ as $\sum_{i=1}^q \alpha_i w_i$, where $\{w_i\}_{i=1}^q$, are a basis for U_k.

Construct a closed convex subset K_k of U_k such that the following conditions are satisfied:

(1) K_k should reduce to a finite number of constraints on the α_i.

(2) K_k should be a "good" approximation to K in a sense to be made clearer in Section 6.

Now pick V_h, a finite-dimensional subspace of a Hilbert space containing the solution of the control equation $y(u)$. Using some method for the numerical solution of elliptic partial differential equations, which we denote method M, obtain approximate solutions $y^h(w_i - f) \in V_h$ to the problems

$$Ay(w_i - f) = Bw_i \qquad i = 1,..., q.$$

Also using method M, obtain an approximate solution $y^h(0) \in V^h$ to the problem

$$Ay(0) = f.$$

For an arbitrary element

$$v_k = \sum_{i=1}^{q} \alpha_i w_i \in U_k$$

define $y^h(v_k) \in V_h$ by

$$y^h(v_k) = y^h(0) + \sum_{i=1}^{q} \alpha_i y^h(w_i - f).$$

Now solve the minimization problem:

Problem (P_k^h)

Find $u_k^h \in K_k$ such that

$$J_h(u_k^h) = \inf_{v_k \in K_k} J_h(v_k),$$

where

$$J_h(v_k) = \| Cy^h(v_k) - Z_d \|_{\mathscr{H}}^2 + (Nv_k, v_k)_U .$$

Note that if assumption (1) is satisfied, this problem simply becomes one of finding the minimum of a quadratic form in α_i, subject to a finite number of constraints on the α_i, a nonlinear programming problem. In the event the constraints are linear, we have a quadratic programming problem.

We would now like to have some estimate for the error that we make in obtaining an approximate solution instead of the true solution. To obtain such an estimate we first observe that Problem (P_k^h) can also be equivalently written as a variational inequality, i.e., Find $u_k^h \in K_k$ such that

$$(Cy^h(u_k^h) - Z_d, C[y^h(v_k) - y^h(u_k^h)])_{\mathscr{H}} + (Nu_k^h, v_k - u_k^h)_U \geqslant 0 \qquad \forall v_k \in K_k .$$

Since the two formulations are equivalent we also denote this variational inequality as Problem (P_k^h). Then we have the following approximation result.

THEOREM 1. *Let u and u_k^h be the respective solutions of Problems (P) and (P_k^h). Then*

$$\begin{aligned}
\nu \| u - u_k^h \|_U^2 + \| C[y(u) - y^h(u_k^h)] \|_{\mathscr{H}}^2
&\leqslant (Nu, v - u_k^h)_U + (Nu, v_k - u)_U + (N(u_k^h - u), v_k - u)_U \\
&\quad + (Cy(u) - Z_d, C[y(v) - y(u_k^h)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\quad + (Cy(u) - Z_d, C[y(u_k^h) - y^h(u_k^h)] + C(y^h(v_k) - y(v_k)])_{\mathscr{H}} \\
&\quad + (C[y^h(u_k^h) - y(u)], C[y^h(v_k) - y(v_k)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\qquad \forall v \in K, \qquad \forall v_k \in K_k .
\end{aligned} \tag{3}$$

Proof. Adding the two variational inequalities, we obtain

$$\begin{aligned}
&(Cy(u) - Z_d, C[y(v) - y(u)])_{\mathscr{H}} \\
&\quad + (Cy^h(u_k^h) - Z_d, C[y^h(v_k) - y^h(u_k^h)])_{\mathscr{H}} \\
&\quad + (Nu, v - u)_U + (Nu_k^h, v_k - u_k^h)_U \geqslant 0 \qquad \forall v \in K, \quad v_k \in K_k .
\end{aligned}$$

Now

$$\begin{aligned}
&(Nu, v - u)_U + (Nu_k^h, v_k - u_k^h)_U \\
&\quad = (Nu, v - u_k^h)_U + (Nu, u_k^h - u)_U \\
&\qquad + (Nu_k^h, v_k - u)_U + (Nu_k^h, u - u_k^h)_U \\
&\quad = (N(u - u_k^h), u_k^h - u)_U + (Nu, v - u_k^h)_U \\
&\qquad + (Nu, v_k - u)_U + (N(u_k^h - u), v_k - u)_U .
\end{aligned}$$

Also

$$\begin{aligned}
&(Cy(u) - Z_d, C[y(v) - y(u)])_{\mathscr{H}} + (Cy^h(u_k^h) - Z_d, C[y^h(v_k) - y^h(u_k^h)])_{\mathscr{H}} \\
&\quad = (Cy(u) - Z_d, C[y(v) - y(u_k^h)] + C[y(u_k^h) - y^h(u_k^h)] \\
&\qquad + C[y^h(u_k^h) - y(u)])_{\mathscr{H}} \\
&\qquad + (Cy(u_k^h) - Z_d, C[y^h(v_k) - y(v_k)] + C[y(v_k) - y(u)] \\
&\qquad + C[y(u) - y^h(u_k^h)])_{\mathscr{H}} \\
&\quad = (Cy(u) - Z_d, C[y(v) - y(u_k^h)] + C[y(u_k^h) - y^h(u_k^h)])_{\mathscr{H}} \\
&\qquad + (Cy^h(u_k^h) - Z_d, C[y^h(v_k) - y(v_k)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\qquad + (C[y(u) - y^h(u_k^h)], C[y^h(u_k^h) - y(u)])_{\mathscr{H}} \\
&\quad = (Cy(u) - Z_d, C[y(v) - y(u_k^h)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\qquad + (Cy(u) - Z_d, C[y(u_k^h) - y^h(u_k^h)] + C[y^h(v_k) - y(v_k)])_{\mathscr{H}} \\
&\qquad + (C[y^h(u_k^h) - y(u)], C[y^h(v_k) - y(v_k)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\qquad + (C[y(u) - y^h(u_k^h)], C[y^h(u_k^h) - y(u)])_{\mathscr{H}} .
\end{aligned}$$

Using the fact that

$$(Nv, v)_U \geqslant \nu \| v \|_U^2 \qquad \forall v \in U,$$

we obtain

$$\begin{aligned}
\nu \| u - u_k^h \|_U^2 &+ \| C[y(u) - y^h(u_k^h)] \|_{\mathscr{H}}^2 \\
&\leqslant (Nu, v - u_k^h)_U + (Nu, v_k - u)_U + (N(u_k^h - u), v_k - u)_U \\
&\quad + (Cy(u) - Z_d, C[y(v) - y(u_k^h)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\quad + (Cy(u) - Z_d, C[y(u_k^h) - y^h(u_k^h)] + C[y^h(v_k) - y(v_k)])_{\mathscr{H}} \\
&\quad + (C[y^h(u_k^h) - y(u)], C[y^h(v_k) - y(v_k)] + C[y(v_k) - y(u)])_{\mathscr{H}} \\
&\qquad \forall v \in K \quad \text{and} \quad \forall v_k \in K_k .
\end{aligned}$$

In applying Theorem 1, we will need a priori estimates for the optimal control u and the solution of the control equation $y(u)$. These follow immediately from the definition of u, i.e., $J(u) \leqslant J(v)$, $\forall v \in K$. Then

$$\| Cy(u) - Z_d \|_{\mathscr{H}}^2 + \nu \| u \|_U^2 \leqslant \| Cy(v) - Z_d \|_{\mathscr{H}}^2 + (Nv, v)_U , \qquad \forall v \in K. \quad (4)$$

Before we can demonstrate the application of Theorem 1, we will first define some function spaces and then reformulate the problem in these spaces.

4. Some Function Spaces

Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We shall assume (for convenience) that $\partial\Omega$ is of class C^∞. Let m be a nonnegative integer, and let $C^\infty(\bar{\Omega})$ denote the set of infinitely differentiable functions on $\bar{\Omega}$. Then $H^m(\Omega)$ will denote the completion of $C^\infty(\bar{\Omega})$ in the norm

$$\| \phi \|_m = \Big(\sum_{|\alpha| \leqslant m} \| D^\alpha \phi \|_0^2 \Big)^{1/2},$$

where

$$\| \phi \|_0 = \Big(\int_\Omega | \phi |^2 \, dx \Big)^{1/2}.$$

Let $C_0^\infty(\Omega)$ be the set of infinitely differentiable functions with compact support in Ω, and denote the completion of $C_0^\infty(\Omega)$ in the above norm by $H_0^m(\Omega)$.

For m a negative integer we define $H^m(\Omega)$ as the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\| \phi \|_m = \sup_{\psi \in C^\infty(\bar{\Omega})} \frac{(\phi, \psi)_0}{\| \psi \|_{-m}},$$

where

$$(\phi, \psi)_0 = \int_\Omega \phi\psi \, dx.$$

For m a negative integer, we also define the space $H_*{}^m(\Omega)$ as the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\| \phi \|_m^* = \sup_{\psi \in C_0{}^\infty(\Omega)} \frac{(\phi, \psi)_0}{\| \psi \|_{-m}}.$$

We note that $H_*{}^m(\Omega) = (H_0^{-m}(\Omega))'$, the dual space of $H_0^{-m}(\Omega)$. Clearly, $H^m(\Omega) \subset H_*{}^m(\Omega)$ for m a negative integer.

With this notation, we will now consider a system governed by the Dirichlet problem with distributed control. Set $V = H_0{}^1(\Omega)$, $H = L^2(\Omega)$, and $V' = H_*^{-1}(\Omega)$. Let $y(u)$ be the solution of the Dirichlet problem

$$\begin{aligned} Ay(u) &= f + u \qquad \text{in } \Omega, \\ y(u) &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where

$$Ay = -\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 y}{\partial x_i \, \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial y}{\partial x_i} + a(x)\, y.$$

We will assume the following two conditions are satisfied.

(6) A is uniformly elliptic with coefficients in $C^\infty(\bar{\Omega})$.

(7) The only solution of problem (5) in $C^\infty(\bar{\Omega})$ with zero data (i.e., $f + u \equiv 0$) is the zero solution.

We now take both U, the Hilbert space of controls and $\mathscr{H}$, the Hilbert space of observations to be $L^2(\Omega)$. Then in terms of our original notation, we have B is the injection of U into V', i.e., $L^2(\Omega) \to H_*^{-1}(\Omega)$, C is the injection of V into $\mathscr{H}$, i.e., $H_0{}^1(\Omega) \to L^2(\Omega)$, and Λ is the identity mapping. Finally, for simplicity, let the mapping $N = \nu I$, where I is the identity mapping.

In this setting, our control problem becomes: Find $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v),$$

where

$$J(v) = \int_\Omega [(y(v) - Z_d)^2 + \nu v^2] \, dx,$$

and K is a closed convex subset of $L^2(\Omega)$.

Notational Remark. In the sections which follow, we will now use the letter C to denote a generic constant, not necessarily the same in any two places.

5. Choice of a Specific Convex Set and the Regularity of the Solution of the Corresponding Optimal Control Problem

We now consider the control problem just defined for a specific choice of the convex set K, namely

$K = \{v \in L^2(\Omega)\colon \xi_0(x) \leqslant v(x) \leqslant \xi_1(x)$ a.e. in Ω, where ξ_0 and ξ_1 are given functions in $L^\infty(\Omega)\}$.

For $f \in H_*^{-1}(\Omega)$ and $Z_d \in L^2(\Omega)$, we have for each $v \in L^2(\Omega)$, the functions $y(v)$ and $p(v) \in H_0^1(\Omega)$ defined as the respective solutions of

$$\begin{aligned} Ay(v) &= f + v \qquad && \text{in } \Omega, \\ y(v) &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{8}$$

$$\begin{aligned} A^*p(v) &= y(v) - Z_d \qquad && \text{in } \Omega, \\ p(v) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{9}$$

We also wish to define what is meant by a weak solution of (8) or (9) under various other assumptions on the regularity of f, v and Z_d. To do so, we apply the following result, which we state only for the special cases in which it will be used. (For the general result, see e.g. Schechter [7].)

Let w be the solution of the problem

$$\begin{aligned} Aw &= F \qquad && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{10}$$

where A is uniformly elliptic with coefficients in $C^\infty(\bar{\Omega})$. Then if zero is the only solution of (10) in $C^\infty(\bar{\Omega})$ when $F = 0$, we have for $s = 1, 2,$ and 3 that

$$\| w \|_s \leqslant C[\| F \|_{s-2}] \qquad \forall w \in C^\infty(\bar{\Omega}), \tag{11}$$

where C is a constant independent of w and F.

Now for any $F \in H^{s-2}(\Omega)$, let $\{F_n\} \in C^\infty(\bar{\Omega})$ converge to F in $H^{s-2}(\Omega)$ as $n \to \infty$. Also, let $w_n \in C^\infty(\bar{\Omega})$ be the corresponding solution of (10) with data F_n (it is well known that such a solution exists and is unique). Then, using (11), we define the weak solution of (10) to be the unique limit in $H^s(\Omega)$ of the sequence $\{w_n\}$.

We are now ready to proceed with the discussion of the regularity of the solution of the optimal control problem. J. L. Lions has shown in [5] that the optimal control u for this problem is given by

$$\Phi(\xi_0, \xi_1, \nu)\, p = \begin{cases} -(1/\nu)\, p(x) & \text{if} \quad \xi_0(x) \leqslant -(1/\nu)\, p(x) \leqslant \xi_1(x), \\ \xi_0(x) & \text{if} \quad -(1/\nu)\, p(x) < \xi_0(x), \\ \xi_1(x) & \text{if} \quad -(1/\nu)\, p(x) > \xi_1(x), \end{cases}$$

where $p(x) = p(x; u)$ is the solution of the adjoint equation, and that if $\xi_i \in L^\infty(\Omega) \cap H^1(\Omega)$, then $u \in H^1(\Omega)$.

Since we know the form of the optimal control u, we are able to prove the following simple result.

LEMMA 2. *Suppose that ξ_0 and $\xi_1 \in L^\infty(\Omega) \cap H^1(\Omega)$. Then*

$$\| u \|_1 \leqslant \{(1/\nu^2) \| p \|_1^2 + \| \xi_0 \|_1^2 + \| \xi_1 \|_1^2\}^{1/2}.$$

Proof. Define sets Ω_1, Ω_2, and Ω_3 contained in Ω as follows:

$$\Omega_1 = \{x \in \Omega: \xi_0(x) \leqslant -(1/\nu)\, p(x) \leqslant \xi_1(x)\},$$
$$\Omega_2 = \{x \in \Omega: -(1/\nu)\, p(x) < \xi_0(x)\},$$
$$\Omega_3 = \{x \in \Omega: -(1/\nu)\, p(x) > \xi_1(x)\}.$$

Then

$$\| u \|_1^2 = \| u \|_{1,\Omega_1}^2 + \| u \|_{1,\Omega_2}^2 + \| u \|_{1,\Omega_3}^2$$
$$= (1/\nu)^2 \| p \|_{1,\Omega_1}^2 + \| \xi_0 \|_{1,\Omega_2}^2 + \| \xi_1 \|_{1,\Omega_3}^2$$
$$\leqslant (1/\nu)^2 \| p \|_1^2 + \| \xi_0 \|_1^2 + \| \xi_1 \|_1^2.$$

Hence

$$\| u \|_1 \leqslant \{(1/\nu^2) \| p \|_1^2 + \| \xi_0 \|_1^2 + \| \xi_1 \|_1^2\}^{1/2}.$$

If f, Z_d, and $u \in H^{-1}(\Omega)$, we have by (11) and the definition following it, the estimates

$$\| y(u) \|_1 \leqslant C \| f + u \|_{-1} \leqslant C[\| f \|_{-1} + \| u \|_{-1}]$$

and

$$\begin{aligned}\| p(u)\|_1 &\leqslant C \| y(u) - Z_d \|_{-1} \\ &\leqslant C[\| y(u)\|_{-1} + \| Z_d \|_{-1}] \\ &\leqslant C[\| f \|_{-1} + \| Z_d \|_{-1} + \| u \|_{-1}].\end{aligned}$$

Applying Lemma 2 gives the estimate

$$\| u \|_1 \leqslant \{(C^2/\nu^2) \| f \|_{-1} + \| Z_d \|_{-1} + \| u \|_{-1}]^2 + \| \xi_0 \|_1^2 + \| \xi_1 \|_1^2\}^{1/2}. \quad (12)$$

Since $\| u \|_{-1} \leqslant \| u \|_0$, we need only apply (4) to obtain on a priori estimate for $\| u \|_1$ in terms of the data $\nu, f, Z_d, \xi_0, \xi_1$, and K.

Note. We shall assume for the remainder of the paper that f is given in $L^2(\Omega)$ and ξ_0 and ξ_1 are given in $L^\infty(\Omega) \cap H^1(\Omega)$.

6. Construction of the Approximating Convex Sets and Application of the Error Estimate

We begin by defining some finite-dimensional subspaces of $L^2(\Omega)$. Let k, $0 < k < 1$, be a parameter. For a given value of k suppose that $\Omega_k{}^j, j \in J_k$, are domains satisfying the following:

(i) $\Omega_k{}^i \cap \Omega_k{}^j = \phi, \forall i, j \in J_k$,

(ii) $\bar{\Omega} = \bigcup_{j \in J_k} \overline{\Omega_k{}^j}$,

(iii) Given a function $\phi \in H^1(\Omega_k{}^j)$, $\exists$ a constant C independent of k, j, and $\phi \ni$

$$\left\| \phi - \frac{1}{\mu(\Omega_k{}^j)} \int_{\Omega_k{}^j} \phi \, dx \right\|_{L^2(\Omega_k{}^j)} \leqslant Ck \| \phi \|_{H^1(\Omega_k{}^j)}.$$

We remark that sufficient conditions for (iii) to hold are that the domains $\Omega_k{}^j$ be convex and satisfy the conditions that $\operatorname{diam}^N(\Omega_k{}^j)/\mu(\Omega_k{}^j) \leqslant C$ where C is a constant independent of k and j, and $\operatorname{diam}(\Omega_k{}^j) \leqslant C'k$ where C' is a constant independent of k and j. For a proof, see Stampacchia [8]. (Note that this last inequality gives a geometric significance to the parameter k, by relating k to the diameters of the elements into which Ω is divided.)

Now define functions $\Phi_k{}^j$: $\mathbb{R}^N \to R$ by

$$\Phi_k{}^j(x) = 1 \quad \text{if} \quad x \in \Omega_k{}^j,$$

$$\Phi_k{}^j(x) = 0 \quad \text{if} \quad x \notin \Omega_k{}^j,$$

i.e., Φ_k^j is the characteristic function of the domain Ω_k^j. We now define a space of piecewise constant functions $S_k(\Omega)$ as

$$\left\{s_k: s_k(x) = \sum_{j \in J_k} s_k^j \Phi_k^j(x), \text{ where } s_k^j \in \mathbb{R}\right\}.$$

Clearly, since Ω is a bounded domain in $\mathbb{R}^N$, $S_k(\Omega)$ is a finite dimensional subspace of $L^2(\Omega)$.

Finally, we define convex sets K_k, approximations to the convex set K by:

$$K_k = \left\{v_k \in S_k(\Omega): \frac{1}{\mu(\Omega_k^j)} \int_{\Omega_k^j} \xi_0 \, dx \leqslant v_k \leqslant \frac{1}{\mu(\Omega_k^j)} \int_{\Omega_k^j} \xi_1 \, dx \text{ on } \Omega_k^j, \forall j \in J_k\right\}.$$

To simplify notation in what follows, define constants

$$M_k^j(\varphi) = \frac{1}{\mu(\Omega_k^j)} \int_{\Omega_k^j} \varphi \, dx$$

for any $\varphi \in H^1(\Omega_k^j)$. Then K_k may be written as

$$\{v_k \in S_k(\Omega): M_k^j(\xi_0) \leqslant v_k \leqslant M_k^j(\xi_1) \text{ on } \Omega_k^j, \forall j \in J_k\}.$$

We observe that with this definition of K_k, condition (1) of Section 3 is satisfied. Furthermore, since the constraints comprising K_k are linear, the approximate solution u_k^h can be found by solving a quadratic programming problem (see Section 3).

We now recall that the approximation procedure described in Section 3 also involves obtaining approximate solutions by some method M to the control equation with various right sides. Suppose that we choose for M a "Rayleigh–Ritz–Galerkin" method.

A method of this type for approximating the solution of (8), for example, may be described generally as follows. Let S be a Sobolev space containing the solution $y(v)$ of (8). If V_h is a finite dimensional subspace of S, we define the approximate solution $y^h(v)$ as a projection of $y(v)$ onto V_h, where the projection is taken in such a way that the approximate solution $y^h(v)$ is computable from the data of the original problem (8).

Part of the Rayleigh–Ritz–Galerkin method consists of the construction of finite dimensional subspaces of S having certain "good" approximation properties. Typically, we have the following situation. Let h, $0 < h < 1$ be a parameter, and H any fixed hypercube containing our domain $\bar{\Omega}$. For m and r

any non-negative integers satisfying $m < r$, let $\{S^h_{m,r}(H)\}$ be any one parameter family of finite dimensional subspaces of $H^m(H)$ (with norm $\|\cdot\|^H_m$) which satisfies the following condition:

(13) For any $y \in H^j(H)$, $\exists$ a constant C independent of h and y such that

$$\inf_{\chi \in S^h_{m,r}} \| y - \chi \|^H_l \leqslant C h^{j-l} \| y \|^H_j$$

$$0 \leqslant l \leqslant m \qquad \text{and} \qquad l \leqslant j \leqslant r.$$

With a condition such as (13), or other similar approximability conditions, a typical error estimate for a Rayleigh–Ritz–Galerkin method for the approximation of (8) will have the form

$$\| y(v) - y^h(v) \|_{L^2(\Omega)} \leqslant C h^\gamma \| f + v \|_{L^2(\Omega)}, \tag{14}$$

where γ is a constant satisfying $0 < \gamma \leqslant 2$ depending on the choice of method, and C is a constant independent of h and $(f + v)$.

For a further discussion of some of these methods, see for example, the papers of Babuška [1], Bramble and Schatz [3], and Strang [9]. Additional references can be found in the bibliographies of these papers.

We remark for readers generally unfamiliar with these methods that an example of subspaces satisfying condition (13) is given by spline functions defined on uniform meshes of width h.

Applying the approximation procedure described in Section 3 with the convex set K_k we have constructed and a Rayleigh–Ritz–Galerkin method satisfying estimate (14), we are able to state the following approximation result.

THEOREM 2. *There exists a constant C depending only on the data f, ξ_0 and ξ_1, such that*

$$(\nu/2) \| u - u_k^h \|^2 + \tfrac{1}{2} \| y(u) - y^h(u_k^h) \|^2 \leqslant C[k^2 + h^\gamma].$$

The proof of Theorem 2 depends on the following approximation results which we now prove.

LEMMA 3. *Let u be the solution of Problem (P) and v_k be given by $\sum_{j \in J_k} M_k^j(u)\, \Phi_k^j(x)$. Then $v_k \in K_k$ and $\exists$ a constant C independent of u and k such that*

$$\| u - v_k \|_{L^2(\Omega)} \leqslant C k \| u \|_{H^1(\Omega)}, \tag{15}$$

$$\| u - v_k \|_{H^{-1}(\Omega)} \leqslant C k^2 \| u \|_{H^1(\Omega)}. \tag{16}$$

Proof. Clearly $v_k \in S_k(\Omega)$. Since $u \in K$, $\xi_0(x) \leqslant u(x) \leqslant \xi_1(x)$ a.e. in Ω. Hence,

$$M_k{}^j(\xi_0) \leqslant M_k{}^j(u) \leqslant M_k{}^j(\xi_1) \qquad \forall j \in J_k .$$

Since $v_k(x) = M_k{}^j(u)$ on $\Omega_k{}^j$, $v_k \in K_k$.

Since the $\Omega_k{}^j$ satisfy the condition (iii), we have

$$\| u - v_k \|_{L^2(\Omega_k{}^j)} \leqslant Ck \| u \|_{H^1(\Omega_k{}^j)} \qquad \forall j \in J_k .$$

Squaring and summing over $\forall j \in J_k$ we obtain

$$\| u - v_k \|_{L^2(\Omega)} \leqslant Ck \| u \|_{H^1(\Omega)} .$$

To obtain an estimate for $\| u - v_k \|_{-1}$, we use the fact that the element v_k we have constructed is actually the best approximation to u in $L^2(\Omega)$ by all elements in $S_k(\Omega)$. To see this, just observe that on each $\Omega_k{}^j$, v_k is the solution of the problem: minimize $\| u - c \|_{L^2(\Omega_k{}^j)}$ over all constants c. It is easy to see that $c = M_k{}^j(u)$ solves this problem. Hence by the characterization of best approximations,

$$(u - v_k , s_k) = 0 \qquad \forall s_k \in S_k(\Omega).$$

Then

$$\| u - v_k \|_{-1} = \sup_{\chi \in C^\infty(\bar{\Omega})} \frac{(u - v_k , \chi)}{\| \chi \|_1}$$

$$= \sup_{\chi \in C^\infty(\bar{\Omega})} \frac{(u - v_k , \chi - s_k)}{\| \chi \|_1} \qquad \forall s_k \in S_k(\Omega)$$

$$\leqslant \sup_{\chi \in C^\infty(\bar{\Omega})} \frac{\| u - v_k \| \, \| \chi - s_k \|}{\| \chi \|_1} \qquad \forall s_k \in S_k(\Omega).$$

Choosing

$$s_k = \sum_{j \in J_k} M_k{}^j(\chi) \, \Phi_k{}^j(x),$$

and again applying condition (iii), we have

$$\| \chi - s_k \|_{L^2(\Omega)} \leqslant Ck \| \chi \|_{H^1(\Omega)} .$$

Hence

$$\| u - v_k \|_{-1} \leqslant \sup_{\chi \in C^\infty(\bar{\Omega})} \frac{Ck \| u \|_1 \, Ck \| \chi \|_1}{\| \chi \|_1} \leqslant Ck^2 \| u \|_1 .$$

LEMMA 4. *Let* s_k *be the function* $\sum_{j\in J_k} s_k{}^j\Phi_k{}^j(x)$. *Suppose* $s_k \in K_k$. *Then*

$$\exists v^* \in K \cap \bigcap_{j\in J_k} H^1(\Omega_k{}^j) \ni M_k{}^j(v^*) = s_k{}^j$$

and

$$\| v^* \|_{H^1(\Omega_k{}^j)} \leqslant (\| \xi_0 \|^2_{H^1(\Omega_k{}^j)} + \| \xi_1 \|^2_{H^1(\Omega_k{}^j)})^{1/2} \qquad \textit{for } j \in J_k .$$

Proof. Given any set of constants $c = \{c_j\}_{j\in J_k}$ define a function $v_k(c, x)$ on Ω by

$$v_k(c, x) = \begin{Bmatrix} \xi_0(x) & c_j < \xi_0(x) & \\ c_j & \xi_0(x) \leqslant c_j \leqslant \xi_1(x) & \text{for } x \in \Omega_k{}^j \\ \xi_1(x) & c_j > \xi_1(x) & \end{Bmatrix}.$$

Clearly $v_k(c, x) \in K$ for all values of c. Furthermore, since ξ_0 and $\xi_1 \in H^1(\Omega)$, $v_k(c, x) \in H^1(\Omega_k{}^j)$, $\forall j \in J_k$. Since

$$| v_k(c, x)| \leqslant \max(| \xi_0(x)| , | \xi_1(x)|)$$

and

$$\left| \frac{\partial v_k(c, x)}{\partial x_i} \right| \leqslant \max \left(\left| \frac{\partial \xi_0(x)}{\partial x_i} \right|, \left| \frac{\partial \xi_1(x)}{\partial x_i} \right| \right),$$

it follows easily that

$$\| v_k(c, x)\|_{H^1(\Omega_k{}^j)} \leqslant (\| \xi_0 \|^2_{H^1(\Omega_k{}^j)} + \| \xi_1 \|^2_{H^1(\Omega_k{}^j)})^{1/2}.$$

For any c_j, define a function

$$G(c_j) = \int_{\Omega_k{}^j} v_k(c, x)\, dx - \mu(\Omega_k{}^j)\, s_k{}^j.$$

Since

$$| v_k(c, x) - v_k(\bar{c}, x)| \leqslant | c_j - \bar{c}_j | \qquad \forall x \in \Omega_k{}^j, G(c_j)$$

is a continuous function of c_j. Define constants

$$m_0 = - \| \xi_0 \|_{L^\infty(\Omega)} - 1 \qquad \text{and} \qquad m_1 = \| \xi_1 \|_{L^\infty(\Omega)} + 1.$$

(Recall that ξ_0 and $\xi_1 \in L^\infty(\Omega)$). Then

$$G(m_0) = \int_{\Omega_k{}^j} \xi_0(x)\, dx - \mu(\Omega_k{}^j)\, s_k{}^j \leqslant 0$$

since $s_k \in K_k$, i.e., $s_k{}^j \geq M_k{}^j(\xi_0)$. Similarly,

$$G(m_1) = \int_{\Omega_k{}^j} \xi_1(x)\, dx - \mu(\Omega_k{}^j)\, s_k{}^j \geq 0$$

(since $s_k{}^j \leq M_k{}^j(\xi_1)$). Since $G(c_j)$ is a continuous function,

$$\exists c_j{}^* \in [m_0, m_1] \ni G(c_j) = 0,$$

i.e., $M_k{}^j(v_k(c^*, x)) = s_k{}^j$. If $c^* = \{c_j{}^*\}_{j \in J_k}$, then $v^* = v_k(c^*, x)$ satisfies the conclusion of the lemma.

LEMMA 5. *Let $u_k{}^h$ be the solution of Problem $(P_k{}^h)$ and v the v^* given by Lemma 4 when $s_k = u_k{}^h$. Then $\exists$ a constant C independent of ξ_0, ξ_1 and k such that*

$$\| u_k{}^h - v \|_{-1} \leq Ck^2[\| \xi_0 \|_1^2 + \| \xi_1 \|^2]^{1/2}. \tag{17}$$

Proof. From the definition of $\| \cdot \|_{-1}$, we have

$$\| u_k{}^h - v \|_{-1} = \sup_{\chi \in C^\infty(\bar{\Omega})} \frac{(u_k{}^h - v, \chi)_0}{\| \chi \|_1} = \sup_{\chi \in H^1(\Omega)} \frac{(u_k{}^h - v, \chi)_0}{\| \chi \|_1}$$

$$= \sup_{\chi \in H^1(\Omega)} \sum_{j \in J_k} \frac{(u_k{}^h - v, \chi)_{L^2(\Omega_k{}^j)}}{\| \chi \|_1}.$$

Now define functions $\phi_k{}^j, j \in J_k$ to be the solutions in $H^1(\Omega_k{}^j)$ of the variational problems

$$(\phi_k{}^j, \psi)_{H^1(\Omega_k{}^j)} = (u_k{}^h - v, \psi)_{L^2(\Omega_k{}^j)} \qquad \forall \psi \in H^1(\Omega_k{}^j).$$

Then

$$\| u_k{}^h - v \|_{-1} = \sup_{\chi \in H^1(\Omega)} \sum_{j \in J_k} \frac{(\phi_k{}^j, \chi)_{H^1(\Omega_k{}^j)}}{\| \chi \|_1}$$

$$\leq \sup_{\chi \in H^1(\Omega)} \sum_{j \in J_k} \frac{\| \phi_k{}^j \|_{H^1(\Omega_k{}^j)} \| \chi \|_{H^1(\Omega_k{}^j)}}{\| \chi \|_1}$$

$$\leq \sup_{\chi \in H^1(\Omega)} \frac{\left(\sum_{j \in J_k} \| \phi_k{}^j \|^2_{H^1(\Omega_k{}^j)} \right)^{1/2} \left(\sum_{j \in J_k} \| \chi \|^2_{H^1(\Omega_k{}^j)} \right)^{1/2}}{\| \chi \|_1}$$

$$= \left(\sum_{j \in J_k} \| \phi_k{}^j \|^2_{H^1(\Omega_k{}^j)} \right)^{1/2}.$$

Now from the definition of $\phi_k{}^j$, we have that

$$\| \phi_k{}^j \|^2_{H^1(\Omega_k{}^j)} = (u_k{}^h - v, \phi_k{}^j)_{L^2(\Omega_k{}^j)}$$

and

$$(\phi_k{}^j, 1)_{H^1(\Omega_k{}^j)} = (u_k{}^h - v, 1)_{L^2(\Omega_k{}^j)} .$$

The latter inequality may be rewritten as

$$\int_{\Omega_k{}^j} \phi_k{}^j \, dx = \int_{\Omega_k{}^j} (u_k{}^h - v) \, dx.$$

Hence we have

$$\begin{aligned} \| \phi_k{}^j \|^2_{H^1(\Omega_k{}^j)} &= \int_{\Omega_k{}^j} (u_k{}^h - v) \, \phi_k{}^j \, dx \\ &= \int_{\Omega_k{}^j} (u_k{}^h - v) \left[\phi_k{}^j - \frac{1}{\mu(\Omega_k{}^j)} \int_{\Omega_k{}^j} \phi_k{}^j \, dx \right] dx \\ &\quad + \frac{1}{\mu(\Omega_k{}^j)} \left[\int_{\Omega_k{}^j} (u_k{}^h - v) \, dx \right]^2. \end{aligned}$$

By the definition of v we have

$$\| \varphi_k{}^j \|^2_{H^1(\Omega_k{}^j)} = \int_{\Omega_k{}^j} [M_k{}^j(v) - v] \, [\varphi_k{}^j - M_k{}^j(\varphi_k{}^j)] \, dx.$$

Applying the Schwartz inequality, we obtain

$$\| \varphi_k{}^j \|^2_{H^1(\Omega_k{}^j)} \leqslant \left\{ \int_{\Omega_k{}^j} | M_k{}^j(v) - v |^2 \, dx \right\}^{1/2} \left\{ \int_{\Omega_k{}^j} | \varphi_k{}^j - M_k{}^j(\varphi_k{}^j) |^2 \, dx \right\}^{1/2}.$$

Applying condition (iii) we have that

$$\| \varphi_k{}^j \|^2_{H^1(\Omega_k{}^j)} \leqslant Ck \, \| v \|_{H^1(\Omega_k{}^j)} \, Ck \, \| \varphi_k{}^j \|_{H^1(\Omega_k{}^j)}$$

or that

$$\| \varphi_k{}^j \|_{H^1(\Omega_k{}^j)} \leqslant Ck^2 \, \| v \|_{H^1(\Omega_k{}^j)} \leqslant Ck^2 (\| \xi_0 \|^2_{H^1(\Omega_k{}^j)} + \| \xi_1 \|^2_{H^1(\Omega_k{}^j)})^{1/2}$$

by Lemma 4. Hence

$$\| u_k{}^h - v \|_{-1} \leqslant \left(\sum_{j \in J_k} Ck^4 [\| \xi_0 \|^2_{H^1(\Omega_k{}^j)} + \| \xi_1 \|^2_{H^1(\Omega_k{}^j)}] \right)^{1/2}$$

or finally,

$$\| u_k{}^h - v \|_{-1} \leqslant Ck^2 [\| \xi_0 \|_1^2 + \| \xi_1 \|_1^2]^{1/2}.$$

With the aid of these three lemmas, we are now ready to prove Theorem 2.

Proof (Theorem 2). In the setting in which we are considering the general problem, Theorem 1 gives the estimate

$$\begin{aligned}\nu \| u - u_k^h \|^2_{L^2(\Omega)} &+ \| y(u) - y^h(u_k^h)\|^2_{L^2(\Omega)} \\ &\leqslant \nu(u, v - u_k^h) + \nu(u, v_k - u) + \nu(u_k^h - u, v_k - u) \\ &\quad + (y(u) - Z_d\,, [y(v) - y(u_k^h)] + [y(v_k) - y(u)]) \\ &\quad + (y(u) - Z_d\,, [y(u_k^h) - y^h(u_k^h)] + [y^h(v_k) - y(v_k)]) \\ &\quad + (y^h(u_k^h) - y(u), [y^h(v_k) - y(v_k)] + [y(v_k) - y(u)]) \\ &\qquad \forall v \in K \quad \text{and} \quad \forall v_k \in K_k\,,\end{aligned} \tag{18}$$

where $(\cdot, \cdot)$ denotes the usual $L^2(\Omega)$ inner product.

Since $u \in H^1(\Omega)$, we have the estimates

$$(u, v - u_k^h) \leqslant \| u \|_1 \| v - u_k^h \|_{-1}\,,$$
$$(u, v_k - u) \leqslant \| u \|_1 \| v_k - u \|_{-1}\,.$$

Applying the Schwartz inequality to each of the remaining terms on the right side, we obtain

$$\begin{aligned}\nu \| u - u_k^h \|^2 &+ \| y(u) - y^h(u_k^h)\|^2 \\ &\leqslant \nu \| u \|_1 [\| v - u_k^h \|_{-1} + \| v_k - u \|_{-1}] \\ &\quad + \nu \| u_k^h - u \| \| v_k - u \| + \| y(u) - Z_d \| [\| y(v) - y(u_k^h)\| \\ &\quad + \| y(v_k) - y(u)\| + \| y(u_k^h) - y^h(u_k^h)\| + \| y^h(v_k) - y(v_k)\|] \\ &\quad + \| y^h(u_k^h) - y(u)\| [\| y^h(v_k) - y(v_k)\| + \| y(v_k) - y(u)\|]\end{aligned}$$

where now $\| \cdot \|$ denotes $\| \cdot \|_{L^2(\Omega)}$.

Applying the arithmetic–geometric mean inequality to the terms

$$\| u_k^h - u \| \| v_k - u \|, \qquad \| y^h(u_k^h) - y(u)\| \| y^h(v_k) - y(v_k)\|,$$

and

$$\| y^h(u_k^h) - y(u)\| \| y(v_k) - y(u)\|,$$

and regrouping terms, we have the estimate:

$$\begin{aligned}(\nu/2) \| u - u_k^h \|^2 &+ \tfrac{1}{2} \| y(u) - y^h(u_k^h)\|^2 \\ &\leqslant \nu \| u \|_1 [\| v - u_k^h \|_{-1} + \| v_k - u \|_{-1}] + (\nu/2) \| v_k - u \|^2 \\ &\quad + \| y(u) - Z_d \| [\| y(v) - y(u_k^h)\| + \| y(v_k) - y(u)\| \\ &\quad + \| y(u_k^h) - y^h(u_k^h)\| + \| y^h(v_k) - y(v_k)\|] \\ &\quad + \| y^h(v_k) - y(v_k)\|^2 + \| y(v_k) - y(u)\|^2 \\ &\qquad \forall v \in K \quad \text{and} \quad \forall v_k \in K_k\,.\end{aligned}$$

Using (11), we have the a priori inequalities

$$\| y(v) - y(u_k)\|_0 \leqslant \| y(v) - y(u_k)\|_1 \leqslant C \| v - u_k \|_{-1},$$
$$\| y(v_k) - y(u)\|_0 \leqslant \| y(v_k) - y(u)\|_1 \leqslant C \| v_k - u \|_{-1}.$$

Inserting these inequalities, our error estimate becomes

$$\begin{aligned}(\nu/2) \| u - u_k^h \|^2 &+ \tfrac{1}{2} \| y(u) - y^h(u_k^h)\|^2 \\ &\leqslant (\nu/2) \| v_k - u \|^2 \\ &\quad + [\nu \| u \|_1 + C \| y(u) - Z_d \|] [\| v - u_k^h \|_{-1} + \| v_k - u \|_{-1}] \\ &\quad + C^2 \| v_k - u \|_{-1}^2 + \| y^h(v_k) - y(v_k)\|^2 \\ &\quad + \| y(u) - Z_d \| [\| y(u_k^h) - y^h(u_k^h)\| + \| y^h(v_k) - y(v_k)\|] \\ &\qquad \forall v \in K, \quad v_k \in K_k .\end{aligned} \tag{19}$$

Using inequalities (4) and (12), we are able to obtain a priori estimates for the quantities $\| u \|_1$ and $\| y(u) - Z_d \|$. Hence the errors $\| u - u_k^h \|$ and $\| y(u) - y^h(u_k^h)\|$ will depend only on how well we can approximate the unknown solutions u and u_k^h by elements of K_k and K respectively, and on how "good" an approximation the method M that we choose gives to the solutions $y(u_k^h)$ and $y(v_k)$ of the control equations

$$\begin{aligned} Ay(u_k^h) &= f + u_k^h && \text{in } \Omega, \\ y(u_k^h) &= 0 && \text{on } \partial\Omega; \end{aligned} \tag{20}$$

$$\begin{aligned} Ay(v_k) &= f + v_k && \text{in } \Omega, \\ y(v_k) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{21}$$

From (15)–(17) of Lemmas 3 and 5, we have estimates for $\| u - v_k \|$, $\| u - v_k \|_{-1}$, and $\| u_k^h - v \|_{-1}$. Estimates for the quantities $\| y(u_k^h) - y^h(u_k^h)\|$ and $\| y(v_k) - y^h(v_k)\|$ follow immediately from (14), i.e.,

$$\| y(u_k^h) - y^h(u_k^h)\| \leqslant Ch^\gamma \| f + u_k^h \|, \tag{22}$$

$$\| y(v_k) - y^h(v_k)\| \leqslant Ch^\gamma \| f + v_k \|. \tag{23}$$

Using estimates (22) and (23) along with estimates (15)–(17) for the quantities $\| u - v_k \|_0$, $\| u - v_k \|_{-1}$, and $\| u_k^h - v \|_{-1}$, inequality (19) becomes

$$\begin{aligned}(\nu/2) \| u - u_k^h \|^2 &+ \tfrac{1}{2} \| y(u) - y^h(u_k^h)\|^2 \\ &\leqslant (\nu/2) [Ck \| u \|_1]^2 \\ &\quad + [\nu \| u \|_1 + C \| y(u) - Z_d \|] [Ck^2 \| u \|_1 + Ck^2(\| \xi_0 \|_1^2 + \| \xi_1 \|_1^2)^{1/2}] \\ &\quad + C^2[Ck^2 \| u \|_1]^2 + [Ch^\gamma \| f + v_k \|]^2 \\ &\quad + \| y(u) - Z_d \| [Ch^\gamma \| f + u_k^h \| + Ch^\gamma \| f + v_k \|].\end{aligned} \tag{24}$$

We have already observed that we have a priori estimates for $\| u \|_1$ and $\| y(u) - Z_d \|$. Since f, ξ_0, and ξ_1 are data, to complete the estimate, we need only to obtain a priori bounds on the quantities $\| u_k^h \|$ and $\| v_k \|$ which are independent of k and h.

Since both u_k^h and $v_k \in K_k$, they must both satisfy $M_k^j(\xi_0) \leqslant u_k^h$, $v_k \leqslant M_k^j(\xi_1)$ on Ω_k^j $\forall j \in J_k$. Hence

$$\| u_k^h \|_0^2 = \int_\Omega | u_k^h |^2 \, dx = \sum_{j \in J_k} \int_{\Omega_k^j} | u_k^h |^2 \, dx = \sum_{j \in J_k} \mu(\Omega_k^j) \, | u_k^h |^2$$

$$\leqslant \sum_{j \in J_k} \mu(\Omega_k^j) \max(| M_k^j(\xi_0)|^2, | M_k^j(\xi_1)|^2)$$

$$\leqslant \sum_{j \in J_k} \mu(\Omega_k^j) \, [| M_k^j(\xi_0)|^2 + | M_k^j(\xi_1)|^2].$$

Now

$$| M_k^j(\xi_i)|^2 = \left[\frac{1}{\mu(\Omega_k^j)}\right]^2 \left| \int_{\Omega_k^j} \xi_i \, dx \right|^2 \leqslant \frac{1}{\mu(\Omega_k^j)} \int_{\Omega_k^j} | \xi_i |^2 \, dx.$$

Hence

$$\| u_k^h \|_0^2 \leqslant \sum_{j \in J_k} \left[\int_{\Omega_k^j} | \xi_0 |^2 \, dx + \int_{\Omega_k^j} | \xi_1 |^2 \, dx\right] \leqslant \| \xi_0 \|_0^2 + \| \xi_1 \|_0^2 ,$$

or finally,

$$\| u_k^h \|_0 \leqslant [\| \xi_0 \|_0^2 + \| \xi_1 \|_0^2]^{1/2} .$$

Obviously, the same estimate holds for v_k. Finally then, we have shown that there exists a constant C depending only on the data f, ξ_0, and ξ_1 such that

$$(\nu/2) \, \| u - u_k^h \|^2 + \tfrac{1}{2} \, \| y(u) - y^h(u_k^h) \|^2 \leqslant C[k^2 + h^\nu]. \tag{25}$$

In the next section we make some observations about this order of convergence estimate and discuss some conditions under which it can be improved.

7. Discussion of the Order of Convergence Estimate

Before we discuss the improvement of estimate (25), we first observe that the errors we are making in solving the approximate problem instead of the original one arise from two sources. One of these is that we are using an approximate solution to the control equation instead of the true solution.

We observe from the derivation of (19) that if $y(u) - Z_d \in L^2(\Omega)$, but does not $\in H^1(\Omega)$, then the last term in (19) cannot be improved. However, if we assume additional regularity for f and Z_d, then $y(u) - Z_d$ can be shown to be an element of $H^j(\Omega)$ for some $0 \leqslant j \leqslant 3$ (i.e., if $f \in H^1(\Omega)$ and $Z_d \in H^3(\Omega)$, then $y(u) - Z_d \in H^3(\Omega)$). From the definition of $\|\cdot\|_{-j}$, we have the estimate

$$(y(u) - Z_d, [y(u_k^h) - y^h(u_k^h)] + [(y^h(v_k) - y(v_k)])$$
$$\leqslant \| y(u) - Z_d \|_j [\| y(u_k^h) - y^h(u_k^h)\|_{-j} + \| y^h(v_k) - y(v_k)\|_{-j}].$$

In addition to inequality (14) we might expect the error in the Raleigh–Ritz–Galerkin method that we select to also satisfy an estimate of the form

$$\| y(v) - y^h(v)\|_{-j} \leqslant Ch^\beta \| f + v \|_{L^2(\Omega)},$$

where now $0 < \beta \leqslant 2 + j$. If so, then instead of (25), our final error will be $\leqslant C(h^{2\gamma} + h^\beta + k^2)$. For $\beta > \gamma$, this will be an improvement. When $\beta \geqslant 2\gamma$, we may say that we have achieved optimality, with respect to a given Raleigh–Ritz–Galerkin method, in the part of the error caused by use of this method.

For example, if $f \in L^2(\Omega)$ and $Z_d \in H^2(\Omega)$, and we use the "least-squares method" of Bramble and Schatz [3] with a subspace $S^h_{2,6}$ (e.g., quintic splines), then we have the estimates

$$\| y(v) - y^h(v)\|_0 \leqslant Ch^2 \| f + v \|_0,$$
$$\| y(v) - y^h(v)\|_{-2} \leqslant Ch^4 \| f + v \|_0.$$

Hence the parts of our error estimate reflecting the use of this method are optimal, i.e., they duplicate up to a multiplicative constant, the error in the method.

We now turn to the error caused by looking for the optimal control not in $L^2(\Omega)$, but rather in some finite dimensional subspace $S_k(\Omega)$ of $L^2(\Omega)$. From inequality (19) we recall that the terms in the error estimate reflecting this part of the error are

$$(\nu/2) \| v_k - u \|^2, \qquad [\nu \| u \|_1 + C \| y(u) - Z_d \|] [\| v_k - u \|_{-1} + \| v - u_k^h \|_{-1}]$$

and

$$C^2 \| v_k - u \|_{-1}^2.$$

From estimates (15)–(17), we have that all these terms are at least $O(k^2)$. Hence the error in the terms $\| u - u_k^h \|$ and $\| y(u) - y^h(u_k^h)\|$ caused by using the approximate convex set K_k instead of the original convex set K is $O(k)$.

Since it is known that the best approximation in $L^2(\Omega)$ to an arbitrary element $u \in H^1(\Omega)$ by elements of $S_k(\Omega)$ is $O(k)$, our estimate is optimal in the sense that it duplicates up to a multiplicative constant, the best approximation properties of the subspace $S_k(\Omega)$.

A practical result of the preceding discussion is that it tells us how to choose the relationship between k and h for computation, i.e., set $k^2 = Ch^\delta$ where C is a constant and $\delta = \min(2\gamma, \beta)$.

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