

ANALYSIS OF A CONTINUOUS FINITE ELEMENT METHOD FOR HYPERBOLIC EQUATIONS*

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Abstract. A finite element method for hyperbolic equations is analyzed in the context of a first order linear problem in R^2 . The method is applicable over a triangulation of the domain, and produces a continuous piecewise polynomial approximation, which can be developed in an explicit fashion from triangle to triangle. In a sense, it extends the basic upwind difference scheme to higher order. The method is shown to be stable, and error estimates are obtained. For n th degree approximation, the errors in the approximate solution and its gradient are shown to be at least of order $h^{n+1/4}$ and $h^{n-1/2}$, respectively, assuming sufficient regularity in the solution.

Key words. finite element method, hyperbolic equation

AMS(MOS) subject classifications. Primary 65M15, 65N30

1. Introduction. In this paper, we analyze a finite element method for the first order scalar hyperbolic equation

$$(1) \quad \begin{aligned} \alpha \cdot \nabla u + \beta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on the inflow boundary } \Gamma_{in}(\Omega), \end{aligned}$$

where α is a unit vector and Ω is a bounded polygonal domain in R^2 . The method produces a continuous piecewise polynomial approximation to u over a triangulation of Ω , and was first reported in the literature by Reed and Hill [8].

A contrasting and more common finite element approach to (1), applicable when the independent variables are time and space, is that of applying a finite element discretization in space only, then solving numerically the resulting system of ordinary differential equations. See, for example, [1] and [4]. Examples of hyperbolic equations of practical interest which do not involve time and are not directly amenable to this approach are the neutron transport equation [7] and the problem of determining the diffusion coefficient $a(x)$ in

$$\nabla p \cdot \nabla a + a \Delta p = f.$$

The latter is an inverse problem arising in flow through porous media [2], [5], [9].

Several techniques for obtaining *full* finite element discretizations of (1) have been reported in the literature. Reed and Hill [8] have provided computational results for the scheme that is the focal point of this paper and also for two other schemes, one of which produces a discontinuous approximation. The discontinuous method has been analyzed by Lesaint and Raviart [7], and more recently by Johnson and Pitkaranta [6], who obtained improved estimates. In a related work, Winther [11] obtained optimal order error estimates for a continuous finite element method applicable over a rectangular mesh.

To describe the method which we shall analyze, we let Δ_h be a quasiuniform triangulation of Ω , constructed so that no triangle has a side parallel to the characteristic direction at any point. For any subdomain Ω_S of Ω , we denote by $\Gamma_{in}(\Omega_S)$ the inflow

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portion of the boundary of Ω_S , i.e., $\{x \in \Gamma(\Omega_S) \mid \alpha \cdot n < 0\}$, where n is the unit outward normal to Ω_S , and by $\Gamma_{out}(\Omega_S)$ the remaining (outflow) portion of $\Gamma(\Omega_S)$. With Δ_h as above, each triangle has one inflow side and two outflow sides (a *type I* triangle) or two inflow sides and one outflow side (a *type II* triangle). Furthermore, the triangles $\{T_i\}$ in Δ_h may be ordered so that

$$(2) \quad \Gamma_{in}(T_k) \subseteq \Gamma_{out}\left(\bigcup_{i < k} T_i\right) \cup \left(\Gamma_{in}(\Omega) - \Gamma_{in}\left(\bigcup_{i < k} T_i\right)\right).$$

Equivalently, for each k , the domain of dependence of T_k contains none of T_{k+1}, T_{k+2}, \dots . This was shown in [7] for constant α and will be proved in the Appendix for smooth variable α (the assumption we make in our analysis). This ordering allows an approximate solution to be developed in an explicit manner, first in T_1 , then in T_2 , etc. At the point when the solution is to be formed in a given triangle, it will be known along the inflow to that triangle.

We seek an approximate solution in the subspace

$$S_h^n = \{v_h \in C^0(\Omega) \text{ such that } v_h|_T \in \mathbf{P}_n(T)\},$$

where $\mathbf{P}_n(T)$ denotes the space of polynomials of degree $\leq n$ over the triangle T . Letting g_I be a suitable interpolant of g in $S_h^n|_{\Gamma_{in}(\Omega)}$ and denoting the L^2 inner product over T by $(\cdot)_T$, we describe the finite element method of interest as follows.

PROBLEM P_h. Find $u_h \in S_h^n$ such that $u_h = g_I$ on $\Gamma_{in}(\Omega)$, and for triangles of type I

$$(3) \quad (\alpha \cdot \nabla u_h + \beta u_h, v_h)_T = (f, v_h)_T \quad \text{for all } v_h \in \mathbf{P}_{n-1}(T),$$

while for triangles of type II

$$(4) \quad (\alpha \cdot \nabla u_h + \beta u_h, v_h)_T = (f, v_h)_T \quad \text{for all } v_h \in \mathbf{P}_{n-2}(T).$$

Note that the approximate solution u_h has a total of $\sigma_n \equiv \sum_{j=1}^{n+1} j$ degrees of freedom in each triangle. In a one-inflow-side triangle, there are $n+1$ degrees of freedom in u_h along the inflow, leaving a total of σ_{n-1} to be determined from (3). In a two-inflow-side triangle, there are $2n+1$ degrees of freedom in u_h along the inflow, leaving σ_{n-2} to be determined from (4). Thus in both (3) and (4), the number of equations equals the number of unknowns.

We shall assume in our analysis that $n \geq 2$, although the case $n = 1$ is also of some interest. The latter is a degenerate case in which (4) is vacuous and (3) completely determines the approximate solution. A simple calculation reveals that for a mesh of right triangles with $\alpha = \text{constant}$, $\beta = 0$, and $f = 0$, u_h at the triangle vertices is given by the upwind difference scheme. Thus the finite element method (3), (4) may be viewed as an extension of the upwind difference scheme to higher order and nonuniform meshes.

We note that other continuous finite element methods besides the one analyzed in this paper may be applied over triangles. For example, in place of low order polynomials, one might use as test functions the Lagrange basis functions that are unity at the unknown points. Numerical experiments in [8] indicate that this method does not perform as well as the one analyzed in this paper.

A natural question is how the continuous method described by (3), (4) compares with the discontinuous Galerkin method, which is fairly well established theoretically. In the case of the latter, one updates the solution on both type I and type II triangles

T as follows: Given u_-^h on $\Gamma_{in}(T)$, find $u^h|_T \in \mathbf{P}_n(T)$ such that

$$\begin{aligned} (\alpha \cdot \nabla u^h + \beta u^h, v^h)_T - \int_{\Gamma_{in}(T)} u_+^h v^h \alpha \cdot n \\ = (f, v^h)_T - \int_{\Gamma_{in}(T)} u_-^h v^h \alpha \cdot n \quad \text{for all } v^h \in \mathbf{P}_n(T) \end{aligned}$$

where $v_-(x) = \lim_{s \rightarrow 0^-} v(x + s\alpha)$, and $v_+(x) = \lim_{s \rightarrow 0^+} v(x + s\alpha)$, for x lying on a side common to two triangles. Hence to update the solution on a triangle requires the solution of a $\sigma_n \times \sigma_n$ linear system of equations. When the case of piecewise quadratics is taken as an example, the discontinuous method yields a 6×6 system on each triangle, whereas the continuous method produces a 1×1 system on type I triangles and a 3×3 system on type II triangles. This illustrates the main advantage of the continuous method: it is less costly to apply. In terms of existing error estimates for the two schemes, using piecewise polynomials of degree $\cong n$, the discontinuous Galerkin method has the error estimates

$$\|u - u^h\|_{L_2(\Omega)} = O(h^{n+1/2})$$

and hence, using inverse properties of the subspace,

$$\|u - u^h\|_{H^1(\Omega)} = O(h^{n-1/2}).$$

In this paper, we prove that the continuous finite element method satisfies the same H^1 estimate and an L_2 estimate with a power of h one quarter less. All this analysis assumes sufficient regularity of the solution. In the last section of the paper, we present the results of some computational experiments, which show that on very smooth solutions, with a regular mesh, it is possible to achieve optimal order approximation. The same is true for the discontinuous Galerkin method. One important feature of the discontinuous Galerkin method is that it produces good results on problems with discontinuous solutions. We have not undertaken a detailed study of the continuous method for such problems; however, we provide computational results for a numerical experiment with discontinuous initial data.

The next section of this paper further describes our notation and the hypotheses under which our results will be established. Section 3 contains an existence and uniqueness proof for the approximate solution and a derivation of some of its local properties. Two basic identities are derived in § 4 and then used in § 5 to prove stability of the method. Error estimates are given in § 6, followed by numerical results in § 7.

2. Assumptions and notation. For D a domain in R^2 , let $\|f\|_D = (\int_D f^2 dx dy)^{1/2}$, and for Γ a line segment, let $\|f\|_\Gamma = (\int_\Gamma f^2 d\tau)^{1/2}$. Let $\|\cdot\|_{n,D}$ denote the norm in the Sobolev space $H^n(D)$ and $\|\cdot\|_{\infty,D}$ and $\|\cdot\|_{1,\infty,D}$ the norms in the Sobolev spaces $L^\infty(D)$ and $[W^{1,\infty}(D)]^2$, respectively. We assume that $\alpha \in [W^{1,\infty}(D)]^2$ and $\beta \in L^\infty(D)$ for the coefficients in (1). Henceforth, we shall omit the subscript D when $D = \Omega$. We also denote by $P_k f$ the L^2 projection over T into $\mathbf{P}_k(T)$ (the space of polynomials of degree $\leq k$ over T).

It will be convenient to have the following notation relative to an arbitrary triangle T of Δ_h . For $i = 1, 2, 3$, we denote by Γ_i the sides of T numbered counterclockwise, by a_i the vertices of T opposite Γ_i , by n_i the unit outward normals to Γ_i , and by τ_i the unit tangents along Γ_i taken in a counterclockwise direction. We shall always take Γ_3 to be the inflow side of a type I triangle or the outflow side of a type II triangle. This notation is illustrated in Fig. 2.1.

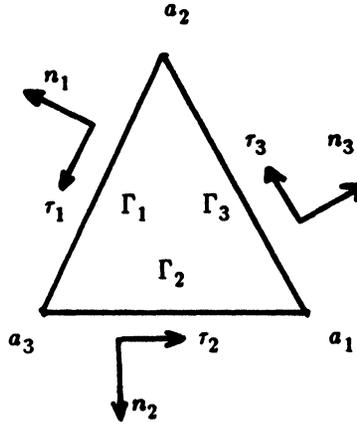


FIG. 2.1

For computational and theoretical purposes, it is advantageous to think of the triangles in a given triangulation Δ_h as partitioned into layers S_i . We define these as follows:

$$S_1 = \{T \in \Delta_h \mid \Gamma_{in}(T) \subseteq \Gamma_{in}(\Omega)\},$$

$$S_{i+1} = \left\{ T \in \Delta_h \mid \Gamma_{in}(T) \subseteq \Gamma_{in} \left(\Omega - \bigcup_{k \leq i} S_k \right) \right\}, \quad i = 1, 2, \dots$$

With this partition of Δ_h , the approximate solution may be obtained in an explicit manner, first in S_1 , then in S_2 , etc. Within each layer, the approximate solution can be obtained in parallel since the solution in any of the triangles within a layer does not depend on the solution in other triangles in that layer. This is illustrated in Fig. 2.2, with the number inside each triangle indicating the layer to which it belongs.

In our analysis, we assume that $\{\Delta_h\}$ is a family of triangulations of Ω satisfying the following hypotheses:

- H1: All angles of all triangles are uniformly bounded away from zero.
- H2: $|\alpha \cdot n|$ is uniformly bounded away from zero along all sides of all triangles.
(This assumption is essential in our analysis but does not appear to be necessary computationally.)

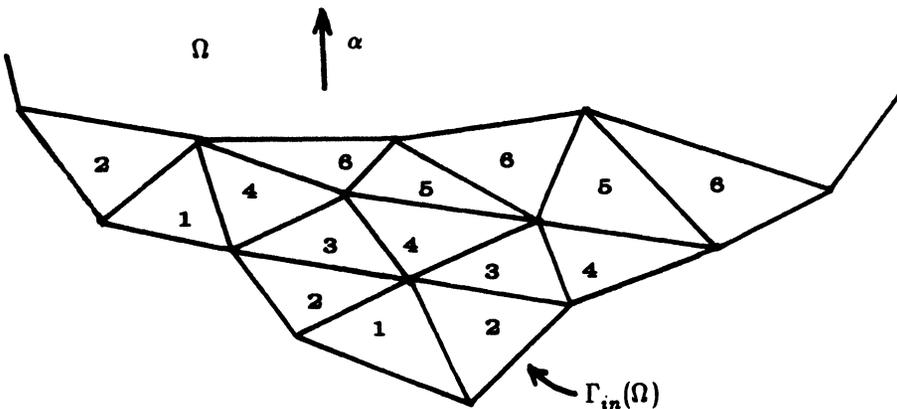


FIG. 2.2

H3: The ratio of the maximum to minimum triangle sides within Δ_h is uniformly bounded.

H4: The triangles in Δ_h can be partitioned into $O(h^{-1})$ layers (where layers are as defined above).

For the sake of brevity and to avoid the introduction of additional variables, we shall use the notation u_α to mean $\nabla u \cdot \alpha$ when α is any unit vector. Equivalently, one may think of u_α as a *partial derivative* of u with respect to a coordinate in the direction α , provided the second variable on which u depends is constant in the direction α . In this notation, we state an integration by parts formula

$$(5) \quad \int_T v_\alpha w = \int_{\Gamma(T)} vw\alpha \cdot n - \int_T vw_\alpha - \int_T (\nabla \cdot \alpha)vw$$

which will be used several times in the paper.

Finally, we shall use the symbol C to denote a generic constant, depending at most on the coefficients α and β in (1) and the bounds in assumptions H1-H4.

3. Existence and uniqueness. In order to establish the results in this section, it will be convenient to introduce a reference triangle \hat{T} with vertices $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, and $\hat{a}_3 = (0, 0)$. The reference triangle \hat{T} can be mapped into the triangle T by the affine transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{F} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + a_3.$$

The matrix \mathbf{F} can be written

$$\mathbf{F} = (|\Gamma_2|\tau_2, -|\Gamma_1|\tau_1)$$

where $|\Gamma_i|$ is the length of Γ_i . Defining $\hat{v}_h(\hat{x}, \hat{y}) = v(x, y)$ (for an arbitrary v defined on T) and

$$\hat{\nabla} = \begin{pmatrix} \partial/\partial\hat{x} \\ \partial/\partial\hat{y} \end{pmatrix},$$

we have

$$\alpha \cdot \nabla u_h = A \cdot \hat{\nabla} \hat{u}_h$$

where

$$A = \mathbf{F}^{-1}\alpha \quad \text{and} \quad \mathbf{F}^{-1} = \frac{1}{n_1 \cdot \tau_2} \begin{pmatrix} n_1^i/|\Gamma_2| \\ n_2^i/|\Gamma_1| \end{pmatrix}.$$

Transforming (3) and (4) to \hat{T} , we obtain

$$(6) \quad \int_{\hat{T}} (A \cdot \hat{\nabla} \hat{u}_h + \hat{\beta} \hat{u}_h - \hat{f}) \hat{v}_h d\hat{x} d\hat{y} = 0 \quad \text{for all } \hat{v}_h \in \mathbf{P}_{n-\rho}(\hat{T})$$

where ρ is the number of inflow sides that \hat{T} has. In \hat{T} , we express \hat{u}_h in the form

$$\hat{u}_h(\hat{x}, \hat{y}) = \sum_{j=1}^{\sigma_n} \hat{u}_h(\hat{P}_j) \hat{\phi}_j(\hat{x}, \hat{y})$$

where \hat{P}_j are the usual equispaced nodes for n th degree interpolation in \hat{T} and $\hat{\phi}_j(\hat{x}, \hat{y})$ are the corresponding Lagrange basis functions for $\mathbf{P}_n(\hat{T})$. We then require that (6)

be satisfied for each of the analogous basis functions $\hat{\psi}_i(\hat{x}, \hat{y})$ for $\mathbf{P}_{n-\rho}(\hat{T})$. The resulting linear algebraic system is equivalent to the following:

$$(7) \quad \sum_{j=1}^{\sigma_n} h(A \cdot \hat{\nabla} \hat{\phi}_j + \hat{\beta} \hat{\phi}_j, \hat{\psi}_i)_{\hat{T}} \hat{u}_h(\hat{P}_j) = h(\hat{f}, \hat{\psi}_i)_{\hat{T}}, \quad i = 1, \dots, \sigma_{n-\rho},$$

$$(8) \quad \hat{u}_h(\hat{P}_j) = \text{given value for } \hat{P}_j \in \Gamma_{in}(\hat{T})$$

or, in matrix form,

$$\mathbf{K} \hat{u}_h = b.$$

The above are σ_n equations in as many variables, the inflow yielding a total of $n+1$ or $2n+1$ conditions, depending on whether T is of type I or II.

Now let Q denote the centroid of T , $\alpha_0 = \alpha(Q)$, and $A_0 = \mathbf{F}^{-1} \alpha_0$. The coefficient matrix \mathbf{K} can then be written

$$\mathbf{K} = \mathbf{K}_0 + h \mathbf{K}_1$$

where \mathbf{K}_0 corresponds to $\alpha = \alpha_0$ and $\beta = 0$. Note that hA_0 involves ratios of sides of triangles in Δ_h which, by H3, are uniformly bounded, and that $|\alpha(x) - \alpha_0| \leq Ch \|\alpha\|_{1,\infty}$ for $x \in T$. Thus both \mathbf{K}_0 and \mathbf{K}_1 are uniformly bounded over all triangles. We next show that \mathbf{K}_0 is invertible under assumption H2. It will then follow that \mathbf{K}_0^{-1} is uniformly bounded and that Problem \mathbf{P}_h has a unique solution for sufficiently small h .

Establishing the invertibility of \mathbf{K}_0 is equivalent to showing that the only solution to (3) or (4) with $\beta = f = 0$, α replaced by α_0 , and $u_h = 0$ on $\Gamma_{in}(T)$ is $u_h \equiv 0$. We now do this.

For type I triangles, we take $v_h = (u_h)_{\alpha_0} \in \mathbf{P}_{n-1}(T)$, and infer that $(u_h)_{\alpha_0} = 0$. This, together with $u_h = 0$ on $\Gamma_{in}(T)$, implies that $u_h \equiv 0$ in T . (We have assumed that h is sufficiently small that replacement of α by α_0 does not alter the inflow and outflow sides of T .)

For type II triangles, we note that $u_h = 0$ on $\Gamma_{in}(T)$ implies that u_h can be written in the form $u_h = \xi \eta w_h$ where ξ and η are coordinates along the two inflow sides, with $\xi = \eta = 0$ at a_3 , $\xi, \eta \geq 0$ in T , and $w_h \in \mathbf{P}_{n-2}(T)$. Taking $v_h = w_h$ in (4), we use the integration by parts formula (5) to obtain

$$\begin{aligned} 0 &= ((u_h)_{\alpha_0}, w_h)_T = (\xi \eta (w_h)_{\alpha_0}, w_h)_T + ((\xi \eta)_{\alpha_0} w_h, w_h)_T \\ &= \frac{1}{2} (\xi \eta, (w_h^2)_{\alpha_0})_T + ((\xi \eta)_{\alpha_0}, w_h^2)_T \\ &= \frac{1}{2} \int_{\Gamma_{out}(T)} w_h^2 \xi \eta \alpha_0 \cdot n \, d\Gamma + \frac{1}{2} ((\xi \eta)_{\alpha_0}, w_h^2)_T. \end{aligned}$$

Now $(\xi \eta)_{\alpha_0}$ is positive in T and $\xi \eta \alpha_0 \cdot n$ is nonnegative on $\Gamma_{out}(T)$. Hence $w_h \equiv 0$ in T and $u_h \equiv 0$ in T . We have thus established the following result.

LEMMA 3.1. *There exists a constant h_0 such that for all $h \leq h_0$, Problem \mathbf{P}_h has a unique solution.*

We shall henceforth implicitly assume that $h \leq h_0$, so that the approximate solution is well defined. We now establish several estimates applying over a single triangle T . They will be used later in the derivation of the main stability result.

LEMMA 3.2. *Let $u_h \in \mathbf{P}_n(T)$ satisfy (3) or (4). Then*

$$(9) \quad \|u_h\|_T \leq C \{h^{1/2} |u_h|_{\Gamma_{in}(T)} + h \|f\|_T\}$$

and

$$(10) \quad \|\nabla u_h\|_T \leq C \left\{ h^{1/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(T)} + h^{1/2} \|\beta\|_{\infty,T} |u_h|_{\Gamma_{in}(T)} + \|f\|_T \right\}$$

where τ denotes arclength along $\Gamma_{in}(T)$.

Proof. From Lemma 3.1 and (7) and (8), it immediately follows that

$$(11) \quad \|\hat{u}_h\|_{\hat{T}} \leq C \{ |\hat{u}_h|_{\Gamma_{in}(\hat{T})} + h \|\hat{f}\|_{\hat{T}} \}.$$

Inequality (9) follows from (11) by transforming coordinates from \hat{T} to T . To prove (10), let P_0 be a point on the inflow to T and write $u_h = u_h(P_0) + v_h$. Then

$$|v_h|_{\Gamma_{in}(T)} \leq Ch \left| \frac{dv_h}{d\tau} \right|_{\Gamma_{in}(T)} = Ch \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(T)}$$

and

$$\|\nabla u_h\|_T = \|\nabla v_h\|_T \leq Ch^{-1} \|v_h\|_T.$$

Now $v_h \in P_n(T)$ satisfies (3) or (4) with f replaced by $f - \beta u_h(P_0)$. Applying the first part of the lemma, we obtain

$$\begin{aligned} \|v_h\|_T &\leq C \{ h^{1/2} |v_h|_{\Gamma_{in}(T)} + h \|f - \beta u_h(P_0)\|_T \} \\ &\leq C \left\{ h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(T)} + h \|f\|_T + h \|\beta\|_{\infty,T} \|u_h(P_0)\|_T \right\} \\ &\leq C \left\{ h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(T)} + h \|f\|_T + h^{3/2} \|\beta\|_{\infty,T} |u_h|_{\Gamma_{in}(T)} \right\}. \end{aligned}$$

The result follows immediately.

LEMMA 3.3. *Let T be a type II triangle and $\phi_h \in P_n(T)$. Then*

$$(12) \quad \|(\phi_h)_\alpha\|_T \leq C \{ h^{1/2} |(\phi_h)_\alpha|_{\Gamma_{out}(T)} + \|P_{n-2}(\phi_h)_\alpha\|_T + h \|\nabla \phi_h\|_T \}.$$

Proof. We first prove the result for the case $\alpha = \alpha_0$, a constant vector. Here $v \equiv (\phi_h)_{\alpha_0} \in P_{n-1}(T)$. Moreover, if $v = 0$ on $\Gamma_{out}(T)$ and $P_{n-2}v = 0$, then $v \equiv 0$ in T . This can be shown by writing $v = p(\tau) + sq(s, \tau)$, where τ is a coordinate measured along $\Gamma_{out}(T)$ and s is a coordinate in the direction of α_0 with $s = 0$ on $\Gamma_{out}(T)$. Then $v = 0$ on $\Gamma_{out}(T)$ implies that $p(\tau) = 0$ and $P_{n-2}v = 0$ therefore implies that

$$\int_T sqw \, dx \, dy = 0 \quad \text{for all } w \in P_{n-2}(T).$$

By taking $w = q \in P_{n-2}(T)$, we conclude that $q \equiv 0$ in T . Thus $v \equiv 0$ in T . In the reference triangle \hat{T} , we therefore have

$$\|\hat{v}\|_{\hat{T}} \leq C \{ |\hat{v}|_{\Gamma_{out}(\hat{T})} + \|P_{n-2}\hat{v}\|_{\hat{T}} \},$$

which, when transformed back to the original triangle, becomes

$$(13) \quad \|(\phi_h)_{\alpha_0}\|_T \leq C \{ h^{1/2} |(\phi_h)_{\alpha_0}|_{\Gamma_{out}(T)} + \|P_{n-2}(\phi_h)_{\alpha_0}\|_T \}.$$

Now suppose α is not constant. Using (13), we obtain

$$(14) \quad \begin{aligned} \|(\phi_h)_\alpha\|_T &\leq \|(\phi_h)_{\alpha_0}\|_T + \|(\alpha - \alpha_0) \cdot \nabla \phi_h\|_T \\ &\leq C \{ h^{1/2} |(\phi_h)_\alpha|_{\Gamma_{out}(T)} + \|P_{n-2}(\phi_h)_\alpha\|_T \\ &\quad + h^{1/2} |(\alpha_0 - \alpha) \cdot \nabla \phi_h|_{\Gamma_{out}(T)} + \|P_{n-2}([\alpha_0 - \alpha] \cdot \nabla \phi_h)\|_T \} \\ &\quad + \|(\alpha - \alpha_0) \cdot \nabla \phi_h\|_T. \end{aligned}$$

Inequality (12) now follows by standard estimates.

4. Basic identities. The test function $v_h = -(u_h)_{\tau_1\tau_2}$ in (3) and (4) will play a key role in our stability analysis. (Recall that τ_1 and τ_2 are the tangents to the two outflow sides of a type I triangle or the two inflow sides of a type II triangle.) In this section, we derive expressions for the two terms in the integral

$$(u_\alpha + \beta u, -u_{\tau_1\tau_2})_T.$$

To facilitate the exposition, we state some obvious identities relating $\alpha = (\alpha_1, \alpha_2)^t$ and τ_i and n_i , the unit tangential and normal vectors to the sides Γ_i of a triangle T (as depicted in Fig. 2.1). Let $\gamma = (-\alpha_2, \alpha_1)^t$. Then

$$(15) \quad \begin{aligned} (a) \quad & \tau_i \cdot n_j = -\tau_j \cdot n_i, & i, j = 1, 2, 3, \\ (b) \quad & \tau_i \cdot \gamma = n_i \cdot \alpha, & i = 1, 2, 3, \\ (c) \quad & \gamma \cdot n_i = -\alpha \cdot \tau_i, & i = 1, 2, 3. \end{aligned}$$

In addition, the following identity is also valid for any unit vector $\alpha = (\alpha_1, \alpha_2)^t$.

LEMMA 4.1.

$$(\alpha \cdot n_2)(\tau_1 \cdot n_3) + (\alpha \cdot n_1)(\tau_3 \cdot n_2) + (\alpha \cdot n_3)(\tau_2 \cdot n_1) = 0.$$

Proof. Let ξ_i denote the angle between α and n_i , measured counterclockwise from α . Since τ_i may be obtained by rotating n_i through an angle $\pi/2$,

$$\tau_i \cdot n_j = \cos [(\xi_i + \pi/2) - \xi_j] = \sin (\xi_j - \xi_i).$$

Thus the lemma may be recast as

$$\cos \xi_2 \sin (\xi_3 - \xi_1) + \cos \xi_1 \sin (\xi_2 - \xi_3) + \cos \xi_3 \sin (\xi_1 - \xi_2) = 0;$$

which can be verified by expanding the sines in the above formula.

We shall now derive the desired identities.

LEMMA 4.2. For any unit vector α and any twice differentiable function u :

$$\begin{aligned} \int_T u_\alpha (-u_{\tau_1\tau_2}) \, dx \, dy &= \frac{1}{2} \int_{\Gamma(T)} \frac{(\alpha \cdot n_1)(\alpha \cdot n_2)}{\alpha \cdot n} u_\tau^2 \, d\tau - \frac{1}{2} \int_{\Gamma_3} \frac{(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)}{\alpha \cdot n_3} u_\alpha^2 \, d\tau \\ &\quad + \frac{1}{2} \int_T \{(A_1)_{\tau_2} u_{\tau_1}^2 + (A_2)_{\tau_1} u_{\tau_2}^2\} \, dx \, dy \end{aligned}$$

where $(A_1, A_2) = (1/\tau_1 \cdot n_2)(\alpha \cdot n_2, -\alpha \cdot n_1)$.

Proof. We first write u_α in terms of derivatives in the directions of τ_1 and τ_2 . From

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \tau_1^t \\ \tau_2^t \end{pmatrix}^{-1} \begin{pmatrix} u_{\tau_1} \\ u_{\tau_2} \end{pmatrix} = \frac{1}{\tau_2 \cdot n_1} (-n_2, n_1) \begin{pmatrix} u_{\tau_1} \\ u_{\tau_2} \end{pmatrix},$$

it follows that

$$u_\alpha = A_1 u_{\tau_1} + A_2 u_{\tau_2},$$

with A_1 and A_2 as defined in the statement of the lemma. Hence, using the integration by parts formula (5) and (15a) from the previous page, we obtain

$$\begin{aligned} \int_T u_\alpha (-u_{\tau_1\tau_2}) \, dx \, dy &= -\frac{1}{2} \int_T \{A_1 (u_{\tau_1}^2)_{\tau_2} + A_2 (u_{\tau_2}^2)_{\tau_1}\} \, dx \, dy \\ &= \frac{1}{2} \int_{\Gamma_1} (\alpha \cdot n_2) u_{\tau_1}^2 \, d\tau + \frac{1}{2} \int_{\Gamma_2} (\alpha \cdot n_1) u_{\tau_2}^2 \, d\tau \\ &\quad - \frac{1}{2} \int_{\Gamma_3} \{A_1 u_{\tau_1}^2 (\tau_2 \cdot n_3) + A_2 u_{\tau_2}^2 (\tau_1 \cdot n_3)\} \, d\tau \end{aligned}$$

$$+\frac{1}{2} \int_T \{(A_1)_{\tau_2} u_{\tau_1}^2 + (A_2)_{\tau_1} u_{\tau_2}^2\} dx dy.$$

We next write u_{τ_1} and u_{τ_2} in terms of the directional derivatives u_{τ_3} and u_α . Let γ be the unit vector $(-\alpha_2, \alpha_1)^t$. Then

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \tau_3^t \\ \alpha^t \end{pmatrix}^{-1} \begin{pmatrix} u_{\tau_3} \\ u_\alpha \end{pmatrix} = \frac{1}{\tau_3 \cdot \gamma} (\gamma, n_3) \begin{pmatrix} u_{\tau_3} \\ u_\alpha \end{pmatrix}$$

and

$$(16) \quad \begin{pmatrix} u_{\tau_1} \\ u_{\tau_2} \end{pmatrix} = \begin{pmatrix} \tau_1^t \\ \tau_2^t \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{\tau_3 \cdot \gamma} \begin{pmatrix} \tau_1 \cdot \gamma & \tau_1 \cdot n_3 \\ \tau_2 \cdot \gamma & \tau_2 \cdot n_3 \end{pmatrix} \begin{pmatrix} u_{\tau_3} \\ u_\alpha \end{pmatrix}.$$

Using this result and the definitions of A_1 and A_2 , we now rewrite the integrand along Γ_3 as follows:

$$\begin{aligned} & A_1 u_{\tau_1}^2 (\tau_2 \cdot n_3) + A_2 u_{\tau_2}^2 (\tau_1 \cdot n_3) \\ &= \frac{1}{(\tau_3 \cdot \gamma)^2 (\tau_1 \cdot n_2)} \\ & \cdot \{ [(\alpha \cdot n_2)(\tau_2 \cdot n_3)(\tau_1 \cdot \gamma)^2 - (\alpha \cdot n_1)(\tau_1 \cdot n_3)(\tau_2 \cdot \gamma)^2] u_{\tau_3}^2 \\ & \quad + [(\alpha \cdot n_2)(\tau_2 \cdot n_3)(\tau_1 \cdot n_3)^2 - (\alpha \cdot n_1)(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)^2] u_\alpha^2 \\ & \quad + 2[(\alpha \cdot n_2)(\tau_2 \cdot n_3)(\tau_1 \cdot \gamma)(\tau_1 \cdot n_3) \\ & \quad \quad - (\alpha \cdot n_1)(\tau_1 \cdot n_3)(\tau_2 \cdot \gamma)(\tau_2 \cdot n_3)] u_{\tau_3} u_\alpha \}. \end{aligned}$$

Using Lemma 4.1 and (15b), we then obtain

$$A_1 u_{\tau_1}^2 (\tau_2 \cdot n_3) + A_2 u_{\tau_2}^2 (\tau_1 \cdot n_3) = \frac{1}{\alpha \cdot n_3} \{ -(\alpha \cdot n_1)(\alpha \cdot n_2) u_{\tau_3}^2 + (\tau_1 \cdot n_3)(\tau_2 \cdot n_3) u_\alpha^2 \}.$$

Lemma 4.2 now follows by combining terms.

LEMMA 4.3. Let $T \in \{\Delta_h\}$, and for $P \in \Gamma$ let $\theta(P)$ be the angle from $\alpha(P)$ to the local tangent vector τ , measured counterclockwise. Then

$$\begin{aligned} \int_T \beta u (-u_{\tau_1 \tau_2}) dx dy &= \int_T \beta u_{\tau_1} u_{\tau_2} dx dy - \int_{\Gamma(T)} \beta (\alpha \cdot n_1) (\alpha \cdot n_2) \cot \theta u u_\tau d\tau \\ &\quad - \int_{\Gamma_3(T)} \beta \frac{(\tau_2 \cdot n_3)(\tau_1 \cdot n_3)}{\alpha \cdot n_3} u u_\alpha d\tau - \frac{1}{2} \int_{\Gamma(T)} \rho_\tau u^2 d\tau \\ &\quad + \int_T u [(\beta \phi_1)_{\tau_2} u_{\tau_1} + (\beta \phi_2)_{\tau_1} u_{\tau_2}] dx dy \end{aligned}$$

where

$$\phi_1 = \frac{(\alpha \cdot n_2)(\tau_1 \cdot n_3)}{(\alpha \cdot n_3)(\tau_1 \cdot n_2)}, \quad \phi_2 = \frac{(\alpha \cdot n_1)(\tau_2 \cdot n_3)}{(\alpha \cdot n_3)(\tau_2 \cdot n_1)}$$

and

$$\rho = \frac{\beta (\alpha \cdot n_1) (\alpha \cdot n_2) (\alpha \cdot \tau_3)}{\alpha \cdot n_3}.$$

Proof. It follows immediately from Lemma 4.1 and the definitions of ϕ_1 and ϕ_2 that $\phi_1 + \phi_2 = 1$. Applying the integration by parts formula (5), we obtain

$$\begin{aligned} \int_T \beta u(-u_{\tau_1 \tau_2}) \, dx \, dy &= \int_T \phi_1 \beta u(-u_{\tau_1 \tau_2}) \, dx \, dy + \int_T \phi_2 \beta u(-u_{\tau_2 \tau_1}) \, dx \, dy \\ &= - \int_{\Gamma} \beta u[\phi_1 u_{\tau_1}(\tau_2 \cdot n) + \phi_2 u_{\tau_2}(\tau_1 \cdot n)] \, d\tau \\ &\quad + \int_T \beta u_{\tau_2} u_{\tau_1} \, dx \, dy + \int_T u[(\beta \phi_1)_{\tau_2} u_{\tau_1} + (\beta \phi_2)_{\tau_1} u_{\tau_2}] \, dx \, dy. \end{aligned}$$

We now derive another form for the integral around $\Gamma(T)$. Using the definitions of ϕ_1 and ϕ_2 , (15b), and formula (16), we have

$$\phi_1 u_{\tau_1}(\tau_2 \cdot n_3) + \phi_2 u_{\tau_2}(\tau_1 \cdot n_3) = \frac{(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)}{\alpha \cdot n_3} u_{\alpha}.$$

We also have the following identity for $i = 1, 2$:

$$\frac{\tau_i \cdot n_3}{(\alpha \cdot n_3)(\alpha \cdot n_i)} = \frac{\tau_i \cdot \alpha}{\alpha \cdot n_i} - \frac{\alpha \cdot \tau_3}{\alpha \cdot n_3}.$$

This may be shown by writing $\tau_i = (\tau_i \cdot \alpha)\alpha + (\tau_i \cdot \gamma)\gamma$, and then applying (15b) and (15c).

Combining the preceding formulas, we obtain

$$\begin{aligned} & - \int_{\Gamma} \beta u[\phi_1 u_{\tau_1}(\tau_2 \cdot n) + \phi_2 u_{\tau_2}(\tau_1 \cdot n)] \, d\tau \\ &= - \int_{\Gamma_3} \beta \frac{(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)}{\alpha \cdot n_3} u u_{\alpha} \, d\tau \\ &\quad - \int_{\Gamma_1 \cup \Gamma_2} \beta (\alpha \cdot n_1)(\alpha \cdot n_2) \frac{\alpha \cdot \tau}{\alpha \cdot n} u u_{\tau} \, d\tau + \int_{\Gamma_1 \cup \Gamma_2} \rho u u_{\tau} \, d\tau \end{aligned}$$

where ρ is as defined in the statement of the lemma. An integration by parts yields for the last term

$$\begin{aligned} \int_{\Gamma_1 \cup \Gamma_2} \rho u u_{\tau} \, d\tau &= \frac{1}{2} \left\{ [\rho u^2]_{a_2}^{a_3} + [\rho u^2]_{a_3}^{a_1} - \int_{\Gamma_1 \cup \Gamma_2} \rho_{\tau} u^2 \, d\tau \right\} \\ &= - \int_{\Gamma_3} \rho u u_{\tau} \, d\tau - \frac{1}{2} \int_{\Gamma} \rho_{\tau} u^2 \, d\tau. \end{aligned}$$

The lemma now follows by combining results and observing that along Γ_i

$$\frac{\alpha \cdot \tau_i}{\alpha \cdot n_i} = \frac{\cos \theta_i}{\cos(\theta_i - \pi/2)} = \cot \theta_i$$

where $\theta_i(P)$ denotes the angle from $\alpha(P)$ to τ_i , measured counterclockwise.

5. Stability. In this section we derive the basic stability estimates for Problem P_h . These will be used to obtain error estimates in the next section. The norm in which we obtain stability is a weighted sum of L^2 norms of u_h and its tangential derivative

taken along *fronts* F_j , which describe the forward boundary of the solution after it has progressed through the first j layers. More specifically, we define

$$F_0 = \Gamma_{in}(\Omega),$$

$$F_j = F_{j-1} \cup \Gamma_{out}(S_j) - \Gamma_{in}(S_j), \quad j = 1, 2, \dots$$

The main result of this section will be the following stability theorem.

THEOREM 5.1. *If u_h is the solution of Problem \mathbf{P}_h , then for h sufficiently small*

$$\int_{F_j} \left\{ \frac{h^{3/2}(du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau + Mh^{1/2} \|(u_h)_\alpha\|_{\Omega_j}^2$$

$$\cong C \left\{ h^{1/2} \|f\|_{\Omega_j}^2 + h^{-1/2} \|P_{n-2}f\|_{\Omega_j}^2 + \int_{F_0} \left\{ \frac{h^{3/2}(du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau \right\}$$

where $\Omega_j = \cup_{k \leq j} S_k$ and M is a positive constant.

To prove this theorem, we first develop local stability results applicable over a single triangle. This is complicated somewhat by the fact that the two different types of triangles require different treatment. For each, we shall obtain a bound on the growth of $du_h/d\tau$ from the identities in the previous section. When these are combined suitably with bounds on the growth of u_h , the desired stability result is obtained. We note the factor $h^{3/2}$, which appears in Theorem 5.1, rather than h^2 which might be expected from approximation theoretic considerations.

The next two lemmas bound the growth in u_h over the two types of triangles.

LEMMA 5.1. *If T is a type I triangle and u_h satisfies (3) in T , then*

$$(17) \quad \int_{\Gamma(T)} u_h^2 \alpha \cdot n \, d\tau \leq C \{ h^{1/2} \|f\|_T^2 + \|P_{n-2}f\|_T^2 + h^{3/2} \|\nabla u_h\|_T^2 + \|u_h\|_T^2 \}.$$

Proof. Omitting the subscript T on the norms and inner products which follow, we apply (5) to $((u_h)_\alpha, u_h)$. This yields

$$\begin{aligned} \frac{1}{2} \int_{\Gamma} u_h^2 \alpha \cdot n \, d\tau &= \frac{1}{2} (u_h^2, \nabla \cdot \alpha) + ((u_h)_\alpha, u_h) \\ &= \frac{1}{2} (u_h^2, \nabla \cdot \alpha) + ((u_h)_\alpha, (I - P_{n-1})u_h) + ((u_h)_\alpha, P_{n-1}u_h) \\ &= \frac{1}{2} (u_h^2, \nabla \cdot \alpha) + ((\alpha - \alpha_0) \cdot \nabla u_h, (I - P_{n-1})u_h) - (f, (I - P_{n-1})u_h) \\ &\quad + (f, (I - P_{n-2})u_h) + (f, P_{n-2}u_h) - (\beta u_h, P_{n-1}u_h). \end{aligned}$$

In the last of these equalities, (3) and the fact that $(u_h)_{\alpha_0} \in \mathbf{P}_{n-1}(T)$ were used. Applying standard estimates we obtain

$$\begin{aligned} \int_{\Gamma} u_h^2 \alpha \cdot n \, d\tau &\leq C \{ [\|\alpha\|_{1,\infty} + \|\beta\|_{\infty}] \|u_h\|^2 + h^2 \|\alpha\|_{1,\infty} \|\nabla u_h\|^2 \\ &\quad + h \|f\| \|\nabla u_h\| + \|P_{n-2}f\| \|u_h\| \} \\ &\leq C \{ h^{1/2} \|f\|^2 + \|P_{n-2}f\|^2 + h^{3/2} \|\nabla u_h\|^2 + \|u_h\|^2 \}. \end{aligned}$$

LEMMA 5.2. *If T is a type II triangle and u_h satisfies (4) on T , then for any $\epsilon > 0$*

$$(18) \quad \int_{\Gamma(T)} u_h^2 \alpha \cdot n \, d\tau \leq \epsilon h^{3/2} |(u_h)_\alpha|_{\Gamma_{out}(T)}^2 + C \{ \|P_{n-2}f\|_T^2 + h^{3/2} \|\nabla u_h\|_T^2 + \|u_h\|_T^2 \}$$

where C depends on ϵ .

Proof. Again omitting the subscript T and using (5), we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Gamma} u_h^2 \alpha \cdot n \, d\tau &= \frac{1}{2} (u_h^2, \nabla \cdot \alpha) + ((u_h)_\alpha, u_h) \\ &= \frac{1}{2} (u_h^2, \nabla \cdot \alpha) + ((u_h)_\alpha, (I - P_{n-2})u_h) + (P_{n-2}f, P_{n-2}u_h) - (\beta u_h, P_{n-2}u_h). \end{aligned}$$

It then follows by standard estimates that

$$\begin{aligned} \int_{\Gamma} u_h^2 \alpha \cdot n \, d\tau &\leq C \{ [\|\alpha\|_{1,\infty} + \|\beta\|_\infty + 1] \|u_h\|^2 + \|P_{n-2}f\|^2 + h \|(u_h)_\alpha\| \|\nabla u_h\| \} \\ &\leq C \{ \|u_h\|^2 + \|P_{n-2}f\|^2 + \varepsilon h^{1/2} \|(u_h)_\alpha\|^2 + \varepsilon^{-1} h^{3/2} \|\nabla u_h\|^2 \} \quad (\text{for any } \varepsilon > 0). \end{aligned}$$

Using (12) and (4) we obtain

$$\begin{aligned} \|(u_h)_\alpha\|^2 &\leq C \{ h |(u_h)_\alpha|_{\Gamma_{out}(T)}^2 + \|P_{n-2}(u_h)_\alpha\|^2 + h^2 \|\nabla u_h\|^2 \} \\ &\leq C \{ h |(u_h)_\alpha|_{\Gamma_{out}(T)}^2 + \|P_{n-2}f\|^2 + \|u_h\|^2 + h^2 \|\nabla u_h\|^2 \}. \end{aligned}$$

Inserting this result in the previous inequality and replacing ε by ε/C establishes the lemma.

We now combine Lemmas 4.2, 4.3, 5.1 and 5.2 into a single local stability result. Theorem 5.1 will then be proved by an appropriate application of this local result.

THEOREM 5.2. *Suppose u_h satisfies (3) on type I triangles and (4) on type II triangles. The following stability results then hold:*

(i) *For a type I triangle T*

$$\begin{aligned} &\int_{\Gamma(T)} \left\{ \frac{h^{3/2} (du_h/d\tau)^2}{\alpha \cdot n} + u_h^2 \alpha \cdot n \right\} d\tau - 2h^{3/2} \int_{\Gamma(T)} \beta \cot \theta u_h (u_h)_\tau \, d\tau \\ &\leq C \left\{ h^{1/2} \|f\|_T^2 + h^{-1/2} \|P_{n-2}f\|_T^2 + h \int_{\Gamma_{in}(T)} [h^{3/2} (u_h)_\tau^2 + u_h^2] \, d\tau \right\}. \end{aligned}$$

(ii) *For a type II triangle T*

$$\begin{aligned} &\int_{\Gamma(T)} \left\{ \frac{h^{3/2} (du_h/d\tau)^2}{\alpha \cdot n} + u_h^2 \alpha \cdot n \right\} d\tau - 2h^{3/2} \int_{\Gamma(T)} \beta \cot \theta u_h (u_h)_\tau \, d\tau + \Lambda h^{3/2} |(u_h)_\alpha|_{\Gamma_{out}(T)}^2 \\ &\leq C \left\{ h^{3/2} \|f\|_T^2 + h^{-1/2} \|P_{n-2}f\|_T^2 + h \int_{\Gamma_{in}(T)} [h^{3/2} (u_h)_\tau^2 + u_h^2] \, d\tau \right\} \end{aligned}$$

where θ is the angle defined in Lemma 4.3 and Λ is a positive constant, independent of h and u_h .

Proof. Let Q be the centroid of triangle T and choose $v_h = -(u_h)_{\tau_1\tau_2}$ in (3) and (4). Using Lemmas 4.2 and 4.3, (3) and (4) become

$$\begin{aligned} &\frac{1}{2} \int_{\Gamma} \frac{(\alpha \cdot n_1)(\alpha \cdot n_2)}{\alpha \cdot n} (u_h)_\tau^2 \, d\tau - \int_{\Gamma} \beta (\alpha \cdot n_1)(\alpha \cdot n_2) \cot \theta u_h (u_h)_\tau \, d\tau \\ (19) \quad &\quad - \frac{1}{2} \int_{\Gamma_3} \frac{(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)}{\alpha \cdot n_3} [(u_h)_\alpha^2 + 2\beta u_h (u_h)_\alpha] \, d\tau \\ &= - \left\{ \int_T f (u_h)_{\tau_1\tau_2} \, dx \, dy + R \right\} \end{aligned}$$

where

$$R = \int_T \beta(u_h)_{\tau_1}(u_h)_{\tau_2} dx dy + \frac{1}{2} \int_T [(A_1)_{\tau_2}(u_h)_{\tau_1}^2 + (A_2)_{\tau_1}(u_h)_{\tau_2}^2] dx dy + \int_T u_h [(\beta\phi_1)_{\tau_2}(u_h)_{\tau_1} + (\beta\phi_2)_{\tau_1}(u_h)_{\tau_2}] dx dy - \frac{1}{2} \int_{\Gamma} (\rho)_{\tau} u_h^2 d\tau$$

and $A_1, A_2, \phi_1, \phi_2,$ and ρ are defined in Lemmas 4.2 and 4.3. First note that by standard estimates

$$(20) \quad |R| \leq C\{\|\nabla u_h\|_T^2 + \|u_h\|_T^2 + |u_h|_{\Gamma(T)}^2\} \leq C\{\|\nabla u_h\|_T^2 + h^{-1}\|u_h\|_T^2\}.$$

Next, since $(u_h)_{\tau_1\tau_2} \in P_{n-2}(T),$

$$(21) \quad \left| \int_T f(u_h)_{\tau_1\tau_2} dx dy \right| \leq C \|P_{n-2}f\|_T \|(u_h)_{\tau_1\tau_2}\|_T \leq Ch^{-1} \|P_{n-2}f\|_T \|\nabla u_h\|_T \leq C\{h^{-2}\|P_{n-2}f\|_T^2 + \|\nabla u_h\|_T^2\}.$$

Using these bounds in (19), then dividing the resulting inequality by $\frac{1}{2}(\alpha(Q) \cdot n_1) \cdot (\alpha(Q) \cdot n_2)$ (a positive quantity by hypothesis H2), we obtain

$$(22) \quad \int_{\Gamma} (1 + \sigma) \frac{(u_h)_{\tau}^2}{\alpha \cdot n} d\tau - 2 \int_{\Gamma} \beta(1 + \sigma) \cot \theta u_h(u_h)_{\tau} d\tau + \int_{\Gamma_3} \omega \{(u_h)_{\alpha}^2 + 2\beta u_h(u_h)_{\alpha}\} d\tau \leq C\{\|\nabla u_h\|_T^2 + h^{-1}\|u_h\|_T^2 + h^{-2}\|P_{n-2}f\|_T^2\}$$

where σ is defined by

$$1 + \sigma = \frac{(\alpha \cdot n_1)(\alpha \cdot n_2)}{(\alpha(Q) \cdot n_1)(\alpha(Q) \cdot n_2)}$$

and

$$\omega = -\frac{(\tau_1 \cdot n_3)(\tau_2 \cdot n_3)}{(\alpha(Q) \cdot n_1)(\alpha(Q) \cdot n_2)} \frac{1}{\alpha \cdot n_3}.$$

Note that σ is of order $h.$ The terms on the left side of (22) which have σ as a factor can therefore be bounded by $C\{\|\nabla u_h\|_T^2 + \|u_h\|_T^2\}.$ We conclude that (22) remains valid with σ deleted.

Note also that $|\omega|$ is bounded away from both 0 (by H1) and ∞ (by H2). Moreover, since $\tau_1 \cdot n_3 < 0$ and $\tau_2 \cdot n_3 > 0,$ ω will have the same sign as $\alpha \cdot n_3,$ which is negative for type I triangles and positive for type II triangles. Thus there exist constants μ, λ such that

$$(23) \quad (a) \quad |\omega| < \mu < \infty \text{ for a triangle of either type,} \\ (b) \quad \omega > \lambda > 0 \text{ for a type II triangle.}$$

This allows us to bound the term

$$\gamma \equiv \int_{\Gamma_3} \omega \{(u_h)_{\alpha}^2 + 2\beta u_h(u_h)_{\alpha}\} d\tau$$

in (22) from above for a type I triangle and from below for a type II triangle.

First consider the case of a type I triangle. Here

$$|\gamma| \leq C\{|(u_h)_\alpha|_{\Gamma_3}^2 + |u_h|_{\Gamma_3}^2\}.$$

But

$$\begin{aligned} |(u_h)_\alpha|_{\Gamma_3} &\leq |(u_h)_{\alpha_0}|_{\Gamma_3} + |(\alpha - \alpha_0) \cdot \nabla u_h|_{\Gamma_3} \\ &\leq C\{h^{-1/2}\|(u_h)_{\alpha_0}\|_T + h^{1/2}\|\nabla u_h\|_T\}. \end{aligned}$$

Now from (3) we have

$$(u_h)_{\alpha_0} = P_{n-1}\{f - \beta u_h - (\alpha - \alpha_0) \cdot \nabla u_h\}.$$

Hence

$$\|(u_h)_{\alpha_0}\|_T \leq C\{\|f\|_T + \|u_h\|_T + h\|\nabla u_h\|_T\}.$$

Thus we obtain

$$(24) \quad |\gamma| \leq C\{h^{-1}\|f\|_T^2 + h^{-1}\|u_h\|_T^2 + h\|\nabla u_h\|_T^2\}.$$

We use (24) in (22), multiply the resulting inequality by $h^{3/2}$, then add (17). Application of the estimates (9) and (10) then leads directly to the desired stability result for type I triangles.

Next consider the case of a type II triangle. Here

$$(25) \quad \begin{aligned} \gamma &> \lambda|(u_h)_\alpha|_{\Gamma_3}^2 - 2\|\beta\|_\infty|u_h|_{\Gamma_3}|(u_h)_\alpha|_{\Gamma_3} \\ &\geq (\lambda - \delta)|(u_h)_\alpha|_{\Gamma_3}^2 - \frac{(\|\beta\|_\infty|u_h|_{\Gamma_3})^2}{\delta} \end{aligned}$$

for any $\delta > 0$. We use (25) in (22), multiply the resulting inequality by $h^{3/2}$, then add (18). We choose $\delta = \varepsilon = \lambda/4$ and apply (9) and (10) again. The desired stability result for type II triangles then follows upon setting $\Lambda = \lambda/2$.

The local results of Theorem 5.2 lead to global stability for u_h and $du_h/d\tau$ along interelement boundaries. Before showing this, however, we reformulate Theorem 5.2 in a way that will enable us to control the growth of $(u_h)_\alpha$ as well.

THEOREM 5.2a. *There exists a positive constant M such that for a triangle T of either type*

$$\begin{aligned} &\int_{\Gamma(T)} \left[\frac{h^{3/2}(du_h/d\tau)^2}{\alpha \cdot n} + u_h^2 \alpha \cdot n \right] d\tau - 2h^{3/2} \int_{\Gamma(T)} \beta \cot \theta u_h (u_h)_\tau d\tau + Mh^{1/2} \|(u_h)_\alpha\|_T^2 \\ &\leq C \left\{ h^{1/2} \|f\|_T^2 + h^{-1/2} \|P_{n-2}f\|_T^2 + h \int_{\Gamma_{in}(T)} [h^{3/2}(u_h)_\tau^2 + u_h^2] d\tau \right\}. \end{aligned}$$

Proof. For a type I triangle T , we write (3) as

$$((u_h)_{\alpha_0}, v_h) = (f - \beta u_h - (\alpha - \alpha_0) \cdot \nabla u_h, v_h) \quad \text{all } v_h \in P_{n-1}(T),$$

which implies that

$$(u_h)_{\alpha_0} = P_{n-1}\{f - \beta u_h - (\alpha - \alpha_0) \cdot \nabla u_h\}.$$

Thus

$$\|(u_h)_{\alpha_0}\|_T \leq C\{\|f\|_T + \|u_h\|_T + h\|\nabla u_h\|_T\}$$

and

$$\begin{aligned} \|(u_h)_\alpha\|_T &\leq \|(u_h)_{\alpha_0}\|_T + \|(\alpha - \alpha_0) \cdot \nabla u_h\|_T \\ &\leq C\{\|f\|_T + \|u_h\|_T + h\|\nabla u_h\|_T\}. \end{aligned}$$

Squaring and applying Lemma 3.2, we obtain

$$\|(\mathbf{u}_h)_\alpha\|_T^2 \leq C\{\|f\|_T^2 + h|\mathbf{u}_h|_{\Gamma_{in}(T)}^2 + h^3|(\mathbf{u}_h)_\tau|_{\Gamma_{in}(T)}^2\}.$$

For any $M > 0$, we may add $Mh^{1/2}$ times the preceding inequality to part (i) of Theorem 5.2 and obtain the desired result for type I triangles.

For a type II triangle T , we obtain from Lemma 3.3

$$(26) \quad \|(\mathbf{u}_h)_\alpha\|_T^2 \leq C\{h|(\mathbf{u}_h)_\alpha|_{\Gamma_{out}(T)}^2 + \|P_{n-2}(\mathbf{u}_h)_\alpha\|_T^2 + h^2\|\nabla \mathbf{u}_h\|_T^2\}.$$

Analogous to what has just been done for type I triangles, we find that

$$P_{n-2}(\mathbf{u}_h)_\alpha = P_{n-2}\{f - \beta \mathbf{u}_h - (\alpha - \alpha_0) \cdot \nabla \mathbf{u}_h\},$$

and that

$$\|P_{n-2}(\mathbf{u}_h)_\alpha\|_T^2 \leq C\{\|f\|_T^2 + h|\mathbf{u}_h|_{\Gamma_{in}(T)}^2 + h^3|(\mathbf{u}_h)_\tau|_{\Gamma_{in}(T)}^2\}.$$

Using this inequality in (26) and applying Lemma 3.2, we obtain

$$\|(\mathbf{u}_h)_\alpha\|_T^2 \leq C\{h|(\mathbf{u}_h)_\alpha|_{\Gamma_{out}(T)}^2 + \|f\|_T^2 + h|\mathbf{u}_h|_{\Gamma_{in}(T)}^2 + h^3|(\mathbf{u}_h)_\tau|_{\Gamma_{in}(T)}^2\}.$$

Adding $Mh^{1/2}$ times the above inequality to part (ii) of Theorem 5.2, where $M \equiv \Lambda/C$, produces the desired result for type II triangles.

Before extending Theorem 5.2a into a global stability result, we establish the following simple lemma.

LEMMA 5.3. *If*

$$(27) \quad x_i + a_i \leq (1 + \kappa h)x_{i-1} + b_i, \quad i = 1, 2, \dots$$

where $\kappa > 0$, $h > 0$ and x_i , a_i , and b_i are nonnegative for all i , then

$$(28) \quad x_n + \sum_{i=1}^n a_i \leq e^{\kappa nh} \left(x_0 + \sum_{i=1}^n b_i \right).$$

Proof. The solution of the inequalities (27) is

$$x_n \leq \sum_{i=1}^n (1 + \kappa h)^{n-i} (b_i - a_i) + (1 + \kappa h)^n x_0.$$

Thus

$$x_n \leq (1 + \kappa h)^n \left(x_0 + \sum_{i=1}^n b_i \right) - \sum_{i=1}^n a_i.$$

The result follows upon noting that

$$(1 + \kappa h)^n < e^{\kappa nh}.$$

We are now ready to prove the main stability result of this section.

Proof of Theorem 5.1. For any triangle T , we infer from Theorem 5.2a that

$$\begin{aligned} & \int_{\Gamma_{out}(T)} \left\{ \frac{h^{3/2}(du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau \\ & \quad - 2h^{3/2} \int_{\Gamma_{out}(T)} \beta \cot \theta u_h (u_h)_\tau d\tau + Mh^{1/2} \|(\mathbf{u}_h)_\alpha\|_T^2 \\ & \cong \int_{\Gamma_{in}(T)} \left\{ \frac{h^{3/2}(du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau + 2h^{3/2} \int_{\Gamma_{in}(T)} \beta \cot \theta u_h (u_h)_\tau d\tau \\ & \quad + C \left\{ h^{1/2} \|f\|_T^2 + h^{-1/2} \|P_{n-2}f\|_T^2 + h \int_{\Gamma_{in}(T)} [h^{3/2}(u_h)_\tau^2 + u_h^2] d\tau \right\}. \end{aligned}$$

Summation over all triangles $T \in S_j$ yields

$$(29) \quad p_j + q_j + Mh^{1/2} \|(u_h)_\alpha\|_{S_j}^2 \cong \{1 + O(h)\} p_{j-1} + q_{j-1} + C\{h^{1/2} \|f\|_{S_j}^2 + h^{-1/2} \|P_{n-2} f\|_{S_j}^2\}$$

where

$$p_j \equiv \int_{F_j} \left\{ \frac{h^{3/2} (du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau$$

and

$$q_j \equiv 2h^{3/2} \int_{F_j} \beta \cot \theta u_h(u_h)_\tau d\tau,$$

with the convention that integrals over F_j are taken *left to right* (thus fixing the sign of $(u_h)_\tau$ in the definition of q_j).

Next, we note that

$$(30) \quad |q_j| \cong Ch^{3/4} p_j$$

(an immediate consequence of the Schwarz and arithmetic-geometric mean inequalities). Hence

$$(31) \quad |q_j| \cong p_j + q_j$$

for h sufficiently small. Thus in (29) we may write

$$\begin{aligned} \{1 + O(h)\} p_{j-1} + q_{j-1} &= \{1 + O(h)\} (p_{j-1} + q_{j-1}) + O(h) q_{j-1} \\ &\cong \{1 + O(h)\} (p_{j-1} + q_{j-1}). \end{aligned}$$

Equation (29) then becomes

$$(32) \quad \begin{aligned} p_j + q_j + Mh^{1/2} \|(u_h)_\alpha\|_{S_j}^2 \\ \cong \{1 + O(h)\} (p_{j-1} + q_{j-1}) + C\{h^{1/2} \|f\|_{S_j}^2 + h^{-1/2} \|P_{n-2} f\|_{S_j}^2\}. \end{aligned}$$

Inequality (31) implies that $p_j + q_j \cong 0$ for h sufficiently small, in which case Lemma 5.3 is applicable to (32). Using hypothesis H4, we get

$$(33) \quad p_j + q_j + Mh^{1/2} \|(u_h)_\alpha\|_{\Omega_j}^2 \cong C\{p_0 + q_0 + h^{1/2} \|f\|_{\Omega_j}^2 + h^{-1/2} \|P_{n-2} f\|_{\Omega_j}^2\}.$$

Using (30) again, we note that

$$p_j + q_j = p_j \{1 + O(h^{3/4})\},$$

and infer that (33) remains valid with q_j and q_0 deleted. The result is Theorem 5.1.

As a consequence of Theorem 5.1 and Lemma 3.2, we can now establish stability for $u_h, \nabla u_h$ and $(u_h)_\alpha$ over Ω .

THEOREM 5.3.

$$\begin{aligned} \|u_h\|_\Omega^2 + h^{3/2} \|\nabla u_h\|_\Omega^2 + h^{1/2} \|(u_h)_\alpha\|_\Omega^2 \\ \cong C \left\{ h^{1/2} \|f\|_\Omega^2 + h^{-1/2} \|P_{n-2} f\|_\Omega^2 + h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(\Omega)}^2 + |u_h|_{\Gamma_{in}(\Omega)}^2 \right\}. \end{aligned}$$

Proof. From Lemma 3.2, we have

$$\|u_h\|_{S_j}^2 \cong C\{h|u_h|_{F_{j-1}}^2 + h^2 \|f\|_{S_j}^2\}$$

and

$$\|\nabla u_h\|_{S_j}^2 \cong C \left\{ h \left| \frac{du_h}{d\tau} \right|_{F_{j-1}}^2 + h|u_h|_{F_{j-1}}^2 + \|f\|_{S_j}^2 \right\}.$$

Thus

$$\begin{aligned} \|u_h\|_{S_j}^2 + h^{3/2} \|\nabla u_h\|_{S_j}^2 &\leq C \left\{ h |u_h|_{F_{j-1}}^2 + h^{5/2} \left| \frac{du_h}{d\tau} \right|_{F_{j-1}}^2 + h^{3/2} \|f\|_{S_j}^2 \right\} \\ &\leq Ch \int_{F_{j-1}} \left\{ \frac{h^{3/2} (du_h/d\tau)^2}{|\alpha \cdot n|} + u_h^2 |\alpha \cdot n| \right\} d\tau + h^{3/2} \|f\|_{S_j}^2. \end{aligned}$$

Applying Theorem 5.1, we obtain

$$\|u_h\|_{S_j}^2 + h^{3/2} \|\nabla u_h\|_{S_j}^2 \leq Ch \left\{ h^{1/2} \|f\|_{\Omega_j}^2 + h^{-1/2} \|P_{n-2} f\|_{\Omega_j}^2 + h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(\Omega)}^2 + |u_h|_{\Gamma_{in}(\Omega)}^2 \right\}.$$

Summation over all layers S_j yields

$$(34) \quad \begin{aligned} &\|u_h\|_{\Omega}^2 + h^{3/2} \|\nabla u_h\|_{\Omega}^2 \\ &\leq C \left\{ h^{1/2} \|f\|_{\Omega}^2 + h^{-1/2} \|P_{n-2} f\|_{\Omega}^2 + h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(\Omega)}^2 + |u_h|_{\Gamma_{in}(\Omega)}^2 \right\}. \end{aligned}$$

To account for $\|(u_h)_\alpha\|_{\Omega}^2$, we apply Theorem 5.1 with $F_j = \Gamma_{out}(\Omega)$ to conclude that

$$(35) \quad h^{1/2} \|(u_h)_\alpha\|_{\Omega}^2 \leq C \left\{ h^{1/2} \|f\|_{\Omega}^2 + h^{-1/2} \|P_{n-2} f\|_{\Omega}^2 + h^{3/2} \left| \frac{du_h}{d\tau} \right|_{\Gamma_{in}(\Omega)}^2 + |u_h|_{\Gamma_{in}(\Omega)}^2 \right\}.$$

The statement in the theorem is the sum of (34) and (35).

6. Error estimates. To obtain error estimates for the method, we define an interpolant $u_I \in S_h^n$ by the following conditions:

- (i) $u_I(a_i) = u(a_i)$ for all triangle vertices a_i ;
- (ii) $\int_{\Gamma} (u_I - u) \tau^l d\tau = 0, l = 0, 1, \dots, n-2$ for all triangle sides Γ ;
- (iii) $\int_T (u_I - u) q dx dy = 0$ for all $q \in P_{n-3}(T)$ and all triangles T .

It is straightforward to show (for example, using the techniques in [3, Chap. 3]) that u_I has the following approximation properties:

$$(36) \quad \|u - u_I\|_{j,T} \leq Ch^{n+1-j} \|u\|_{n+1,T}, \quad j = 0, 1$$

and

$$(37) \quad |u - u_I|_{j,\Gamma(T)} \leq Ch^{n+1/2-j} \|u\|_{n+1,T}, \quad j = 0, 1.$$

Rewriting (3) and (4) in the form

$$((u_h - u_I)_\alpha + \beta(u_h - u_I), v_h)_T = ((u - u_I)_\alpha + \beta(u - u_I), v_h)_T,$$

we may apply Theorems 5.1 and 5.3 with u_h replaced by $u_h - u_I$, and f replaced by $r \equiv (u - u_I)_\alpha + \beta(u - u_I)$.

From (36), it follows immediately that

$$\|r\|_{\Omega_j} \leq Ch^n \|u\|_{n+1,\Omega_j}.$$

Moreover, for all $v_h \in P_{n-2}(T)$,

$$((u - u_I)_{\alpha_0}, v_h)_T = \int_{\Gamma} (u - u_I) v_h \alpha_0 \cdot n d\Gamma - ((u - u_I), (v_h)_{\alpha_0})_T = 0$$

since $(v_h)_{\alpha_0} \in P_{n-3}(T)$. Hence

$$\begin{aligned} \|P_{n-2} r\|_{\Omega_j} &\leq \|P_{n-2}(\alpha - \alpha_0) \cdot \nabla(u - u_I)\|_{\Omega_j} + \|\beta(u - u_I)\|_{\Omega_j} \\ &\leq Ch^{n+1} \|u\|_{n+1,\Omega_j}. \end{aligned}$$

We also assume, for convenience, that $u_h = u_I$ on $\Gamma_{in}(\Omega)$. Insertion of these bounds into Theorems 5.1 and 5.3 now yields the following theorem.

THEOREM 6.1. *Let u be the solution of (1) and u_h the solution of Problem P_h . If $u \in H^{n+1}(\Omega)$, there exists a constant C independent of h such that*

$$\begin{aligned} \|u - u_h\|_{\Omega} &\leq Ch^{n+1/4} \|u\|_{n+1,\Omega}, \\ \|\nabla(u - u_h)\|_{\Omega} &\leq Ch^{n-1/2} \|u\|_{n+1,\Omega}, \\ \|(u - u_h)_{\alpha}\| &\leq Ch^n \|u\|_{n+1,\Omega}, \end{aligned}$$

and for $j = 1, 2, \dots$

$$\left\{ \int_{F_j} (u - u_h)^2 d\tau \right\}^{1/2} \leq Ch^{n+1/4} \|u\|_{n+1,\Omega_j}$$

and

$$\left\{ \int_{F_j} \left[\frac{d}{d\tau}(u - u_h) \right]^2 d\tau \right\}^{1/2} \leq Ch^{n-1/2} \|u\|_{n+1,\Omega_j}.$$

7. Computational results. We shall present numerical results for three test problems, each involving approximation by piecewise quadratics. The first problem is

$$.6u_x + .8u_y - u = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

with initial data along the x and y axes chosen to make the solution be $u = \exp(.6x + .8y)$. A triangulation was obtained by dividing the domain uniformly into N^2 squares, then dividing each square into two right triangles by drawing a diagonal parallel to the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. In Table 6.1, values are presented for the quantities

$$\begin{aligned} E &\equiv \sqrt{\int_{\Gamma_{out}(\Omega)} (u_h - u)^2 \alpha \cdot n d\tau}, \\ E' &\equiv \sqrt{\int_{\Gamma_{out}(\Omega)} \frac{(du_h/d\tau)^2}{\alpha \cdot n} d\tau} \end{aligned}$$

and ratios of consecutive values of these quantities as N is doubled.

Our theory predicts errors of order $h^{n+1/4}$ in u_h and $h^{n-1/2}$ in $du_h/d\tau$. If these estimates correctly described the rates of convergence of u_h and $du_h/d\tau$, the ratios of consecutive errors in these quantities would have limiting values $2^{2.25} = 4.76$ and $2^{1.5} = 2.83$, respectively. In fact, the ratios in Table 6.1 are consistently larger than these values, indicating a convergence rate similar to that of an interpolant of the exact solution. This example illustrates the tendency of the method to achieve the optimal

TABLE 6.1
Numerical results for first test problem.

N	E	Ratio	E'	Ratio
2	.735 (-3)	—	.132 (-1)	—
4	.985 (-4)	7.46	.355 (-2)	3.73
8	.136 (-4)	7.23	.917 (-3)	3.87
16	.182 (-5)	7.48	.233 (-3)	3.94
32	.235 (-6)	7.37	.587 (-4)	3.97
64	.299 (-7)	7.86	.147 (-4)	3.98
128	.379 (-8)	7.93	.369 (-5)	3.99

order of approximation on problems with a very smooth u and a regular triangulation. A partial explanation may be found in [10].

The second test problem is

$$u_x + u_y = 0, \quad 0 < x < 1, \quad 0 < y < 1$$

with initial data chosen to make the solution be $u = |t|^{2.5+\epsilon}/2.5$, $\epsilon \ll 1$, where $t = (y - x)/\sqrt{2}$ is a coordinate orthogonal to the characteristic direction. The domain and triangulation are as in the previous problem. Here there is no extra differentiability in u beyond what is needed for applicability of our theoretical estimates. In this case, the ratios appear to be consistent with the theoretical predictions (see Table 6.2).

The last test problem is

$$\sqrt{3}u_t + u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

with a discontinuous initial condition:

$$u(x, 0) = \begin{cases} 1, & x < 0, \\ .5, & x = 0, \\ 0, & x > 0. \end{cases}$$

The discontinuity propagates away from $(x, t) = (0, 0)$ along a line making an angle 60° with the x -axis. We again use a uniform set of right isosceles triangles in the triangulation, but here the hypotenuses are taken parallel to the x -axis.

Both the continuous and discontinuous finite element methods were applied to this problem. Table 6.3 indicates the L^2 error in u_h at $t = 1$ and $t = 2$, displayed as functions of the hypotenuse length h (the time step is $\Delta t = h/2$). For the range of

TABLE 6.2
Numerical results for second test problem.

N	E	Ratio	E'	Ratio
2	.509 (-2)	—	.412 (-1)	—
4	.785 (-3)	6.49	.110 (-1)	3.73
8	.141 (-3)	5.58	.312 (-2)	3.53
16	.286 (-4)	4.93	.977 (-3)	3.20
32	.598 (-5)	4.78	.297 (-3)	3.29
64	.129 (-5)	4.65	.100 (-3)	2.97
128	.276 (-6)	4.66	.338 (-4)	2.96

TABLE 6.3
 L^2 errors for third test problem.

h	Continuous method		Discontinuous method	
	$t = 1$	$t = 2$	$t = 1$	$t = 2$
1	.298	.377	.247	.231
.5	.267	.263	.163	.182
.25	.186	.194	.129	.137
.125	.137	.157	.0970	.103
.0625	.111	.117	.0725	.0784
.03125	.0830	.0894	.0554	.0577
.015625	.0632	.0680	.0408	.0434
.0078125	.0481	.0519	.0307	.0327

discretizations presented, the discontinuous method yields an error 29–39% less than the continuous method. However, the discontinuous piecewise quadratic has *three* times as many degrees of freedom over the triangulation as does its continuous counterpart. Provision for computational cost would thus reduce the advantage of the discontinuous method on this problem.

Finally, in Figs. 6.1 and 6.2, the continuous and discontinuous solutions are displayed for $h = .03125$ at time $t = 2$ (128 time steps). The discontinuity has been smoothed more by the continuous method than by the discontinuous method.

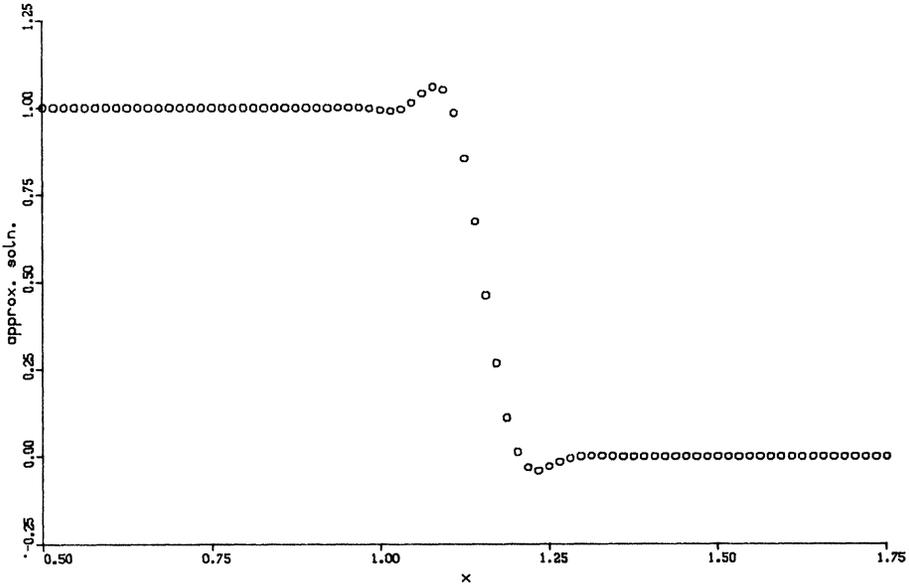


FIG. 6.1. *Continuous method: $t = 2$.*

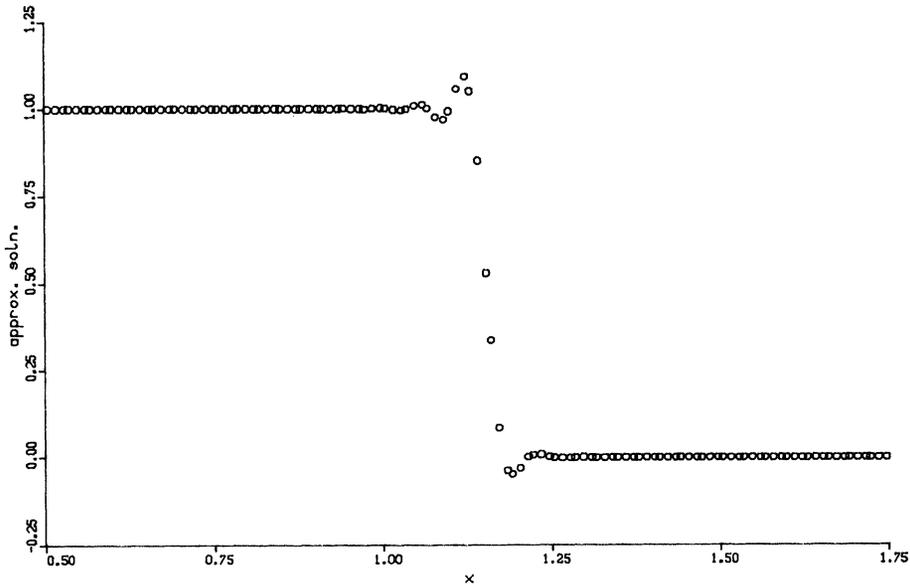


FIG. 6.2. *Discontinuous method: $t = 2$.*

Appendix. Suppose $\Delta_h = \{T_1, T_2, \dots, T_N\}$ is a triangulation of Ω such that $|\alpha \cdot n| > 0$ along every side of every triangle, where α is continuous and of unit length over Ω . We will show that the triangles in Δ_h can be ordered such that

$$(38) \quad \Gamma_{in}(T_k) \subseteq \Gamma_{out}\left(\bigcup_{i < k} T_i\right) \cup \left(\Gamma_{in}(\Omega) - \Gamma_{in}\left(\bigcup_{i < k} T_i\right)\right).$$

DEFINITION A1. A *path* from $T_j \in \Delta_h$ to $T_k \in \Delta_h$ is an ordered set of 2 or more triangles

$$T_{i_1}, \dots, T_{i_m}$$

such that $T_{i_1} = T_j$, $T_{i_m} = T_k$ and for $l = 1, \dots, m - 1$, T_{i_l} and $T_{i_{l+1}}$ have a common side which serves as outflow from T_{i_l} and inflow to $T_{i_{l+1}}$.

DEFINITION A2. A *cycle* is a path from a triangle to itself.

LEMMA A.1.

- (i) Δ_h has no cycles;
- (ii) There exists a triangle whose inflow is a subset of $\Gamma_{in}(\Omega)$;
- (iii) The triangles in Δ_h can be ordered in a manner consistent with (38).

Proof. (i). Suppose there were a cycle with enclosed triangles as depicted in Fig. A.1, and consider any characteristic C inside the enclosed region. C must enter the enclosed region from some triangle A of the cycle, and leave it via another triangle B of the cycle. Moreover, the set of triangles intersecting with this portion of C forms a path through the interior from A to B . This path permits formation of a new cycle having fewer enclosed triangles than the original one. Continuing in this manner, we eventually obtain a cycle with no enclosed triangles, as shown in Fig. A.2. However, Fig. A.2 describes a situation where, at the common vertex P , the value of α is such that $\alpha(P) \cdot n_i$ has the same sign for each of the normals n_i depicted in the figure. But α , by assumption, is well defined at P , so $\alpha(P) \cdot x$ will be of constant sign only for vectors x lying in some half plane. It is geometrically impossible for all the normals n_i to lie in a common half plane. Thus there cannot be any cycles.

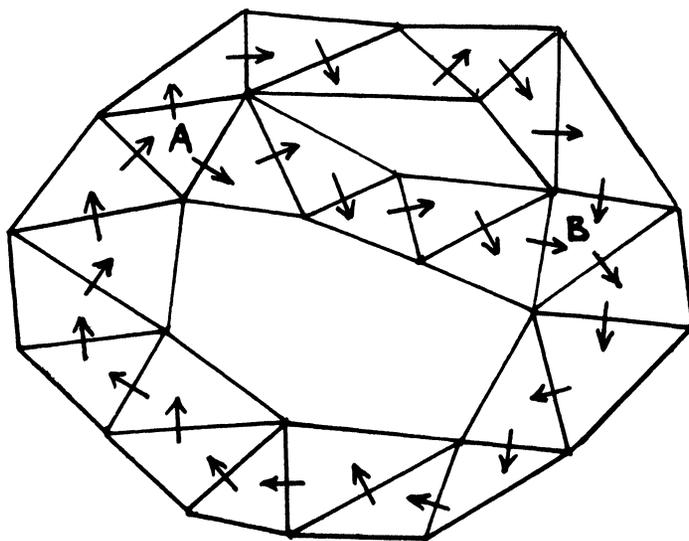


FIG. A.1

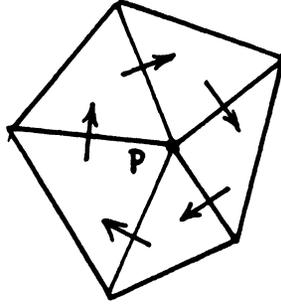


FIG. A.2

(ii) If the inflow to T_i is not a subset of $\Gamma_{in}(\Omega)$, there must be a path to T_i from another triangle T_j , $j \neq i$. If this were true for all triangles $T_i \in \Delta_h$, there would have to be a cycle, which was ruled out in part (i) of the lemma.

(iii) This follows by induction, using part (ii) of the lemma.

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