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Geometric decompositions and local bases for spaces of finite element differential forms ☆

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ABSTRACT

We study the two primary families of spaces of finite element differential forms with respect to a simplicial mesh in any number of space dimensions. These spaces are generalizations of the classical finite element spaces for vector fields, frequently referred to as Raviart–Thomas, Brezzi–Douglas–Marini, and Nédélec spaces. In the present paper, we derive geometric decompositions of these spaces which lead directly to explicit local bases for them, generalizing the Bernstein basis for ordinary Lagrange finite elements. The approach applies to both families of finite element spaces, for arbitrary polynomial degree, arbitrary order of the differential forms, and an arbitrary simplicial triangulation in any number of space dimensions. A prominent role in the construction is played by the notion of a consistent family of extension operators, which expresses in an abstract framework a sufficient condition for deriving a geometric decomposition of a finite element space leading to a local basis.

1. Introduction

The study of finite element exterior calculus has given increased insight into the construction of stable and accurate finite element methods for problems appearing in various applications, ranging from electromagnetics to elasticity. Instead of considering the design of discrete methods for each particular problem separately, it has proved beneficial to simultaneously study approximations of a family of problems, tied together by a common differential complex.

To be more specific, let $\Omega \subset \mathbb{R}^n$ and let $H\Lambda^k(\Omega)$ be the space of differential k forms ω on Ω, which is in L^2, and where its exterior derivative, $d\omega$, is also in L^2. The L^2 version of the de Rham complex then takes the form

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0.$$

The basic construction in finite element exterior calculus is of a corresponding subcomplex

$$0 \to \Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_h^n \to 0,$$

where the spaces Λ_h^k are finite-dimensional subspaces of $H\Lambda^k(\Omega)$ consisting of piecewise polynomial differential forms with respect to a partition of the domain Ω. In the theoretical analysis of the stability of numerical methods constructed from this discrete complex, bounded projections $\Pi_h : H\Lambda^k(\Omega) \to \Lambda_h^k$ are utilized, such that the diagram

$$\begin{array}{ccccccccc}
0 \to & H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) & \to 0 \\
 & \downarrow \Pi_h & & \downarrow \Pi_h & & & & \downarrow \Pi_h & \\
0 \to & \Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda_h^n & \to 0
\end{array}$$

commutes. For a general reference to finite element exterior calculus, we refer to the survey paper [2], and references given therein. As is shown there, the spaces Λ_h^k are taken from two main families. Either Λ_h^k is one of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ consisting of all elements of $H\Lambda^k(\Omega)$ which restrict to polynomial k-forms of degree at most r on each simplex T in the partition $\mathcal{T}$, or $\Lambda_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T})$, which is a space which sits between $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_{r-1}\Lambda^k(\mathcal{T})$ (the exact definition will be recalled below). These spaces are generalizations of the Raviart–Thomas and Brezzi–Douglas–Marini spaces used to discretize $H(\mathrm{div})$ and $H(\mathrm{rot})$ in two space dimensions and the Nédélec edge and face spaces of the first and second kind, used to discretize $H(\mathrm{curl})$ and $H(\mathrm{div})$ in three space dimensions.

A key aim of the present paper is to explicitly construct geometric decompositions of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ for arbitrary values of $r \geqslant 1$ and $k \geqslant 0$, and an arbitrary simplicial

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partition $\mathscr{T}$ of a polyhedral domain in an arbitrary number of space dimensions. More precisely, we will decompose the space into a direct sum with summands indexed by the faces of the mesh (of arbitrary dimension), such that the summand associated to a face is the image under an explicit extension operator of a finite-dimensional space of differential forms on the face. Such a decomposition is necessary for an efficient implementation of the finite element method, since it allows an assembly process that leads to local bases for the finite element space. The construction of explicit local bases is the other key aim of this work.

The construction given here leads to a generalization of the so-called Bernstein basis for ordinary polynomials, i.e., 0-forms on a simplex T in $\mathbb{R}^n$, and the corresponding finite element spaces, the Lagrange finite elements. See Section 2.3 below. This polynomial basis is a well known and useful theoretical tool both in finite element analysis and computational geometry. For low order piecewise polynomial spaces, it can be used directly as a computational basis, while for polynomials of higher order, this basis can be used as a starting point to construct a basis with improved conditioning or other desired properties. The same will be true for the corresponding bases for spaces of piecewise polynomial differential forms studied in this paper.

This paper continues the development of geometric decompositions begun in [2, Section 4]. In the present paper, we give a prominent place to the notion of a *consistent family of extension operators*, and show that such a family leads to a direct sum decomposition of the piecewise polynomial space of differential forms with proper interelement continuity. The explicit notion of a consistent family of extension operators is new to this paper. We also take a more geometric and coordinate-independent approach in this paper than in [2], and so are able to give a purely geometric characterization of the decompositions obtained here. The geometric decomposition we present for the spaces $\mathcal{P}_r^-\Lambda^k$ here turns out to be the same as obtained in [2], but the decomposition of the spaces $\mathcal{P}_r\Lambda^k$ obtained here is new. It improves upon the one obtained in [2], since it no longer depends on a particular choice of ordering of the vertices of the simplex T, and leads to a more canonical basis for $\mathcal{P}_r\Lambda^k$.

The construction of implementable bases for some of the spaces we consider here has been considered previously by a number of authors. Closest to the present paper is the work of Gopalakrishnan et al. [4]. They give a basis in barycentric coordinates for the space $\mathcal{P}_r^-\Lambda^1$, where T is a simplex in any number of space dimensions. In this particular case, their basis is the same as we present in Section 9. In fact, Table 3.1 of [4] is the same, up to a change in notation, as the left portion of Table 9.2 of this paper. As will be seen below, explicit bases for the complete polynomial spaces $\mathcal{P}_r\Lambda^k$ are more complicated than for the $\mathcal{P}_r^-\Lambda^k$ spaces. To our knowledge, the basis we present here for the $\mathcal{P}_r\Lambda^k$ spaces have not previously appeared in the literature, even in two dimensions or for small values of r.

Other authors have focused on the construction of p-hierarchical bases for some of the spaces considered here. We particularly note the work of Ainsworth and Coyle [1], Hiptmair [5], and Webb [7]. In [1], the authors construct hierarchical bases of arbitrary polynomial order for the spaces we denote $\mathcal{P}_r\Lambda^k$, $k = 0,\ldots,3$, $r \geqslant 1$, and T a simplex in three dimensions. In Section 5 of [5], Hiptmair considers hierarchical bases of $\mathcal{P}_r^-\Lambda^k$ for general r, k, and simplex dimension. In [7], Webb constructs hierarchical bases for both $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$, for $k = 0,1$ in one, two, and three space dimensions. The approaches of these three sets of authors differ. Even when adapted to the simple case of zero-forms, i.e., Lagrange finite elements, they produce different hierarchical bases, from among the many that have been proposed. Our approach is quite distinct from these in that we are not trying to find hierarchical bases, but rather we generalize the explicit Bernstein basis to the full range of spaces $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$.

In the present work, by treating the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ families together, and adopting the framework of differential forms, we are able to give a presentation that shows the close connection of these two families, and is valid for all order polynomials and all order differential forms in arbitrary space dimensions. Moreover, the viewpoint of this paper is that the construction of basis functions is a straightforward consequence of the geometric decomposition of the finite element spaces, which is the key ingredient needed to construct spaces with the proper interelement continuity. Thus, the main results of the paper focus on these geometric decompositions.

An outline of the paper is as follows: In the next section, we define our notation and review material we will need about barycentric coordinates, the Bernstein basis, differential forms, and simplicial triangulations. The $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ families of polynomial and piecewise polynomial differential forms are described in Section 3. In Section 4, we introduce the concept of a consistent family of extension operators and use it to construct a geometric decomposition of a finite element space in an abstract setting. In addition to the Bernstein decomposition, a second familiar decomposition which fits this framework is the dual decomposition, briefly discussed in Section 5. Barycentric spanning sets and bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ and the corresponding subspaces $\mathring{\mathcal{P}}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ with vanishing trace are presented in Section 6. The main results of this paper, the geometric decompositions and local bases, are derived in Sections 7 and 8 for $\mathcal{P}_r^-\Lambda^k(\mathscr{T})$ and $\mathcal{P}_r\Lambda^k(\mathscr{T})$. Finally, in Section 9, we discuss how these results can be used to obtain explicit local bases, and tabulate such bases in the cases of 2 and 3 space dimensions and polynomial degree at most 3.

2. Notation and preliminaries

2.1. Increasing sequences and multi-indices

We will frequently use increasing sequences, or increasing maps from integers to integers, to index differential forms. For integers j,k,l,m, with $0 \leqslant k-j \leqslant m-l$, we will use $\Sigma(j:k,l:m)$ to denote the set of increasing maps $\{j,\ldots,k\} \to \{l,\ldots,m\}$, i.e.,

$$\Sigma(j:k,l:m) = \{\sigma : \{j,\ldots,k\} \to \{l,\ldots,m\} | \sigma(j) < \sigma(j+1) < \cdots < \sigma(k)\}.$$

Furthermore, $[\![\sigma]\!]$ will denote the range of such maps, i.e., for $\sigma \in \Sigma(j:k,l:m)$, $[\![\sigma]\!] = \{\sigma(i) | i = j,\ldots,k\}$. Most frequently, we will use the sets $\Sigma(0:k,0:n)$ and $\Sigma(1:k,0:n)$ with cardinality $\binom{n+1}{k+1}$ or $\binom{n+1}{k}$, respectively. Furthermore, if $\sigma \in \Sigma(0:k,0:n)$, we denote by $\sigma^* \in \Sigma(1:n-k,0:n)$ the complementary map characterized by

$$[\![\sigma]\!] \cup [\![\sigma^*]\!] = \{0,1,\ldots,n\}. \tag{2.1}$$

On the other hand, if $\sigma \in \Sigma(1:k,0:n)$, then $\sigma^* \in \Sigma(0:n-k,0:n)$ is the complementary map such that (2.1) holds.

We will use the multi-index notation $\alpha \in \mathbb{N}_0^n$, meaning $\alpha = (\alpha_1,\cdots,\alpha_n)$ with integer $\alpha_i \geqslant 0$. We define $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, and $|\alpha| := \sum \alpha_i$. We will also use the set $\mathbb{N}_0^{0:n}$ of multi-indices $\alpha = (\alpha_0,\cdots,\alpha_n)$, with $x^\alpha := x_0^{\alpha_0}\cdots x_n^{\alpha_n}$. The support $[\![\alpha]\!]$ of a multi-index α is $\{i | \alpha_i > 0\}$. It is also useful to let

$$[\![\alpha,\sigma]\!] = [\![\alpha]\!] \cup [\![\sigma]\!], \quad \alpha \in \mathbb{N}_0^{0:n}, \quad \sigma \in \Sigma(j:k,l:m).$$

If $\Omega \subset \mathbb{R}^n$ and $r \geqslant 0$, then $\mathcal{P}_r(\Omega)$ denotes the set of real valued polynomials defined on Ω of degree less than or equal to r. For simplicity, we let $\mathcal{P}_r = \mathcal{P}_r(\mathbb{R}^n)$. Hence, if Ω has nonempty interior, then $\dim \mathcal{P}_r(\Omega) = \dim \mathcal{P}_r = \binom{r+n}{n}$. The case where Ω consists of a single point is allowed: then $\mathcal{P}_r(\Omega) = \mathbb{R}$ for all $r \geqslant 0$. For any Ω, when $r < 0$, we take $\mathcal{P}_r(\Omega) = \{0\}$.

2.2. Simplices and barycentric coordinates

Let $T \in \mathbb{R}^n$ be an n-simplex with vertices $x_0, x_1, \ldots, x_n$ in general position. We let $\Delta(T)$ denote all the subsimplices, or faces, of T, while $\Delta_k(T)$ denotes the set of subsimplices of dimension k. Hence, the cardinality of $\Delta_k(T)$ is $\binom{n+1}{k+1}$. We will use elements of the set $\Sigma(j:k, 0:n)$ to index the subsimplices of T. For each $\sigma \in \Sigma(j:k, 0:n)$, we let $f_\sigma \in \Delta(T)$ be the closed convex hull of the vertices $x_{\sigma(j)}, \ldots, x_{\sigma(k)}$, which we henceforth denote by $[x_{\sigma(j)}, \ldots, x_{\sigma(k)}]$. Note that there is a one-to-one correspondence between $\Delta_k(T)$ and $\Sigma(0:k, 0:n)$. In fact, the face f_σ is uniquely determined by the range of σ, $[\![\sigma]\!]$. If $f = f_\sigma$ for $\sigma \in \Sigma(j:k, 0:n)$, we let the index set associated to f be denoted by $\mathscr{I}(f)$, i.e., $\mathscr{I}(f) = [\![\sigma]\!]$. If $f \in \Delta_k(T)$, then $f^* \in \Delta_{n-k-1}(T)$ will denote the subsimplex of T opposite f, i.e., the subsimplex whose index set is the complement of $\mathscr{I}(f)$ in $\{0, 1, \ldots, n\}$. Note that if $\sigma \in \Sigma(0:k, 0:n)$ and $f = f_\sigma$, then $f^* = f_{\sigma^*}$.

We denote by $\lambda_0^T, \lambda_1^T, \ldots, \lambda_n^T$ the barycentric coordinate functions with respect to T, so $\lambda_i^T \in \mathcal{P}_1(T)$ is determined by the equations $\lambda_i^T(x_j) = \delta_{ij}$, $0 \leqslant i,j \leqslant n$. The functions λ_i^T form a basis for $\mathcal{P}_1(T)$, are non-negative on T, and sum to 1 identically on T. Moreover, the subsimplices of T correspond to the zero sets of the barycentric coordinates, i.e., if $f = f_\sigma$ for $\sigma \in \Sigma(0:k, 0:n)$, then f is characterized by

$$f = \{x \in T | \lambda_i^T(x) = 0,\ i \in [\![\sigma^*]\!]\}.$$

For a subsimplex $f \in \Delta(T)$, the barycentric coordinates functions with respect to f, $\{\lambda_i^f\}_{i \in \mathscr{I}(f)} \subset \mathcal{P}_1(f)$, satisfy

$$\lambda_i^f = \mathrm{tr}_{T,f}\lambda_i^T,\ i \in \mathscr{I}(f). \tag{2.2}$$

Here the trace map $\mathrm{tr}_{T,f} : \mathcal{P}_1(T) \to \mathcal{P}_1(f)$ is the restriction of the function to f. Due to the relation (2.2), we will sometimes omit the superscript T or f, and simply write λ_i instead of λ_i^T or λ_i^f. Note that, by linearity, the map $\lambda_i^f \to \lambda_i^T$, $i \in \mathscr{I}(f)$, defines a *barycentric extension operator* $E_{f,T}^1 : \mathcal{P}_1(f) \to \mathcal{P}_1(T)$, which is a right inverse of $\mathrm{tr}_{T,f}$. The barycentric extension $E_{f,T}^1 p$ can be characterized as the unique extension of the linear polynomial p on f to a linear polynomial on T which vanishes on f^*.

2.3. The Bernstein decomposition

Let $T = [x_0, x_1, \ldots, x_n] \subset \mathbb{R}^n$ be as above and $\{\lambda_i\}_{i=0}^n \subset \mathcal{P}_1(T)$ the corresponding barycentric coordinates. For $r \geqslant 1$, the Bernstein basis for the space $\mathcal{P}_r(T)$ consists of all monomials of degree r in the variables λ_i, i.e., the basis functions are given by

$$\{\lambda^\alpha = \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\cdots\lambda_n^{\alpha_n} | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r\}. \tag{2.3}$$

(It is common to take the scaled barycentric monomials $(n!/\alpha!)\lambda^\alpha$ as the Bernstein basis elements, as in [6], but the scaling is not relevant here, and so we use the unscaled monomials.) Of course, for $f \in \Delta(T)$, the space $\mathcal{P}_r(f)$ has the corresponding basis

$$\{(\lambda^f)^\alpha | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r,\ [\![\alpha]\!] \subseteq \mathscr{I}(f)\}.$$

Hence, from this Bernstein basis, we also obtain a barycentric extension operator, $E = E_{f,T}^r : \mathcal{P}_r(f) \to \mathcal{P}_r(T)$, by simply replacing λ_i^f by λ_i^T in the bases and using linearity.

We let $\mathring{\mathcal{P}}_r(T)$ denote the subspace of $\mathcal{P}_r(T)$ consisting of polynomials which vanish on the boundary of T or, equivalently, which are divisible by the corresponding bubble function $\lambda_0 \cdots \lambda_n$ on T. Alternatively, we have

$$\mathring{\mathcal{P}}_r(T) = \mathrm{span}\{\lambda^\alpha | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r,\ [\![\alpha]\!] = \{0, \ldots, n\}\}. \tag{2.4}$$

Note that multiplication by the bubble function establishes an isomorphism $\mathcal{P}_{r-n-1}(T) \cong \mathring{\mathcal{P}}_r(T)$.

The Bernstein basis (2.3) leads to an explicit geometric decomposition of the space $\mathcal{P}_r(T)$. Namely, we associate to the face f, the subspace of $\mathcal{P}_r(T)$ that is spanned by the basis functions λ^α with $[\![\alpha]\!] = \mathscr{I}(f)$. We then note that this subspace is precisely $E[\mathring{\mathcal{P}}_r(f)]$, i.e.,

$$E[\mathring{\mathcal{P}}_f(f)] = \mathrm{span}\{\lambda^\alpha | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r,\ [\![\alpha]\!] = \mathscr{I}(f)\}. \tag{2.5}$$

Clearly,

$$\mathcal{P}_r(T) = \bigoplus_{f \in \Delta(T)} E[\mathring{\mathcal{P}}_r(f)], \tag{2.6}$$

which we refer to as the *Bernstein decomposition* of the space $\mathcal{P}_r(T)$. This is an example of a geometric decomposition, as discussed in the introduction. An illustration of the decomposition (2.6) is given in Fig. 2.1.

Moreover, the extension operator E may also be characterized geometrically, without recourse to barycentric coordinates. To obtain such a characterization, we first recall that a smooth function $u : T \to \mathbb{R}$ is said to *vanish to order r* at a point x if

$$(\partial^\alpha u)(x) = 0,\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \leqslant r - 1.$$

We also say that u vanishes to order r on a set g if it vanishes to order r at each point of g. Note that the extension operator $E = E_{f,T}^r$ has the property that for any $\mu \in \mathcal{P}_r(f)$, $E\mu$ vanishes to order r on f^*. In fact, if we set

$$\mathcal{P}_r(T,f) = \{\omega \in \mathcal{P}_r(T) | \omega \text{ vanishes to order } r \text{ on } f^*\},$$

we can prove

Lemma 2.1. *$\mathcal{P}_r(T,f) = E[\mathcal{P}_r(f)]$ and for $\mu \in \mathcal{P}_r(f)$, $E\mu = E_{f,T}^r\mu$ can be characterized as the unique extension of μ to $\mathcal{P}_r(T,f)$.*

Proof. It is easy to see that $E[\mathcal{P}_r(f)] \subseteq \mathcal{P}_r(T,f)$. To establish the reverse inclusion, we observe that if $f^* = \{x_i\}$, then $\omega \in \mathcal{P}_r(T)$ vanishes at f^* if and only if it can be written in the form

$$\omega = \sum_{|\alpha|=r} c_\alpha \lambda^\alpha,$$

where the sum is restricted to multi-indices i for which $\alpha_i = 0$. For a more general set f^*, this fact will be true for any $i \in \mathscr{I}(f^*)$ and hence

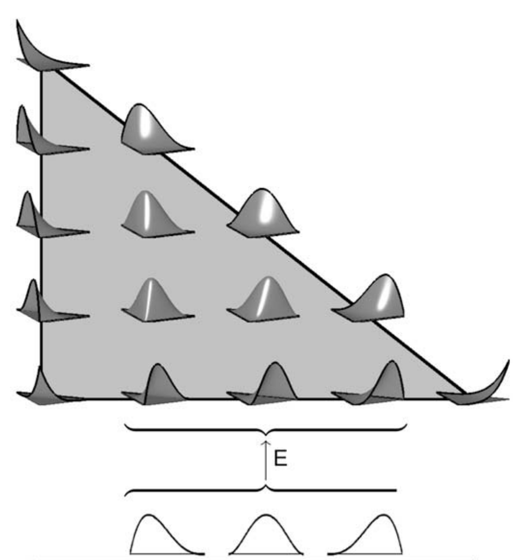

Fig. 2.1. The Bernstein basis of $\mathcal{P}_4(T)$ for a triangle T. One basis function is associated with each vertex, three with each edge, and three with the triangle. The basis functions associated with any face f are obtained by extending basis functions for $\mathring{\mathcal{P}}_4(f)$ to the triangle.

$$\omega = \sum_{\substack{|\alpha|=r \\ [\![\alpha]\!]\subseteq \mathcal{I}(f)}} c_\alpha \lambda^\alpha,$$

and so $\omega \in E[\mathcal{P}_r(f)]$. □

We also note that it follows immediately from Lemma 2.1 that the space $E[\mathring{\mathcal{P}}_r(f)]$ appearing in the Bernstein decomposition (2.6) is characterized by

$$E[\mathring{\mathcal{P}}_r(f)] = \{\omega \in \mathcal{P}_r(T) | \omega \text{ vanishes to order } r \text{ on } f^*, \mathrm{tr}_{T,f}\omega \in \mathring{\mathcal{P}}_r(f)\}.$$

In this paper, we will establish results analogous to those of this section for spaces of polynomial differential forms, and in particular, direct sum decompositions of these spaces analogous to the Bernstein decomposition (2.6).

2.4. Differential forms

Next we indicate the notations we will be using for basic concepts related to differential forms. See [2, Section 2] or the references indicated there for a more detailed treatment. For $k \geqslant 0$, we denote by $\mathrm{Alt}^k V$ the set of real-valued, alternating k-linear maps on a vector space V (with $\mathrm{Alt}^0 V = \mathbb{R}$). Hence, $\mathrm{Alt}^k V$ is a vector space of dimension $\binom{\dim V}{k}$. The exterior product, or the wedge product, maps $\mathrm{Alt}^j V \times \mathrm{Alt}^k V$ into $\mathrm{Alt}^{j+k} V$. If $\omega \in \mathrm{Alt}^k V$ and $v \in V$, then the contraction of ω with v, $\omega \lrcorner v \in \mathrm{Alt}^{k-1} V$, is given by $\omega \lrcorner v(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1})$.

If Ω is a smooth manifold (e.g., an open subset of Euclidean space), a differential k-form on Ω is a map which assigns to each $x \in \Omega$ an element of $\mathrm{Alt}^k T_x\Omega$, where $T_x\Omega$ is the tangent space to Ω at x. In case f is an open subset of an affine subspace of Euclidean space, all the tangents spaces $T_x f$ may be canonically identified, and we simply write them as T_f.

We denote by $\Lambda^k(\Omega)$ the space of all smooth differential k-forms on Ω. The exterior derivative d maps $\Lambda^k(\Omega)$ to $\Lambda^{k+1}(\Omega)$. It satisfies $d \circ d = 0$, so defines a complex

$$0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0,$$

the de Rham complex. If $F : \Omega \to \Omega'$, is a smooth map between smooth manifolds, then the pullback $F^* : \Lambda^k(\Omega') \to \Lambda^k(\Omega)$ is given by

$$(F^*\omega)_x(v_1, v_2, \ldots, v_k) = \omega_{F(x)}(DF_x(v_1), DF_x(v_2), \ldots, DF_x(v_k)),$$

where the linear map $DF_x : T_x\Omega \to T_{F(x)}\Omega'$ is the derivative of F at x. The pullback commutes with the exterior derivative, i.e.,

$$F^*(d\omega) = d(F^*\omega), \quad \omega \in \Lambda^k(\Omega'),$$

and distributes with respect to the wedge product:

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta.$$

We also recall the integral of a k-form over an orientable k-dimensional manifold is defined, and

$$\int_\Omega F^*\omega = \int_{\Omega'} \omega, \quad \omega \in \Lambda^n(\Omega'), \tag{2.7}$$

when F is an orientation-preserving diffeomorphism.

If Ω' is a submanifold of Ω, then the pullback of the inclusion $\Omega' \hookrightarrow \Omega$ is the *trace map* $\mathrm{tr}_{\Omega,\Omega'} : \Lambda^k(\Omega) \to \Lambda^k(\Omega')$. If the domain Ω is clear from the context, we may write $\mathrm{tr}_{\Omega'}$ instead of $\mathrm{tr}_{\Omega,\Omega'}$, and if Ω' is the boundary of Ω, $\partial\Omega$, we just write tr. Note that if Ω' is a submanifold of positive codimension and $k > 0$, then the vanishing of $\mathrm{Tr}_{\Omega,\Omega'}\omega$ on Ω' for $\omega \in \Lambda^k(\Omega)$ does *not* imply that $\omega_x \in \mathrm{Alt}^k T_x\Omega$ vanishes for $x \in \Omega'$, only that it vanishes when applied to k-tuples of vectors tangent to Ω', or, in other words, that the tangential part of ω_x with respect to $T_x\Omega'$ vanishes.

If Ω is a subset of $\mathbb{R}^n$ (or, more generally, a Riemannian manifold), we can define the Hilbert space $L^2\Lambda^k(\Omega) \supset \Lambda^k(\Omega)$ of L^2 differential k-forms, and the Sobolev space

$$H\Lambda^k(\Omega) := \{\omega \in L^2\Lambda^k(\Omega) | d\omega \in L^2\Lambda^{k+1}(\Omega)\}.$$

The L^2 de Rham complex is the sequence of mappings and spaces given by

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0. \tag{2.8}$$

We remark that for $\Omega \subset \mathbb{R}^n$, $H\Lambda^0(\Omega)$ is equal to the ordinary Sobolev space $H^1(\Omega)$ and, via the identification of $\mathrm{Alt}^n\mathbb{R}^n$ with $\mathbb{R}$, $H\Lambda^n(\Omega)$ can be identified with $L^2(\Omega)$. Furthermore, in the case $n = 3$, the spaces $\mathrm{Alt}^1\mathbb{R}^3$ and $\mathrm{Alt}^2\mathbb{R}^3$ can be identified with $\mathbb{R}^3$, and the complex (2.8) may be identified with the complex

$$0 \to H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0.$$

2.5. Simplicial triangulations

Let Ω be a bounded polyhedral domain in $\mathbb{R}^n$ and $\mathcal{T}$ a finite set of n-simplices. We will refer to $\mathcal{T}$ as a *simplicial triangulation* of Ω if the union of all the elements of $\mathcal{T}$ is the closure of Ω, and the intersection of two is either empty or a common subsimplex of each. For $0 \leqslant j \leqslant n$, we let

$$\Delta_j(\mathcal{T}) = \bigcup_{T\in\mathcal{T}} \Delta_j(T) \quad \text{and} \quad \Delta(\mathcal{T}) = \bigcup_{j=0}^{n} \Delta_j(\mathcal{T}).$$

In the finite element exterior calculus, we employ spaces of differential forms ω which are piecewise smooth (usually polynomials) with respect to $\mathcal{T}$, i.e., the restriction $\omega|_T$ is smooth for each $T \in \mathcal{T}$. Then for $f \in \Delta_j(\mathcal{T})$ with $j \geqslant k$, $\mathrm{tr}_f\omega$ may be multi-valued, in that we can assign a value for each $T \in \mathcal{T}$ containing f by first restricting ω to T and then taking the trace on f. If all such traces coincide, we say that $\mathrm{tr}_f\omega$ is single-valued. The following lemma, a simple consequence of Stokes' theorem, cf. [2, Lemma 5.1], is a key result.

Lemma 2.2. *Let $\omega \in L^2\Lambda^k(\Omega)$ be piecewise smooth with respect to the triangulation $\mathcal{T}$. The following statements are equivalent:*

(1) $\omega \in H\Lambda^k(\Omega)$,
(2) $\mathrm{tr}_f\omega$ *is single-valued for all* $f \in \Delta_{n-1}(\mathcal{T})$,
(3) $\mathrm{tr}_f\omega$ *is single-valued for all* $f \in \Delta_j(\mathcal{T})$, $k \leqslant j \leqslant n-1$.

As a consequence of this lemma, in order to construct subspaces of $H\Lambda^k(\Omega)$, consisting of differential forms ω which are piecewise smooth with respect to the triangulation $\mathcal{T}$, we need to build into the construction that $\mathrm{tr}_f\omega$ is single-valued for each $f \in \Delta_j(\mathcal{T})$ for $k \leqslant j \leqslant n-1$.

3. Polynomial and piecewise polynomial differential forms

In this section, we formally define the two families of spaces of polynomial differential forms $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$. These polynomial spaces will then be used to define piecewise polynomial differential forms with respect to a simplicial triangulation of a bounded polyhedral domain in $\mathbb{R}^n$. In fact, as explained in [2, Section 3.4], the two families presented here are nearly the only affine invariant spaces of polynomial differential forms.

3.1. The space $\mathcal{P}_r\Lambda^k$

Let Ω be a subset of $\mathbb{R}^n$. For $0 \leqslant k \leqslant n$, we let $\mathcal{P}_r\Lambda^k(\Omega)$ be the subspace of $\Lambda^k(\Omega)$ consisting of all $\omega \in \Lambda^k(\Omega)$ such that

$\omega(v_1, v_2, \ldots, v_k) \in \mathcal{P}_r(\Omega)$ for each choice of vectors $v_1, v_2, \ldots v_k \in \mathbb{R}^n$. Frequently, we will write $\mathcal{P}_r\Lambda^k$ instead of $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$. The space $\mathcal{P}_r\Lambda^k$ is isomorphic to $\mathcal{P}_r \otimes \mathrm{Alt}^k$ and

$$\dim \mathcal{P}_r\Lambda^k = \dim \mathcal{P}_r \times \dim \mathrm{Alt}^k\mathbb{R}^n = \binom{r+n}{n}\binom{n}{k} = \binom{r+k}{r}\binom{n+r}{n-k}. \tag{3.1}$$

Furthermore, if $\Omega \subset \mathbb{R}^n$ with nonempty interior, then $\dim \mathcal{P}_r\Lambda^k(\Omega) = \dim \mathcal{P}_r\Lambda^k$.

If T is a simplex, we define

$$\mathring{\mathcal{P}}_r\Lambda^k(T) = \{\omega \in \mathcal{P}_r\Lambda^k(T) | \mathrm{tr}\,\omega = 0\}.$$

In the case $k = 0$, this space simply consists of all the polynomials divisible by the bubble function $\lambda_0 \cdots \lambda_n$, so

$$\mathring{\mathcal{P}}_r\Lambda^0(T) \cong \mathcal{P}_{r-n-1}\Lambda^0(T). \tag{3.2}$$

For $k = n$, the trace map vanishes, so we have

$$\mathring{\mathcal{P}}_r\Lambda^n(T) = \mathcal{P}_r\Lambda^n(T). \tag{3.3}$$

3.2. The space $\mathcal{P}_r^-\Lambda^k$

The Koszul differential κ of a differential k-form ω on $\mathbb{R}^n$ is the $(k-1)$-form given by

$$(\kappa\omega)_x(v_1, \ldots, v_{k-1}) = \omega_x(X(x), v_1, \ldots, v_{k-1}),$$

where $X(x)$ is the vector from the origin to x. For each r, κ maps $\mathcal{P}_{r-1}\Lambda^k$ to $\mathcal{P}_r\Lambda^{k-1}$, and the Koszul complex

$$0 \to \mathcal{P}_{r-n}\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{P}_r\Lambda^0 \to \mathbb{R} \to 0,$$

is exact. Furthermore, the Koszul operator satisfies the Leibniz relation

$$\kappa(\omega \wedge \eta) = (\kappa\omega) \wedge \eta + (-1)^k \omega \wedge (\kappa\eta), \quad \omega \in \Lambda^k, \ \eta \in \Lambda^l. \tag{3.4}$$

We define

$$\mathcal{P}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}.$$

From this definition, we easily see that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$. However, if $0 < k < n$, then

$$\mathcal{P}_{r-1}\Lambda^k \subsetneq \mathcal{P}_r^-\Lambda^k \subsetneq \mathcal{P}_r\Lambda^k.$$

An important property of the spaces $\mathcal{P}_r^-\Lambda^k$ is the closure relation

$$\mathcal{P}_r^-\Lambda^k \wedge \mathcal{P}_s^-\Lambda^l \subseteq \mathcal{P}_{r+s}^-\Lambda^{k+l}. \tag{3.5}$$

A key identity relating the Koszul operator κ with the exterior derivative d is the homotopy relation

$$(d\kappa + \kappa d)\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k, \tag{3.6}$$

where $\mathcal{H}_r\Lambda^k$ is the space of *homogeneous* polynomial k-forms of degree r.

Using the homotopy relation and the exactness of the Koszul complex, we can inductively compute the dimension of $\mathcal{P}_r^-\Lambda^k$ as

$$\dim \mathcal{P}_r^-\Lambda^k = \binom{r+k-1}{k}\binom{n+r}{n-k}. \tag{3.7}$$

If $\Omega \subset \mathbb{R}^n$, then $\mathcal{P}_r^-\Lambda^k(\Omega)$ denotes the restriction of functions in $\mathcal{P}_r^-\Lambda^k$ to Ω, which implies that the space $\mathcal{P}_r^-\Lambda^k(\Omega)$ is isomorphic to $\mathcal{P}_r^-\Lambda^k$ if Ω has nonempty interior. Finally, we remark that although the Koszul operator κ depends on the choice of origin used to associate a point in $\mathbb{R}^n$ with a vector, the space $\mathcal{P}_r^-\Lambda^k$ is unaffected by the choice of origin. We refer to [2] for more details on the spaces $\mathcal{P}_r^-\Lambda^k$. In particular, if $T \subset \mathbb{R}^n$ is a simplex and $f \in \Delta_j(T)$, then $\mathrm{tr}_f\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_r^-\Lambda^k(f)$, where the space $\mathcal{P}_r^-\Lambda^k(f) \cong \mathcal{P}_r^-\Lambda^k(\mathbb{R}^j)$ depends on f, but is independent of T.

For a simplex T, we define

$$\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \{\omega \in \mathcal{P}_r^-\Lambda^k(T) | \mathrm{tr}\,\omega = 0\}.$$

From the Hodge star isomorphism, we have that $\mathcal{P}_{r-n-1}\Lambda^0(T) \cong \mathcal{P}_{r-n-1}\Lambda^n(T) = \mathcal{P}_{r-n}^-\Lambda^n(T)$ and that $\mathcal{P}_r\Lambda^n(T) \cong \mathcal{P}_r\Lambda^0(T) = \mathcal{P}_r^-\Lambda^0(T)$. Therefore (3.2) and (3.3) become

$$\mathring{\mathcal{P}}_r\Lambda^0(T) \cong \mathcal{P}_{r-n}^-\Lambda^n(T), \quad \mathring{\mathcal{P}}_r\Lambda^n(T) \cong \mathcal{P}_r^-\Lambda^0(T). \tag{3.8}$$

These are the two extreme cases of the relation

$$\mathring{\mathcal{P}}_r\Lambda^k(T) \cong \mathcal{P}_{r-n+k}^-\Lambda^{n-k}(T), \quad 0 \leqslant k \leqslant n. \tag{3.9}$$

But (3.8) can also be written

$$\mathring{\mathcal{P}}_r^-\Lambda^0(T) \cong \mathcal{P}_{r-n-1}\Lambda^n(T), \quad \mathring{\mathcal{P}}_r^-\Lambda^n(T) \cong \mathcal{P}_{r-1}\Lambda^0(T),$$

(where we have substituted $r-1$ for r in the second relation), which are the extreme cases of

$$\mathring{\mathcal{P}}_r^-\Lambda^k(T) \cong \mathcal{P}_{r-n+k-1}\Lambda^{n-k}(T), \quad 0 \leqslant k \leqslant n. \tag{3.10}$$

That the isomorphisms in (3.9) and (3.10) do indeed exist for all k follows from Corollary 5.2 below.

3.3. The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$

For $\mathcal{T}$ a simplicial triangulation of a domain $\Omega \in \mathbb{R}^n$, we define

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\omega \in L^2\Lambda^k(\Omega) | \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}, \ \mathrm{tr}_f\omega \text{ is single-valued for } f \in \Delta_j(\mathcal{T}), \ k \leqslant j \leqslant n-1\},$$

and define $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ similarly. In view of Lemma 2.2, we have

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\omega \in H\Lambda^k(\Omega) | \omega|_T \in \mathcal{P}_r\Lambda^k(T) \ \forall T \in \mathcal{T}\},$$
$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) = \{\omega \in H\Lambda^k(\Omega) | \omega|_T \in \mathcal{P}_r^-\Lambda^k(T) \ \forall T \in \mathcal{T}\}.$$

4. Consistent extension operators and geometric decompositions

Let $\mathcal{T}$ be a simplicial triangulation of $\Omega \subset \mathbb{R}^n$, and let there be given a finite-dimensional subspace $X(T)$ of $\Lambda^k(T)$ for each $T \in \mathcal{T}$. In this section, we shall define the notion of a *consistent family of extension operators*, and show that it leads to the construction of a geometric decomposition and a local basis of the finite element space

$$X(\mathcal{T}) = \{\omega \in L^2\Lambda^k(\Omega) | \omega|_T \in X(T) \ \forall T \in \mathcal{T}, \ \mathrm{tr}_f\omega \text{ is single-valued for } f \in \Delta(\mathcal{T})\}. \tag{4.1}$$

We note that as a result of Lemma 2.2, $X(\mathcal{T}) \subset H\Lambda^k(\Omega)$.

For the Lagrange finite element space $\mathcal{P}_r(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T})$, both the Bernstein basis discussed in Section 2.3 and the dual basis discussed in the next section arise from this construction. One of the main goals of this paper is to generalize these bases to the two families of finite element spaces of k-forms.

We require that the family of spaces $X(T)$ fulfills the following consistency assumption:

$$\mathrm{tr}_{T,f}X(T) = \mathrm{tr}_{T',f}X(T') \quad \text{whenever } T, T' \in \mathcal{T} \quad \text{with } f \in \Delta(T) \cap \Delta(T'). \tag{4.2}$$

In this case, we may define for any $f \in \Delta(\mathcal{T})$, $X(f) =_\circ \mathrm{tr}_{T,f}X(T)$ where $T \in \mathcal{T}$ is any simplex containing f. We also define $\mathring{X}(f)$ as the subspace of $X(f)$ consisting of all $\omega \in X(f)$ such that $\mathrm{tr}_{f,\partial f}\omega = 0$. Note that

$$\mathrm{tr}_{g,f}X(g) = X(f) \quad \text{for all } f, g \in \Delta(\mathcal{T}) \quad \text{with } f \subseteq g. \tag{4.3}$$

Consequently, for each such f and g we may choose an *extension operator* $E_{f,g} : X(f) \to X(g)$, i.e., a right inverse of $\mathrm{tr}_{g,f} : X(g) \to X(f)$.

We say that a family of extension operators $E_{f,g}$, defined for all $f, g \in \Delta(\mathcal{T})$ with $f \subseteq g$, is *consistent* if

$$\mathrm{tr}_{h,g} E_{f,h} = E_{f\cap g,g} \mathrm{tr}_{f,f\cap g} \quad \text{for all } f,g,h \in \Delta(\mathcal{T}) \quad \text{with } f,g \subseteq h. \tag{4.4}$$

In other words, we require that the diagram

$$\begin{array}{ccc} X(f) & \xrightarrow{E} & X(h) \\ \downarrow \mathrm{tr} & & \downarrow \mathrm{tr} \\ X(f\cap g) & \xrightarrow{E} & X(g) \end{array}$$

commutes.

One immediate implication of (4.4) is that for $\omega \in X(f)$,

$$\mathrm{tr}_{h,g} E_{f,h}\omega = E_{f,g}\omega \quad \text{for all } f,g,h \in \Delta(\mathcal{T}) \quad \text{with } f \subseteq g \subseteq h. \tag{4.5}$$

A second implication is:

Lemma 4.1. *Let* $h \in \Delta(\mathcal{T})$, *and* $f, g \in \Delta(h)$ *with* $f \not\subseteq g$. *Then* $\mathrm{tr}_{h,g}\omega = 0$ *for all* $\omega \in E_{f,h}\mathring{X}(f)$.

Proof. Let $\omega = E_{f,h}\mu$ with $\mu \in \mathring{X}(f)$. Since $f \not\subseteq g$, we have $f \cap g \subset \partial f$, and therefore $\mathrm{tr}_{f,f\cap g}\mu = 0$. Then, by (4.4), $\mathrm{tr}_{h,g}\omega = \mathrm{tr}_{h,g}E_{f,h}\mu = E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}\mu = 0$. □

We now define an extension operator $E_f : \mathring{X}(f) \to X(\mathcal{T})$ for each $f \in \Delta(\mathcal{T})$. Given $\mu \in \mathring{X}(f)$, we define $E_f\mu$ piecewise:

$$(E_f\mu)|_T = \begin{cases} E_{f,T}\mu & \text{if } f \subseteq T, \\ 0, & \text{otherwise}. \end{cases} \tag{4.6}$$

We claim that for each $g \in \Delta(\mathcal{T})$, $\mathrm{tr}_g E_f\mu$ is single-valued, so $E_f\mu$ does indeed belong to $X(\mathcal{T})$. To see this, we consider separately the cases $f \subseteq g$ and $f \not\subseteq g$. In the former case, if $T \in \mathcal{T}$ is any simplex containing g, then $f \subseteq T$, and so

$$\mathrm{tr}_{T,g}[(E_f\mu)|_T] = \mathrm{tr}_{T,g}E_{f,T}\mu = E_{f,g}\mu$$

by (4.5). Thus $\mathrm{tr}_{T,g}[(E_f\mu)|_T]$ does not depend on the choice of T containing g, so in this case we have established that $\mathrm{tr}_g E_f\mu$ is single-valued. On the other hand, if $f \not\subseteq g$ then $\mathrm{tr}_{T,g}[(E_f\mu)|_T] = 0$ for any T containing g, either because $f \not\subseteq T$ and so $(E_f\mu)|_T = 0$, or by Lemma 4.1 if $f \subseteq T$. Thus we have established that all traces of $E_f\mu$ are single-valued, and so we have defined extension operators $E_f : \mathring{X}(f) \to X(\mathcal{T})$ for each $f \in \Delta(\mathcal{T})$. We refer to E_f as the global extension operator determined by the consistent family of extension operators.

We easily obtain this variant of Lemma 4.1.

Lemma 4.2. *Let* $f, g \in \Delta(\mathcal{T})$, $f \not\subseteq g$. *Then* $\mathrm{tr}_g\omega = 0$ *for all* $\omega \in E_f\mathring{X}(f)$.

Proof. Pick $T \in \mathcal{T}$ containing g. If $f \in \Delta(T)$, then we can apply Lemma 4.1 with $h = T$. Otherwise, $(E_f\mu)|_T = 0$ for all $\mu \in \mathring{X}(f)$. □

The following theorem is the main result of this section.

Theorem 4.3. *Let* $\mathcal{T}$ *be a simplicial triangulation and suppose that for each* $T \in \mathcal{T}$, *a finite-dimensional subspace* $X(T)$ *of* $\Lambda^k(T)$ *is given fulfilling the consistency assumption* (4.2). *Assume that there is a consistent family of extensions operators* $E_{f,g}$ *for all* $f, g \in \Delta(\mathcal{T})$ *with* $f \subseteq g$. *Define* E_f, $f \in \Delta(\mathcal{T})$ *by* (4.6). *Then the space* $X(\mathcal{T})$ *defined in* (4.1) *admits the direct sum decomposition*

$$X(\mathcal{T}) = \bigoplus_{f\in\Delta(\mathcal{T})} E_f\mathring{X}(f). \tag{4.7}$$

Proof. To show that the sum is direct, we assume that $\sum_{f\in\Delta(\mathcal{T})}\omega_f = 0$, where $\omega_f \in E_{f,T}\mathring{X}(f)$, and prove by induction that $\omega_f = 0$ for all $f \in \Delta(\mathcal{T})$ with $\dim f \leqslant j$. This is certainly true for $j < k$, (since then $\Lambda^k(f)$ and, *a fortiori*, $X(f)$ vanishes), so we assume it is true and must show that $\omega_g = 0$ for $g \in \Delta_{j+1}(\mathcal{T})$. By Lemma 4.2,

$$0 = \mathrm{tr}_g\left(\sum_{f\in\Delta(\mathcal{T})}\omega_f\right) = \mathrm{tr}_g\omega_g.$$

Hence, $\omega_g = E_g\mathrm{tr}_g\omega_g = 0$. We thus conclude that the sum is direct, and $X(\mathcal{T}) \supseteq \oplus_{f\in\Delta(\mathcal{T})}E_f\mathring{X}(f)$.

To show that this is an equality, we write any $\omega \in X(\mathcal{T})$ in the form

$$\omega = \omega^n - \sum_{j=k}^{n-1}(\omega^{j+1} - \omega^j),$$

where $\omega^k = \omega$, and for $k < j \leqslant n$, $\omega^j \in X(\mathcal{T})$ is defined recursively by

$$\omega^{j+1} = \omega^j - \sum_{f\in\Delta_j(\mathcal{T})} E_f\mathrm{tr}_f\omega^j.$$

We shall prove by induction that for $k \leqslant j \leqslant n$

$$\mathrm{tr}_f\omega^j \in \mathring{X}(f), \quad f \in \Delta_j(\mathcal{T}). \tag{4.8}$$

Assuming this momentarily, we get that $\omega^{j+1} - \omega^j \in \sum_{f\in\Delta_j(\mathcal{T})}E_f\mathring{X}(f)$. Also, $\omega^n|_T = \mathrm{tr}_T\omega^n \in \mathring{X}(T)$ for all $T \in \mathcal{T}$, and $\omega^n = \sum_{T\in\mathcal{T}}\mathrm{tr}_T(\omega^n|_T)$. Thus, $\omega \in \oplus_{f\in\Delta(T)}E_{f,T}\mathring{X}(f)$ as desired.

To prove (4.8) inductively, we first note it is certainly true if $j = k$, since $\mathring{X}(f) = X(f)$ for $f \in \Delta_k(\mathcal{T})$. Now assume (4.8) and let $g \in \Delta_{j+1}(\mathcal{T})$. We show that $\mathrm{tr}_g\omega^{j+1} \in \mathring{X}(g)$, by showing that $\mathrm{tr}_h\omega^{j+1} = 0$ for $h \in \Delta_j(g)$. In fact,

$$\mathrm{tr}_h\omega^{j+1} = \mathrm{tr}_h\omega^j - \sum_{f\in\Delta_j(\mathcal{T})}\mathrm{tr}_h E_f\mathrm{tr}_f\omega^j.$$

Now $\mathrm{tr}_f\omega^j \in \mathring{X}(f)$ by the inductive hypothesis, and therefore, by Lemma 4.2, $\mathrm{tr}_h E_f\mathrm{tr}_f\omega^j = 0$ unless $f = h$, in which case $\mathrm{tr}_h E_f\mathrm{tr}_f\omega^j = \mathrm{tr}_h\omega^j$. Thus,

$$\mathrm{tr}_h\omega^{j+1} = \mathrm{tr}_h\omega^j - \mathrm{tr}_h\omega^j = 0.$$

This completes the proof of the theorem. □

Remark. By considering the case of a mesh consisting of a single simplex T, we see that

$$X(T) = \bigoplus_{f\in\Delta(T)} E_{f,T}\mathring{X}(f). \tag{4.9}$$

The decomposition (4.7) is very important in practice. It leads immediately to a local basis for the large space $X(\mathcal{T})$ consisting of elements $E_f\mu$, where f ranges over $\Delta(\mathcal{T})$ and μ ranges over a basis for the space $\mathring{X}(f)$.

We close this section with the simplest example of this theory. Let $X(T) = \mathcal{P}_r(T) = \mathcal{P}_r\Lambda^0(T)$ be the polynomial space discussed in Section 2.3. Then (4.2) is fulfilled and the trace spaces $X(f)$ are simply $\mathcal{P}_r(f)$ for $f \in \Delta(\mathcal{T})$. For $f, g \in \Delta(\mathcal{T})$ with $f \subseteq g$, the trace operator $\mathrm{tr}_{g,f}$ and barycentric extension operator $E_{f,g}$ are given in barycentric coordinates as follows. If $\alpha \in \mathbb{N}_0^{0:n}$ with $[\![\alpha]\!] \subseteq \mathcal{I}(g)$, then

$$\mathrm{tr}_{g,f}(\lambda^g)^\alpha = \begin{cases} (\lambda^f)^\alpha & \text{if } [\![g]\!] \subseteq \mathcal{I}(f), \\ 0, & \text{otherwise}. \end{cases}$$

For $\alpha \in \mathbb{N}_0^{0:n}$ with $|\alpha| = r$ and $[\![\alpha]\!] \subseteq \mathcal{I}(f)$, then $E_{f,g}(\lambda^f)^\alpha = (\lambda^g)^\alpha$. We now check that the family of barycentric extension operators is consistent, i.e., we verify (4.4). We must show that if $f, g, h \in \Delta(\mathcal{T})$ with $f, g \subseteq h$, then

$$\mathrm{tr}_{h,g}E_{f,h}(\lambda^f)^\alpha = E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}(\lambda^f)^\alpha$$

for all multi-indices α with $|\alpha| = r$ and $[\![\alpha]\!] \subseteq \mathcal{I}(f)$. Indeed, it is easy to check that both sides are equal to $(\lambda^g)^\alpha$ if $[\![\alpha]\!] \subseteq \mathcal{I}(g)$ and zero otherwise. Note that, in this case, the decomposition (4.9) is simply the Bernstein decomposition (2.6). If we then define, as in the general definition (4.1) above,

$$\mathcal{P}_r\Lambda^0(\mathscr{T}) = \{\omega \in L^2(\Omega) | \omega|_T \in \mathcal{P}_r(T)\ \forall T \in \mathscr{T},\ \mathrm{tr}_f\omega \text{ is single-valued for } f \in \Delta(\mathscr{T})\},$$

then the decomposition (4.7) gives a decomposition of the space $\mathcal{P}_r\Lambda^0(\mathscr{T})$, i.e., the space of continuous piecewise polynomials of degree $\leqslant r$.

5. Degrees of freedom and the dual decomposition

Although our main interest in this paper is obtaining direct sum decompositions for polynomial differential forms that are analogous to the Bernstein decomposition for ordinary polynomials, we include here a discussion of another decomposition, referred to as the dual decomposition, for completeness and as an illustration of the general theory developed in the previous section.

Before we consider the case of differential forms, we review the corresponding decomposition for polynomials. For the construction of finite element spaces based on the local space $\mathcal{P}_r(T)$, a basis for the dual space $\mathcal{P}_r(T)^*$ is given, with each basis element associated to a subsimplex of T. This is referred to as a set of *degrees of freedom* for $\mathcal{P}_r(T)$. The degrees of freedom then determine the interelement continuity imposed on the finite element space. Indeed, in the classical approach of Ciarlet [3], the degrees of freedom and their association to subsimplices is used to define a finite element space. For this purpose, what matters is not the particular basis of $\mathcal{P}_r(T)^*$, but rather the decomposition of this space into the spaces spanned by the basis elements associated to each simplex. For the standard Lagrange finite elements, this geometric decomposition of the dual space is

$$\mathcal{P}_r(T)^* = \bigoplus_{f\in\Delta(T)} W_r(T,f), \tag{5.1}$$

where

$$W_r(g,f) := \{\psi \in \mathcal{P}_r(g)^* | \psi(\omega) = \int_f (\mathrm{tr}_{g,f}\omega)\eta,\ \eta \in \mathcal{P}_{r-\dim f-1}(f)\}.$$

We note that for $\omega \in \mathcal{P}_r(h)$, $\mathrm{tr}_{h,f}\omega$ is uniquely determined by $\oplus_{g\in\Delta(f)} W_r(h,g)$.

Consequently, if $f \subseteq h \in \Delta(T)$, we may define an extension operator $F_{f,h} = F^r_{f,h} : \mathcal{P}_r(f) \to \mathcal{P}_r(h)$, determined by the conditions:

$$\int_g (\mathrm{tr}_{h,g}F_{f,h}\omega)\eta = \int_g (\mathrm{tr}_{f,g}\omega)\eta, \quad \eta \in \mathcal{P}_{r-\dim g-1}(g), \quad g \in \Delta(f),$$
$$\psi(F_{f,h}\omega) = 0, \quad \psi \in W_r(h,g), \quad g \in \Delta(h), \quad g \not\subseteq f.$$

To apply the theory developed in Section 4, we need to check that the extension operator is consistent, i.e., that it satisfies (4.4). For $f,g \subseteq h$, let $\omega \in \mathcal{P}_r(f)$, and set $\mu := F_{f\cap g,g}\mathrm{tr}_{f,f\cap g}\omega \in \mathcal{P}_r(g)$, $\nu := \mathrm{tr}_{h,g}F_{f,h}\omega \in \mathcal{P}_r(g)$. For any face $e \subseteq g \cap f$, $\mathrm{tr}_{g,e}\mu = \mathrm{tr}_{f,e}\omega = \mathrm{tr}_{g,e}\nu$. Therefore, $\psi(\mu) = \psi(\nu)$ for all $\psi \in \mathbb{W}_r(g,e)$ with $e \in \Delta(g)$ such that $e \subseteq f$. Also, for $e \in \Delta(g)$ with $e \not\subseteq f$, it follows from the definition of the extension that for all $\psi \in W_r(g,e)$, $\psi(\mu) = 0 = \psi(\nu)$. Thus, we have shown that the extension operators $F_{f,h}$ form a consistent family. The decomposition

$$\mathcal{P}_r(T) = \bigoplus_{f\in\Delta(T)} F_{f,T}[\mathring{\mathcal{P}}_r(f)],$$

corresponding to (4.9), is now called the decomposition dual to (5.1). Furthermore, from Theorem 4.3 we obtain a corresponding direct sum decomposition for the assembled space $\mathcal{P}_r(\mathscr{T}) = \mathcal{P}_r\Lambda^0(\mathscr{T})$ of the form (4.7).

In the remainder of this section, we present analogous results for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$. This will be based on the following decompositions of the dual spaces $\mathcal{P}_r\Lambda^k(T)^*$ and $\mathcal{P}^-_r\Lambda^k(T)^*$, established in [2, Section 4, Theorems 4.10 and 4.14].

Theorem 5.1.

1. *For each $f \in \Delta(T)$ define*

$$W^k_r(T,f) := \left\{\psi \in \mathcal{P}_r\Lambda^k(T)^* | \psi(\omega) = \int_f \mathrm{tr}_{T,f}\omega \wedge \eta \quad \textit{for some } \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f)\right\}.$$

Then the obvious mapping $\mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f) \to W^k_r(T,f)$ is an isomorphism, and

$$\mathcal{P}_r\Lambda^k(T)^* = \bigoplus_{f\in\Delta(T)} W^k_r(T,f).$$

2. *For each $f \in \Delta(T)$ define*

$$W^{k-}_r(T,f) := \left\{\psi \in \mathcal{P}^-_r\Lambda^k(T)^* | \psi(\omega) = \int_f \mathrm{tr}_{T,f}\omega \wedge \eta \textit{ for some } \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f)\right\}.$$

Then the obvious mapping $\mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f) \to W^{k-}_r(T,f)$ is an isomorphism, and

$$\mathcal{P}^-_r\Lambda^k(T)^* = \bigoplus_{f\in\Delta(T)} W^{k-}_r(T,f).$$

Note that as in the polynomial case, if $\omega \in \mathcal{P}_r\Lambda^k(T)$, then $\mathrm{tr}_{T,f}\omega$ is determined by the degrees of freedom in $W^k_r(T,g)$ for $g \in \Delta(f)$. In particular, if $\omega \in \mathcal{P}_r\Lambda^k(T)$ such that all the degrees of freedom associated to the subsimplices of T with dimension less than or equal to $n-1$ vanish, then $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$. The corresponding property holds for the spaces $\mathcal{P}^-_r\Lambda^k(T)$ as well.

An immediate consequence of this theorem are the following isomorphisms, that will be used in the following section.

Corollary 5.2.

$$\mathring{\mathcal{P}}_r\Lambda^k(T)^* \cong \mathcal{P}^-_{r+k-n}\Lambda^{n-k}(T) \quad \textit{and} \quad \mathring{\mathcal{P}}^-_r\Lambda^k(T)^* \cong \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T).$$

As in the case of 0-forms, if $f \subset h \in \Delta(T)$, we define an extension operator $F^{k,r}_{f,h} : \mathcal{P}_r\Lambda^k(f) \to \mathcal{P}_r\Lambda^k(h)$, determined by the conditions:

$$\int_g (\mathrm{tr}_{h,g}F^{k,r}_{f,h}\omega) \wedge \eta = \int_g (\mathrm{tr}_{f,g}\omega) \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-\dim g}\Lambda^{\dim g-k}(g),$$
$$g \in \Delta(f), \quad \psi(F^{k,r}_{f,h}\omega) = 0, \quad \psi \in W^k_r(h,g), \quad g \in \Delta(h), \quad g \not\subseteq f.$$

We may similarly define an extension operator $F^{k,r,-}_{f,T} : \mathcal{P}^-_r\Lambda^k(f) \to \mathcal{P}^-_r\Lambda^k(h)$. The verification of the consistency of these families of extension operators is essentially the same as for the space $\mathcal{P}_r(T)$ given above, and so we do not repeat the proof.

6. Barycentric spanning sets

Let $T = [x_0,\ldots,x_n] \subset \mathbb{R}^n$ be a nondegenerate n-simplex. The Bernstein basis described in Section 2.3 above is given in terms of the barycentric coordinates $\{\lambda_i\}^n_{i=0} \subset \mathcal{P}_1(T)$. The main purpose of this paper is to construct the generalization of the Bernstein basis for the polynomial spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$. In the present section, we will give spanning sets and bases for these spaces and for the corresponding spaces with vanishing trace expressed in barycentric coordinates. Note that the bases given in this section depend on the ordering of the vertices. These are *not* the bases we suggest for computation.

For convenience we summarize the results of the section in the following theorem, referring not only to the n-dimensional simplex T, but, more generally, to any subsimplex f of T. Here, we use the notation

$$d\lambda_\sigma^f = d\lambda_{\sigma(1)}^f \wedge \cdots \wedge d\lambda_{\sigma(k)}^f \in \mathrm{Alt}^k T_f \tag{6.1}$$

for $f \in \Delta(T)$, $\sigma \in \Sigma(1:k, 0:n)$ with $[\![\sigma]\!] \subseteq \mathscr{I}(f)$, and ϕ_σ^f for the Whitney form defined in (6.3).

Theorem 6.1. *Let $f \in \Delta(T)$.*

1. *Spanning set and basis for $\mathcal{P}_r\Lambda^k(f)$. The set*
$$\{(\lambda^f)^\alpha d\lambda_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n),\ [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f)\}$$
is a spanning set for $\mathcal{P}_r\Lambda^k(f)$, and
$$\{(\lambda^f)^\alpha d\lambda_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n),\ [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f), \min[\![\sigma]\!] > \min \mathscr{I}(f)\}$$
is a basis.
2. *Spanning set and basis for $\mathring{\mathcal{P}}_r\Lambda^k(f)$. The set*
$$\{(\lambda^f)^\alpha d\lambda_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n),\ [\![\alpha,\sigma]\!] = \mathscr{I}(f)\}$$
is a spanning set for $\mathring{\mathcal{P}}_r\Lambda^k(f)$, and
$$\{(\lambda^f)^\alpha d\lambda_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n), [\![\alpha,\sigma]\!] = \mathscr{I}(f)\ \alpha_i = 0 \text{ if } i < \min[\mathscr{I}(f) \setminus [\![\sigma]\!]]\}$$
is a basis.
3. *Spanning set and basis for $\mathcal{P}_r^-\Lambda^k(f)$. The set*
$$\{(\lambda^f)^\alpha \phi_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k, 0:n),\ [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f)\}$$
is a spanning set for $\mathcal{P}_r^-\Lambda^k(f)$, and
$$\{(\lambda^f)^\alpha \phi_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k, 0:n), [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f),\ \alpha_i = 0 \text{ if } i < \min[\![\sigma]\!]\} \tag{6.2}$$
is a basis.
4. *Spanning set and basis for $\mathring{\mathcal{P}}_r^-\Lambda^k(f)$. The set*
$$\{(\lambda^f)^\alpha \phi_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k, 0:n),\ [\![\alpha,\sigma]\!] = \mathscr{I}(f)\}$$
is a spanning set for $\mathring{\mathcal{P}}_r^-\Lambda^k(f)$, and
$$\{(\lambda^f)^\alpha \phi_\sigma^f \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k, 0:n), [\![\alpha,\sigma]\!] = \mathscr{I}(f),\ \alpha_i = 0 \text{ if } i < \min[\![\sigma]\!]\}$$
is a basis.

6.1. Barycentric spanning set and basis for $\mathcal{P}_r\Lambda^k(T)$

Observe that $d\lambda_i \in \mathrm{Alt}^1\mathbb{R}^n$. Furthermore, $d\lambda_i(x_j - y) = \delta_{ij}$ for any y in the subsimplex opposite x_i. In particular, $\mathrm{tr}_{T,f} d\lambda_i = 0$ for any subsimplex $f \in \Delta(T)$ with $x_i \notin f$ or equivalently $i \in \mathscr{I}(f^*)$. Furthermore, $\{d\lambda_i\}_{i=0}^n$ is a spanning set for $\mathrm{Alt}^1\mathbb{R}^n$, and any subset of n elements is a basis. Therefore, writing $d\lambda_\sigma$ for $d\lambda_\sigma^T$, the set

$$\{d\lambda_\sigma \mid \sigma \in \Sigma(1:k, 0:n)\}$$

is a spanning set for $\mathrm{Alt}^k\mathbb{R}^n$, and the set

$$\{d\lambda_\sigma \mid \sigma \in \Sigma(1:k, 1:n)\}$$

is a basis. The forms $d\lambda_\sigma \in \mathrm{Alt}^k\mathbb{R}^n$ have the property that for any $f \in \Delta(T)$ with $\dim f \geqslant k$,

$$\mathrm{tr}_{T,f} d\lambda_\sigma = 0 \quad \text{if and only if } [\![\sigma]\!] \cap \mathscr{I}(f^*) \neq \emptyset.$$

More generally, for polynomial forms of the form $\lambda^\alpha d\lambda_\sigma \in \mathcal{P}_r\Lambda^k(T)$ and $\dim f \geqslant k$, we observe that

$$\mathrm{tr}_{T,f}(\lambda^\alpha d\lambda_\sigma) = 0 \quad \text{if and only if } [\![\alpha,\sigma]\!] \cap \mathscr{I}(f^*) \neq \emptyset.$$

In particular, if $k < n$, then $\lambda^\alpha d\lambda_\sigma \in \mathring{\mathcal{P}}_r\Lambda^k(T)$ if and only if $[\![\alpha,\sigma]\!] = \{0,\ldots,n\}$.

Taking the tensor product of the Bernstein basis for $\mathcal{P}_r(T)$, given by (2.3), with the spanning set and basis given above for $\mathrm{Alt}^k\mathbb{R}^n$, we get that

Proposition 6.2. *The set*

$$\{\lambda^\alpha d\lambda_\sigma \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n)\}$$

is a spanning set for $\mathcal{P}_r\Lambda^k(T)$, and

$$\{\lambda^\alpha d\lambda_\sigma \mid \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 1:n)\}$$

is a basis.

Restricting to a face $f \in \Delta(T)$, we obtain the spanning set and basis for $\mathcal{P}_r\Lambda^k(f)$ given in the first part of Theorem 6.1.

6.2. Barycentric spanning set and basis for $\mathcal{P}_r^-\Lambda^k(T)$

For $f \in \Delta(T)$ and $\sigma \in \Sigma(0:k, 0:n)$ with $[\![\sigma]\!] \subseteq \mathscr{I}(f)$, define the associated Whitney form by

$$\phi_\sigma^f = \sum_{i=0}^k (-1)^i \lambda_{\sigma(i)}^f d\lambda_{\sigma(0)}^f \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}^f} \wedge \cdots \wedge d\lambda_{\sigma(k)}^f. \tag{6.3}$$

Just as we usually write λ_i rather than λ_i^T when the simplex is clear from context, we will usually write ϕ_σ instead of ϕ_σ^T. We note that if $k = 0$, so that the associated subsimplex f_σ consists of a single point x_i, then $\phi_\sigma = \lambda_i$. It is evident that the Whitney forms belong to $\mathcal{P}_1\Lambda^k(T)$. In fact, they belong to $\mathcal{P}_1^-\Lambda^k(T)$. This is a direct consequence of the identity

$$\kappa d\lambda_\sigma = \phi_\sigma - \phi_\sigma(0), \tag{6.4}$$

which can be easily established by induction on k, using the Leibniz rule (3.4). In fact, the set

$$\{\phi_\sigma \mid \sigma \in \Sigma(0:k, 0:n)\}$$

is a basis for $\mathcal{P}_1^-\Lambda^k(T)$. Furthermore, $\mathrm{tr}_{T,f}\phi_\sigma = d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k)}$ is a nonvanishing constant k-form on $f = f_\sigma$, while $\mathrm{tr}_{T,f}\phi_\sigma = 0$ for $f \in \Delta_k(T), f \neq f_\sigma$. Therefore, we refer to ϕ_σ as *the Whitney form associated to the face f_σ*.

For $\sigma \in \Sigma(0:k, 0:n)$ and $0 \leqslant j \leqslant k$, we let $\phi_{\sigma\hat{j}}$ be the Whitney form corresponding to the subsimplex of f_σ obtained by removing the vertex $\sigma(j)$. Hence,

$$\phi_{\sigma\hat{j}} = \sum_{i=0}^{j-1} (-1)^i \lambda_{\sigma(i)} d\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(j)}} \wedge \cdots d\lambda_{\sigma(k)} - \sum_{i=j+1}^{k} (-1)^i \lambda_{\sigma(i)} d\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(j)}} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}} \wedge \cdots d\lambda_{\sigma(k)}.$$

From this expression, we easily obtain the identity

$$\sum_{j=0}^k (-1)^j \lambda_{\sigma(j)} \phi_{\sigma\hat{j}} = 0, \quad \sigma \in \Sigma(0:k, 0:n). \tag{6.5}$$

Correspondingly, for $j \notin [\![\sigma]\!]$, we define

$$\phi_{j\sigma} = \lambda_j d\lambda_\sigma - d\lambda_j \wedge \phi_\sigma.$$

Thus, modulo a possible factor of -1, $\phi_{j\sigma}$ is the Whitney form associated to the simplex $[x_j, f_\sigma]$. For these functions, we obtain

$$\sum_{j \notin [\![\sigma]\!]} \phi_{j\sigma} = \left(\sum_{j \notin [\![\sigma]\!]} \lambda_j\right) d\lambda_\sigma - \left(\sum_{j \notin [\![\sigma]\!]} d\lambda_j\right) \wedge \phi_\sigma = \left(\sum_{j \notin [\![\sigma]\!]} \lambda_j\right) d\lambda_\sigma + \left(\sum_{j \in [\![\sigma]\!]} d\lambda_j\right) \wedge \phi_\sigma = \left(\sum_{j=0}^n \lambda_j\right) d\lambda_\sigma = d\lambda_\sigma. \tag{6.6}$$

Now consider functions of the form $\lambda^\alpha\phi_\sigma$, where $\alpha \in \mathbb{N}_0^{0:n}$, $|\alpha| = r-1$, $\sigma \in \Sigma(0:k,0:n)$. It follows from the relation (3.5) that these functions belong to $\mathcal{P}_r^-\Lambda^k(T)$. In fact, they span. From the identity (6.5) we know that these forms are not, in general, linearly independent. The following lemma, cf. [2, Lemma 4.2], enables us to extract a basis.

Lemma 6.3. *Let x be a vertex of T. Then the Whitney forms corresponding to the k-subsimplices that contain x are linearly independent over the ring of polynomials $\mathcal{P}(T)$.*

Using these results, we are able to prove:

Proposition 6.4. *The set*

$$\{\lambda^\alpha\phi_\sigma | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k,0:n)\}. \tag{6.7}$$

is a spanning set for $\mathcal{P}_r^-\Lambda^k(T)$, and

$$\{\lambda^\alpha\phi_\sigma | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r-1,\ \sigma \in \Sigma(0:k,0:n),\ \alpha_i = 0 \text{ if } i < \min[\![\sigma]\!]\} \tag{6.8}$$

is a basis.

Proof. Let $\alpha \in \mathbb{N}_0^{0:n}$, with $|\alpha| = r-1$, and $\rho \in \Sigma(0:k-1,0:n)$. The identity (6.6) implies that

$$\lambda^\alpha d\lambda_\rho = \sum_{j\notin[\![\rho]\!]} \lambda^\alpha\phi_{j\rho},$$

and hence all forms in $\mathcal{P}_{r-1}\Lambda^k(T)$ are in the span of the set given by (6.7). Furthermore, if $\sigma \in \Sigma(0:k,0:n)$, we obtain from (6.4) that

$$\kappa(\lambda^\alpha d\lambda_\sigma) + \lambda^\alpha\phi_\sigma(0) = \lambda^\alpha\phi_\sigma,$$

and therefore all of $\kappa[\mathcal{P}_{r-1}\Lambda^{k+1}(T)]$ is also in the span. By the definition of the space $\mathcal{P}_r^-\Lambda^k(T)$, it follows that (6.7) is a spanning set. To show that (6.8) is a basis, we use the identity (6.5) to see that any form given in the span of (6.7) is in the span of the forms in (6.8). Then we use Lemma 6.3, combined with a simple inductive argument, to show that the elements of the asserted basis are linearly independent. For details, see the proof of Theorem 4.4 of [2]. □

Restricting to a face $f \in \Delta(T)$, we obtain the spanning set and basis for $\mathcal{P}_r^-\Lambda^k(f)$ given in the third part of Theorem 6.1.

6.3. Spaces of vanishing trace

In this subsection, we will derive spanning sets and bases for the corresponding spaces of zero trace. This will be based on the results obtained above and Corollary 5.2, which leads to the dimension of these spaces. We first characterize the space $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$.

Proposition 6.5. *The set*

$$\{\lambda^\alpha\phi_\sigma | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r-1, \sigma \in \Sigma(0:k,0:n), [\![\alpha,\sigma]\!] = \{0,\ldots,n\}\}$$

is a spanning set for $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ and

$$\{\lambda^\alpha\phi_\sigma | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r-1,\ \sigma \in \Sigma(0:k,0:n),\ [\![\alpha,\sigma]\!] = \{0,\ldots,n\},\ \alpha_i = 0 \text{ if } i < \min[\![\sigma]\!]\}$$

is a basis.

Proof. Since $[\![\alpha,\sigma]\!] = \{0,\ldots,n\}$, each of the forms $\lambda^\alpha\phi_\sigma$ is contained in $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. Moreover, the condition $\alpha_i = 0$ if $i < \min[\![\sigma]\!]$ reduces to $\sigma(0) = 0$ in this case. Lemma 6.3 implies that the forms $\lambda^\alpha\phi_\sigma$ for which $\sigma(0) = 0$ are linearly independent. The cardinality of this set is equal to $\binom{n}{k}\dim\mathcal{P}_{r-n+k-1}$ which is equal to $\dim\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ by Corollary 5.2. This completes the proof. □

Restricting to a face $f \in \Delta(T)$, we obtain the spanning set and basis for $\mathring{\mathcal{P}}_r^-\Lambda^k(f)$ given in the fourth part of Theorem 6.1.

Finally, we obtain a characterization of the space $\mathring{\mathcal{P}}_r\Lambda^k(T)$.

Proposition 6.6. *The set*

$$\{\lambda^\alpha d\lambda_\sigma | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r,\ \sigma \in \Sigma(1:k,0:n),\ [\![\alpha,\sigma]\!] = \{0,\ldots,n\}\}$$

is a spanning set for $\mathring{\mathcal{P}}_r\Lambda^k(T)$, and

$$\{\lambda^\alpha d\lambda_\sigma | \alpha \in \mathbb{N}_0^{0:n},\ |\alpha| = r,\ \sigma \in \Sigma(1:k,0:n),\ [\![\alpha,\sigma]\!] = \{0,\ldots,n\},\ \alpha_i = 0 \text{ if } i < \min[\![\sigma^*]\!]\} \tag{6.9}$$

is a basis.

Proof. Since $[\![\alpha,\sigma]\!] = \{0,\ldots,n\}$, each of the forms $\lambda^\alpha d\lambda_\sigma$ is contained in $\mathring{\mathcal{P}}_r\Lambda^k(T)$. Furthermore, we have seen in Corollary 5.2, that $\mathring{\mathcal{P}}_r\Lambda^k(T) \cong \mathcal{P}_{r+k-n}^-\Lambda^{n-k}(T)$, whence, $\dim\mathring{\mathcal{P}}_r\Lambda^k(T) = \binom{r-1}{n-k}\binom{r+k}{r}$. On the other hand, the cardinality of the set given by (6.9) can be computed as $\sum_j A_j \cdot B_j$, where A_j is the number of elements $\sigma \in \Sigma(1:k,0:n)$ with $\min[\![\sigma^*]\!] = j$, and for each fixed such σ, B_j is the number of multi-indices α satisfying the conditions of (6.9), namely

$$A_j = \binom{n-j}{k-j} \quad \text{and} \quad B_j = \binom{r+k-j-1}{n-j}.$$

Hence, the cardinality of the set is given by

$$\sum_{j=0}^{k}\binom{n-j}{k-j}\binom{r+k-j-1}{n-j} = \binom{r-1}{n-k}\sum_{j=0}^{k}\binom{r+k-j-1}{r-1} = \binom{r-1}{n-k}\binom{r+k}{r}.$$

Here the first identity follows from a binomial identity of the form

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c},$$

while the second is a standard summation formula. Hence, the cardinality of the set given by (6.9) is equal to the dimension of $\mathring{\mathcal{P}}_r\Lambda^k(T)$. To complete the proof, we show that the elements of the set (6.9) are linearly independent. Denote the index set by

$$S := \{(\alpha,\sigma) \in \mathbb{N}_0^{0:n} \times \Sigma(1:k,0:n) | |\alpha| = r,\ [\![\alpha,\sigma]\!] = \{0,\ldots,n\},\ \alpha_i = 0 \text{ if } i < \min[\![\sigma^*]\!]\},$$

so we must show that if

$$\sum_{(\alpha,\sigma)\in S} c_{\alpha\sigma}\lambda^\alpha d\lambda_\sigma = 0, \tag{6.10}$$

for some real coefficients $c_{\alpha\sigma}$, then all the coefficients vanish. Since the Bernstein monomials λ^α are linearly independent, (6.10) implies that for each $\alpha \in \mathbb{N}_0^{0:n}$ with $|\alpha| = r$,

$$\sum_{\{\sigma|(\alpha,\sigma)\in S\}} c_{\alpha\sigma} d\lambda_\sigma = 0. \tag{6.11}$$

First consider a multi-index α with $\alpha_0 > 0$. Then the definition of the index set S implies that $\min[\![\sigma^*]\!] = 0$ for all the summands in (6.11). Since the corresponding $d\lambda_\sigma$ are linearly independent, we conclude that all the $c_{\alpha\sigma}$ vanish when $\alpha_0 > 0$. Next consider α with $\alpha_0 = 0$ but $\alpha_1 > 0$. If $(\alpha,\sigma) \in S$, then $\min[\![\sigma^*]\!] = 1$, and again we conclude that $c_{\alpha\sigma} = 0$. Continuing in this way we find that all the $c_{\alpha\sigma}$ vanish, completing the proof. □

Restricting to a face $f \in \Delta(T)$, we obtain the spanning set and basis for $\mathring{\mathcal{P}}_r\Lambda^k(f)$ given in the second part of Theorem 6.1.

7. A geometric decomposition of $\mathcal{P}_r^-\Lambda^k(T)$

In this section, we will apply the theory developed in Section 4 with $X(T) = \mathcal{P}_r^-\Lambda^k(T)$ to obtain a geometric decomposition of

$\mathcal{P}_r\Lambda^k(\mathcal{T})$ into subspaces $E_f[\mathring{\mathcal{P}}_r^-\Lambda^k(f)]$, where E_f is the global extension operator constructed as in Section 4 from a consistent family of easily computable extension operators. The resulting decomposition reduces to the Bernstein decomposition (2.6) in the case $k=0$.

We first note that if $T,T'\in\mathcal{T}$ with $f\in\Delta(T)\cap\Delta(T')$ then $\mathrm{tr}_{T,f}\mathcal{P}_r^-\Lambda^k(T)=\mathrm{tr}_{T',f}\mathcal{P}_r^-\Lambda^k(T')=\mathcal{P}_r^-\Lambda^k(f)$. Hence, the assumption (4.2) holds. Furthermore, for $f,g\in\Delta(\mathcal{T})$ with $f\subseteq g$, we define $E=E_{f,g}^{k,r,-}:\mathcal{P}_r^-\Lambda^k(f)\to\mathcal{P}_r^-\Lambda^k(g)$ as the barycentric extension:

$$(\lambda^f)^\alpha\phi_\sigma^f\mapsto(\lambda^g)^\alpha\phi_\sigma^g,\quad [\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f). \tag{7.1}$$

This generalizes to k-forms, the barycentric extension operator $E_{f,T}^r$ on $\mathcal{P}_r$, introduced in Section 2.3. Since the forms $(\lambda^f)^\alpha\phi_\sigma^f$ are not linearly independent, it is not clear that (7.1) well-defines E. We show this in the following theorem.

Theorem 7.1. *There is a unique mapping $E=E_{f,g}^{k,r,-}$ from $\mathcal{P}_r^-\Lambda^k(f)$ to $\mathcal{P}_r^-\Lambda^k(g)$ satisfying (7.1).*

Proof. We first recall from part 3 of Theorem 6.1 that the set

$$\{(\lambda^f)^\alpha\phi_\sigma^f|\alpha\in\mathbb{N}_0^{0:n},|\alpha|=r-1,\sigma\in\Sigma(0:k,0:n),\ [\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f),\ \alpha_i=0\text{ if }i<\min[\![\sigma]\!]\}$$

is a basis for $\mathcal{P}_r^-\Lambda^k(f)$. Hence, we can uniquely define an extension E by (7.1), if we restrict to these basis functions. We now show that (7.1) holds for all $(\lambda^f)^\alpha\phi_\sigma^f$ with $[\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f)$. To see this, we use the identity (6.5). Consider forms $(\lambda^f)^\alpha\phi_\sigma^f$ which do not belong to the given basis, i.e., $s:=\min[\![\alpha]\!]<\min[\![\sigma]\!]$. Write $\lambda^\alpha=\lambda^\beta\lambda_s$ and let $\rho\in\Sigma(0:k+1,0:n)$ be determined by $[\![\rho]\!]=\{s\}\cup[\![\sigma]\!]$. Then, by (6.5),

$$(\lambda)^\alpha\phi_\sigma=\sum_{j=1}^{k+1}(-1)^{j-1}\lambda^\beta\lambda_{\rho(j)}\phi_{\rho\hat{j}},$$

and so

$$(\lambda^f)^\alpha\phi_\sigma^f=\sum_{j=1}^{k+1}(-1)^{j-1}(\lambda^f)^\beta\lambda_{\rho(j)}^f\phi_{\rho\hat{j}}^f,$$

$$(\lambda^g)^\alpha\phi_\sigma^g=\sum_{j=1}^{k+1}(-1)^{j-1}(\lambda^g)^\beta\lambda_{\rho(j)}^g\phi_{\rho\hat{j}}^g.$$

Hence

$$E[(\lambda^f)^\alpha\phi_\sigma^f]=\sum_{j=1}^{k+1}(-1)^{j-1}E[(\lambda^f)^\beta\lambda_{\rho(j)}^f\phi_{\rho\hat{j}}^f]=\sum_{j=1}^{k+1}(-1)^{j-1}(\lambda^g)^\beta\lambda_{\rho(j)}^g\phi_{\rho\hat{j}}^g=(\lambda^g)^\alpha\phi_\sigma^g,$$

and the proof is completed. □

Theorem 7.2. *The family of extension operators E is consistent, i.e., for all $f,g,h\in\Delta(\mathcal{T})$ with $f,g\subseteq h$, and all $\omega\in\mathcal{P}_r^-\Lambda^k(f)$,*

$$\mathrm{tr}_{h,g}E_{f,h}\omega=E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}\omega.$$

Proof. It is enough to establish this result for $\omega=(\lambda^f)^\alpha\phi_\sigma^f$, with $[\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f)$, since such ω span $\mathcal{P}_r^-\Lambda^k(f)$. Now for such pairs (α,σ), $E_{f,h}[(\lambda^f)^\alpha\phi_\sigma^f]=(\lambda^h)^\alpha\phi_\sigma^h$ and then

$$\mathrm{tr}_{h,g}E_{f,h}[(\lambda^f)^\alpha\phi_\sigma^f]=\begin{cases}(\lambda^g)^\alpha\phi_\sigma^g, & \text{if }[\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f\cap g),\\ 0, & \text{otherwise}.\end{cases}$$

On the other hand,

$$\mathrm{tr}_{f,f\cap g}(\lambda^f)^\alpha\phi_\sigma^f=\begin{cases}(\lambda^{f\cap g})^\alpha(\phi^{f\cap g})_\sigma, & \text{if }[\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f\cap g),\\ 0, & \text{otherwise},\end{cases}$$

and hence

$$E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}(\lambda^f)^\alpha\phi_\sigma^f=\begin{cases}(\lambda^g)^\alpha\phi_\sigma^g, & \text{if }[\![\alpha,\sigma]\!]\subseteq\mathcal{I}(f\cap g),\\ 0, & \text{otherwise}.\end{cases}\qquad\square$$

From Theorem 4.3, we obtain the desired geometric decomposition of $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$.

Theorem 7.3

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T})=\bigoplus_{\substack{f\in\Delta(\mathcal{T})\\ \dim f\geqslant k}}E_f[\mathring{\mathcal{P}}_r^-\Lambda^k(f)].$$

where $E_f:\mathring{\mathcal{P}}_r^-\Lambda^k(f)\to\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ denotes the global extension operator determined by the family $E_{f,g}^{k,r,-}$.

The final part of Theorem 6.1 furnishes an explicit spanning set and basis for $\mathring{P}_r^-\Lambda^k(f)$, and so this theorem gives an explicit spanning set and basis for $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$. We discuss these explicit representations further in Section 9.

We now turn to a geometric characterization of the extension operator $E:\mathcal{P}_r^-\Lambda^k(f)\to\mathcal{P}_r^-\Lambda^k(T,f)$. To this end, we say that a smooth k-form $\omega\in\Lambda^k(T)$ vanishes to order r at a point x if the function $x\mapsto\omega_x(v_1,\ldots,v_k)$ vanishes to order r at x for all $v_1,\ldots,v_k\in\mathbb{R}^n$, and that it vanishes to order r on a set g if it vanishes to order r at each point of the set. Note that the extension operator $E=E_{f,T}^{k,r,-}$ has the property that for any $\mu\in\mathcal{P}_r^-\Lambda^k(f)$, $E_{f,T}^{k,r,-}\mu$ vanishes to order r on f^*. In fact, if we set

$$\mathcal{P}_r^-\Lambda^k(T,f)=\{\omega\in\mathcal{P}_r^-\Lambda^k(T)|\omega\text{ vanishes to order }r\text{ on }f^*\},$$

we can prove

Theorem 7.4. *$\mathcal{P}_r^-\Lambda^k(T,f)=E[\mathcal{P}_r^-\Lambda^k(f)]$ and for $\mu\in\mathcal{P}_r^-\Lambda^k(f)$, $E\mu=E_{f,T}^{k,r,-}\mu$ can be characterized as the unique extension of μ to $\mathcal{P}_r^-\Lambda^k(T,f)$.*

Proof. We note that the second statement of the theorem follows from the first, since $\mathrm{tr}_{T,f}$ from $E[P_r^-\Lambda^k(f)]$ to $P_r^-\Lambda^k(f)$ has a unique right inverse. Since $E[P_r^-\Lambda^k(f)]\subseteq P_r^-\Lambda^k(T,f)$, we only need to prove the opposite inclusion. Without loss of generality we may assume that $f=[x_{m+1},\ldots,x_n]$, $f^*=[x_0,\ldots,x_m]$, for some $0\leqslant m<n$. We proceed by induction on m. When $m=0$, we may assume without loss of generality that the vertex x_0 is at the origin. Now $\mathcal{P}_r^-\Lambda^k=\mathcal{P}_{r-1}\Lambda^k+\kappa\mathcal{H}_{r-1}\Lambda^{k+1}$ (where $\mathcal{H}_r$ denotes the homogeneous polynomials of the degree r). Since $\kappa\mathcal{H}_{r-1}\Lambda^{k+1}\subseteq\mathcal{H}_r\Lambda^k$, every element $\omega\in\kappa\mathcal{H}_{r-1}\Lambda^{k+1}$ vanishes to order r at the origin. On the other hand, no non-zero element of $\mathcal{P}_{r-1}\Lambda^k$ vanishes to order r at the origin. Thus $\mathcal{P}_r^-\Lambda^k(T,f)=\kappa\mathcal{H}_{r-1}\Lambda^{k+1}$. It follows from [2, Theorem 3.3] that

$$\begin{aligned}\dim\mathcal{P}_r^-\Lambda^k(T,f)&=\dim\kappa\mathcal{H}_{r-1}\Lambda^{k+1}(T)\\&=\binom{r+n-1}{n-k-1}\binom{r+k-1}{k}=\dim\mathcal{P}_r^-\Lambda^k(f)\\&=\dim E[P_r^-\Lambda^k(f)],\end{aligned}$$

and since $E[P_r^-\Lambda^k(f)]\subseteq\mathcal{P}_r^-\Lambda^k(T,f)$, $E[P_r^-\Lambda^k(f)]=\mathcal{P}_r^-\Lambda^k(T,f)$.

Now suppose that ω vanishes to order r on the m-dimensional face $[x_0,\ldots,x_m]$ with $m>0$. Let $T'=[x_1,\ldots,x_n]$, $\omega'=\mathrm{tr}_{T,T'}\omega$. Then $\omega'\in\mathcal{P}_r^-\Lambda^k(T')$ vanishes to order r on the $(m-1)$-dimensional face $[x_1,\ldots,x_m]$, so, by induction, $\omega'=E_{f,T'}\mu$ for some $\mu\in P_r^-\Lambda^k(f)$. Furthermore, since ω vanishes to order r at x_0, we can use the result established above for $m=0$ to conclude that $\omega=E_{T',T}\omega'=E_{T',T}E_{f,T'}\mu$. However, it follows immediately from (7.1) that $E_{T',T}E_{f,T'}=E_{f,T}$, and hence the two spaces are equal, and the theorem is established. □

8. A geometric decomposition of $P_r{}^k(T)$

In this section, we again apply the theory developed in Section 4, this time with $X(T) = \mathcal{P}_r\Lambda^k(T)$. In this case condition (4.2) is obvious, since $\mathrm{tr}_{T,f}\mathcal{P}_r\Lambda^k(T) = \mathcal{P}_r\Lambda^k(f)$. In view of the previous section, one might hope that we could define the extension operator as

$$(\lambda^f)^\alpha d\lambda^f_\sigma \mapsto \lambda^\alpha d\lambda_\sigma.$$

However, this does not lead to a well-defined operator. To appreciate the problem, consider the space $\mathcal{P}_2\Lambda^1(T)$, where $T \subset \mathbb{R}^2$ is a triangle spanned by the vertices x_0, x_1, x_2, and let $f = [x_1, x_2]$. Then $\lambda^f_1\lambda^f_2(d\lambda^f_1 + d\lambda^f_2) = 0$, but $\lambda_1\lambda_2(d\lambda_1 + d\lambda_2) = -\lambda_1\lambda_2 d\lambda_0 \neq 0$.

To remedy this situation, we will show that for $f, g \in \Delta(T)$ with $f \subseteq g$, a consistent extension operator $E = E^{k,r}_{f,g} : \mathcal{P}_r\Lambda^k(f) \to \mathcal{P}_r\Lambda^k(g)$ is given by

$$(\lambda^f)^\alpha d\lambda^f_\sigma \mapsto (\lambda^g)^\alpha \psi^{\alpha,f,g}_\sigma, \quad [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f), \tag{8.1}$$

where $\psi^{\alpha,f,g}_\sigma$ is defined as follows. We first introduce forms $\psi^{\alpha,f,g}_i \in \mathrm{Alt}^1 T_g$ defined by

$$\psi^{\alpha,f,g}_i = d\lambda^g_i - \frac{\alpha_i}{|\alpha|}\sum_{j\in\mathscr{I}(f)} d\lambda^g_j, \quad i \in \mathscr{I}(f), \tag{8.2}$$

and then define $\psi^{\alpha,f,g}_\sigma \in \mathrm{Alt}^k T_g$ by

$$\psi^{\alpha,f,g}_\sigma = \psi^{\alpha,f,g}_{\sigma(1)} \wedge \cdots \wedge \psi^{\alpha,f,g}_{\sigma(k)}, \quad \sigma \in \Sigma(1:k, 0:n), \quad [\![\sigma]\!] \subseteq \mathscr{I}(f). \tag{8.3}$$

A geometric interpretation of $\psi^{\alpha,f,g}_\sigma$ will be given below.

First we show that E is well-defined and is, in fact, an extension operator.

Theorem 8.1. *There is a unique mapping $E = E^{k,r}_{f,g} : \mathcal{P}_r\Lambda^k(f) \to \mathcal{P}_r\Lambda^k(g)$ satisfying (8.1). Moreover, it is an extension operator: $\mathrm{tr}_{g,f}E^{k,r}_{f,g}\omega = \omega$ for $\omega \in \mathcal{P}_r\Lambda^k(f)$.*

Proof. By the first part of Theorem 6.1, the set

$$\{(\lambda^f)^\alpha d\lambda^f_\sigma | \alpha \in \mathbb{N}^{0:n}_0, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n),\ [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f),\ \min[\![\sigma]\!] > \min\mathscr{I}(f)\}$$

is a basis for $\mathcal{P}_r\Lambda^k(f)$. Hence, we can define an extension E by (8.1), if we restrict to the basis functions. We now show that (8.1) holds for all $(\lambda^f)^\alpha d\lambda^f_\sigma$ with $[\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f)$, i.e., also when $\min[\![\sigma]\!] = \min\mathscr{I}(f)$. Writing $d\lambda_\sigma = d\lambda_{\sigma(1)} \wedge d\lambda_\rho$, and using the fact that $\sum_{j\in\mathscr{I}(f)} d\lambda^f_j = 0$ on the face f, we can write

$$d\lambda^f_\sigma = -\sum_{\substack{j\in\mathscr{I}(f)\\ j\neq\sigma(1)}} d\lambda^f_j \wedge d\lambda^f_\rho.$$

Hence,

$$E[(\lambda^f)^\alpha d\lambda^f_\sigma] = -E[(\lambda^f)^\alpha \sum_{\substack{j\in\mathscr{I}(f)\\ j\neq\sigma(1)}} d\lambda^f_j \wedge d\lambda^f_\rho] = -(\lambda^g)^\alpha \sum_{\substack{j\in\mathscr{I}(f)\\ j\neq\sigma(1)}} \psi^{\alpha,f,g}_j \wedge \psi^{\alpha,f,g}_\rho = (\lambda^g)^\alpha\psi^{\alpha,f,g}_\sigma,$$

where in the last step we have used the fact that $\sum_{i\in\mathscr{I}(f)}\psi^{\alpha,f,g}_i = 0$.

That E is an extension operator follows directly from the observation

$$\mathrm{tr}_{T,f}\psi_\sigma = d\lambda^f_\sigma,$$

which holds since $\sum_{j\in\mathscr{I}(f)} d\lambda^f_j = 0$ on the face f. □

Theorem 8.2. *The family of extension operators E is consistent, i.e., for all $f, g, h \in \Delta(\mathscr{T})$ with $f, g \subseteq h$ and all $\omega \in \mathcal{P}_r\Lambda^k(f)$,*

$$\mathrm{tr}_{h,g}E_{f,h}\omega = E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}\omega.$$

Proof. It is enough to establish this result for $\omega = (\lambda^f)^\alpha d\lambda^f_\sigma$, with $[\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f)$. Now for such pairs (α,σ), $E_{f,h}[(\lambda^f)^\alpha (d\lambda^f)_\sigma] = (\lambda^h)^\alpha\psi^{\alpha,f,h}_\sigma$. To determine $\mathrm{tr}_{h,g}[(\lambda^h)^\alpha\psi^{\alpha,f,h}_\sigma]$, we consider three cases. When $[\![\alpha]\!] \subseteq \mathscr{I}(f)$, but $[\![\alpha]\!] \not\subseteq \mathscr{I}(g)$, $\mathrm{tr}_{h,g}[(\lambda^h)^\alpha\psi^{\alpha,f,h}_\sigma] = 0$, since $\mathrm{tr}_{h,g}[(\lambda^h)^\alpha] = 0$. If $[\![\alpha]\!] \subseteq \mathscr{I}(f\cap g)$, then $\mathrm{tr}_{h,g}[(\lambda^h)^\alpha] = (\lambda^g)^\alpha$, so we need only compute $\mathrm{tr}_{h,g}\psi^{\alpha,f,h}_\sigma$. We do this by first considering $\mathrm{tr}_{h,g}\psi^{\alpha,f,h}_i$ for $i \in \mathscr{I}(f)$. If $i \in \mathscr{I}(f\cap g)$, we have

$$\mathrm{tr}_{h,g}\psi^{\alpha,f,h}_i = \mathrm{tr}_{h,g}\left(d\lambda^h_i - \frac{\alpha_i}{|\alpha|}\sum_{j\in\mathscr{I}(f)} d\lambda^h_j\right) = d\lambda^g_i - \frac{\alpha_i}{|\alpha|}\sum_{j\in\mathscr{I}(f\cap g)} d\lambda^g_j = \psi^{\alpha,f\cap g,g}_i.$$

On the other hand, if $i \in \mathscr{I}(f)\setminus\mathscr{I}(f\cap g)$, then since $\alpha_i = 0$, $\psi^{\alpha,f,h}_i = d\lambda^h_i$ and so $\mathrm{tr}_{h,g}\psi^{\alpha,f,h}_i = 0$. Combining these results, we obtain

$$\mathrm{tr}_{h,g}E_{f,h}[(\lambda^f)^\alpha d\lambda^f_\sigma] = \begin{cases} (\lambda^g)^\alpha\psi^{\alpha,f\cap g,g}_\sigma, & \text{if } [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f\cap g), \\ 0, & \text{otherwise.}\end{cases}$$

But

$$\mathrm{tr}_{f,f\cap g}[(\lambda^f)^\alpha d\lambda^f_\sigma] = \begin{cases} (\lambda^{f\cap g})^\alpha d\lambda^{f\cap g}_\sigma, & \text{if } [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f\cap g), \\ 0, & \text{otherwise,}\end{cases}$$

and hence

$$E_{f\cap g,g}\mathrm{tr}_{f,f\cap g}[(\lambda^f)^\alpha d\lambda^f_\sigma] = \begin{cases} (\lambda^g)^\alpha\psi^{\alpha,f\cap g,g}_\sigma, & \text{if } [\![\alpha,\sigma]\!] \subseteq \mathscr{I}(f\cap g), \\ 0, & \text{otherwise.}\end{cases} \quad \square$$

From Theorem 4.3, we obtain the desired geometric decomposition $\mathcal{P}_r\Lambda^k(\mathscr{T})$.

Theorem 8.3

$$\mathcal{P}_r\Lambda^k(\mathscr{T}) = \bigoplus_{\substack{f\in\Delta(\mathscr{T})\\ \dim f \geqslant k}} E_f[\mathring{\mathcal{P}}_r\Lambda^k(f)].$$

where $E_f : \mathring{\mathcal{P}}_r\Lambda^k(f) \to \mathcal{P}_r\Lambda^k(\mathscr{T})$ denotes the global extension operator determined by the family $E^{k,r}_{f,g}$.

Combining this result with the second part of Theorem 6.1, we obtain an explicit spanning set and basis for $\mathcal{P}^-_r\Lambda^k(\mathscr{T})$ (see Section 9).

We now turn to a geometric characterization of the extension operator $E = E^{k,r}_{f,g} : \mathcal{P}_r\Lambda^k(f) \to \mathcal{P}_r\Lambda^k(T,f)$. First, we will motivate the choice of E, and in particular the forms $\psi^{\alpha,f,g}_\sigma$, by establishing some additional properties of these forms. Observe that any multi-index α determines a convex combination of the vertices x_i of T, namely

$$x_\alpha := |\alpha|^{-1}\sum_m \alpha_m x_m \in T,$$

and if $[\![\alpha]\!] \subset \mathscr{I}(f)$, then $x_\alpha \in f$. For each such multi-index α, we then define the vectors

$$t_{\alpha l} = x_\alpha - x_l = \frac{1}{|\alpha|}\sum_{m\in\mathscr{I}(f)} \alpha_m(x_m - x_l), \quad l \in \mathscr{I}(f^*).$$

Clearly, for each such α, $\mathbb{R}^n$ decomposes as the direct sum $T_f \oplus \mathrm{span}\{t_{\alpha l} | l \in \mathscr{I}(f^*)\}$, where T_f denotes the tangent space of f. See Fig. 8.1. This decomposition defines a projection operator $P = P_{f,\alpha} : \mathbb{R}^n \to T_f$ determined by the equations $Pv = v$ for $v \in T_f$ and $Pt_{\alpha l} = 0$ for $l \in \mathscr{I}(f^*)$. Hence, we have

$$P^*_{f,\alpha}\mathrm{Alt}^k T_f = \{a \in \mathrm{Alt}^k\mathbb{R}^n | a \lrcorner t_{\alpha l} = 0, l \in \mathscr{I}(f^*)\}. \tag{8.4}$$

Furthermore, since $d\lambda_j(x_m - x_l) = \delta_{jm}$ for any $j \in \mathscr{I}(f)$, $m \in \mathscr{I}(f)$, and $l \in \mathscr{I}(f^*)$, we get for $[\![\alpha]\!] \subset \mathscr{I}(f)$,

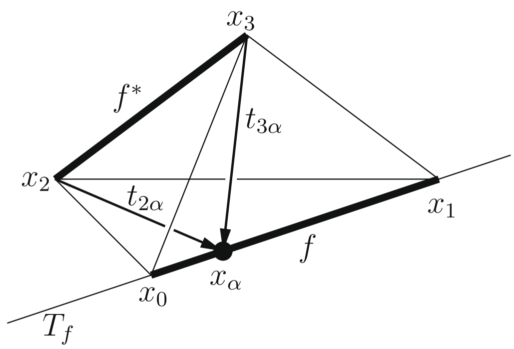

Fig. 8.1. $T = [x_0, x_1, x_2, x_3]$, $f = [x_0, x_1]$, $\alpha = (3, 1, 0, 0)$, $\mathbb{R}^3 = T_f \oplus \mathrm{span}\{t_{2\alpha}, t_{3\alpha}\}$.

$$\psi_i^{\alpha,f,T}(t_{\alpha l}) = d\lambda_i(t_{\alpha l}) - \frac{\alpha_i}{|\alpha|}\sum_{j\in\mathscr{I}(f)} d\lambda_j(t_{\alpha l}) = \frac{1}{\alpha}\left(\alpha_i - \frac{\alpha_i}{|\alpha|}\sum_{j\in\mathscr{I}(f)}\alpha_j\right) = 0. \tag{8.5}$$

It follows that

$$\begin{aligned}\psi_\sigma^{\alpha,f,T}(v_1,\cdots,v_k) &= \psi_\sigma^{\alpha,f,T}(Pv_1,\cdots,Pv_k)\\ &= \mathrm{tr}_{T,f}\psi_\sigma^{\alpha,f,T}(Pv_1,\cdots,Pv_k)\\ &= d\lambda_\sigma^f(Pv_1,\cdots,Pv_k).\end{aligned} \tag{8.6}$$

Hence, in the language of pullbacks,

$$\psi_\sigma^{\alpha,f,T} = P_{f,\alpha}^* d\lambda_\sigma^f,$$

where $P_{f,\alpha}^*$ is the pullback of $P_{f,\alpha}$, and so

$$E_{f,g}[(\lambda^f)^\alpha d\lambda_\sigma^f] = (\lambda^g)^\alpha P_{f,\alpha}^* d\lambda_\sigma^f.$$

Recall that the geometric characterization in the previous section hinged upon the fact that a form in $\mathcal{P}_r^-\Lambda^k(T)$ which vanishes to order r on f^* and has vanishing trace on f must vanish identically. Now this is not true for an arbitrary element of the larger space $\mathcal{P}_r\Lambda^k(T)$. Returning to the example given at the beginning of this section, where $T = [x_0, x_1, x_2]$ and $f = [x_1, x_2]$, the form

$$\omega = \lambda_1\lambda_2[d\lambda_1 + d\lambda_2] = -\lambda_1\lambda_2 d\lambda_0 \in \mathcal{P}_2\Lambda^1(T)$$

vanishes to second order at $f^* = \{x_0\}$. However, $\mathrm{tr}_{T,f}\omega$ also vanishes. Thus, additional conditions on ω will be needed in order to insure that ω is uniquely determined by $\mathrm{tr}_{T,f}\omega$. We say that ω *vanishes to order* r^+ *on* f^*, if ω vanishes to order r on f^* and the following conditions hold:

$$\partial_{t_l}^\alpha \omega \lrcorner t_{\alpha l} = 0,\ l \in \mathscr{I}(f^*),\ [\![\alpha]\!] \subseteq \mathscr{I}(f),\ |\alpha| = r. \tag{8.7}$$

Here $\partial_{t_l}^\alpha := \prod_{j\in\mathscr{I}(f)}\partial_{t_{jl}}^{\alpha_j}$ with $\partial_{t_{jl}} = t_{jl}\cdot\nabla$, the directional derivative along the vector $t_{jl} := x_j - x_l$. The contraction operator $\lrcorner$ is defined at the start of Section 2.4.

Note that $\partial_{t_{jl}}\lambda_i = \delta_{ij}$ for $i, j \in \mathscr{I}(f)$, $j \in \mathscr{I}(f^*)$. It follows that if $\alpha, \beta \in \mathbb{N}_0^n$, with $|\alpha| = |\beta| = r$, $[\![\alpha]\!], [\![\beta]\!] \subseteq \mathscr{I}(f)$, and $l \in \mathscr{I}(f^*)$, then

$$\partial_{t_l}^\beta \lambda^\alpha = 0 \quad \text{for } \alpha \neq \beta \quad \text{and} \quad \partial_{t_l}^\alpha\lambda^\alpha = \alpha!. \tag{8.8}$$

Setting

$$\mathcal{P}_r\Lambda^k(T,f) = \{\omega \in \mathcal{P}_r\Lambda^k(T) | \omega \text{ vanishes to order } r^+ \text{ on } f^*\},$$

we can now give the geometric description of the extension operator E.

Theorem 8.4. $\mathcal{P}_r\Lambda^k(T,f) = E[\mathcal{P}_r\Lambda^k(f)]$ *and for* $\mu \in \mathcal{P}_r\Lambda^k(f)$, $E\mu = E_{f,T}^{k,r}\mu$ *can be characterized as the unique extension of* μ *to* $\mathcal{P}_r\Lambda^k(T,f)$.

Proof. We note that the second statement of the theorem follows from the first, since $\mathrm{tr}_{T,f}$ from $E[P_r\Lambda^k(f)]$ to $P_r\Lambda^k(f)$ has a unique right inverse. To prove the first statement, we first show that $E[\mathcal{P}_r\Lambda^k(f)] \subseteq \mathcal{P}_r\Lambda^k(T,f)$. Observe first that $E[(\lambda^f)^\alpha(d\lambda^f)_\sigma] = \lambda^\alpha\psi_\sigma^{\alpha,f,T}$ vanishes to order r on f^* since $(\lambda)^\alpha$ does. Next, note that (8.8) tells us that $\partial_{t_l}^\beta[\lambda^\alpha\psi_\sigma^{\alpha,f,T}] = \alpha!\psi_\sigma^{\alpha,f,T}$ if $\beta = \alpha$ and vanishes if β is any other multi-index of order r with $[\![\beta]\!] \subseteq \mathscr{I}(f)$. Therefore, the conditions (8.7) for vanishing of order r^+ are reduced to verifying the conditions $\psi_\sigma^{\alpha,f,T}\lrcorner t_{\alpha l} = 0$ for all $l \in \mathscr{I}(f^*)$. However, this follows immediately from the definition of the wedge product and (8.5).

To show that $\mathcal{P}_r\Lambda^k(T,f) \subseteq E[\mathcal{P}_r\Lambda^k(f)]$, we use Lemma 2.1 to see that any element $\omega \in \mathcal{P}_r\Lambda^k(T,f)$ admits a representation of the form

$$\omega = \sum_{\substack{[\![\alpha]\!]\subseteq\mathscr{I}(f)\\ |\alpha|=r}} a_\alpha\lambda^\alpha$$

for some $a_\alpha \in \mathrm{Alt}^k\mathbb{R}^n$. However, invoking (8.8) and (8.7), we conclude that, if ω vanishes to the order r^+ on f^*, then $a_\alpha \lrcorner t_{\alpha l} = 0$ for all $l \in \mathscr{I}(f^*)$, and hence by (8.4), $a_\alpha \in P_{f,\alpha}^*\mathrm{Alt}^k T_f$. It therefore follows from (8.6), that $\omega \in E[\mathcal{P}_r\Lambda^k(f)]$. □

9. Construction of bases

From Theorem 7.3, (4.6), Theorem 7.1, and part 4 of Theorem 6.1, one immediately obtains explicit formulas for a spanning set and basis for $\mathcal{P}_r^-\Lambda^k(\mathscr{T})$, with each spanning and basis form associated to a particular face $f \in \Delta(\mathscr{T})$. The forms associated to f vanish

Table 9.1
Bases for the spaces $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r\Lambda^1$, $n = 2$.

r	$\mathcal{P}_r^-\Lambda^1$		$\mathcal{P}_r\Lambda^1$	
	Edge $[x_i, x_j]$	Triangle $[x_i, x_j, x_k]$	Edge $[x_i, x_j]$	Triangle $[x_i, x_j, x_k]$
1	ϕ_{ij}		$\lambda_i\, d\lambda_j$, $\lambda_j\, d\lambda_i$	
2	$\{\lambda_i, \lambda_j\}\phi_{ij}$	$\lambda_k\phi_{ij}$, $\lambda_j\phi_{ik}$	$\lambda_i^2\, d\lambda_j$, $\lambda_j^2\, d\lambda_i$ $\lambda_i\lambda_j d(\lambda_j - \lambda_i)$	$\lambda_i\lambda_j\, d\lambda_k$, $\lambda_i\lambda_k\, d\lambda_j$ $\lambda_j\lambda_k\, d\lambda_i$
3	$\{\lambda_i^2, \lambda_j^2, \lambda_i\lambda_j\}\phi_{ij}$	$\{\lambda_i, \lambda_j, \lambda_k\}\lambda_k\phi_{ij}$ $\{\lambda_i, \lambda_j, \lambda_k\}\lambda_j\phi_{ik}$	$\lambda_i^3\, d\lambda_j$, $\lambda_j^3 d\lambda_i$ $\lambda_i^2\lambda_j d(2\lambda_j - \lambda_i)$ $\lambda_i\lambda_j^2 d(\lambda_j - 2\lambda_i)$	$\{\lambda_i, \lambda_j, \lambda_k\}\lambda_i\lambda_j d\lambda_k$, $\{\lambda_i, \lambda_j, \lambda_k\}\lambda_i\lambda_k d\lambda_j$ $\{\lambda_j, \lambda_k\}\lambda_j\lambda_k d\lambda_i$

Table 9.2
Bases for the spaces $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r^-\Lambda^2$, $n = 3$.

r	$\mathcal{P}_r^-\Lambda^1$			$\mathcal{P}_r^-\Lambda^2$	
	Edge $[x_i, x_j]$	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	ϕ_{ij}			ϕ_{ijk}	
2	$\{\lambda_i, \lambda_j\}\phi_{ij}$	$\lambda_k\phi_{ij}$, $\lambda_j\phi_{ik}$		$\{\lambda_i, \lambda_j, \lambda_k\}\phi_{ijk}$	$\lambda_l\phi_{ijk}$, $\lambda_k\phi_{ijl}$ $\lambda_j\phi_{ikl}$
3	$\{\lambda_i^2, \lambda_j^2, \lambda_i\lambda_j\}\phi_{ij}$	$\{\lambda_i, \lambda_j, \lambda_k\}\lambda_k\phi_{ij}$ $\{\lambda_i, \lambda_j, \lambda_k\}\lambda_j\phi_{ik}$	$\lambda_k\lambda_l\phi_{ij}$ $\lambda_j\lambda_l\phi_{ik}$ $\lambda_j\lambda_k\phi_{il}$	$\{\lambda_i^2, \lambda_j^2, \lambda_k^2\}\phi_{ijk}$ $\{\lambda_i\lambda_j, \lambda_i\lambda_k, \lambda_j\lambda_k\}\phi_{ijk}$	$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_l\phi_{ijk}$ $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_k\phi_{ijl}$ $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_j\phi_{ikl}$

Table 9.3
Basis for the space $\mathcal{P}_r\Lambda^1$, $n = 3$.

r	Edge $[x_i, x_j]$	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	$\lambda_i d\lambda_j$, $\lambda_j d\lambda_i$		
2	$\lambda_i^2 d\lambda_j$, $\lambda_j^2 d\lambda_i$, $\lambda_i\lambda_j d(\lambda_j - \lambda_i)$	$\lambda_i\lambda_j d\lambda_k$, $\lambda_i\lambda_k d\lambda_j$, $\lambda_j\lambda_k d\lambda_i$	
3	$\lambda_i^3 d\lambda_j$, $\lambda_i^2\lambda_j d(2\lambda_j - \lambda_i)$	$\{\lambda_i, \lambda_j\}\lambda_i\lambda_j d\lambda_k$, $\lambda_i\lambda_j\lambda_k d(2\lambda_k - \lambda_i - \lambda_j)$	$\lambda_i\lambda_j\lambda_k d\lambda_l$, $\lambda_i\lambda_j\lambda_l d\lambda_k$
	$\lambda_j^3 d\lambda_i$, $\lambda_i\lambda_j^2 d(\lambda_j - 2\lambda_i)$	$\{\lambda_i, \lambda_k\}\lambda_i\lambda_k d\lambda_j$, $\lambda_i\lambda_j\lambda_k d(2\lambda_j - \lambda_i - \lambda_k)$	$\lambda_i\lambda_k\lambda_l d\lambda_j$, $\lambda_j\lambda_k\lambda_l d\lambda_i$
		$\{\lambda_j, \lambda_k\}\lambda_j\lambda_k d\lambda_i$	

Table 9.4
Basis for the space $\mathcal{P}_r\Lambda^2$, $n = 3$.

r	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	$\lambda_k d\lambda_i \wedge d\lambda_j$, $\lambda_j d\lambda_i \wedge d\lambda_k$, $\lambda_i d\lambda_j \wedge d\lambda_k$	
2	$\lambda_k^2 d\lambda_i \wedge d\lambda_j$, $\lambda_j\lambda_k d\lambda_i \wedge d(\lambda_k - \lambda_j)$	$\lambda_k\lambda_l d\lambda_i \wedge d\lambda_j$, $\lambda_j\lambda_l d\lambda_i \wedge d\lambda_k$
	$\lambda_j^2 d\lambda_i \wedge d\lambda_k$, $\lambda_i\lambda_j d(\lambda_j - \lambda_i) \wedge d\lambda_k$	$\lambda_j\lambda_k d\lambda_i \wedge d\lambda_l$, $\lambda_i\lambda_l d\lambda_j \wedge d\lambda_k$
	$\lambda_i^2 d\lambda_j \wedge d\lambda_k$, $\lambda_i\lambda_k d\lambda_j \wedge d(\lambda_k - \lambda_i)$	$\lambda_i\lambda_k d\lambda_j \wedge d\lambda_l$, $\lambda_i\lambda_j d\lambda_k \wedge d\lambda_l$
3	$\lambda_k^3 d\lambda_i \wedge d\lambda_j$, $\lambda_j^3 d\lambda_i \wedge d\lambda_k$, $\lambda_i^3 d\lambda_j \wedge d\lambda_k$	$\{\lambda_k, \lambda_l\}\lambda_k\lambda_l d\lambda_i \wedge d\lambda_j$
	$\lambda_j^2\lambda_k d\lambda_i \wedge d(2\lambda_k - \lambda_j)$, $\lambda_j\lambda_k^2 d\lambda_i \wedge d(\lambda_k - 2\lambda_j)$	$\{\lambda_j, \lambda_k, \lambda_l\}\lambda_j\lambda_l d\lambda_i \wedge d\lambda_k$
	$\lambda_i^2\lambda_j d(2\lambda_j - \lambda_i) \wedge d\lambda_k$, $\lambda_i^2\lambda_k d\lambda_j \wedge d(2\lambda_k - \lambda_i)$	$\{\lambda_j, \lambda_k, \lambda_l\}\lambda_j\lambda_k d\lambda_i \wedge d\lambda_l$
	$\lambda_i\lambda_j^2 d(\lambda_j - 2\lambda_i) \wedge d\lambda_k$, $\lambda_i\lambda_k^2 d\lambda_j \wedge d(\lambda_k - 2\lambda_i)$	$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_i\lambda_l d\lambda_j \wedge d\lambda_k$
	$\lambda_i\lambda_j\lambda_k d(2\lambda_j - \lambda_i - \lambda_k) \wedge d(2\lambda_k - \lambda_i - \lambda_j)$	$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_i\lambda_k d\lambda_j \wedge d\lambda_l$
		$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_i\lambda_j d\lambda_k \wedge d\lambda_l$

on simplices $T \in \mathscr{T}$ that do not contain f, while for T containing f, the spanning and basis forms are given by

$$\{(\lambda^T)^\alpha \phi_\sigma^T | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r - 1, \sigma \in \Sigma(0:k, 0:n), [\![\alpha, \sigma]\!] = \mathscr{I}(f)\}$$

and

$$\{(\lambda^T)^\alpha \phi_\sigma^T | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r - 1, \sigma \in \Sigma(0:k, 0:n), [\![\alpha, \sigma]\!] = \mathscr{I}(f), \\ \alpha_i = 0 \text{ if } i < \min [\![\sigma]\!]\},$$

respectively. Note that the spanning set is independent of the ordering of the vertices, while our choice of basis depends on the ordering of the vertices. Other choices of basis are possible as well, but there is no one canonical choice.

The same considerations give an explicit spanning set and basis for $\mathcal{P}_r\Lambda^k(\mathscr{T})$, based on Theorem 8.3, (4.6), Theorem 8.1, and part 2 of Theorem 6.1. The corresponding formulas for the spanning set and basis are:

$$\{(\lambda^T)^\alpha \psi_\sigma^{\alpha,f,T} | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n),$$

and

$$\{(\lambda^T)^\alpha \psi_\sigma^{\alpha,f,T} | \alpha \in \mathbb{N}_0^{0:n}, |\alpha| = r, \sigma \in \Sigma(1:k, 0:n), [\![\alpha, \sigma]\!] = \mathscr{I}(f), \\ \alpha_i = 0 \text{ if } i < \min[I(f) \setminus [\![\sigma]\!]]\},$$

respectively, where $\psi_\sigma^{\alpha,f,T}$ is defined by (8.2) and (8.3).

Bases for the spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ are summarized in Tables 9.1–9.4 for $n = 2, 3$, $0 < k < n$, $r = 1, 2, 3$. In the tables, we assume $i < j < k < l$, and recall that the Whitney forms ϕ_{ij} and ϕ_{ijk} are given by:

$$\phi_{ij} = \lambda_i d\lambda_j - \lambda_j d\lambda_i, \qquad \phi_{ijk} = \lambda_i d\lambda_j \wedge d\lambda_k - \lambda_j d\lambda_i \wedge d\lambda_k + \lambda_k d\lambda_i \wedge d\lambda_j.$$

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